SIEVES AND THE MINIMAL RAMIFICATION PROBLEM

LIOR BARY-SOROKER[1] AND TOMER M. SCHLANK[2]

[1] *Tel Aviv University, School of Mathematical Sciences, Israel*
(barylior@post.tau.ac.il)

[2] *The Hebrew University, Einstein Institute of Mathematics, Israel*
(tomer.schlank@mail.huji.ac.il)

Abstract   The minimal ramification problem may be considered as a quantitative version of the inverse Galois problem. For a nontrivial finite group G, let $m(G)$ be the minimal integer m for which there exists a G-Galois extension $N/\mathbb{Q}$ that is ramified at exactly m primes (including the infinite one). So, the problem is to compute or to bound $m(G)$.

In this paper, we bound the ramification of extensions $N/\mathbb{Q}$ obtained as a specialization of a branched covering $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$. This leads to novel upper bounds on $m(G)$, for finite groups G that are realizable as the Galois group of a branched covering. Some instances of our general results are:

$$1 \leqslant m(S_k) \leqslant 4 \quad \text{and} \quad n \leqslant m(S_k^n) \leqslant n + 4,$$

for all $n, k > 0$. Here S_k denotes the symmetric group on k letters, and S_k^n is the direct product of n copies of S_k. We also get the correct asymptotic of $m(G^n)$, as $n \to \infty$ for a certain class of groups G.

Our methods are based on sieve theory results, in particular on the Green–Tao–Ziegler theorem on prime values of linear forms in two variables, on the theory of specialization in arithmetic geometry, and on finite group theory.

Keywords: Hilbert's irreducibility theorem; inverse Galois problem; specializations; ramification

2010 *Mathematics subject classification*: 11-XX number theory; 12-XX eld theory and polynomials; 14-XX algebraic geometry

1. Introduction

This study is motivated by a problem in inverse Galois theory. We first describe the problem and the new results we obtain. Then we discuss the methods that needed to be developed which are of interest by themselves.

1.1. The minimal ramification problem

The *inverse Galois problem*, which is one of the central problems in Galois theory, asks whether every finite group G can be realized as the Galois group $G \cong \mathrm{Gal}(N/\mathbb{Q})$ of a Galois extension N of $\mathbb{Q}$. This problem is widely open. There are several different approaches to attack this problem that yield realizations of certain families of groups. The three main approaches found in the literature are:

(I) Specializations of geometrically irreducible branched coverings of $\mathbb{P}^1_{\mathbb{Q}}$ using Hilbert's irreducibility theorem; see [18, 26, 28].

(II) Class field theory; see [26, § 2.1.1] or [22, § 9.6.1].

(III) Galois representations; see [26, § 5] or [14, 29, 30] for some recent results.

The *minimal ramification problem* is a quantitative version of the inverse Galois problem: For a nontrivial finite group G, let $m(G)$ be the minimal integer m for which there exists a Galois extension $N/\mathbb{Q}$ that is ramified at exactly m primes such that $\mathrm{Gal}(N/\mathbb{Q}) \cong G$. If no such N exists, put $m(G) = \infty$. Here, we include the infinite prime, and the extension is unramified at infinity if and only if it is totally real.

The minimal ramification problem asks to calculate or to bound $m(G)$. Boston and Markin [2, Theorem 1.1] prove that if $G \neq 1$ is abelian, then $m(G) = d(G)$, where $d(G)$ is the minimal number of generators of G. It is convenient to put $d(1) = 1$, and then since $\mathbb{Q}$ has no unramified extensions, one gets the lower bound

$$m(G) \geqslant d(G^{ab}), \tag{1}$$

for any nontrivial G, where $G^{ab} = G/[G, G]$ is the abelianization of G. Boston and Markin [2] conjecture that equality actually holds:

Conjecture 1.1 (Boston–Markin). $m(G) = d(G^{ab})$ *for all nontrivial finite groups G.*

This conjecture has a lot of evidence in the literature mostly for solvable groups; for example, Jones and Roberts [13] build certain number fields ramified at one prime.

For solvable groups G, one can use Approach II, to obtain upper bounds on $m(G)$ and for some subclasses of solvable groups, the full conjecture, see [2, 15, 16, 20, 23, 24]. For example, Kisilevsky, Neftin, and Sonn [15] establish the conjecture for semi-abelian p-groups. However, to date, the conjecture is widely open for p-groups.

For linear groups, Approach III is very effective in giving bounds on ramification. For example, for every prime $p \geqslant 5$, Zywina [30] realizes $\mathrm{PSL}_2(\mathbb{F}_p)$ with ramification $\{2, p\}$. (This work is the first realization of these groups as Galois groups *for all p*.)

For the special case, $G = S_k$, the symmetric group, the literature contains both theoretical and computational bounds on $m(S_k)$ using Approach I: Plans [24, Remark 3.10] remarks that under the deep conjecture in number theory, the Schinzel Hypothesis H, $m(S_k) = 1$, as the conjecture predicts; however, an unconditional uniform bound for $m(S_k)$ does not seem to be in the literature. Malle and Roberts [19] construct S_k-extensions that are unramified outside $\{2, 3\}$ for some k's between 9 and 33.

An analogue of the minimal ramification problem for function fields; that is, when one replaces $\mathbb{Q}$ by $\mathbb{F}_q(T)$ is also treated in the literature; see e.g., [3, 12]. In this case, it closely relates to the Abhyankar conjecture about the finite quotients of the étale fundamental group of an affine curve over an algebraically closed field of positive characteristic that was resolved by Harbater [10] and Raynaud [25].

The methods used for non-solvable groups that were discussed above yield a specific extension that realizes the group with a few ramified primes. This is reflected by the fact that proving the conjecture for G and H does not yield a solution for $G \times H$. We propose

to study the conjecture, in the following asymptotical formulation:

$$m(G^n) = \begin{cases} d(G^{ab}) \cdot n, & G^{ab} \neq 1 \\ 1, & G^{ab} = 1. \end{cases} \tag{2}$$

To the best of our knowledge, there is no strong evidence for the case of perfect G and large n.

In this work we propose an attack on the minimal ramification problem using Approach I. Our method produces novel results for groups having a realization as the Galois group of a branched covering and it may be applied to direct products; hence in the asymptotic formulation (2) we get new strong upper bounds, and sometimes asymptotic formulas. The results are discussed in detail below. This attack necessitates developing the theory of specializations, and combining it with sieve theory results on prime values of polynomials, such as combinatorial sieve [9] and the Green–Tao–Ziegler theorem [8].

1.2. Statement of main results

Here we gather our results on the minimal ramification problem, the proofs of which appear in § 7. Recall that a group G is said to be *regularly realizable* if it is isomorphic to the Galois group of a geometrically irreducible branched covering $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ defined over $\mathbb{Q}$.

Theorem 1.2. *Let G be a regularly realizable group. Then*

$$m(G^n) = O(n), \quad n \to \infty. \tag{3}$$

The implied constant in (3) is given explicitly in terms of the underlying branched covering ϕ.

If G is not perfect, then by the simple observation (1) one gets that (3) gives the correct order of magnitude in the sense that

$$m(G^n) = \Theta(n).$$

Here, for non-negative functions f, g, we write $f(n) = \Theta(g(n))$ if and only if there are constants $0 < c \leqslant C$ such that $cg(n) \leqslant f(n) \leqslant Cg(n)$ for all n sufficiently large.

When the ramification locus is rational (i.e., the branch points are in $\mathbb{P}^1(\mathbb{Q})$), we have a more explicit bound:

Theorem 1.3. *Assume G is the Galois group of a geometrically irreducible branched covering $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ defined over $\mathbb{Q}$. Assume that the branch locus of ϕ consists of r rational points. Then*

$$m(G^n) \leqslant (r-1)n + O(1), \quad n \to \infty. \tag{4}$$

We get a better bound if our group G satisfies the so called $E(p)$-condition for some prime number p: *all* the nontrivial simple quotients of G are p-groups, but *none* of the quotients of the commutator $[G, G]$ are (see Definition 6.2 and the examples that follow; e.g., the symmetric group is $E(2)$).

Theorem 1.4. *Let G, ϕ, and r be as in Theorem 1.3. Assume G is $E(p)$ and that $d(G^{ab}) \leqslant r - 2$. Then*

$$d(G^{ab}) \cdot n \leqslant m(G^n) \leqslant (r-2)n + O(1), \quad n \to \infty. \tag{5}$$

In the special case when $G = S_k$, which is of particular interest, we can find ϕ with $r = 3$ so that (1) and (5) give $n \leqslant m(S_k^n) \leqslant n + O(1)$. With more work we can obtain a sharper result:

Theorem 1.5. *We have*

$$n \leqslant m(S_k^n) \leqslant n + 4, \quad \forall n \geqslant 1, \ k > 0 \tag{6}$$

and

$$m(S_k) \leqslant 4, \quad \forall m > 0. \tag{7}$$

Thus we immediately obtain the following

Corollary 1.6. $m(S_k)$ *is bounded.*

We emphasize that in (6) and (7) the infinite prime is ramified; that is to say, the minimal number of prime numbers that ramify in S_k^n and S_k extensions is at most $n + 3$ and 3, respectively.

Our bounds in (6) and (7) are independent of k and are unconditional. This comes in contrast to the hitherto known results [24] that were conditional on the Schinzel Hypothesis H and restricted to $n = 1$.

In general, constructing branched covering $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ with specific Galois group G is notoriously difficult. The classical method of rigidity, reduces this problem to the group theoretical problem of finding a *rigid* tuple; see § 6.1 or the books [18, 26, 28].

Theorem 1.7. *Assume G has a rational rigid r-tuple. Then*

$$m(G) \leqslant r + \#(\mathrm{Prms}(|G|) \cup \{p \leqslant r\}). \tag{8}$$

Here $\mathrm{Prms}(|G|)$ denotes the set of prime divisors of the order of G.

If in addition G satisfies the $E(p)$-condition, then $d((G^n)^{ab}) = d(G^{ab})n$ and we obtain a sharp asymptotic formula:

Theorem 1.8. *Let G be an $E(p)$ group that has a rational rigid r-tuple. Then*

$$m(G^n) = d(G^{ab}) \cdot n + O\left(\frac{n}{\log(n)}\right). \tag{9}$$

Finally we remark that the methods for the results above work also for general direct products of groups and we have restricted the discussion to direct powers merely for simplicity of presentation. For example, the same proof of (6) gives that

$$m\left(\prod_{i=1}^n S_{m_i}\right) \leqslant n + 4.$$

1.3. Methods

We always write elements of $\mathbb{P}^1(\mathbb{Q})$ as pairs $[a : b]$ with $a, b \in \mathbb{Z}$ relatively prime. This presentation is unique up to a sign. For us a prime p denotes either a prime number or the infinite prime of $\mathbb{Q}$. The completion at p is denoted by $\mathbb{Q}_p$, so in particular, $\mathbb{Q}_\infty = \mathbb{R}$. Every finite set of primes S defines the S-adic topology on $\mathbb{P}^1(\mathbb{Q})$ induced by the diagonal embedding $\mathbb{P}^1(\mathbb{Q}) \to \prod_{p \in S} \mathbb{P}^1(\mathbb{Q}_p)$. For a finite set of primes S that contains ∞ and for an integer $n \in \mathbb{Z}$ we denote

$$\mathrm{Prms}_S(n) = \{p : p \mid n\} \smallsetminus S. \tag{10}$$

The following function plays a key role in the investigation.

Definition 1.9. Let $D_1, \ldots, D_r \in \mathbb{Z}[t, s]$ be non-associate irreducible homogeneous polynomials and $D = \prod_i D_i$. We defined $B(D_1, \ldots, D_r)$ to be the minimal positive integer B for which there exists a finite set of primes $S_0 = S_0(B)$ that contains ∞ such that for every finite set of primes $S_0 \subseteq S$ and a nonempty S-adic open set $V_S \subseteq \mathbb{P}^1(\mathbb{Q})$ there exists $[a : b] \in V_S$ such that

$$\#\mathrm{Prms}_S(D(a, b)) \leqslant B.$$

It follows that there exist infinitely many such $[a : b]$ in each V_S.

For an r-tuple $\mathbf{d} = (d_1, \ldots, d_r)$ of positive integers, we let

$$B(\mathbf{d}) = \max_{(D_1, \ldots, D_r)} B(D_1, \ldots, D_r), \tag{11}$$

where $(D_1, \ldots, D_r)$ runs over all non-associate irreducible homogeneous polynomials of degrees $\deg D_i = d_i$.

It is far from being obvious that $B(\mathbf{d})$ is finite. However sieve methods may be used to derive effective bounds in terms of r and $d = \sum_i d_i$.

From [9, Theorem 10.11] the general bound

$$B(\mathbf{d}) \leqslant d - 1 + r \sum_{j=1}^r \frac{1}{j} + r \log \left(\frac{2d}{r} + \frac{1}{r+1} \right) \tag{12}$$

may be derived. Schinzel Hypothesis H on prime values of polynomials implies

$$B(\mathbf{d}) \leqslant r. \tag{13}$$

When all $d_i = 1$, the Green–Tao–Ziegler theorem [8] achieves this bound:

$$B(1, \ldots, 1) \leqslant r. \tag{14}$$

The formal derivations of all of these results appear in §4.

Another key notion is the universally ramified primes: Let $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ be a geometrically irreducible branched covering. For each point $[a : b] \in \mathbb{P}^1(\mathbb{Q})$, we let $A^{\phi}_{[a:b]}$ be the specialized algebra at $[a : b]$ which is defined by

$$\phi^{-1}([a : b]) = \mathrm{Spec}(A^{\phi}_{[a:b]}).$$

Note that $A_{[a:b]}^{\phi}$ is a finite $\mathbb{Q}$-algebra of degree $[A_{[a:b]}^{\phi} : \mathbb{Q}] = \deg \phi$ and it is étale resp. a field if and only if $[a : b]$ is not a branch point of ϕ resp. $\phi^{-1}([a : b])$ is $\mathbb{Q}$-irreducible. The set of *universally ramified primes* is defined as

$$U = U(\phi) = \bigcap_{[a:b] \in \mathbb{P}^1(\mathbb{Q})} \mathrm{Ram}(A_{[a:b]}^{\phi}/\mathbb{Q}),$$

where for a finite $\mathbb{Q}$-algebra A we let $\mathrm{Ram}(A/\mathbb{Q}) = \{p \mid A \otimes \mathbb{Q}_p^{ur} \not\cong (\mathbb{Q}_p^{ur})^n\}$. Here $\mathbb{Q}_p^{ur}$ is the maximal unramified extension of $\mathbb{Q}_p$, where for the infinite prime we set $\mathbb{Q}_\infty^{ur} = \mathbb{R}$, i.e., we consider $\mathbb{C}/\mathbb{R}$ as ramified. We also write

$$\mathrm{Ram}_S(A/\mathbb{Q}) = \mathrm{Ram}(A/\mathbb{Q}) \smallsetminus S,$$

where S is a finite set of primes. We have that $p \notin \mathrm{Ram}(A/\mathbb{Q})$ if and only if A is isomorphic to a product of number fields that are unramified at p. Thus U is the set of the primes that ramify under every specialization. In practice it is easy to bound U from above, simply by taking some arbitrarily chosen points $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ and calculating the greatest common divisor of the discriminants of the specialized algebras $A_{[a:b]}^{\phi}/\mathbb{Q}$. However, to calculate U exactly, may be difficult.

We denote by $\mathrm{Branch}(\phi) \subset \mathbb{P}_{\mathbb{Q}}^1$ the closed subscheme of branch points of ϕ. So $\mathrm{Branch}(\phi)$ is the zero locus of some nonzero homogeneous polynomial $D(t, s) \in \mathbb{Z}[t, s]$.

Definition 1.10. Let G be a finite group, U a finite set of primes of $\mathbb{Q}$ and $\mathbf{d} = (d_1, \ldots, d_r)$ a tuple of positive integers. We say that G has $(U; \mathbf{d})$ realization if there exists a geometrically irreducible branched covering $\phi \colon C \to \mathbb{P}_{\mathbb{Q}}^1$ such that

- $\mathbb{Q}(C)/\mathbb{Q}(\mathbb{P}^1)$ is Galois with Galois group G,
- $U(\phi) \subseteq U$,
- $\mathrm{Branch}(\phi) = \{(D_1), \ldots, (D_r)\}$ with $D_i(t, s) \in \mathbb{Z}[t, s]$ homogeneous of degree d_i.

The key technical tool in proving the results mentioned above is a bound on $m(G)$ in terms of U and $B(\mathbf{d})$:

Proposition 1.11. *Let G be a finite group that has a $(U; \mathbf{d})$ realization and let $L/\mathbb{Q}$ be a finite extension. Then there exists a Galois extension $N/\mathbb{Q}$ with Galois group G such that $N \cap L = \mathbb{Q}$ and $\#\mathrm{Ram}_U(N/\mathbb{Q}) \leqslant B(\mathbf{d})$, where $B(\mathbf{d})$ is defined in (11). In particular,*

$$m(G) \leqslant B(\mathbf{d}) + \#U.$$

The proof of Proposition 1.11 is based on the existence of many good specializations. To formally state this we introduce the notion of thin sets [26] in the sense of Serre: A thin set of type 1 in $\mathbb{P}^1(\mathbb{Q})$ is a finite set. A thin set of type 2 is $\phi(C(\mathbb{Q}))$, where $\phi \colon C \to \mathbb{P}_{\mathbb{Q}}^1$ is an irreducible branched covering of degree $\geqslant 2$. A **thin** set in $\mathbb{P}^1(\mathbb{Q})$ is a set contained in a finite union of thin sets of types 1 and 2. The Hilbert irreducibility theorem is the statement that $\mathbb{P}^1(\mathbb{Q})$ is not thin.

Theorem 1.12. *Let $\phi\colon C \to \mathbb{P}^1_{\mathbb{Q}}$ be a geometrically irreducible branched covering. Let $U = U(\phi)$ be the set of universally ramified primes and $\mathrm{Branch}(\phi) = \{(D_1), \ldots, (D_r)\} \subseteq \mathbb{P}^1_{\mathbb{Q}}$ be the branch locus of ϕ, where $D_i \in \mathbb{Z}[t, s]$ are non-associate homogeneous irreducible polynomials. Then the set Ω of all $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ such that $\#\mathrm{Ram}_U(A^{\phi}_{[a:b]}/\mathbb{Q}) \leqslant B(D_1, \ldots, D_r)$ is not thin.*

Theorem 1.12 follows from a strong version of Hilbert's irreducibility theorem and the following result on ramification under specialization.

Theorem 1.13. *Under the notation of Theorem 1.12 and with $D = D_1 \cdots D_r$ there exists a finite set of primes T_ϕ containing $U \cup \{\infty\}$ such that for every finite set of primes S with $T_\phi \subseteq S$ there exists a nonempty S-adic open set V_S of $\mathbb{P}^1(\mathbb{Q})$ satisfying the following property: For every $[a : b] \in V_S$ we have*

(1) $\mathrm{Ram}(A^{\phi}_{[a:b]}/\mathbb{Q}) \cap S = U$.

(2) $\mathrm{Ram}_S(A^{\phi}_{[a:b]}/\mathbb{Q}) \subseteq \mathrm{Prms}_S(D(a, b))$.

Outline of The Paper

In §2 we discuss the universally ramified primes and obtain some auxiliary results that are then used in §3 to prove Proposition 1.11 and Theorems 1.12 and 1.13.

Then we develop the necessary machinery in order to apply Proposition 1.11 to the minimal ramification problem: In §4, we establish the above bounds on $B(\mathbf{d})$ and in §§5 and 6, we construct covers ϕ with small $U(\phi)$.

Finally, in §7, we prove the results stated in §1.2.

2. Universally ramified primes

2.1. Preliminaries in commutative algebra

We start with some notation: Recall that by $[a : b] \in \mathbb{P}^1(\mathbb{Q})$, we always mean that a and b are co-prime integers and that this uniquely defines the pair a, b, up to a sign. Given a homogeneous ideal $I \lhd \mathbb{Z}[t, s]$ and $[a : b] \in \mathbb{P}^1(\mathbb{Q})$, we denote

$$I([a : b]) = \{f(a, b) : f \in I\} \lhd \mathbb{Z}$$

which is an ideal in $\mathbb{Z}$. For a prime number p we denote by $v_p(n)$ the p-adic valuation of n. We extend the functions $v_p(\bullet)$ and $\mathrm{Prms}_S(\bullet)$ (defined in (10)) from the integers to ideals in $\mathbb{Z}$ in the obvious way: If $J = (n) \lhd \mathbb{Z}$, then

$$\mathrm{Prms}_S(J) = \mathrm{Prms}_S(n),$$

and

$$v_p(J) = v_p(n).$$

For a prime p of $\mathbb{Q}$, recall that $\mathbb{Q}^{ur}_p$ denotes the maximal unramified extension of $\mathbb{Q}_p$ with $\mathbb{Q}^{ur}_\infty = \mathbb{R}$. If p is a prime number, then $\mathbb{Z}^{ur}_p$ denotes the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}^{ur}_p$; i.e., the subring of elements with non-negative valuation (w.r.t. the unique lifting of v_p to $\mathbb{Q}^{ur}_p$).

The following technical lemma is needed in the proof of Theorem 1.13.

Lemma 2.1. *Let $I \lhd \mathbb{Z}[t, s]$ be a nonzero homogeneous ideal and let $D(t, s) \in \mathbb{Z}[t, s]$ be a homogeneous polynomial such that $D\mathbb{Q}[t, s] = I\mathbb{Q}[t, s]$. Then, there exists a finite set of primes S that contains the infinite prime such that for every $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ and for every $p \notin S$ we have $v_p(I(a, b)) = v_p(D(a, b))$.*

Proof. The ideal I is generated by finitely many homogeneous polynomials, say $I = \sum_{i=1}^{k} g_i \mathbb{Z}[t, s]$. Thus $D\mathbb{Q}[t, s] = \sum_{i=1}^{k} g_i \mathbb{Q}[t, s]$, which implies that there exist homogeneous polynomials

$$c_1(t, s), \ldots, c_k(t, s), d_1(t, s), \ldots, d_k(t, s) \in \mathbb{Q}[t, s]$$

such that $g_i = c_i D$, $i = 1, \ldots, k$ and $D = \sum_{i=1}^{k} d_i g_i$. Let S' be the set of primes dividing the denominators of the coefficients of $c_1, \ldots, c_k, d_1, \ldots, d_k$. Let $S := S' \cup \{\infty\}$. Then, for $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ and $p \notin S$, we have that $c_i, d_i \in \mathbb{Z}_p[t, s]$ for all $1 \leqslant i \leqslant k$; thus $I\mathbb{Z}_p[t, s] = D\mathbb{Z}_p[t, s]$. For every $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ we thus have $\mathbb{Z}_p I(a, b) = \mathbb{Z}_p(D(a, b))$, hence the desired assertion. $\qquad\square$

2.2. Unramified specializations

The goal of this section is to give two results about unramified specializations. The first shows that there are specializations which are unramified in a finite set of non-universally ramified primes.

Lemma 2.2. *Let $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ be a branched covering, and let S be a finite set of primes. Then there exists a nonempty S-adic open set V_S of $\mathbb{P}^1(\mathbb{Q})$ such that for every $\zeta \in V_S$ we have $\mathrm{Ram}(A^\phi_\zeta/\mathbb{Q}) \cap S \subseteq U(\phi)$.*

Proof. By the Chinese Remainder Theorem, if $S_1 \cap S_2 = \emptyset$ and if V_{S_i} is a nonempty open S_i-adic open set, $i = 1, 2$, then $V_{S_1} \cap V_{S_2}$ is a nonempty $S_1 \cup S_2$-adic set. Thus it suffices to consider the case where $S = \{p\}$; i.e., S contains only one prime.

If $p \in U(\phi)$, then the assertion is trivial. Otherwise, there exists $\zeta \in \mathbb{P}^1(\mathbb{Q})$ such that A^ϕ_ζ is unramified at p. In particular, A^ϕ_ζ is reduced, so $\zeta \notin \mathrm{Branch}(\phi)$. Consider the map

$$\phi_{\mathbb{Q}^{ur}_p} \colon C(\mathbb{Q}^{ur}_p) \to \mathbb{P}^1(\mathbb{Q}^{ur}_p).$$

Since ζ is not a branch point, $\#\phi^{-1}_{\mathbb{Q}^{ur}_p}(\zeta)(\mathbb{Q}^{ur}_p) = \deg \phi$, and so as $\mathbb{Q}^{ur}_p$ is Henselian, by the inverse function theorem (see e.g. [5, Corollary 9.5]) there exists some p-adic neighborhood V of ζ such that for every $\zeta' \in V$

$$\#\phi^{-1}_{\mathbb{Q}^{ur}_p}(\zeta')(\mathbb{Q}^{ur}_p) = \#\phi^{-1}_{\mathbb{Q}^{ur}_p}(\zeta)(\mathbb{Q}^{ur}_p) = \deg \phi.$$

The proof is done with $V_S = V \cap \mathbb{P}^1(\mathbb{Q})$. $\qquad\square$

For the second result we need some preparations.

Lemma 2.3. *Let p be a finite prime and let*

$$\phi \colon F \to \mathrm{Spec}\,\mathbb{Z}^{ur}_p$$

be an étale map of degree n. Then $F \cong \mathrm{Spec}(\mathbb{Z}^{ur}_p)^n$.

Proof. The ring $\mathbb{Z}_p^{ur}$ is a Henselian ring with algebraically closed residue field. Thus the assertion follows from [21, Proposition I.4.4]. □

Let $\phi \colon C \to \mathbb{P}_{\mathbb{Q}}^1$ be a branched covering. The branch locus $\mathsf{Branch}(\phi) \subset \mathbb{P}_{\mathbb{Q}}^1$ is a closed subscheme of dimension 0, so $\mathsf{Branch}(\phi)$ is the zero locus of some nonzero homogeneous polynomial $D(t, s) \in \mathbb{Z}[t, s]$.

The following fact on the closeness of the branch locus over $\mathbb{Z}$ is well known. We include a proof of it for the reader's convenient.

Lemma 2.4. *Let $\phi_{\mathbb{Z}} \colon \mathfrak{C} \to \mathbb{P}_{\mathbb{Z}}^1$ be the normalization of $\mathbb{P}_{\mathbb{Z}}^1$ in the generic point of C via ϕ. Then, the branch locus $\mathfrak{R}_\phi \subset \mathbb{P}_{\mathbb{Z}}^1$ of $\phi_{\mathbb{Z}}$ is closed.*

Proof. Since $\mathbb{P}_{\mathbb{Z}}^1$ is Nagata [27, Tag 035B], hence universally Japanese [27, Tag 033Z], and since $\mathbb{P}_{\mathbb{Z}}^1$ is integral, it is Japanese which means by definition that $\phi_{\mathbb{Z}}$ is finite. Thus, by [27, Tag 024P], the ramification locus consists of all x at which the stalk of the coherent sheaf $\Omega_{\mathfrak{C}/\mathbb{P}_{\mathbb{Z}}^1}$ is nontrivial. The sheaf $\Omega_{\mathfrak{C}/\mathbb{P}_{\mathbb{Z}}^1}$ is locally of finite type by [27, Tag 01V2] hence [27, Tag 01BA] implies that the ramification locus is closed. Thus we conclude that the branch locus $\mathfrak{R}_\phi$, which is the image of the ramification locus under $\phi_{\mathbb{Z}}$ is closed in $\mathbb{P}_{\mathbb{Z}}^1$ as finite morphisms are closed. □

Away from $\mathfrak{R}_\phi$, the morphism $\phi_{\mathbb{Z}}$ is étale. We denote by $d_{\phi, \mathbb{Z}} \lhd \mathbb{Z}[t, s]$ the homogeneous ideal that defines $\mathfrak{R}_\phi$. We have:

$$d_{\phi, \mathbb{Z}} \mathbb{Q}[t, s] = D(t, s) \mathbb{Q}[t, s]. \tag{15}$$

Now we are ready for the second result of the section:

Proposition 2.5. *Let $\phi \colon C \to \mathbb{P}_{\mathbb{Q}}^1$ be a branched covering, let $[a : b] \in \mathbb{P}^1(\mathbb{Q})$, and let p be a prime number. Assume that $v_p(d_{\phi, \mathbb{Z}}(a, b)) = 0$. Then $A_{[a:b]}^\phi$ is unramified at p.*

Proof. We identify $\mathbb{P}_{\mathbb{F}_p}^1 = \mathbb{P}_{\mathbb{Z}}^1 \times_{\mathbb{Z}} \mathbb{F}_p$ and $\mathbb{P}_{\mathbb{Q}}^1 = \mathbb{P}_{\mathbb{Z}}^1 \times_{\mathbb{Z}} \mathbb{Q}$ as subschemes of $\mathbb{P}_{\mathbb{Z}}^1$: the *special* and the *generic* fibers, respectively. Write $\zeta = [a : b] \in \mathbb{P}_{\mathbb{Q}}^1$ and $\zeta_p = [\bar{a} : \bar{b}] \in \mathbb{P}_{\mathbb{F}_p}^1$, where the overline denotes reduction modulo p. By assumption, there exists $f \in d_{\phi, \mathbb{Z}}$ such that $f(a, b) \not\equiv 0 \mod p$, which implies that

$$\zeta_p \notin \mathfrak{R}_\phi.$$

So $\phi_{\mathbb{Z}}$ is étale at ζ_p. We base change with $\mathbb{Z}_p^{ur}$ to get the following diagram:

Since $\phi_{\mathbb{Z}_p^{ur}}$ is étale in a neighborhood of ζ_p, the fiber F_ζ is étale over $\mathrm{Spec}\mathbb{Z}_p^{ur}$. By Lemma 2.3,

$$F_\zeta \cong \mathrm{Spec}(\mathbb{Z}_p^{ur})^{\deg \phi};$$

so

$$\operatorname{Spec}(A_\zeta^\phi \otimes_{\mathbb{Z}} \mathbb{Q}_p^{ur}) = F_\zeta \times_{\operatorname{Spec}(\mathbb{Z}_p^{ur})} \operatorname{Spec}(\mathbb{Q}_p^{ur}) = \operatorname{Spec}(\mathbb{Q}_p^{ur})^{\deg \phi}.$$

This implies that A_ζ^ϕ is unramified at p, as needed. □

2.3. Universally and Vertically Ramified Primes

We keep the notation of the previous section. We say that a prime number p is *vertically ramified* in ϕ if

$$\mathbb{P}_{\mathbb{F}_p}^1 \subset \mathfrak{R}_\phi \subset \mathbb{P}_{\mathbb{Z}}^1$$

(under the natural embedding induced from $\mathbb{Z} \to \mathbb{F}_p$). This notion is consistent with the one in [17]. We denote the set of vertically ramified primes by $V(\phi)$. We use that the property of being unramified is preserved under fiber products. In particular, for $\phi_i \colon C_i \to \mathbb{P}^1(\mathbb{Q})$, $i = 1, 2$, we have

$$V(\phi_1 \times_{\mathbb{P}^1} \phi_2) = V(\phi_1) \cup V(\phi_2). \tag{16}$$

Let $\operatorname{Branch}(\phi) = \{(D_1), \ldots, (D_r)\}$ and let $B(\phi)$ be the set of prime numbers p for which for every $[a : b] \in \mathbb{P}^1(\mathbb{F}_p)$ there is i with $D_i(a, b) = 0$ (in $\mathbb{F}_p$). Since the D_i are not the zero polynomials modulo p, for all p, we get that

$$p \in B(\phi) \Rightarrow p + 1 \leqslant \deg(\operatorname{Branch}(\phi)) := \sum_{i=1}^r \deg D_i. \tag{17}$$

In general, we conclude

$$U_\infty(\phi) \subseteq V(\phi) \cup B(\phi), \tag{18}$$

see [17, Specialization Inertia Theorem (1)].

3. Proofs for the methods

In this section, we prove Proposition 1.11 and Theorems 1.12 and 1.13.

Proof of Theorem 1.13. Recall that we are given a geometrically irreducible branched covering $\phi \colon C \to \mathbb{P}_{\mathbb{Q}}^1$ with $U = U(\phi)$ the set of universally ramified primes and with branch locus $\operatorname{Branch}(\phi) = \{(D_1), \ldots, (D_r)\}$. Put $D = D_1 \cdots D_r$. We have to find a finite set of primes T_ϕ that contains $U \cup \{\infty\}$ such that for every finite set of primes S containing T_ϕ there exists nonempty S-adic open set V_S with properties 1 and 2 of the theorem.

Let $I = d_{\phi, \mathbb{Z}} \lhd \mathbb{Z}[t, s]$. Since $D\mathbb{Q}[t, s] = I\mathbb{Q}[t, s]$, by Lemma 2.1 there exists a finite set of primes S_1 such that for all $p \notin S_1$ and for all $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ we have

$$v_p(I(a, b)) = v_p(D(a, b)). \tag{19}$$

Let $T_\phi = S_1 \cup U \cup \{\infty\}$ and let S be a finite set of primes containing T_ϕ. By Lemma 2.2, there exists a nonempty S-adic open set V_S of $\mathbb{P}^1(\mathbb{Q})$ such that for all $\zeta \in V_S$ we have $\operatorname{Ram}(A_\zeta^\phi/\mathbb{Q}) \cap S \subseteq U$, so $\operatorname{Ram}(A_\zeta^\phi/\mathbb{Q}) \cap S = U$.

Let $\zeta = [a : b] \in V_S$, let $p \notin S$, hence $p \notin S_1$, and assume that $p \nmid D(a, b)$. By (19), we have $v_p(I(a, b)) = v_p(D(a, b)) = 0$, so p is prime to $I(a, b)$. This implies, by Proposition 2.5, that $p \notin \operatorname{Ram}(A_\zeta^\phi/\mathbb{Q})$. □

In order to prove Theorem 1.12 we need the following strong variant of Hilbert's irreducibility theorem that gives a nonempty S'-adic open set in the complement of any thin set:

Lemma 3.1. *Let Z be a thin set in $\mathbb{P}^1(\mathbb{Q})$. For every finite set of primes S there exist a finite set of primes S' and a nonempty S'-adic open set $V_{S'}$ such that $S \cap S' = \emptyset$ and $Z \cap V_{S'} = \emptyset$.*

Proof. By [26, Theorem 3.5.3] there exists S' with $S' \cap S = \emptyset$ such that Z is not S'-adic dense in $\prod_{p \in S'} \mathbb{P}^1(\mathbb{Q}_p)$. So there exists an open subset U of $\prod_{p \in S'} \mathbb{P}^1(\mathbb{Q}_p)$ with $Z \cap U = \emptyset$. Since $\mathbb{P}^1_\mathbb{Q}$ has the weak approximation property (p. 30 in *loc.cit.*) $V_{S'} := U \cap \mathbb{P}^1(\mathbb{Q}) \neq \emptyset$, as needed. $\square$

Proof of Theorem 1.12. Recall that Ω is the set of $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ such that $\#\mathrm{Ram}_U(A^\phi_{[a:b]}/\mathbb{Q}) \leqslant B(D_1, \ldots, D_r)$ and it suffices to show that Ω is not contained in any thin set Z. Put $B = B(D_1, \ldots, D_r)$, and let T_ϕ be as in Theorem 1.13. Let $S_0 = S_0(B)$ be the set of primes from Definition 1.9. By Theorem 1.13, for $S_1 = T_\phi \cup S_0$, there exists a nonempty S_1-adic open set V_{S_1} such that for every $\zeta = [a : b] \in V_{S_1}$ we have

$$\#\mathrm{Ram}_U(A^\phi_\zeta) = \#\mathrm{Ram}_{S_1}(A^\phi_\zeta) \leqslant \#\mathrm{Prms}_{S_1}(D(a, b)). \tag{20}$$

By Lemma 3.1 there exists a finite set of primes S'_1 such that $S_1 \cap S'_1 = \emptyset$ and there exists a nonempty S'_1-adic open set $V_{S'_1}$ such that $V_{S'_1} \cap Z = \emptyset$. Thus $V_S = V_{S_1} \cap V_{S'_1}$ is an S-adic open set, for $S = S_1 \cup S'_1$ which is nonempty by the Chinese Remainder Theorem and that satisfies

$$V_S \cap Z = \emptyset. \tag{21}$$

Since $S_0 \subseteq S$, by Definition 1.9 and by (20), there exists $\zeta \in V_S$ such that

$$\#\mathrm{Ram}_U(A^\phi_\zeta) \leqslant B.$$

This together with (21) implies that $\zeta \in \Omega \smallsetminus Z$, so $\Omega \nsubseteq Z$. $\square$

Now we are ready to the

Proof of Proposition 1.11. Let $\phi \colon C \to \mathbb{P}^1(\mathbb{Q})$ be a branched covering from Definition 1.10 and let Z be the set of $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ such that $A^\phi_{[a:b]} \otimes L$ is not a field. Then Z is thin (see [4, Corollary 12.2.3] and note that a subset of $\mathbb{Q}$ is thin if and only if its complement contains a Hilbert set by Lemma 13.1.2 in *loc.cit.*). By Theorem 1.12 and by (11), there exists $[a : b] \in \mathbb{P}^1(\mathbb{Q}) \smallsetminus Z$ such that for $N = A^\phi_{[a:b]}$ we have

$$\#\mathrm{Ram}_U(N/\mathbb{Q}) \leqslant B(D_1, \ldots, D_r) \leqslant B(\mathbf{d}).$$

As $N \otimes L$ is a field, it follows that N is a field that is linearly disjoint from L, and so $N \cap L = \mathbb{Q}$. Clearly $N/\mathbb{Q}$ is Galois with Galois group G. $\square$

Remark 3.2. The proof above allows us to choose the primes in $\mathrm{Ram}_U(N/\mathbb{Q})$ to be disjoint from any chosen finite set (indeed, the primes one gets from sieve methods are big, since we sieve out the small primes). In particular, we can make the ramification at the primes

in $\mathrm{Ram}_U(N/\mathbb{Q})$ to be tame by avoiding the primes smaller than $\deg(\phi)$. Our analysis does not give any control on the tameness of the ramification of the primes in U. However, in some cases this suffices, for example, if our branched covering ϕ is Galois of degree $|G|$ and all the primes in $U(\phi)$ do not divide $|G|$, then our specialized extensions are tame. In our construction of S_n^k, this happens, because the only universally ramified prime is the infinite one.

Similarly, if $\infty \notin U(\phi)$, then the extension we get in the theorem can be made totally real.

4. Prime values of polynomials

The goal of this section is to formally deduce the bounds (12) and (14) from sieve theoretical results and (13) conditionally on Schinzel Hypothesis H. These bounds are needed in order to apply Proposition 1.11 to the minimal ramification problem.

4.1. Local obstructions

Since many of the results in this theory are stated in the literature for univariate polynomials we first deal with those, and then move to bivariate homogeneous polynomials.

We say that $f(x) \in \mathbb{Z}[x]$ has a *local obstruction at p* if p divides $f(n)$ for all $n \in \mathbb{Z}$. We denote the set of primes at which there is a local obstruction by O_f.

Lemma 4.1. *If f is primitive (i.e., the greatest common divisor of its coefficients is 1), then $p \leqslant \deg f$ for all $p \in O_f$.*

Proof. By assumption $f \mod p \in \mathbb{F}_p[x]$ is not the zero polynomial, hence has at most $\deg f$ roots modulo p. $\qquad \square$

Definition 4.2. Let $d_1, \ldots, d_r$ be positive integers. Define

$$B_0 = B_0(d_1, \ldots, d_r)$$

to be the minimum positive integer B_0 such that for every $f = f_1 \cdots f_r$, with $f_i(x) \in \mathbb{Z}[x]$ irreducible of degree d_i with $O_f = \emptyset$ there exist infinitely many $n > 0$ such that $\#\mathrm{Prms}(f(n)) \leqslant B_0$.

Sieve methods are effective in bounding B_0 in terms of r and $d = \sum_{i=1}^r d_i$: By the beta-sieve, [9, Theorem 10.11] we have

$$B_0(d_1, \ldots, d_r) \leqslant b \tag{22}$$

for every

$$b > d - 1 + r \sum_{j=1}^r \frac{1}{j} + r \log\left(\frac{2d}{r} + \frac{1}{r+1}\right).$$

Schinzel Hypothesis H is the conjecture that says that if $f_1, \ldots, f_r$ are non-associate irreducible polynomials with $O_{f_1 \cdots f_r} = \emptyset$, then there are infinitely many n with

$f_1(n), \ldots, f_r(n)$ being simultaneously prime. In particular, it implies that

$$B_0(d_1, \ldots, d_r) \leqslant r.$$

(Note that one cannot do better.) Hence to obtain (12) and (13) it suffices to prove that

$$B(\mathbf{d}) \leqslant B_0(\mathbf{d}), \tag{23}$$

which we now pursue. First we remove the restriction of having no local obstructions:

Lemma 4.3. *Let* $f_1, \ldots, f_r \in \mathbb{Z}[x]$ *be irreducible polynomials of positive leading coefficients and of respective degrees* $d_1, \ldots, d_r$, $f = f_1 \cdots f_r$, *and* S *a finite set of primes such that* f *has no local obstructions outside of* S. *Then, there exist infinitely many* n *such that* $\#\mathrm{Prms}_S(f(n)) \leqslant B_0(d_1, \ldots, d_r)$.

Proof. For each $p \in S$ let α_p be the maximal non-negative integer such that the function $n \mapsto f(n) \mod p^{\alpha_p}$ is the zero function. Put $N = \prod_p p^{\alpha_p}$ and choose an integer a_p such that $f(a_p) \not\equiv 0 \mod p^{\alpha_p+1}$. By the Chinese Remainder Theorem, we have an integer a with $a \equiv a_p \pmod{p^{\alpha_p+1}}$ for all $p \in S$ and let $g(y) = \frac{f(Ny+a)}{N}$.

We claim that $g(y)$ is an integral polynomial with no local obstructions. Indeed, since $(x-a)$ divides $f(x) - f(a)$ in $\mathbb{Z}[x]$ we get, by substitution $x = Ny + a$, that Ny divides $f(Ny+a) - f(a)$ in $\mathbb{Z}[y]$. Since $N \mid f(a)$, N divides the coefficients of $f(Ny+a) = (f(Ny+a) - f(a)) + f(a)$, so $g(y) \in \mathbb{Z}[y]$. To show that $g(y)$ has no local obstruction at a prime p, we note that if $p \in S$, then $g(0) \not\equiv 0 \mod p$ and if $p \notin S$, then f does not have local obstruction at p, hence there exists m with $f(m) \not\equiv 0 \pmod{p}$, and since $p \nmid N$, there is n such that $m \equiv Nn + a \pmod{p}$, hence $g(n) \not\equiv 0 \pmod{p}$.

Next we apply the definition of $B_0 = B_0(d_1, \ldots, d_r)$ to g (which has the same factorization type as f) and the trivial observation that $\mathrm{Prms}_S(f(Nn+a)) = \mathrm{Prms}_S(g(n))$ to conclude that for infinitely many n we have

$$\#\mathrm{Prms}_S(f(Nn+a)) \leqslant \#\mathrm{Prms}(g(n)) \leqslant B_0,$$

as needed. $\qquad \square$

Let N be a positive integer and $S := \mathrm{Prms}(N) \cup \{\infty\}$ we define V_N to be the following S-adic neighborhood of $[1:0] \in \mathbb{P}^1(\mathbb{Q})$:

$$V_N := \left\{ [a : bN] \in \mathbb{P}^1(\mathbb{Q}) : a, b \in \mathbb{Z} \text{ and } \left| \frac{bN}{a} \right| \leqslant \frac{1}{N} \right\}. \tag{24}$$

Note that by our notational agreement, $\gcd(a, bN) = 1$.

Lemma 4.4. *For every* $D = D_1 \cdots D_r$ *with* $D_1, \ldots, D_r \in \mathbb{Z}[t, s]$ *homogeneous irreducible polynomials of respective positive degrees* $d_1, \ldots, d_r$, *there exists a finite set of primes* $S_0 = S_0(d_1, \ldots, d_r)$ *depending only on* $d_1, \ldots, d_r$ *such that for every positive integer* N *there exists* $[a : b] \in V_N$ *such that*

$$\#\mathrm{Prms}_S(D(a, b)) \leqslant B_0(d_1, \ldots, d_r), \quad S = S_0 \cup \mathrm{Prms}(N).$$

Proof. Let S_0 be the set of all primes p such that $p \leqslant \deg D$. If $p \nmid N$, then

$$\{[1 + xN : N] \in \mathbb{P}^1(\mathbb{F}_p) \mid x \in \mathbb{F}_p\} = \mathbb{A}^1(\mathbb{F}_p).$$

Thus if $D(1 + xN, N) \equiv 0 \pmod{p}$ for all x, then $p \in S_0$ by Lemma 4.1 (note that D is primitive as the product of irreducible polynomials in $\mathbb{Z}[t, s]$). Therefore for $p \notin S$, the function $n \mapsto D(1 + nN, N) \pmod{p}$ is nonzero.

Denote $g_i(x) = D_i(1 + xN, N)$. If $D_i(t, s) \neq s$, then g_i is an irreducible polynomial of degree d_i in $\mathbb{Q}[x]$. Moreover, we may write $g_i(x) = c_i f_i(x)$, where $c_i \in \mathbb{Z}$ and $f_i(x) \in \mathbb{Z}[x]$ is irreducible. By the above $\mathrm{Prms}(c_i) \subseteq S$. If $D_i(t, s) = s$, we denote $f_i(x) = x$.

Now $f_1, \ldots, f_r$ are irreducible in $\mathbb{Z}[x]$, $f = f_1 \ldots f_r$ has no local obstruction outside of S, and $\deg f_i = \deg D_i$. By Lemma 4.3, there exists $n \geqslant N$ such that $\#\mathrm{Prms}_S(f(n)) \leqslant B_0(d_1, \ldots, d_r)$. This finishes the proof since $\frac{N}{1+nN} < \frac{1}{N}$, so $[1 + nN : N] \in V_N$. $\square$

Note that $\mathrm{GL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ by

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} [a : b] = [x_1 a + x_2 b : y_1 a + y_2 b]. \tag{25}$$

Lemma 4.5. *Let S be a finite set of primes containing the infinite prime and let V_S be a nonempty S-adic open set. Then, there exist a positive integer N and a matrix $g \in \mathrm{GL}_2(\mathbb{Z})$ such that $gV_N \subseteq V_S$.*

Proof. Let $[a : b] \in V_S$ and choose $g \in \mathrm{GL}_n(\mathbb{Z})$ such that $g[1 : 0] = [a : b]$. Then $g^{-1}(V_S)$ is a neighborhood of $[1 : 0]$. Hence there exists N with $\mathrm{Prms}(N) \subseteq S$ such that $V_N \subseteq g^{-1}V_S$, so $gV_N \subseteq V_S$. $\square$

Proof of (23). Let $D_1, \ldots, D_r \in \mathbb{Z}[t, s]$ be non-associate irreducible homogeneous polynomials of positive degrees $d_1, \ldots, d_r$. Let $S_0 = S_0(d_1, \ldots, d_r)$ be as in Lemma 4.4. Let S be a finite set of primes containing S_0 and V_S a nonempty S-adic open set. By Lemma 4.5, there exists N with $\mathrm{Prms}(N) \subseteq S$ and $g \in \mathrm{GL}_2(\mathbb{Z})$ such that $gV_N \subseteq V_S$. We let $D_i' = D_i \circ g$ and $D' = D_1' \cdots D_r'$. Then each D_i' is irreducible of degree d_i. By Lemma 4.4, there exists $[a' : b'] \in V_N$ with $\#\mathrm{Prms}_S(D'(a', b')) \leqslant B_0(d_1, \ldots, d_r)$ (note that $S = S \cup \mathrm{Prms}(N)$). Therefore, for $[a : b] = g[a' : b']$ we get that $\#\mathrm{Prms}_S(D(a, b)) \leqslant B_0(d_1, \ldots, d_r)$, which proves (23) by the definition of B. $\square$

Equation (14) immediately follows from the following form of [8, Corollary 1.9] (which essentially appears in Proposition [11, Proposition 1.2]).

Proposition 4.6. *Let $L_i(s, t) = \beta_i t - \alpha_i s$ be distinct primitive integral linear forms, $i = 1, \ldots, r$. Let S be a finite set of primes containing all primes $p \leqslant r$ and let V_S be a nonempty S-adic open set. Then there exists $[a : b] \in V_S$ such that for all $i = 1, \ldots, r$ the value $L_i(a, b)$ as an element of $\mathbb{Z}[S^{-1}]$ is either a prime or a unit.*

Proof. As $r = 1$ follows from Dirichlet's theorem on primes in arithmetic progressions, we may assume w.l.o.g. that $r \geqslant 2$. By Lemma 4.5, it suffices to show the following assertion: Let $L_i(s, t) = \beta_i t - \alpha_i s$ be distinct primitive integral linear forms, $i = 1, \ldots, r$. Let S_0 be the set of primes $p \leqslant r$. Then, for every positive integer N there exists $[a : b] \in V_N$ such that $\#\mathrm{Prms}_S(L_i(a, b)) \leqslant 1$, for all $1 \leqslant i \leqslant r$, with $S = S_0 \cup \mathrm{Prms}(N)$.

Let N be a positive integer and $S = S_0 \cup \mathrm{Prms}(N)$. For every $p \in S$, we let

$$a_p := \max_{i, \beta_i \neq 0} v_p(\beta_i)$$

and

$$C := \prod_{p \in S} p^{a_p + 1}.$$

For every $1 \leqslant i \leqslant r$, we set $c_i := \gcd(\beta_i, C)$ and

$$M_i(t, s) = \frac{\beta_i t - \alpha_i C N s}{c_i}$$

if $L_i(t, s) \neq \pm s$, and

$$M_i(t, s) = s$$

if $L_i(t, s) = \pm s$.

We claim that there are no local obstructions; namely, for every prime p there exists $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ such that for all $1 \leqslant i \leqslant r$ we have $p \nmid M_i(a, b)$. Indeed, if $p \notin S$, then $\deg \prod M_i = r < p + 1$, so such $[a : b]$ exists. Otherwise, we take $a = b = 1$.

Let

$$K := \left\{ (x, y) \in \mathbb{R}^2 \,\Big|\, 0 < y < \frac{x}{CN^2} \right\}.$$

The convex set K and the linear forms $M_i(t, s)$ satisfy the conditions of a theorem of Green–Tao–Ziegler [6, Cor 1.9][1] (after replacing M_i by $-M_i$ is necessary). So, we have infinitely many $(a, b) \in \mathbb{Z}^2 \cap K$ such that $M_i(a, b)$ is prime for every $1 \leqslant i \leqslant r$. Since S is finite, we may choose (a, b) such that $M_i(a, b)$ is also not in S. This implies that a has no prime factors from S. Thus $\gcd(a, N) = 1$. Let $\gamma = \gcd(a, NCb) = \gcd(a, Cb)$. So

$$[a/\gamma : NCb/\gamma] \in V_N.$$

Note that

$$L_i(a/\gamma, NCb/\gamma) = c_i/\gamma M_i(a, b).$$

As c_i is a unit in $\mathbb{Z}[S^{-1}]$ and $M_i(a, b)$ is a prime in $\mathbb{Z}[S^{-1}]$, we get that $L_i(a/\gamma, NCb/\gamma)$ divides a prime and so either a prime or a unit in $\mathbb{Z}[S^{-1}]$. $\qquad\square$

5. Universally ramified primes in fiber products

5.1. The action of $\mathrm{GL}_2(\mathbb{Q})$

Recall that we view an element g of $\mathrm{GL}_2(\mathbb{Q})$ as an automorphism $g \colon \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ via the action (25). Given $\phi \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ denote by $\phi^g \colon C \to \mathbb{P}^1_{\mathbb{Q}}$ the composition $g \circ \phi$. If ϕ is generically Galois, then so is ϕ^g, and

$$\mathrm{Gal}(\phi) \cong \mathrm{Gal}(\phi^g).$$

---

[1] This theorem is stated in [6, Cor 1.9] conditionally on two conjectures, one of which is proved in [7] and the other in [8].

From its definition, the set of universally ramified primes is stable under the action of $\mathrm{GL}_2(\mathbb{Q})$, that is, for all $g \in \mathrm{GL}_2(\mathbb{Q})$

$$U(\phi^g) = U(\phi).$$

However, the branch locus is not invariant:

$$\mathrm{Branch}(\phi^g) = g \cdot \mathrm{Branch}(\phi). \tag{26}$$

We set

$$U_\infty(\phi) = U(\phi) \smallsetminus \{\infty\}.$$

Let $g \in \mathrm{GL}_2(\mathbb{Z}_p)$. As an automorphism of $\mathbb{P}^1_{\mathbb{Z}_p}$, it follows that

$$p \in V(\phi) \Longleftrightarrow p \in V(\phi^g), \quad \text{if } g \in \mathrm{GL}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}). \tag{27}$$

However, for general $g \in \mathrm{GL}_2(\mathbb{Q})$ it may happen that $V(\phi) \neq V(\phi^g)$.

5.2. Constructions of Fiber Products

For the applications to the minimal ramification problem, we are especially interested in controlling the universally ramified primes in fiber products. For an element $x \in \mathbb{Q}^\times$, we let

$$g_{x_\times} := \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}) \quad \text{and} \quad \phi^{x_\times} := \phi^{g_{x_\times}}.$$

For an element $b \in \mathbb{Q}$, we let

$$g_{b^+} := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}) \quad \text{and} \quad \phi^{b^+} := \phi^{g_{b^+}}.$$

Lemma 5.1. *Let $\phi \colon C \to \mathbb{P}^1$ be a branched covering and let $p \neq q$ be prime numbers such that $p \notin U(\phi)$.*

(1) *There exists a positive integer A such that for every sequence of integers $k_1, \dots, k_r$ that are multiples of A we have that $p \notin U(\prod_{\mathbb{P}^1} \phi^{(q^{k_i})_\times})$.*

(2) *There exists a positive integer B such that for every sequence of integers $k_1, \dots, k_r$ that are multiples of B, we have $p \notin U(\prod_{\mathbb{P}^1} \phi^{k_i+})$.*

Proof. By Lemma 2.2 with $S = \{p\}$, there exists a nonempty p-adic open set V such that p is unramified in A^ϕ_ζ, for all $\zeta \in V$. Fix some $\zeta \in V$. By the p-adic continuity of the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathbb{P}^1(\mathbb{Q}_p)$, there exists a neighborhood $W \subset \mathrm{GL}_2(\mathbb{Q}_p)$ of the identity matrix I such that for any $g \in W \cap \mathrm{GL}_2(\mathbb{Q})$ we have that $g\zeta \in V$. In particular, p is unramified at $\phi^g(\zeta)$. Since unramification is preserved in fiber products, given any set of elements $g_1, \dots, g_n \in W$, p is unramified at $\psi^{-1}(\zeta)$, for

$$\psi = \prod_{\mathbb{P}^1} \phi^{g_i}.$$

Thus, for (1), it suffices to find a positive integer A such that if k is a multiple of A, then $g^{(q^k)_\times} \in W$. For this we take $A = (p-1)p^m$ for a sufficiently large m.

Similarly, for (2), it suffices to find a positive integer B such that if k is a multiple of B, then $g^{k+} \in W$. For this we take $B = p^\ell$ for a sufficiently large ℓ. $\qquad\square$

Lemma 5.2. *Let $\phi\colon C \to \mathbb{P}^1$ be a branched covering with rational branch locus. Let $n \geqslant 1$ be an integer and S a finite set of primes. Then, there exist a sequence of integers $k_1, \ldots, k_n$ such that*

$$\mathrm{Branch}(\phi^{k_i+}) \cap \mathrm{Branch}(\phi^{k_j+}) \subset \{\infty\}, \quad \text{for } i \neq j, \tag{28}$$

$$U\left(\prod_{\mathbb{P}^1} \phi^{k_i+}\right) \cap S \subset U(\phi). \tag{29}$$

Proof. Let $R \subset \mathbb{Q} = \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{\infty\}$ be the finite branch points and

$$M = \max R - \min R$$

the diameter of R. As the set of finite branch points of $\phi^{k_i^+}$ is $R + k_i$, to obtain (28), it suffices to take the k_i's such that $k_i - k_{i-1} > M$.

For every $p \in S \smallsetminus U(\phi)$, we let B_p be the constant from Lemma 5.1(2) (applied to ϕ and p). Then, to obtain (29), it suffices to take the k_i's to be multiples of $B_0 = \prod_{p \in S \smallsetminus U(\phi)} B_p$. Clearly, these two sufficient conditions can be simultaneously satisfied; e.g., take $k_i = iB$, where B is a multiple of B_0 that is larger than M. $\qquad\square$

Lemma 5.3. *Let $\phi\colon C \to \mathbb{P}^1$ be a branched covering with rational branch locus and let $n \geqslant 1$ be an integer. Then there exists a sequence of integers $k_1, \ldots, k_n$ such that both (28) and*

$$U_\infty\left(\prod_{\mathbb{P}^1} \phi^{k_i+}\right) \subset U_\infty(\phi) \tag{30}$$

hold true.

Proof. Denote

$$d = \#\mathrm{Branch}(\phi)$$

and let S be a finite set of primes that contains $V(\phi)$ and all the prime numbers $p \leqslant nd$. Now choose $k_1, \ldots, k_n$ as in Lemma 5.2 and denote

$$\psi = \prod_{\mathbb{P}^1} \phi^{k_i+}.$$

Thus (28) holds true.

By (29) to obtain (30), it suffices to show that

$$p \notin S \implies p \notin U_\infty(\psi).$$

Indeed, given $p \notin S$, as $p > nd \geqslant \#\mathrm{Branch}(\psi)$, by (17) we have $p \notin B(\psi)$. Thus, by (18) it remains to show that $p \notin V(\psi)$: By (16),

$$V(\psi) = \bigcup V(\phi^{k_i+})$$

and since $p \notin V(\phi)$ and

$$g_{k_i+} \in \mathrm{GL}_2(\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}),$$

we also have $p \notin V(\phi^{k_i+})$ by (27). Therefore, $p \notin V(\psi)$ and by (18) $p \notin U_\infty(\psi)$. $\qquad\square$

Lemma 5.4. *Let $\phi\colon C \to \mathbb{P}^1$ be a dominant map of curves with rational branch locus. Let $n \geqslant 1$ be an integer, q a rational prime, and S be finite set of primes not containing q. Then, there exists a sequence of integers $k_1, \dots, k_n$ such that*

$$\mathrm{Branch}(\phi^{q^{k_i}\times}) \cap \mathrm{Branch}(\phi^{q^{k_j}\times}) \subset \{0, \infty\}, \quad for\ i \neq j, \tag{31}$$

$$U\left(\prod_{\mathbb{P}^1} \phi^{q^{k_i}\times}\right) \cap S \subset U(\phi). \tag{32}$$

Proof. Denote by $R \subset \mathbb{Q}^\times = \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0, \infty\}$ the finite nonzero branch points and set

$$M = \max_{x\in R} \log_q |x| - \min_{x\in R} \log_q |x|.$$

By (26), (31) would follow if $k_i - k_{i-1} > M$.

For every $p \in S \smallsetminus U(\phi)$, we let $A_p = A$ be the constant from Lemma 5.1(1) (applied to ϕ and $p \neq q$). Then, (32) would follow if the k_i's to be multiples of $A_0 := \prod_{p\in S\smallsetminus U(\phi)} A_p$. We thus put $k_i = i \cdot A$, where A is a multiple of A_0 that is larger than M to finish the proof. $\qquad\square$

Lemma 5.5. *Let $\phi\colon C \to \mathbb{P}^1$ be a dominant map of curves with branch locus defined over $\mathbb{Q}$, let $n \geqslant 1$ be an integer, and let q be a rational prime. Then, there exists a sequence of integers $k_1, \dots, k_n$ such that both (31) and*

$$U_\infty\left(\prod_{\mathbb{P}^1} \phi^{q^{k_i}\times}\right) \subset U_\infty(\phi) \cup \{q\} \tag{33}$$

hold true.

Proof. Denote

$$d = \#\mathrm{Branch}(\phi).$$

Let S be a finite set of primes $\neq q$ that contains $V(\phi) \cup \{p \leqslant nd\} \smallsetminus \{q\}$. Take $k_1, \dots, k_n$ as in Lemma 5.4 and denote

$$\psi = \prod_{\mathbb{P}^1} \phi^{q^{k_i}\times}.$$

As (31) holds true, it suffices to prove (33). For this, by (32), it suffices to show that if $p \notin S$ and $p \neq q$, then

$$p \notin U_\infty(\psi).$$

Indeed, given $p \notin S$ and $p \neq q$, we have $p > nd \geqslant \#\mathrm{Branch}(\psi)$, so by (17), $p \notin B(\psi)$. By (16),

$$V(\psi) = \bigcup V(\phi^{q^{k_i}\times}).$$

As $p \notin V(\phi)$ and

$$g_{q^{k_i}\times} \in \mathrm{GL}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}),$$

(27) gives that $p \notin V(\phi^{(q^{k_i})\times})$, so by (18), $p \notin U_\infty(\psi)$, as needed. $\qquad\square$

6. Irreducibility of fiber products and group theory

We shall use the following function field criterion for irreducibility: Let $\phi_1 \colon C_1 \to \mathbb{P}^1_{\mathbb{Q}}$ and $\phi_2 \colon C_2 \to \mathbb{P}^1_{\mathbb{Q}}$ be geometrically irreducible branched coverings with function field extensions $F_1/\mathbb{Q}(T)$ and $F_2/\mathbb{Q}(T)$, respectively, in some fixed algebraically closed field of $\mathbb{Q}(T)$. The fiber product $C_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} C_2$ is irreducible (respectively geometrically irreducible) if and only if F_1, F_2 are linearly disjoint over $\mathbb{Q}(T)$ (respectively $F_1\bar{\mathbb{Q}}$ and $F_2\bar{\mathbb{Q}}$ are linearly disjoint over $\bar{\mathbb{Q}}(T)$).

Lemma 6.1. *Let $\phi_i \colon C_i \to \mathbb{P}^1_{\mathbb{Q}}$ be a geometrically irreducible branched covering, $i = 1, 2$. Assume that $\mathrm{Branch}(\phi_1) \cap \mathrm{Branch}(\phi_2) \subseteq \{\alpha\}$ for some $\alpha \in \mathbb{P}^1(\mathbb{Q})$. Then $C_1 \times_{\mathbb{P}^1} C_2$ is geometrically irreducible.*

Proof. Let $F_1/\mathbb{Q}(T)$ and $F_2/\mathbb{Q}(T)$ be the function fields extensions corresponding to ϕ_1, ϕ_2 in some algebraic closure of $\mathbb{Q}(T)$. Let $E_i = F_i\bar{\mathbb{Q}}$ be the base change to an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and let N_i be the Galois closure of E_i over $\bar{\mathbb{Q}}(T)$, $i = 1, 2$.

Since unramification is preserved under compositions, N_i has the same branch locus as F_i, and so $N_1 \cap N_2$ is ramified at $\mathrm{Branch}(\phi_1) \cap \mathrm{Branch}(\phi_2)$ which consists, by assumption, of at most one point. By the Riemann–Hurwitz formula, $N_1 \cap N_2 = \bar{\mathbb{Q}}(T)$.

Thus N_1, N_2 are linearly disjoint over $\bar{\mathbb{Q}}(T)$, which implies that the subextensions E_1, E_2 are also linearly disjoint. Thus, $C_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} C_2$ is geometrically irreducible. $\qquad\square$

In the applications below, we need to relax the condition of Lemma 6.1 that the branch loci of ϕ_1 and of ϕ_2 have at most one rational point in common. For this we need some group theory.

Definition 6.2. For a prime number p, we say that a finite group G satisfies *condition-$E(p)$* if all the nontrivial simple quotients of G are of order p, but none of the quotients of the commutator $[G, G]$ are.

We give a few examples and basic properties and we omit the details:

(1) Let G be an $E(p)$-group. Then G is a p-group if and only if G is abelian.

(2) The symmetric group S_k is $E(2)$.

(3) Let m be a positive integer with $v_2(m) \leqslant 1$. Then, the Dihedral group D_m of order $2m$ is $E(2)$.

(4) If G, H satisfy condition-$E(p)$, then so does $G \times H$.

(5) Let G be a group satisfying condition-$E(p)$ and N a normal subgroup. Then G/N satisfies condition-$E(p)$. (Indeed, $[G/N, G/N] = [G, G]N/N$.)

(6) Let G be an $E(p)$-group and H a perfect group, then the wreath product $H \wr G$ satisfies $E(p)$. The proof of this fact is slightly involved, but we omit it, as we do not use it.

We study irreducibility of fiber products of covers with $E(p)$-Galois groups. For this we need an auxiliary result from group theory.

Lemma 6.3. *Let p be a prime, $G_1, \ldots, G_n$ groups that satisfy condition-$E(p)$, put $\Phi(G_i) := G_i^p[G_i, G_i]$ and*

$$\psi \colon G_1 \times \cdots \times G_n \to (G_1/\Phi(G_1)) \times \cdots \times (G_n/\Phi(G_n))$$

the quotient map. Let $H \leqslant G_1 \times \cdots \times G_n$ be such that the restriction of the projection on the i-th coordinate to H, $\pi_i \colon H \to G_i$ is surjective, for every $i = 1, \ldots, n$ and the restriction of ψ to H is surjective. Then $H = G_1 \times \cdots \times G_n$.

Proof. Since the family of finite groups satisfying condition-$E(p)$ is close under direct products and since Φ respects direct products, by induction argument, we may assume that $n = 2$. Let $K_i = \ker \pi_i$ and $C_i = \pi_i^{-1}([G_i, G_i])$, $i = 1, 2$. Note that $K_i \leqslant C_i$ are normal in H and that $H/K_i \cong G_i$ and $C_i/K_i = [H/K_i : H/K_i]$. Let

$$\rho \colon G_1 \times G_2 \to G_1^{ab} \times G_2^{ab}$$

be the abelianization map. We break the proof into several parts.

Part 1: $\rho|_H$ is surjective. Indeed, by assumption, $\Phi(G_i)/[G_i, G_i]$ is the Frattini subgroup of G_i^{ab}. Thus the assumption gives that $\rho(H)$ generates $G_1^{ab} \times G_2^{ab}$ modulo the Frattini subgroup; hence $\rho(H) = G_1^{ab} \times G_2^{ab}$.

Part 2: $C_1C_2 = H$. Indeed, it is immediate that $C_1 = \rho|_H^{-1}(1 \times G_2^{ab})$ and $C_2 = \rho|_H^{-1}(G_1^{ab} \times 1)$. Hence, as $\rho|_H$ is surjective, $C_1C_2 = \rho|_H^{-1}(G_1^{ab} \times G_2^{ab}) = H$.

Part 3: $H = K_1C_2$. Indeed, as $H = C_1C_2 = C_1(K_1C_2)$, the second isomorphism theorem gives that

$$H/K_1C_2 \cong C_1/C_1 \cap (K_1C_2).$$

Assume by contradiction that H/K_1C_2 is nontrivial; then H/K_1C_2 has a simple quotient S. As H/K_1C_2 is a quotient of $H/C_2 \cong G_2^{ab}$, S is of order p. On the other hand, $C_1/C_1 \cap (K_1C_2)$ is a quotient of $C_1/K_1 \cong [G_1, G_1]$, which contradicts the assumption that G satisfies condition-$E(p)$.

Part 4: $H = K_1K_2$. We argue in a similar fashion as in Part 3: As $H = K_1C_2 = (K_1K_2)C_2$, the second isomorphism theorem gives that

$$H/K_1K_2 = C_2/(C_2 \cap K_1K_2).$$

Assume by contradiction that H/K_1K_2 is nontrivial, then it has a simple quotient S. Since H/K_1K_2 is a quotient of $H/K_2 \cong G_2$ and G_2 satisfies condition-$E(p)$, the order of S is p. On the other hand, $C_2/(C_2 \cap K_1K_2)$ is quotient of $C_2/K_2 \cong [G_2, G_2]$, which contradicts the assumption that G_2 satisfies condition-$E(p)$.

Conclusion of the proof: Since $H = K_1K_2$ and $K_1 \cap K_2 = 1$, we get that

$$H \cong K_2 \times K_1 \cong H/K_1 \times H/K_2 \cong G_1 \times G_2,$$

as needed. $\qquad\square$

Lemma 6.4. *Let p be a prime and for each $i = 1, \ldots, n$ let $\phi_i \colon C_i \to \mathbb{P}^1_{\mathbb{Q}}$ be a geometrically irreducible branched covering that is generically Galois with Galois group G_i. Let $D_i = C_i/\Phi(G_i)$, where $\Phi(G_i) = G_i^p[G_i, G_i]$. Assume that G_i satisfies condition-$E(p)$ for all i and that $\prod_{\mathbb{P}^1_{\mathbb{Q}}} D_i$ is geometrically irreducible. Then $\prod_{\mathbb{P}^1_{\mathbb{Q}}} C_i$ is geometrically irreducible.*

Proof. For each i, let $\bar{\mathbb{Q}}(T) \subseteq E_i \subseteq F_i$ be the function field extensions corresponding to the maps $\mathbb{P}^1_{\bar{\mathbb{Q}}} \leftarrow (D_i)_{\bar{\mathbb{Q}}} \leftarrow (C_i)_{\bar{\mathbb{Q}}}$. Since $(C_i)_{\bar{\mathbb{Q}}}$ is irreducible by assumption, it follows that $(D_i)_{\bar{\mathbb{Q}}}$ is also irreducible. Hence by Galois correspondence and since $\Phi(G_i) \lhd G_i$ it follows that these extensions are Galois with Galois groups

$$\mathrm{Gal}(F_i/\bar{\mathbb{Q}}(T)) = G_i, \quad \mathrm{Gal}(F_i/E_i) = \Phi(G_i), \quad \mathrm{Gal}(E_i/\bar{\mathbb{Q}}(T)) \cong G_i/\Phi(G_i).$$

Let $E = E_1 \cdots E_n$ be the composition of the E_i, $i = 1, \ldots, n$. The assumption that $\prod_{\mathbb{Q}(T)} D_i$ is absolutely irreducible, implies that

$$[E : \bar{\mathbb{Q}}(T)] = \prod[E_i : \bar{\mathbb{Q}}(T)] = \prod[G_i : \Phi(G_i)].$$

Hence, $\mathrm{Gal}(E/\bar{\mathbb{Q}}(T)) \cong \prod_i G_i/\Phi(G_i)$. We put $F = F_1 \cdots F_n$. We summarize the above in Diagram 1.
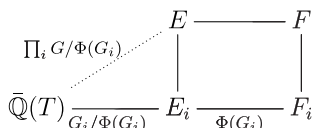

Diagram 1. Function fields and Galois groups.

Let $H = \mathrm{Gal}(F/\bar{\mathbb{Q}}(T))$. Then H embeds into $\prod_i G_i$ via the restriction maps; namely, $\sigma \mapsto (\sigma|_{F_i})_i$. The restriction of the projection onto the jth coordinate $\prod_i G_i \to G_j$ to H is surjective for every j. Also, by Galois correspondence, the image of H under the quotient map $\prod_i G_i \to \prod_i G_i/\Phi(G_i)$ is $\mathrm{Gal}(E/\bar{\mathbb{Q}}(T)) = \prod_i G_i/\Phi(G_i)$. Thus the conditions of Lemma 6.3 are satisfied, so $H = \prod_i G_i$. This implies that, $[F : \bar{\mathbb{Q}}(T)] = \deg \phi_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} \cdots \times_{\mathbb{P}^1_{\mathbb{Q}}} \phi_n$, so $C_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} \cdots \times_{\mathbb{P}^1_{\mathbb{Q}}} C_n$ is geometrically irreducible. $\square$

6.1. The $E(p)$-condition and rational rigid tuples

Let G be a finite group. We say that a k-tuple $\mathbf{g} = (g_1, \ldots, g_k) \in G^k$ is a *good generating k-tuple* for G if G is generated by $g_1, \ldots, g_k$ and $g_1 \cdots g_k = 1$. Two good generating k-tuples $\mathbf{g} = (g_1, \ldots, g_k)$ and $\mathbf{g}' = (g_1', \ldots, g_k')$ for G are *semi-conjugate* if for every $1 \leqslant i \leqslant k$ there exists $h_i \in G$ such that $g_i' = h_i^{-1} g_i h_i$. We say that $\mathbf{g}$ and $\mathbf{g}'$ are *conjugate* if there exists $h \in G$ such that $g_i' = h^{-1} g_i h$ for all $1 \leqslant i \leqslant k$.

Let G be a finite group, a k-tuple $\mathbf{g} = (g_1, \ldots, g_k) \in G^k$ is called *rigid* if the following conditions hold:

(1) G has a trivial center.

(2) $\mathbf{g}$ is a good generating tuple.

(3) Every good generating k-tuple $\mathbf{g}'$ which is semi-conjugate to $\mathbf{g}$ is conjugate to $\mathbf{g}$.

Recall that an element g in a group G is called *rational* if for every integer n which is relatively prime to the order of G, g^n is conjugated to g. A rigid tuple is called *rational rigid* if in addition:

(4) Every g_i is rational.

Lemma 6.5. *If $\mathbf{g} = (g_1, \ldots, g_k)$ is a rational rigid k-tuple for G, then $\mathbf{g}' = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_k)$ is a rational rigid $k+1$-tuple.*

Proof. Clear.   □

Lemma 6.6. *Let G and H be finite groups. Let $\mathbf{g} = (g_1, \ldots, g_k)$ be a rational rigid k-tuple for G and $\mathbf{h} = (h_1, \ldots, h_k)$ be a rational rigid k-tuple for H. Assume that the collection of elements $(g_i, h_i) \in G \times H$ generates $G \times H$. Then $\mathbf{g} \times \mathbf{h} = ((g_1, h_1), \ldots, (g_k, h_k)) \in (G \times H)^k$ is a rational rigid k-tuple for $G \times H$.*

Proof. The rationality is clear. Condition (1) is clear since the center of a product is the product of centers. Condition (2) holds true by assumption.

Hence it suffices to show Condition (3): Indeed, let $\mathbf{g}' \times \mathbf{h}' \in (G \times H)^k$ be a good generating tuple which is semi-conjugate to $\mathbf{g} \times \mathbf{h}$. Then $\mathbf{g}'$ is a good generating tuple which is semi-conjugate to $\mathbf{g}$ and $\mathbf{h}'$ is a good generating tuple which is semi-conjugate to $\mathbf{h}$. Thus, $\mathbf{g}'$ is conjugate to $\mathbf{g}$ and $\mathbf{h}'$ is conjugate to $\mathbf{h}$. This implies that $\mathbf{g}' \times \mathbf{h}'$ is conjugate to $\mathbf{g} \times \mathbf{h}$.   □

Proposition 6.7. *Let G_1, G_2 be groups satisfying the $E(p)$-condition. Assume that G_i admits a rational rigid k_i-tuple for each $i = 1, 2$. Let $d_i = d(G_i^{ab})$. Then $G_1 \times G_2$ admits a rational rigid s-tuple, for $s = d_1 + d_2 + \max(k_1 - d_1, k_2 - d_2)$.*

Proof. Since G_i is $E(p)$ we have that G_i^{ab} is a p-group and $G_i/\Phi(G_i) = G_i/G_i^p[G_i, G_i] = (\mathbb{Z}/p\mathbb{Z})^{d_i}$. Let

$$\rho_i : G_i \to G_i/\Phi(G_i) = (\mathbb{Z}/p\mathbb{Z})^{d_i}$$

be the quotient map. By Lemma 6.5, we may assume w.l.o.g. that $r := k_1 - d_1 = k_2 - d_2$, so $s = d_1 + d_2 + r$. Let $\mathbf{g}^{(i)} = (g_1^{(i)}, \ldots, g_{k_i}^{(i)})$ be a rational rigid k_i-tuple for G_i. Let $A_i \subset \{1, \ldots, k_i\}$ be a set of size $d_i = |A_i|$ such that $\{\rho_i(g_a^{(i)}) : a \in A_i\}$ generates $G_i/\Phi(G_i)$ and $B_i = \{1, \ldots, k_i\} \smallsetminus A_i$ the complement. Write the elements of B_i as

$$b_{i,1} < b_{i,2} < \ldots < b_{i,r}.$$

Consider the set V of all the pairs

$$v_a = (g_a^{(1)}, 1), \quad v_{a'}' = (1, g_{a'}^{(2)}), \quad w_j = (g_{b_{1,j}}^{(1)}, g_{b_{2,j}}^{(2)}),$$

for $a \in A_1$, $a' \in A_2$, and $j = 1, \ldots, r$. One may order them such that the resulting s-tuple of elements in $(G_1 \times G_2)^s$ has the property that the projection to each of the coordinates G_i gives the original tuple diluted by 1's.

Let $H \leqslant G_1 \times G_2$ be the subgroup generated by V. By Lemma 6.6, it suffices to show that $H = G_1 \times G_2$. Indeed, on the one hand, H maps onto each of the G_i's. On the other hand, by the construction of V, $(\rho_1 \times \rho_2)(V)$ contains a basis of $G_1/\Phi(G_1) \times G_2/\Phi(G_2)$, so by Lemma 6.3, $H = G_1 \times G_2$, as needed for rigidity. The rationality is immediate.   □

Applying the previous proposition repeatedly gives:

Corollary 6.8. *Let G be a group satisfying the $E(p)$-condition, $d = d(G^{ab})$, and $n \geqslant 1$. Assume that G admits a rational rigid r-tuple. Then G^n admits a rational rigid s-tuple, for $s = (n-1)d + r$.*

7. The minimal ramification problem

In this section, we use the methods developed so far and prove the bounds on $m(G)$ stated in §1.2.

Proof of Theorem 1.2. Let G be a regularly realizable group, which in terms of Definition 1.10 means that G has a $(U; \mathbf{d})$ realization for some $U, \mathbf{d}$. We claim that we can realize G^n with at most $B(\mathbf{d})n$ ramified primes outside of U. And indeed, assume by induction that $G^{n-1} = \mathrm{Gal}(L/\mathbb{Q})$ and $\#\mathrm{Ram}_U(L/\mathbb{Q}) \leqslant B(\mathbf{d})(n-1)$. Then by Proposition 1.11 we have a Galois extension $N/\mathbb{Q}$ with Galois group G such that $N \cap L = \mathbb{Q}$ and $\#\mathrm{Ram}_U(N/\mathbb{Q}) \leqslant B(\mathbf{d})$, so $NL/\mathbb{Q}$ is a Galois extension with Galois group $G^n = G \times G^{n-1}$ and

$$\#\mathrm{Ram}_U(NL/\mathbb{Q}) = \#\mathrm{Ram}_U(N/\mathbb{Q}) + \#\mathrm{Ram}_U(L/\mathbb{Q})$$
$$\leqslant B(\mathbf{d}) + B(\mathbf{d})(n-1) = B(\mathbf{d})n.$$

In particular we have

$$m(G^n) \leqslant B(\mathbf{d})n + \#U = O(n), \tag{34}$$

as needed. $\qquad\square$

If G has a $(U; \mathbf{d})$ realization with $\mathbf{d} = \mathbf{1}_r = \overbrace{(1, \ldots, 1)}^{r \text{ times}}$, then since $B(\mathbf{d}) \leqslant r$ by (14), the inequality (34) immediately gives that

$$m(G^n) \leqslant rn + O(1).$$

However this is not sufficient for Theorem 1.3, as we need to reduce r to $r - 1$. So one requires an extra construction:

Proposition 7.1. *Let $\mathbf{1}_r = (1, \ldots, 1)$ be an r-tuple of ones, let $G \neq 1$ be a finite group having a $(U; \mathbf{d})$ realization, and let $n \geqslant 1$ be an integer. Then G^n has a $(U, \mathbf{1}_R)$ realization, where $R = (r-1)n + 1$.*

Proof. Let $\phi \colon C \to \mathbb{P}^1$ be the $(U; \mathbf{d})$ realization of G with $\mathrm{Branch}(\phi) = \{(D_1), \ldots, (D_r)\}$, $\deg D_i = 1$. Since $G \neq 1$, the morphism ϕ must be ramified, so $r \geqslant 1$. Without loss of generality we may assume that ∞ is a branch point (otherwise we compose ϕ with a matrix in $\mathrm{GL}_2(\mathbb{Q})$ that maps a branch point to infinity).

Put $S = V(\phi) \cup \{p \leqslant R\} \cup \{\infty\}$, where $V(\phi)$ is the set of vertically ramified primes of a model of ϕ over $\mathbb{Z}$. We apply Lemma 5.2 to get integers $k_1, \ldots, k_n$ satisfying (28) and (29). Put $\hat{\phi} = \prod_{\mathbb{P}^1} \phi^{k_i} \colon \hat{C} \to \mathbb{P}^1_{\mathbb{Q}}$.

By (26), $\mathrm{Branch}(\phi^{k_i+}) = \{\infty, p_{1,i}, \ldots, p_{r-1,i}\}$. By (28), $p_{j,i} \neq p_{j',i'}$ for all $(j, i) \neq (j', i')$. Since unramification is preserved in fiber products, we conclude that $\mathrm{Branch}(\hat{\phi}) =$

$\{\infty, p_{1,1}, \ldots, p_{r-1,n}\}$. In particular, $\hat{\phi}$ has exactly R branch points which are all $\mathbb{Q}$-rational.

The conditions of Lemma 6.1 are satisfied by (28), thus the curve $\hat{C}$ is geometrically irreducible. This in particular implies that the extensions $E_i/\mathbb{Q}(\mathbb{P}^1)$ defined by ϕ^{k_i+} are linearly disjoint Galois extensions of $\mathbb{Q}(\mathbb{P}^1)$, and so the Galois group of $\mathbb{Q}(\hat{C}) = \prod E_i$ over $\mathbb{Q}(\mathbb{P}^1)$ is the direct product of the Galois groups of the extensions; i.e., G^n.

By (27), $V(\phi^{k_i+}) = V(\phi)$; so by (16) we have

$$V(\hat{\phi}) = V(\phi) \subseteq S. \tag{35}$$

By (17), $B(\hat{\phi}) \subseteq \{p \leqslant R\} \subseteq S$. Together with (35) and (18) this gives that

$$U(\hat{\phi}) \subseteq S.$$

Since $U(\phi) \subseteq U$ and by (29) we conclude that

$$U(\hat{\phi}) = U(\hat{\phi}) \cap S \subseteq U(\phi) \cap S \subseteq U,$$

and so $\hat{\phi}$ is a $(U, \mathbf{1}_R)$ realization of G^n, as needed. $\qquad\square$

Proof of Theorem 1.3. Assume G has a $(U; \mathbf{1}_r)$ realization. Then by Proposition 7.1, G^n has a $(U; \mathbf{1}_R)$, $R = (r-1)n + 1$ realization. By Proposition 1.11 and the bound (14) we get

$$m(G^n) \leqslant \#U + B(\mathbf{1}_R) \leqslant R + \#U \leqslant (r-1)n + \#U + 1 = (r-1)n + O(1).$$

This finishes the proof. $\qquad\square$

Next we prove Theorem 1.5, which reduces the number of ramification to $(r-2)n$ under certain group theoretical conditions.

Proof of Theorem 1.4. Let $\phi\colon C \to \mathbb{P}^1_{\mathbb{Q}}$ be a non-constant map of smooth connected projective $\mathbb{Q}$-curves that is generically Galois with group G. Assume that the branch locus consists of r rational points. We assume that $[G, G]$ is simple non-abelian, $d = d(G^{ab}) \leqslant r - 2$, and that there exists a prime number p such that every maximal normal subgroup has index p. This implies that $d(G^{ab})$ is a p-group, and that G satisfies condition $E(p)$. We note that in this case the Frattini quotient of G^{ab} is $G/M(G) \cong (\mathbb{Z}/p\mathbb{Z})^d$ with $M(G) = G^p[G, G]$; and thus a subgroup $H \leqslant G^n$ maps onto $(G^{ab})^n$ if and only if it maps onto $(G/M(G))^n$.

We let $C_M = C/M(G)$; so $\phi_M\colon C_M \to G$ is Galois with Galois group $(\mathbb{Z}/p\mathbb{Z})^d$. Choose d branch points $x_1, \ldots, x_d$, such that the inertia groups above the x_i's generate $(\mathbb{Z}/p\mathbb{Z})^d$. By assumption, there exist at least two other branch points y_1, y_2. By applying a Mobius transformation, we may assume w.l.o.g. that $y_1 = [0 : 1]$ and $y_2 = [1 : 0]$.

We pick an auxiliary prime q. By Lemma 5.5 there exist $k_1, \ldots, k_n$ such that if we write $C_i = C$ and $\phi_i = \phi^{q^{k_i} \times}\colon C_i \to \mathbb{P}^1_{\mathbb{Q}}$, then we have (for $i \neq j$)

$$\text{Branch}(\phi_i) \cap \text{Branch}(\phi_j) \subseteq \{0, \infty\} \quad \text{and} \quad U_\infty\left(\prod_{\mathbb{P}^1} \phi_i\right) \subseteq U_\infty(\phi) \cup \{q\}.$$

Let $F_i/\mathbb{Q}(x)$ be the function field extension corresponding to $\phi_i\colon C \to \mathbb{P}^1_{\mathbb{Q}}$, $i = 1, \ldots, n$. Then $G \cong \text{Gal}(F_i/\mathbb{Q}(x))$; denote by L_i and L_i' the fixed fields of $[G, G]$ and $G^p[G, G]$

(respectively) in F_i. Since each $\mathrm{Gal}(L_i'/\mathbb{Q}(x))$ is generated by the inertia groups over distinct points, the L_i' are linearly disjoint over $\mathbb{Q}(x)$. By Lemma 6.4, $\hat{C} = \prod_{\mathbb{P}^1} C_i$ is geometrically irreducible. Now, as we chose the k_i as in Lemma 5.5, we have that $U_\infty(\prod_{\mathbb{P}^1} \phi_i) \subseteq U_\infty(\phi) \cup \{q\}$. By construction, the branch locus of $\hat{\phi}$ consists of $(r-2)n+2$ rational branch points. So, if we put $U = U_\infty(\phi) \cup \{q\} \cup \{\infty\}$ and $s = (r-2)n+2$, we have obtained a

$$(U; \mathbf{1}_s)$$

realization of G^n. By (14) and Proposition 1.11,

$$m(G^n) \leqslant s + \#U \leqslant (r-2)n + \#U_\infty(\phi) + 4 = (r-2)n + O(1),$$

which proves (5). $\qquad\square$

Now we consider the special case $G = S_k$ and we prove Theorem 1.5, that is $m(S_k^n) \leqslant n+4$ and $m(S_k) \leqslant 4$. For this we first need to recall a concrete realization of S_k over $\mathbb{P}^1_{\mathbb{Q}}$.

Lemma 7.2. *Let $a, b, c \in \mathbb{P}^1(\mathbb{Q})$ be distinct and $m > 3$. There exists a cover $\phi: C \to \mathbb{P}^1_{\mathbb{Q}}$ with Galois group S_k such that $\mathrm{Branch}(\phi) = \{a, b, c\}$ and the inertia groups at a, b, c are generated by cycles of length $n, n-1, 2$ respectively and $U(\phi) = \{\infty\}$.*

Proof. By applying Mobius transformation, we see that it suffices to find ϕ for one triplet (a, b, c). Consider $\mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto x^m - x^{m-1}$, i.e., generated by $f(X, Y) = X^m - X^{m-1} - Y$, let F be the splitting field of f over $\mathbb{Q}(Y)$, and let $\phi: C \to \mathbb{P}^1_{\mathbb{Q}}$ be the branch covering corresponding to $F/\mathbb{Q}(Y)$. It is an exercise to show that the Galois group is S_k and that ramification points are $0, u, \infty$, with $u = \frac{m-1}{m}$ and that the inertia groups are generated by cycles of lengths $2, n-1, n$, respectively. For details see [26, p. 42].

It now remains to calculate $U = U(\phi)$. For any $y \in \mathbb{Q} \smallsetminus \{0, u\}$, let A_y be the algebra at y. Since the X-derivative of $f(X, y)$ has only 2 roots (0 and u), $f(X, y)$ has at most 3 real roots. Thus A_y has at most three embeddings into $\mathbb{R}$, which implies as $m > 3$ that $A_y \otimes \mathbb{R} \not\cong \mathbb{R}^3$. Thus $\infty \in U$.

A direct application of the discriminant formula $\mathrm{disc}\, f = \pm m^m \prod f(\alpha)$, where α runs on the set of zeros of f' with multiplication, shows that

$$\mathrm{disc}\, f(X, y) = \pm Y^m \big((m-1)^{m-1} + m^m Y\big).$$

Let p be a prime; we show that there exists $y \in \mathbb{Z}$ with $p \nmid \mathrm{disc}\, f(X, y)$, and thus $p \notin U$. This will show that $U = \{\infty\}$. If $p \nmid m$ and $p > 2$, then $m^m y^m$ takes $p - 1 > 1$ values for $y \not\equiv 0 \pmod{p}$, and so we can take $y \in \mathbb{Z}$ with $m^m y \not\equiv -(m-1)^{m-1}, 0 \mod p$; so $p \nmid \mathrm{disc}\, f(X, y)$, as needed. If $p \mid m$, then $p \nmid \mathrm{disc}\, f(X, 1)$. We are left with the case $p = 2$ and m odd; then $p \mid m - 1$, so $p \nmid \mathrm{disc}\, f(X, 1)$. $\qquad\square$

Proof of Theorem 1.5. We just apply the construction appeared in the proof of Theorem 1.4 to the cover $\phi: C \to \mathbb{P}^1$ given in Lemma 7.2 that is ramified at $(\infty, 0, 1)$ with the inertia group at 1 being generated by a transposition.

This gives a $(U, \mathbf{d}_{n+2})$ realization of S_k^n, with $U = \{\infty\}$ if $n = 1$ and $U \subseteq \{\infty, q\}$ if $n \geqslant 2$. Thus by (14), Proposition 1.11 gives that

$$m(S_k^n) \leqslant n+4,$$

as needed for (6), and that

$$m(S_k) \leqslant 4,$$

as needed for (7). □

We conclude by proving our results for rational rigid groups.

Proof of (8). Let G be a group with a rational rigid r-tuple. By [26, Theorem 8.1.1], there exists a geometrically irreducible branched covering $\phi\colon C \to \mathbb{P}^1_{\mathbb{Q}}$ with $\mathrm{Branch}(\phi) = \{1, \ldots, r\}$. Let $T = \{p \leqslant r\} \cup \mathrm{Prms}(|G|)$. If $p \notin T$, by [1, Theorem 1.2], p is unramified at A^ϕ_{r+1}, $r + 1 \in \mathbb{A}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{Q})$. So $U(\phi) \subseteq T$. Now Proposition 1.11 and (14) immediately give $m(G) \leqslant r + \#T$. □

Proof of Theorem 9. By Corollary 6.8, G^n has a rational rigid s-tuple with $s = d(G^{ab})(n-1) + r = d(G^{ab})n + O(1)$. Note that by the prime number theorem $\#\{p \leqslant s\} = O(n/\log n)$ and that $\mathrm{Prms}(|G|^n) = \mathrm{Prms}(|G|) = O(1)$. Hence (8) gives that

$$m(G) \leqslant d(G^{ab})n + O\left(\frac{n}{\log(n)}\right).$$

As needed. □

References

1. S. BECKMANN, On extensions of number fields obtained by specializing branched coverings, *J. Reine Angew. Math.* **419** (1991), 27–53.
2. N. BOSTON AND N. MARKIN, The fewest primes ramified in a G-extension of $\mathbb{Q}$, *Ann. Sci. Math. Québec* **33**(2) (2009), 145–154.
3. M. DE WITT, Minimal ramification and the inverse Galois problem over the rational function field $\mathbb{F}_p(t)$, *J. Number Theory* **143** (2014), 62–81.
4. M. D. FRIED AND M. JARDEN, *Field Arithmetic*, 3rd edn, Revised by Jarden, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Volume 11 (Springer, Berlin, 2008).
5. B. GREEN, F. POP AND P. ROQUETTE, On Rumely's local-global principle, *Jahresber. Dtsch. Math.-Ver.* **97**(2) (1995), 43–74.
6. B. GREEN AND T. TAO, Linear equations in primes, *Ann. of Math. (2)* **171**(3) (2010), 1753–1850.
7. B. GREEN AND T. TAO, The Möbius function is strongly orthogonal to nilsequences, *Ann. of Math. (2)* **175**(2) (2012), 541–566.
8. B. GREEN, T. TAO AND T. ZIEGLER, An inverse theorem for the Gowers $U^{s+1}[N]$-norm, *Ann. of Math. (2)* **176**(2) (2012), 1231–1372.

9. H. HALBERSTAM AND H.-E. RICHERT, *Sieve Methods*, London Mathematical Society Monographs, Volume 4 (Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1974).

10. D. HARBATER, Abhyankar's conjecture on Galois groups over curves, *Invent. Math.* **117**(1) (1994), 1–25.

11. Y. HARPAZ, A. N. SKOROBOGATOV AND O. WITTENBERG, The Hardy-Littlewood conjecture and rational points, *Compositio Mathematica* **150**(12) (2014), 2095–2111.

12. J. L. HOELSCHER, Galois extensions ramified only at one prime, *J. Number Theory* **129**(2) (2009), 418–427.

13. J. W. JONES AND D. P. ROBERTS, Number fields ramified at one prime, in *Algorithmic Number Theory, Vol. 5011*, Lecture Notes in Computational Science, pp. 226–239 (Springer, Berlin, 2008).

14. C. KHARE, M. LARSEN AND G. SAVIN, Functoriality and the inverse Galois problem, *Compos. Math.* **144**(3) (2008), 541–564.

15. H. KISILEVSKY, D. NEFTIN AND J. SONN, On the minimal ramification problem for semiabelian groups, *Algebra Number Theory* **4**(8) (2010), 1077–1090.

16. H. KISILEVSKY AND J. SONN, On the minimal ramification problem for ℓ-groups, *Compos. Math.* **146**(3) (2010), 599–606.

17. F. LEGRAND, Specialization results and ramification conditions, *Israel Journal of Mathematics* **214**(2) (2016), 621–650.

18. G. MALLE AND B. HEINRICH MATZAT, *Inverse Galois Theory*, Springer Monographs in Mathematics (Springer, Berlin, 1999).

19. G. MALLE AND D. P. ROBERTS, Number fields with discriminant $\pm 2^a 3^b$ and Galois group A_n or S_n, *LMS J. Comput. Math.* **8** (2005), 80–101 (electronic).

20. N. MARKIN AND S. V. ULLOM, Minimal ramification in nilpotent extensions, *Pacific J. Math.* **253**(1) (2011), 125–143.

21. J. S. MILNE, *Étale Cohomology*, Princeton Mathematical Series, Volume 33 (Princeton University Press, Princeton, NJ, 1980).

22. J. NEUKIRCH, A. SCHMIDT AND K. WINGBERG, *Cohomology of Number Fields*, 2nd edn, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Volume 323 (Springer, Berlin, 2008).

23. A. NOMURA, Notes on the minimal number of ramified primes in some *l*-extensions of **Q**, *Arch. Math. (Basel)* **90**(6) (2008), 501–510.

24. B. PLANS, On the minimal number of ramified primes in some solvable extensions of $\mathbb{Q}$, *Pacific J. Math.* **215**(2) (2004), 381–391.

25. M. RAYNAUD, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, *Invent. Math.* **116**(1–3) (1994), 425–462.

26. J.-P. SERRE, *Topics in Galois Theory*, 2nd edn, Research Notes in Mathematics, (A K Peters, Ltd., 2008).

27. The Stacks Project Authors. *stacks project*. http://stacks.math.columbia.edu, 2015.

28. H. VÖLKLEIN, *Groups as Galois Groups, An Introduction*, Cambridge Studies in Advanced Mathematics, Volume 53 (Cambridge University Press, Cambridge, 1996).

29. G. WIESE, On projective linear groups over finite fields as Galois groups over the rational numbers, in *Modular Forms on Schiermonnikoog*, pp. 343–350 (Cambridge University Press, Cambridge, 2008).

30. D. ZYWINA, The inverse Galois problem for $\mathrm{PSL}_2(\mathbb{F}_p)$, *Duke Mathematical Journal* **164**(12) (2015), 2253–2292.