Math. Struct. in Comp. Science (2007), *vol.* 17, *pp.* 289–305.
doi:10.1017/S0960129507006007 Printed in the United Kingdom

Randomness and determinism in the interplay between the continuum and the discrete†

FRANCIS BAILLY‡ and GIUSEPPE LONGO§

‡*Physique, CNRS, Meudon*
Email: bailly@cnrs-bellevue.fr

§*LIENS, CNRS ENS and CREA, Paris*
Web site: http://www.di.ens.fr/users/longo

Received 20 May 2006; revised 19 November 2006

This paper provides a conceptual analysis of the role of the mathematical continuum *versus* the discrete in the understanding of randomness as a notion with a physical meaning or origin. The presentation is 'informal' as we will not write formulas; however, we will refer to non-obvious technical results from various scientific domains, and we will also propose a conceptual framework for understanding randomness (and predictability), which we believe is, essentially, original. As a matter of fact, unpredictability and randomness may be conveniently identified in various physico-mathematical contexts. This will allow us to explore these concepts in continuous *versus* discrete frameworks, with particular emphasis on the relationships and differences between classical approaches and quantum theories in Physics.

1. Introduction

Our starting point is the idea hinted at in Bailly and Longo (2006) and Longo (2007) that the mathematical structures that have been constructed to understand physical phenomena may suggest different ways of understanding Nature according to their continuous (mostly in Physics) or discrete (generally in Computing) natures. In particular, the *causal relations*, as explanatory structures (we use them to 'understand Nature'), are mathematically related to the use of the continuum or the discrete.

But what discrete (mathematical) structures are we talking about? We believe that there is one clear mathematical definition of 'discrete', and we will use it in this paper: a structure is *discrete* when the discrete topology on it is 'natural'. Of course, this is not a formal definition, but in mathematics we all know what 'natural' means. For example, one can endow Cantor's real line with the discrete topology, but this is not 'natural' (you cannot do much with it, and it does not help you better understand the reals or the notion of 'continuous functions'); on the other hand, the integer numbers or a digital data base are naturally endowed with the discrete topology (though one may have good reasons to work with them using a different structuring).

† A preliminary French version of part of this paper formed the Appendix of Bailly and Longo (2006).

Later in the paper we will also discuss the randomness/unpredictability issue in quantum mechanics. Our approach will then stress that, in the *space-time* of modern microscopic physics, one may not consider the discrete topology as in any way 'natural'. Our work will be based specifically on quantum non-locality and non-separability results, which, in our view, suggest the exact opposite of an underlying discrete space-time. Indeed, the discrete topology 'separates' and 'localises' the elements of mathematical structures – this is its job. Of course, quantum mechanics originated from the discovery of a fundamental (and unexpected) discretisation of the light absorption or emission spectra of atoms (specifically, the hydrogen atom). Then, a few dared to propose a discrete lower bound to the measurement of *action*, that is, of the product energy×time. It is this physical dimension that bears a discrete structure. Clearly, one can then compute, by assuming the relativistic maximum for the speed of light, a Planck length and time. But in no way are space and time organised in small 'quantum boxes' in this way. And this is the most striking and crucial feature of quantum mechanics: the global and entanglement effects (Bell 1964; Bohm 1951; Aspect *et al.* 1982). These effects are the opposite of a discrete, separated organisation of space and time, and are at the core of its scientific originality. In particular, they motivate quantum computing (as well as our analysis of quantum randomness).

As for the continuum, its role will be stressed in understanding classical determinism, as mathematised in the geometry of dynamical systems. In fact, the notion of randomness lies at the centre of dynamic unpredictability: a deterministic system is unpredictable precisely when it presents random evolutions. In quantum mechanics, this notion is also evoked, but in a very different and *intrinsic* way (we will give a more precise meaning to this term later). We will attempt to examine the following issues closely:

— How does randomness present itself in the natural sciences today?
— Is there a randomness specific to the various different fields of physics?
— What impact does this possible differentiation/unification of the notion of physical uncertainties have on the common/scientific concepts of randomness?
— Is it correlated with the various mathematical tools established (the continuous *versus* the discrete, for example)?

We will not return here to the terrain of specifically biological randomness (see Bailly and Longo (2006, Chapter 6)), which remains, in our opinion, unexplored. We will first consider classical physics in order to look at the problem of the meaning of randomness in the context of different theoretical frameworks. We will see that dynamical systems and quantum physics independently suggest very important, but different, notions of *randomness*, giving rise to a different role for probabilities in their various contexts. Our distinguishing criterion will refer to the notion of 'epistemicity' as being in contraposition to that of 'objectivity', the difference being related to the role of the knowing subject in the construction of scientific objectivity. This is an essential role within modern physics, particularly when it is a question of probability and randomness. In short, we will first state the problem of knowing whether a disordered sequence (or more generally, of a disordered state) is the effect of chaotic determinism or of pure random processes, that is, is it of an epistemic or an objective nature. In the absence of very general theorems on the matter (which would contribute to a constitution of objectivity – as is the case for 'mixing' random sequences, or Bernoulli's dynamics, which can be demonstrated as being equivalent to, and thus interpreted as, 'heads or tails' type sequences; we will return to this), we can develop an argument that will have the effect of somewhat displacing the problem, while at the same time highlighting some of the constraints it implies.

We will develop the thesis that in classical physics, randomness is of an 'epistemic' nature; in particular, we mean that one will always be able to interpret an apparently random sequence as stemming from either a chaotic determinism or from 'pure' randomness that is analysable in *statistical terms* (independent of any possible determinism). Thus, two different approaches are possible according to the viewpoint one assumes. In contrast, we will say the randomness specific to quantum mechanics is 'objective'. Our argument will base itself on two elements justifying this distinction. One is centred on the role of theoretical determinism, of the 'view' (constituted by the proposed theoretical framework) and of measurement. The other, which is stronger in a certain sense, is based on the properties of quantum non-separability and of continuous mathematics, which we will discuss at length.

2. Deterministic chaos and mathematical randomness: the classical physics case

Thus, we will begin by analysing classical randomness. To this end, we assume the idea that, for example, the *result* of the throw of a die, which we can legitimately consider to be random, may be interpreted as random given that the system of equations and constraints we can use to describe it (and thus to *determine* it) is (very) sensitive to the initial conditions. Note also that the reverse interpretation is quite different: a dynamic system defined by its equations and presenting a chaotic behaviour has an objective character associated with these equations themselves and with their intrinsic properties (it is described/given by physical objects, with their properties, their invariants, their symmetries, and so on, deduced from the equations). In short, from the modern (post-Laplacian) point of view, even this system, the paradigm of randomness, is indeed deterministic, in the sense that a sufficient number (in fact a very great number) of equations could, theoretically, describe all the forces at play, all of them (gravitation, various frictions, and so on) being *theoretically* well-known. However, this would tell us very little about its evolution: this classical system is so sensitive to the slightest variation in the boundary conditions, which are, moreover, very numerous, that the mathematical effort required to describe the (very numerous!) equations determining it will not help us in practice, not even qualitatively. Thus its evolution remains unpredictable: it is this that leads us to consider it, within a classical framework, as random, while at the same time being deterministic (chaotic).

As a less familiar but simpler example, we may consider a double pendulum, by which we mean a pendulum where a second pendulum is articulated on the first, that is, a weight is placed at an articulation point of the broken stick of the pendulum. This simple mechanism, which is perfectly determined by the two equations (it has two degrees of freedom), has a chaotic behaviour: its trajectories are dense (the weights go everywhere, within the limits of their constraints) and it is sensitive to the initial conditions (see Lighthill (1986)). Once again, the system's evolution appears random to all observers, despite the apparent simplicity of the determining equations and conditions. Here we have another case of epistemic randomness, which could, in fact, also be analysed in terms of a random sequence (for example, by writing 0 or 1 depending on whether the smaller weight finds itself to the right or to the left after 10 oscillations). On the other hand, a simple pendulum is deterministic and, in principle, predictable (we can be grateful that Galileo came across a simple pendulum, otherwise we might still be far from understanding the law of falling bodies in its basic simplicity!).

As a final classical example, which is highly relevant given that it triggered all the deterministic unpredictability analysis, we will consider the gravitational 'three-body problem'. Poincaré demonstrated the impossibility of resolving the system of nine Newton–Laplace equations describing its movements in either an elementary and direct manner (by using 'simple' functions say) or analytically (by means of convergent series). In doing so, he provided an analysis of what we have just done: classical determinism may fail to imply predictability. With this result, he opened the way to the integration/comprehension of classical randomness within the framework of mathematical determinism: the 'laws' concerned are all expressed by means of equations, but the evolution remains unpredictable, and *thus*, epistemically random. Modern results confirm the scientific relevance of this approach: Laskar has demonstrated in his numerous articles (Laskar 1990; Laskar 1994) that the solar system, our good old planetary system, is chaotic. So, apart from a few differences regarding the time scales (demonstrable time of unpredictability: 1 million years for Pluto, 100 million years for the Earth), it is not, from the mathematical point of view, very different from the double pendulum case, or the throwing of dice. In the long term we could also address it in purely statistical terms, like dice (and, for instance, make a bet at 2/1 that the Earth will no longer be revolving around the Sun in 500 million years).

The dynamic system we are considering may, therefore, have a great number of parameters, like dice, or a medium number of parameters, like the solar system, or even a very small number of degrees of freedom, like the double pendulum. On the other hand, a random representation of the statistics of the results obtained in all of the above-mentioned cases assumes a large number of parameters. Finally, note that this alternative possibility (the analysis of chaotic determinism or randomness in purely statistical terms) is intimately related to the fact that we can give local descriptions for the system (those associated with the underlying equations and their initial conditions), while the statistical and probabilistic representations generally involve a global representation of the system (an example of this is the physical behaviour of a gas, for which the thermodynamics consists precisely of taking global statistical averages – statistical mechanics – of local mechanical behaviours).

Finally, we can summarise the two reasons for us saying that classical randomness is *epistemic*. First, we can choose a purely statistical mathematical approach *or* an analysis in terms of (equational) determinism. Of course, both approaches, though theoretically equivalent (and there are plenty of theorems demonstrating this equivalence), may be more or less effective or even relevant: normally, we do not analyse planetary evolutions statistically, just as we do not analyse dice in equational terms; each system will have its own best-adapted method of analysis (in the case of a double pendulum, the difference is less clear, and depends on the aims of the analysis: this pendulum could very well be used for a little family game of chance). Second, the unpredictability of a deterministic system is due to a (classical) physical *principle*: measurement is always an interval. That is, even if we have a system of equations that determines a system point by point (in Euclid's sense of points, or of real numbers *à la* Cantor), as Laplace quite soundly tells us, only God knows the world through (mathematical) points and can thus predict (and retrodict) its future (and past) states. As far as we humans are concerned, our physics uses approximate measurements, and does so as a matter of principle, because, in the worst (or best) of cases, there are classical thermal fluctuations that force approximation (the interval of measurement). Only discrete state machines (our digital computers, in particular, when they are sequential and thus theoretically independent of physical contexts) and their access to *exact* data bases, in their discrete, well separated, topologies, can thus possess predictable evolutions. This observation has already been made by Turing, who viewed his machine as 'Laplacian' (see Longo (2007) for details and references). In this way, if we consider a deterministic system that is sensitive to the boundary conditions,[†] one gets that unpredictability *is the joint result* of this sensitive dependency to the boundary conditions and the *theoretical* properties of classical measurement. Classical (and, of course, relativistic) theories thus simultaneously give us perfect determinism (from God's viewpoint, but even conceivable by us, mathematically) and unpredictability. The key role of the mathematical continuum in these frames will be analysed in Section 4.

In conclusion, the two following arguments lead us to consider that classical randomness is epistemic:

1 the possible equivalence of the statistical view and equational determinism;
2 the role of measurement (performed by the knowing subject).

Thus, we could propose a 'Poincaré thesis' for classical dynamics:

'*Any classical random process is a trajectory given by a system that is, in principle, deterministic and in a chaotic regime.*'

This thesis does not necessarily correspond to Poincarés actual thinking, but to what we can say 120 years after his great theorem. It is at best just a thesis, since it is clearly unprovable (where is this list of *all* classical processes?), though it is falsifiable.

3. The objectivity of quantum randomness

One may argue that, in contrast with classical physics, quantum physics contains an 'objective' randomness that is intrinsic to the theory and conceptually and mathematically quite different from classical randomness (in Mugur-Schachter (2006) it is called 'primordial'). This randomness is *intrinsic* inasmuch as it is associated with any operation of measurement, because, in quantum physics, a measurement only returns a *probability* as

[†] Boundary conditions are typically non-linear, even chaotic, and would require a more general definition than that suggested and use the notions of topological transitivity and density of periodic points, which we will not develop here, see Devaney (1989)

https://doi.org/10.1017/S0960129507006007 Published online by Cambridge University Press

its result. More specifically, the objective randomness of quantum physics is due jointly (but in each of the cases listed, quite differently) to:

1 the non-null value of the Planck constant h (which constitutes a lower limit for the product of the possible precisions in the simultaneous measurement of two conjugated variables, namely, for the volume of the phase space, position and impulse);
2 the process of measurement specific to quantum physics (the 'projection of the state vector', which does not depend upon the value of h, while being 'non-determined' and returning a value as a probability);
3 the complex nature of the wave function (it is complex values that are added through the superimposition principle, but they are not what we measure, which are real numbers; the probability amplitudes do not coincide with the probabilities themselves).

Concerning the first two points, the difference compared with classical measurement is well understood. Quantum theory is centred on this essential role played by *non-determinism* in measurement: by the theoretical choice inherent in the approach itself, determinism by points (that of Euclid–Cantor, as in the mathematics of classical determinism) is not always possible; such a general determinism is theoretically inconceivable, and even proscribed, in direct contrast with what is presumed by the mathematics of classical deterministic systems (and which we have called the 'Poincaré thesis'). The mathematics of quantum physics begins with Planck's h, and is developed through an analysis of quantum states in terms of vectors (state vectors or, in other words, wave functions) within a space of infinite dimension (Hilbert space: the space of complex functions the squares of which are integrable). It then assumes a linear field where these wave functions are given in terms of complex components. Measurement, which always provides real numbers as a result, is thus associated with an essential loss of information, which is due to the passage from a complex variable to its absolute value. The physical relevance of the mathematical representation by complex numbers (or phase) is shown by phenomena such as quantum interference, while a representation as point particles was 'classically' expected (*cf.* Young's experiments). Similar reasons are at the root of quantum entanglement (see below).

So we already have a few good reasons for considering randomness to be intrinsic to the theory: it stems from the measurement, the mathematics, and the evolution of the system (the wave function).

To these we should add the intrinsic character of quantum fluctuations (as opposed to classical fluctuations related to temperature, for instance). In the measurements, this characteristic is manifested through both the zero-point energy of the harmonic oscillator (at absolute zero, $0°K$), for instance, as well as through resonance widths in particle theory. In short, $0°K$ in classical physics corresponds to the complete absence of energy, while in quantum mechanics, zero-point energy is admitted – in fact, we may speculate that zero-point energy could have played a remarkable role in cosmology through the destabilisation of the 'quantum void' (fundamental energetic state) during the 'big-bang'. As an aside, regarding the big-bang, one may notice (with a few word plays) that this representation also resolves the *enigma* of the Lucretian *clinamen* (we have always wondered what the origin of this could be in the absence of any external influence: we find it here in the intrinsic fluctuations) as well as resolving Leibniz's perplexity (*why is there something rather than nothing?*). Well, this is because 'nothing', the quantum void, is unstable, all the while being subjected to these same intrinsic fluctuations: unstable quantum void, fluctuation and ... big bang.

3.1. *Separability versus non-separability*

We will now present our main, and we believe new, argument for these different analyses of randomness (classical *versus* quantum). Objective (quantum) randomness appears to be deeply coupled to the properties of *non-separability* (the equation that describes the evolution of the system produces an entangled result for the quantons after they have interacted), as well as with the properties of *non-locality* (the measurement performed on one of the quantons after they have interacted produces instantaneous 'information' regarding the other's state). It is this, indeed, that leads us to refute the possibility of any *local* causal representation intended to account for specific quantum properties (technically, this refutation is ensured by the Bell inequalities (Bell 1964), which are, in turn, validated by the Aspect experiments (Aspect *et al.* 1982)). In this sense, therefore, the situation of a quantum system may always be considered as *purely* global, and without the possibility of reducing it to a combination of local components (which appears to be indissociable from the possibility of establishing causal/deterministic evolution equations). Thus, we will not here address Schrödinger's *local* equation, which describes the evolution of a *complex* state vector and is *beyond* the operation of measurement (the latter concerning another state vector involving the measurement device itself, *cf.*, among other things, decoherence theory (Zurek 1991), which, from the mathematical standpoint, corresponds to the passage from complex values to *real* numbers).

This point of view could indicate that in order to account for the situations we have just evoked, rather than resorting to the concepts of 'chaotic determinism' on the one hand and 'intrinsic randomness' on the other, it would be even more enlightening to use the concepts of 'separability' (to characterise the deterministic side) and 'non-separability' (to characterise the intrinsically or objectively random aspect in the role it plays in the construction of scientific objectivity in quantum physics). In short, it is the ability to separate the different 'objects' that participate in a classical process (each planet of an astronomical system, each die, a simple or double pendulum) that enables the mathematical description/determination of observable evolutions. In systems that are sufficiently sensitive to boundary conditions, these objects may evolve in an unpredictable way, although individually always theoretically determined by the dynamic equations. For quantum physics, on the other hand, the fact of its non-separability (an aspect of globality associated with non-locality) definitely confers a different character to randomness, which we call *intrinsic*. Typically, if two flipping coins interact in some classically possible way, and then separate while flipping in the air and fall, their analysis may be based on independent probabilities: the observation of one coin sets no limitation on the observation of the other. In contrast to this, two interacting quantons are *entangled*, that is, the measurement of one of them sets limitations on the measurement of the other (they cannot be 'separated'; they are entangled). So, in general, a set of n classical random events may be analysed in statistical terms, and follows classical laws of probability distribution, while quantum observables may violate them, for example, when they depend on entangled particles.

Later in the paper we will return to the role of trajectories in a space-time continuum as in the classical notion of determinism. Now, for a quantum system there are no hidden variables or equations that could determine its evolution (or the 'trajectory' of a quanton in an underlying continuum), and this corresponds to the absence of any possibility of local determinism. Note that the so-called 'hidden variable' theories do not escape this type of analysis inasmuch as these variables must be considered as non-local: they cannot describe a purely *local* dependence on a separated quanton, but require a global dependence (in view of entanglement, see next section).

In conclusion, the classical deterministic chaotic processes (dice, a double pendulum, as opposed to a simple pendulum, and even the solar system) normally also enable another description in purely statistical terms, and this follows classical probabilities. Thus, given that classical physical reality depends on this double description and on the physical (and not mathematical) nature of the limits of measurement, we have insisted on the epistemic character of classical randomness (it would depend on the approach, as is the case for the relationship between statistical mechanics and thermodynamics). On the other hand, in quantum physics, there is no possible double representation: any data that would enable us to access (to construct) knowledge, any measurement, is a probability, but of a different nature from the classical one.

Of course, it is a question here of the approach we should now take, which would need to base itself upon somewhat general theorems in order to be argued further. It nevertheless remains the case that the quantum situation differs essentially from the classical situation with respect to the status of the probabilities and randomness it involves, and that we cannot avoid taking this into consideration.

3.2. *Possible objections*

We should ask what objections there may be to such a point of view?

1 *It could be that, in a way comparable (though different) to the quantum situation, there is an intrinsic classical randomness, which is linked to precisely this collective effect of kinetic energy (temperature) and which would not be reducible to a dynamic system.*
  There are at least two possible responses to this objection. First, it is clear that the role of the initial conditions is determinant: there exists, classically, from the mathematical standpoint, a null set of measurements of the initial conditions that produces an ordered situation for a gas, for example, all speed vectors are parallel. In contrast to this, quantum non-locality and spontaneous fluctuations necessarily generate a '*clinamen*'. Second, for a classical system, the Nernst principle (the third law of thermodynamics) states that as the temperature approaches zero (no kinetic energy, which is conceivable for a classical system), the entropy of the system becomes zero (complete order). Conversely, in a quantum system the non-deterministic relationships prohibit such a complete order even at this limit (inasmuch as one may conceive of attaining it): Planck's h forces a zero-point energy, and, thus, the null kinetic energy is inconceivable.

We stress, and we will do so even more below, that the theoretical possibility of limit cases is crucial in classical approaches: this is the core of Cantor's continuum: a limit construction, and differential equations on it.

These two counter-objections make it rather implausible (or even impossible) that we could transfer the quantum viewpoint to the classical frameworks (and would confirm the meaning of our 'Poincaré thesis'): a mathematically coherent, limit-based, classical situation that has no quantum meaning. However, a simple example (of the inverse) of this limit passage may be given. Classically, it is theoretically conceivable that we could place a needle on its tip and leave it there for ever (if we are very lucky): it is the epistemic nature of your shaking (approximation inducing) hands that make it difficult. In the quantum case, however, a quantum fluctuation would make it fall in a random direction, under all theoretical circumstances.

2 *A more profound objection is that, despite indications to the contrary, the quantum randomness manifested through these fluctuations is also an effect of the 'viewpoint' (it is epistemic) and that the underlying agitation within an environment that one could describe as sub-quantum (cf. Vigier, Bohm, Halbwachs, Hillion, Lochak) could account for it in as deterministic a way as for the case of chaotic dynamic systems: for example, Louis de Broglie's double solution theory, that is, a superimposition of a regular and extended solution of Schrödingers (wave) equation and a singular and localised (particle) solution – corresponding to a (still unknown) non-linear operator inserted within this equation; or Bohm's version of hidden variables theory. This amounts to considering quantum mechanics as only 'providing a statistically exact but incomplete description of physical phenomena' (see de Broglie (1961)), an incompleteness that was also the perspective of the early Schrödinger or of Einstein – and more generally of E.P.R.*

A possible response to this objection lies in the fact that this environment itself, if one formalises its effect, intervenes at quantum magnitudes and their measurements in a global, non-local fashion, as we have noted earlier. Moreover, in order to justify the probabilistic properties of quantum physics and the presence of zero-point energy, fluctuations, virtual particles, and so on, these authors referred to a sub-quantum medium or to a new concept of 'ether', which is in principle a continuum, presenting these properties. If one cannot totally discard the possibility of such an approach, it is the case that it is not in accord with the heuristic principle of conceptual economy, in that it adds an underlying ether to phenomena (which, in the end, is not determinant). But also, and most importantly from our point of view, it remains non-local, and thus non-classical, when applied to the issues raised by Bell and Aspect, in requiring continuous, yet non-separable, hidden variables.

3 *One could still argue that this non-locality is similar to the globality presented by a classical system (for instance, the above-mentioned gas) and therefore reinforces the representation of the effect of a statistical disorder beneath the quantum level itself.*

However, our response is that the lines of research seeking the unification of physical theories, which lead to the theories of quantum gravitation or superstrings, seem to fit badly with this representation. In particular, supersymmetries require us to replace the point-like (zero size) character of the structures considered as elementary by a non-null dimensionality (strings or p-branes). On the other hand, the issues regarding

https://doi.org/10.1017/S0960129507006007 Published online by Cambridge University Press

the 'mass of particles' open up a way of thinking and a view of objectivity that are of a different nature (Higgs fields, or supplementary 'compactified' dimensions), and are classically inconceivable. All this is a consequence of breaking away from the point-like classical representation and the usual four-dimensional space-time, which raises these 'paradoxes'.

4 *Another point to take into consideration, and which could nullify our distinction, is what happens in the passage to the classical limit from quantum mechanics, that is, in reconstructing the classical regime by making h tend towards 0.*
Now, this passage requires at least two conditions. We will only hint at what is involved, as we know that in reality the passage from the quantum to the semi-classical and classical is much more delicate than is presented here, and the fundamental problems are still not completely resolved. On the one hand, the Planck constant h must be removed (its value going to 0), and, on the other hand, we also need to consider large quantum numbers (simply removing h does not always enable us to construct the classical limit). Once these conditions are assumed, it is tempting to consider them as a means of passing to a classical limit, with its epistemic notion of randomness. In particular, it would be possible, starting from quantum non-separability (in the framework of decoherence theory, for instance, which allows us to understand how the interactions with the environment – such as through the measurement apparatus – destroy entanglement and non-separability), to move away from an objective quantum randomness (if, as we hypothesise, there is a sense to this) and get to a classical epistemic dual random (purely statistical) representation and deterministic chaos. This would be compatible with so-called 'quantum chaos', which precisely corresponds to a chaotic classical limit. Thus, many see, in taking the zero limit of h, the possibility of a reduction to a dynamical system (thus obtaining the classical equations of mechanics) and, by resorting to large quantum numbers, the possibility of a reduction to epistemic probability. This double passage to the limit is a long way from being accomplished, and constitutes one of the great challenges for the much sought after 'unification' between the classical (and relativistic) theories and quantum frameworks.

Having attempted to respond to a few possible strong objections to our approach, there is still another aspect favouring the objective character of quantum randomness: the profound difference between quantum and classical statistics. This difference does not only stem from a non-zero h, but also from the 'observational' properties associated with classical particles in comparison with quantons, and to the nature of the symmetries to which bosons and fermions respond (symmetry constraints that do not exist in the classical framework). Classical particles are distinguishable, while quantum particles are not, which leads to different expressions for the statistics that these entities obey: Fermi–Dirac for fermions (which cannot simultaneously occupy the same quantum state, and are considered as the matter quantons, endowed with spin half); Bose–Einstein for bosons (which can simultaneously occupy the same quantum state, and are considered as the interaction quantons, endowed with spin one or zero); and Maxwell–Boltzmann statistics for classical particles. The indistinguishability of quantum particles (which is moreover related to non-separability) harks back, in our opinion, to the objective character of quantum randomness, while the distinguishability of classical particles (the well-isolated particles, each with an individual identity, whose integral sum of free energy we are considering) would refer to the epistemic character of classical randomness.

Moreover, we would like to add another element to this, which is of a quite different nature, but one that, in our opinion, has the effect of reinforcing the objective character of quantum probabilities. As we have previously mentioned, the domain of physics in which probabilities and statistics are most present is, beyond any doubt, statistical mechanics and thermodynamics, be it at equilibrium or in the study of irreversible phenomena.

To account for these phenomena, and because fluctuations (which are not necessarily quantum) also fill our universe at the same time as there being a certain degree of disorder, Boltzmann was lead to introduce the constant k_B, which bears his name. In a way, this measures an entropy, which is a physical quantity generally related to averages taken from collections. The elements of these collections (a gas, for example, formed of atoms in movement and subject to random collisions) are animated by disordered movements, of which the average effect of the interactions is represented by the temperature T (in kelvins: K). In the simplest of cases, each of the system's degrees of freedom is associated with an energy, which is given by the formula $k_B T/2$. It is worth noting that other physical theories (gravitational, quantum, electromagnetic, and so on) usually address isolated elements (two masses, an electron, a photon, in any event, elements with a limited number of degrees of freedom), while thermodynamics (domain *par excellence* for the relevance, appearance and utilisation of k_B) addresses situations where these elements are very numerous (a large number of degrees of freedom), making it dependent on a statistical mechanics approach (however, see point 2 above for a brief commentary regarding what Louis de Broglie has called the 'thermodynamics of the isolated particle').

3.3. *Final remarks on quantum randomness*

Clearly, it is not possible for us to provide a real conclusion: our approach is partly conjectural, and there is a lack of general mathematical theorems (on the relationships between classical randomness and chaos) to give it sufficient support. Further developments in the physical theories themselves are also needed, and these could nurture a vision of contemporary physics that is thoroughly unified, or at least sufficiently objective and discriminating with regard to the processes at play in quantum interactions. Currently, the dominant paradigm (but this is not to say that it is necessarily definitely established) is generally in line with the conceptions we have presented. But the search for a causal interpretation of quantum physics referring, for instance, to a (continuous) sub-quantum environment is not finished, even if one might consider that, since its inception, it has not been particularly fruitful, while progress in other directions has proved quite rich. We would above all like to avoid finding ourselves overdetermined by *a priori* views that are too ideological in nature, as this might make us seek and thus defend a total determinism or, conversely, an essential non-determinism for which, in any case, the relevance, in terms of philosophical implications remains, to be demonstrated. Admitting that, nevertheless, we want to go beyond the operational aspect of the scientific approach in order to question the set of interpretations that it raises (which is also one of our preoccupations), it appears to us to be much more interesting to argue for the spatial/energetical/temporal irreducibility introduced by the non-nullity of the h constant. In fact, this irreducibility is the harbinger of considerable developments that are yet to be established, because we believe it to be correlated with the intrinsic probabilities we have been considering. These are themselves the expression, it appears, of the conditions imposed by non-determinism, by the complex character of the wave function, by non-commutativity, non-separability, and so on, in short, by quantum specificities. It seems that it might be more heuristic and fruitful to argue for a specificity arising from living phenomena, which we know can be reduced at a functional level to physico-chemical processes, but whose mode of existence is not susceptible to the same analysis (however, in order to understand this mode of existence we are led to introduce concepts as specific as those of metabolism, normal/pathological, living/dead, phylogenesis/ontogenesis, and even ‘contingent finality’, see Bailly and Longo (2006) for a discussion along these lines).

Finally, it seems important to emphasise the fact that the objective character we are considering here is a constructed, or *constituted* objectivity. Its construction obviously depends on the standpoint (preparation, measurements, choice of formulations, convenient mathematical structures and principles, and so on). But once constituted, it becomes independent of the viewpoint in the sense that it has recourse to abstract mathematical constants or structures that henceforth prescribe it as much as they describe it. Invariance and stability with respect to a change of coordinate system and measure (view-point) correspond to constructed objectivity in science.

4. Determinism and continuous mathematics

In the preceding sections we have identified ‘random’ and ‘unpredictable’, in both classical and quantum physics. In the classical case, dynamic unpredictability has, in fact, provided us with the definition of randomness, as a consequence of the relationship between mathematical determinism on the one hand and physical measurement on the other. In microscopic physics, we have stressed the fact that randomness is integrated within the theory itself, that it is, in a way, the starting point, being rooted in the peculiar polarity between a knowing subject and the object, and given by means of mathematics, the preparation of the experiments and measurement, all of which lead to an ‘unpredictable’ result, where only a probability is given as the result. In turning now to the role of continuous mathematics we will return to the difference between the two theoretical (and phenomenal) fields, which, when discussing randomness, we have separated into the epistemic and objective (or intrinsic for quantum theory).

When we use the classical viewpoint or tools for the analysis or production of randomness (we observe turbulence, or throw a coin or a die, and so on), a preliminary analysis of the phenomenon or object is possible: we analyse the irregularities and the stabilities of a fluid, we look at the physical structure and symmetries of a coin or a die, and so on. This enables us to ascribe probabilities to the ensuing processes and to make physically well-founded estimates for some elements of these processes. In the case of a coin or a die, the object’s *symmetries* and the set of physical properties enable us to ascribe probabilities to the occurrence of the various possible outcomes before they take place (1/2, 1/6, respectively). In short, it is possible to *separate* the physical object and its properties from the process, to study it before making the measurement relating to the dynamics of interest to us. This is impossible in microscopic physics: prior to measurement, that is, before the process where randomness will be produced, it is impossible to completely determine the physical object and its list of properties; in particular, quantum non-separability prevents us from isolating the 'properties' of a quanton. In microscopic physics, our only form of access to the world lies through the measurement of processes. It is impossible to 'look' at the photon or the electron, in the same way as we can a coin or a die, independently of the processes of production, evolution and measurement, where probabilities are intrinsic to observation.

Even when some information about possible measurements is given at the beginning of the process, for example, when preparing an electron for the measurement of its spin (by setting a direction, one knows *a priori* that it will be 'up' or 'down'), there is no underlying classical theory enabling us to determine, even theoretically, the *exact* spin, before and independently of measurement, as the result of a determined state or even of a trajectory. On the other hand, we can conceive of an analysis of the classical dynamics of a coin that, as the solution to the equations of motion, would describe the coin's exact trajectory, in terms, for example, of the Euclidean lines of the barycentre and of a point on the edge. The different 'hidden variables' approaches in quantum mechanics assumed that these classical theories of real underlying trajectories exist but, as we recalled earlier, they do not escape the non-local aspect of their specifications.

Moreover, the classical theory enables us, at least conceptually, to go to the limit of measurement. Once again, we know very well that measurement in classical physics is always an interval; however, the theory of dynamical systems is given within a framework of continuous mathematics, with Euclidean points and trajectories, which are considered to be zero-width lines. For this reason, as Laplace rightly stated and we have already mentioned, an infinite and perfect intelligence knowing the world point by point could predict everything, including throws of dice. In theory, this boundary continuous framework is, to this day, essential, because the imposing of an *a priori mathematically* finite limit for measurement makes no physical sense classically. This then enables us to *conceive* of determining the whole motion, in particular, the trajectory beginning from a point; it is in this sense that the classical world is deterministic. Thus randomness, as unpredictability, remains epistemic: it is in the relationship between, on the one hand, the tool of knowledge and determinism given by mathematics, and, on the other hand, the actual object, which we assume to be independent and measurable only in a humanly approximate fashion. It is this (presumed) independence that does not occur in quantum physics, where the object and objectivity itself are constituted in the practice of knowing (the preparation of the experiment and measurement, and their mathematisation, or even their principal mathematical consequence: the quantum object).

In order to better highlight the correlated roles of determinism and continuous mathematics within the classical frameworks, and their autonomy relative to the (preconsitituted) physical object, we will return to a previous remark. Mathematics performs the passage to the continuous limit in many abstract constructions that are quite independent of (are at the external boundary of) the 'rational' (ratios between integer magnitudes, for Greek thought). We are thinking, for instance, of the sequences of rational numbers converging towards an irrational, for example, $\sqrt{2}$: this theoretical limit *produces* $\sqrt{2}$, which is not a rational number. It therefore exists within a conceptual universe of actual limits (the geometric construction of $\sqrt{2}$, which so deeply troubled our colleagues in Greece that it led some of them to the brink of suicide, is the true beginning of mathematics). The understanding stemming from real numbers *à la* Cantor–Dedekind provided a definitive foundation for the mathematics of continua, or provided this continuum made of points beyond this world with an immense mathematical stability and conceptual invariance. It has also enabled the developments of modern physics, where the infinite and the passage to the limit play a crucial role well beyond differential calculus (see Bailly and Longo (2006)). However, it is clear that this does not have any 'physical sense' if we are referring to physical measurement, despite the fact that differential and algebraic calculus, in the continuous framework, are at the centre of classical physico-mathematical *determinism*. In short, classical determination is a 'limit notion' and has long sat at the core of the mathematics of continua and mathematical physics.

The dimensionless points, Euclid told us, are the exact departing points for trajectories, for, he continued, *zero-width* lines determined by equations, as Newton, Laplace and Einstein explained. It is, therefore, continuous mathematics that enables us to conceive, on the one hand, of theoretically perfect classical determinations, and on the other, of the unpredictability of physical evolutions that are sensitive to the boundary conditions and affected by the unavoidable imprecision of physical measurement. It is the very idea of a *conceptually* possible continuous substrate that highlights the approximation of measurement: a universe in which the spacetime is discrete, digital for instance, would be exact, because it would allow for exact measurements, digit by digit, as *separable* points, with exact access to information, just as the digital machine accesses its databases.

Let us note that even in turbulence theory, the framework provided by the Navier–Stokes equations is continuous, and therefore deterministic, in this limited sense, although specific to highly unpredictable phenomena (with a few underlying important difficulties for prediction, even of the theoretical type, due to the absence, even today, of a proof of the uniqueness of solutions and therefore for the uniqueness of the possible trajectory, once we are given the boundary conditions).

Yet none of this carries over to quantum physics, as we have argued: it seems impossible to pass to the threshold of possible measurement, to refer to a continuous substrate, even a purely conceptual one. The theory begins with Planck's h constant, which provides a theoretical lower bound for measurement and the non-determinism intrinsic in the mathematics of quantum mechanics. There are not, in quantum space, any dimensionless points to act as possible departure points for zero-width trajectories: they are proscribed by the theory. In fact, there are, in the classical sense, no trajectories whatsoever within space-time: it is here we find the radical watershed constituted by quantum physics, after two thousand years of the physics of trajectories, from Aristotle to Galileo and Newton to Einstein. In this sense, randomness becomes intrinsic to the theory and participates in the construction of objectivity, and itself becomes 'objective'.

5. Conclusion: towards computability

In this paper we have tried to understand the notion of randomness as unpredictability in two different theoretical frames (classical dynamics and quantum mechanics). We have stressed fact that the epistemic nature of classical chaotic dynamics is based on:

— possible alternative understandings (deterministic or purely statistical);

— an underlying continuum structure that allows one to conceive of perfect determinism, in the sense of complete infinitary predictability (God, the theory says, who knows the world by Cantorian points, one by one, would be able to predict all future events).

In contrast to this, in the (prevailing) interpretation of quantum mechanics, there is no alternative to measurement as probability value, nor are there any (hidden) continuous variables, thus we are led to the 'theoretically objective' nature of quantum randomness (there is no such conceivable God in quantum theories). Moreover, the entanglement effects (non-separability and non-locality) are at the core of the prevailing interpretation of quantum physics and, as stressed above, they contribute the peculiar nature of quantum randomness. This interpretation is also the basis of current approaches to quantum computing and cryptography (deterministic hidden variables are incompatible with current security quantum protocols).

Many computer scientists are familiar with the mathematical definition of randomness proposed by Martin-Löf, which is related to the Kolmogorof approach and has been widely developed by Chaitin and others. Briefly, Martin-Löf used classical computability theory to give a notion of 'passing any effective statistical test' and used this to define the so-called 'infinite ML-random sequences'. We will not give a formal presentation here, but just quote one of the major consequences, which highlights the sense of the approach:

An infinite ML-random sequence has no infinite recursively enumerable subsequence.

The meaning of this strong non-computability property should be clear: there is no way to predict or compute infinitely many values of the sequence. As a matter of fact, if you had a total recursive function that could output the date and the results of infinitely many Lotto or Bingo games, you would be very happy. Infinitely many, of course, otherwise your sequence would be 'just' eventually random, which is mathematically the same, for infinite sequences.

How does this mathematical definition relate to our analysis within physics in terms of dynamics or *processes*? In the mathematical approach based on computability, there is no reference whatsoever to an underlying physical process generating the sequence, be it the Lotto, dice or coin tossing. In this way it is truly general, and it applies just as well to a quantum sequence of, say, an electron's spin-up/spin-down, interpreted as 0s and 1s (of course, this latter sequence is random or unpredictable for rather different reasons, as we have dsicussed at length: in particular, the measurement itself contributes to producing the states).

In other words, if we are given a chaotic deterministic dynamics or a quantum phenomenon to which we can associate an infinite sequence of integers, we can say it is ML-random. Clearly, one has to specify how we obtain a sequence from the intended process (for example, by writing numbers or signs on the dice or coins, by quantum measurement, and so on). A precise statement of this implication is still to be given (that is, a characterisation of the dynamics or the processes whose chaotic behaviour produces exactly ML-random sequences). In the case of a deterministic dynamics in a chaotic regime, one has to wait long 'enough' for unpredictability to pop out, once we have associated a measure and values to the process (some techniques, such as the analysis of Lyapounov exponents, can give an estimate of this time: for an ago-antagonistic process modelled by the logistic functions with a minimal level of observability of, say, 10^{-15}, one has to wait at least 50 iterations before the kneading sequence becomes ML-random; for the evolution of the solar system, after 1 million years the iterated analysis of the position of Pluto still being *in* or *out* of the system is an ML-random sequence).

The use of recursion theory allowed Martin-Löf to put previous work on randomness (which was done before the invention or a sufficiently widespread use of this theory) onto firm foundations. The notion of 'passing any *effective* statistical test', and its consequence in terms of the non-existence of recursively enumerable subsequences, clarifies two major aspects of randomness *as* unpredictability.

First, it provides a framework in which one can show that unpredictability is *stronger* than undecidability. As we have already mentioned, both classical and quantum unpredictability, which we identified with physical randomness in their contexts, yield ML-randomness as a mathematical notion. In turn, the latter implies a strong form of undecidability for a sequence: it cannot contain any recursively enumerable subsequence. It is then fair to say that unpredictability is stronger than undecidability, as *non*-recursive enumerability, in a context where these two notions can be compared.

Second, the Martin-Löf approach allows us to better understand the interplay between subject and object in (scientific) knowledge. There is neither randomness *in* nature, nor unpredictability. The world is not random nor unpredictable, *per se*: this simply makes no sense. Both randomness and unpredictability pop out of the relationship between the world and a knowing subject: in order to predict, one needs someone to *pre-dicere* (latin for *to say in advance*). If nobody says, there is no unpredictability nor physical randomness. In quantum physics, we do not even have a 'physical object' without measurement and mathematics. Now, recursion theory is an eminently linguistic theory: it was born as and is a matter of algorithms, given in words within formal systems, over sequences of letters or of 0s and 1s. In this way, it also makes an important contribution to the analysis of physical randomness, when it is defined, as we did, in terms of unpredictability. As a matter of fact, ML-randomness is based on the notion of effectiveness for a *test*, that is, for the activity of someone who wants to test or try to predict the evolution of a sequence. However, in no way does the relevance of computability for the analysis of this interplay between a knowing subject and the world prove that there is anything intrinsically computational in the world, as many claim. On the contrary, it serves only to set a limit on our linguistic (algorithmic) effort *to say* something about the world (to predict), and it does this through a *negative* notion of mathematical *un*predictability or ML-randomness.

References

Adler, R. L. (1979) *Topological entropy and equivalence of dynamical systems*, American Mathematical Society.

Alligood, K., Sauer, T. and Yorke, J. (2000) *Chaos: an introduction to Dynamical Systems*, Springer.

Anandan, J. (2002) Causality, Symmetries and Quantum Mechanics. *Foundations of Physics Letters* **15** (5) 415–438.

Aspect, A., Grangier, P. and Roger, G. (1982) Experimental Realization of the Einstein–Podolsky–Rosen–Bohm Gedankenexperiment: A New Violation of Bell's Inequalities. *Phys. Rev. Let.* **49** 91.

Bailly, F. and Longo, G. (2006) *Mathématiques et Sciences de la Nature. La singularité physique du vivant*, Hermann, Paris. (English introduction available at `http://www.di.ens.fr/users/longo`.)

Bell, J. S. (1964) On the Einstein–Podolsky–Rosen Paradox. *Physics* **1** 195.

Bitbol, M. (2000a) *L'aveuglante proximité du réel*, Flammarion.

Bitbol, M. (2000b) *Physique et philosophie de l'esprit*, Flammarion.

Bohm, D. (1951) The Paradox of Einstein, Rosen and Podolsky. *Quantum Th.* 611–623.

Bohm, D. (1987) *La plénitude de l'univers*, Le Rocher, Paris.

de Broglie, L. (1961) *Introduction à la nouvelle théorie des particules de M. Jean-Pierre Vigier et de ses collaborateurs*, Gauthier-Villars.

Devaney, R. L. (1989) *An Introduction to chaotic dynamical systems*, Addison-Wesley.

Einstein, A., Podolsky, B. and Rosen, N. (1935) Can Quantum-Mechanical Description of Physical Reality be Considered complete? *Phys. Rev.* **41** 777.

van Fraassen, B. (1994) *Lois et symétrie*, Vrin.

Laskar, J. (1990) The chaotic behaviour of the solar system. *Icarus* **88** 266–291.

Laskar, J. (1994) Large scale chaos in the solar system. *Astron. Astrophys.* **287** L9 L12.

Lebowitz, J. (1999) Microscopic Origins of Irreversible Macroscopic Behavior. *Physica A* **CCLXIII** 516.

Lighthill, J. (1986) The recent recognized failure of predictability in Newtonian dynamics. *Proc. R. Soc. Lond. A* **407** 35–50.

Longo, G. (2007) Laplace, Turing and the 'imitation game' impossible geometry: randomness, determinism and programs in Turing's test. In: Epstein, R., Roberts, G. and Beber, G. (eds.) *The Turing Test Sourcebook*, Kluwer.

Mugur-Schachter, M. (2006) *Sur le tissage des connaissances*, Hermès-Lavoisier.

Pilyugin, S. Yu. (1999) *Shadowing in dynamical systems*, Springer.

Poincaré, H. (1902) *La Science et l'Hypothèse*, Flammarion, Paris.

Sauer, T. (2003) Shadowing breakdown and large errors in dynamical simulations of physical systems. Preprint, George Mason University.

Weyl, H. (1918a) *Das Kontinuum*, 1918a.

Weyl, H. (1918b) Raum–Zeit–Materie, Springer.

Zurek, W. H. (1991) Decoherence and the Transition from the Quantum to the Classical. *Physics Today* **44** 36–44.

End of the Special Issue on 3-body, classical quantum, discrete-continuous

https://doi.org/10.1017/S0960129507006007 Published online by Cambridge University Press