Ergod. Th. & Dynam. Sys. (2021), **41**, 1156–1159
doi:10.1017/etds.2019.111
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A note on the topology of escaping endpoints

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(*Received* 31 *August* 2019 *and accepted in revised form* 12 *November* 2019)

Abstract. We study topological properties of the escaping endpoints and fast escaping endpoints of the Julia set of complex exponential $\exp(z)+a$ when $a\in(-\infty,-1)$. We show neither space is homeomorphic to the whole set of endpoints. This follows from a general result stating that for every transcendental entire function f, the escaping Julia set $I(f)\cap J(f)$ is first category.

Key words: Julia set, escaping endpoints, almost zero-dimensional
2010 Mathematics Subject Classification: 37F10 (Primary); 30D05, 54E52 (Secondary)

1. *Introduction*

For each $a\in(-\infty,-1)$, define $f_a:\mathbb{C}\to\mathbb{C}$ by $f_a(z)=e^z+a$. The Julia set $J(f_a)$ is known to be a *Cantor bouquet* consisting of an uncountable union of pairwise disjoint rays, each joining a finite endpoint to the point at infinity [**7**, p. 50]; see Figure 1. Let $E(f_a)$ denote the set of finite endpoints of these rays. Mayer proved $E(f_a)\cup\{\infty\}$ is connected, even though $E(f_a)$ is totally disconnected [**13**]. The one-point compactification $J(f_a)\cup\{\infty\}$ is a Lelek fan [**1**], so $E(f_a)$ is actually homeomorphic to the 'irrational Hilbert space' $\mathfrak{E}_c:=\{x\in\ell^2:x_i\notin\mathbb{Q}\text{ for each }i<\omega\}$ [**12**], which is *almost zero-dimensional* [**9**, **14**]. This means $E(f_a)$ has a basis of open sets whose closures are intersections of clopen sets. We note that by [**8**, Theorem 3.1] and [**1**, **5**, **6**], an almost zero-dimensional space X has a one-point connectification if and only if X is homeomorphic to a dense set $X'\subseteq E(f_a)$ with the property that $X'\cup\{\infty\}$ is connected.

Alhabib and Rempe-Gillen recently discovered that $\dot{E}(f_a)\cup\{\infty\}$ is connected, where $\dot{E}(f_a)$ is the set of *escaping endpoints* of $J(f_a)$ [**2**, Theorem 1.4]. The even smaller set of *fast escaping endpoints* $\ddot{E}(f_a)$ also has the property that its union with $\{\infty\}$ is connected [**2**, Remark p. 68]. More can be said about the topologies of $\dot{E}(f_a)\cup\{\infty\}$ and $\ddot{E}(f_a)\cup\{\infty\}$ based on [**8**]. For example, $\dot{E}(f_a)\cup\{\infty\}\setminus K$ is connected for every σ-compact set $K\subseteq\dot{E}(f_a)$ [**8**, Theorem 4.6]. The primary goal of this paper is to investigate whether $\dot{E}(f_a)$ and $\ddot{E}(f_a)$ are topologically equivalent to $\mathfrak{E}_c$ or the 'rational Hilbert space' $\mathfrak{E}:=\{x\in\ell^2:x_i\in$

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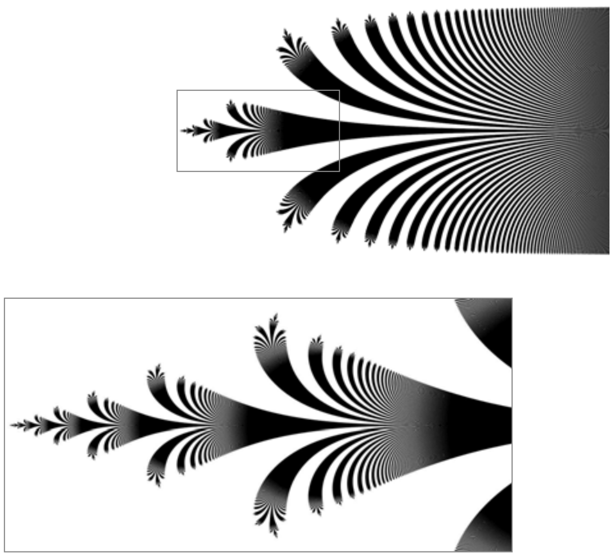

FIGURE 1. Partial images of $J(f_{-2})$.

$\mathbb{Q}$ for each $i < \omega\}$. We show both sets are first category in themselves, implying neither space is homeomorphic to $\mathfrak{E}_c$. We also show $\ddot{E}(f_a) \not\simeq \mathfrak{E}$. It is presently unknown whether $\dot{E}(f_a)$ is homeomorphic to $\mathfrak{E}$.

2. *Preliminaries*

Let f be an entire function.

- A set $X \subseteq \mathbb{C}$ is:

  backward-invariant under f provided $f^{-1}[X] \subseteq X$;
  forward-invariant under f provided $f[X] \subseteq X$; and
  completely invariant under f if $f^{-1}[X] \cup f[X] \subseteq X$.
- The *backward orbit* of a point $z \in \mathbb{C}$ is the union of pre-images

  $$O^-(z) = \bigcup \{f^{-n}\{z\} : n < \omega\}.$$

  The *forward orbit* of z is the set $O^+(z) = \{f^n(z) : n < \omega\}$.
- A point $z \in \mathbb{C}$ is *exceptional* if $O^-(z)$ is finite. There is at most one exceptional point [**4**, p. 6].
- $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$ is called the *escaping set* for f.
- Define the maximum modulus function $M(r) := M(r, f) = \max\{|f(z)| : |z| = r\}$ for $r \geq 0$. Choose $R > 0$ sufficiently large such that $M^n(R) \to +\infty$ as $n \to \infty$ and let $A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R)$ for all $n \geq 0\}$. The *fast escaping set* for f is defined to be the increasing union of closed sets

  $$A(f) = \bigcup_{n \geq 0} f^{-n}[A_R(f)].$$

  It can be shown that the definition of $A(f)$ is independent of the choice of R when f is transcendental [**15**, Theorem 2.2].
- Note that $J(f)$, $I(f)$, and $A(f)$ are completely invariant under f.

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Recall that for each $a \in (-\infty, -1)$, the endpoint set of $J(f_a)$ is denoted by $E(f_a)$. We let

$$\dot{E}(f_a) = I(f_a) \cap E(f_a),$$
$$\ddot{E}(f_a) = A(f_a) \cap E(f_a)$$

denote the *escaping endpoints* and *fast escaping endpoints* of $J(f_a)$, respectively.

3. *Results for transcendental entire functions*

In this section we assume f is a transcendental entire function, so that $I(f) \cap J(f) \neq \varnothing$ [**10**, Theorem 2]. We will make repeated use of [**4**, Lemma 4], which states that $\overline{O^-(z)} = J(f)$ for each non-exceptional point $z \in J(f)$. This is a simple consequence of Montel's theorem. A topological space X is *first category* if X can be written as the union of countably many (closed) nowhere dense subsets.

THEOREM 3.1. *Every completely invariant subset of $I(f) \cap J(f)$ is first category.*

Proof. Let $X \subseteq I(f) \cap J(f)$ be completely invariant under f. Let $R = |z_0| + 1$ for some $z_0 \in J(f)$. For each $n < \omega$, let $X_n = \{z \in X : |f^k(z)| \geq R$ for all $k \geq n\}$. Since $X \subseteq I(f)$, we have $X = \bigcup\{X_n : n < \omega\}$. It remains to show each X_n is nowhere dense in X. To that end, fix $n < \omega$. Let U be any open subset of $J(f)$ such that $U \cap X \neq \varnothing$. We will show $U \cap X \not\subseteq X_n$.

For any point $z \in I(f)$, the forward orbit $O^+(z)$ is infinite. Since X is forward-invariant, it contains $O^+(z)$ when $z \in X$. We assume X is non-empty, so X is infinite. There is at most one exceptional point by Picard's theorem, so there is a non-exceptional point $z_1 \in X$. By [**4**, Lemma 4], $O^-(z_1)$ contains a dense subset of $\{z \in J(f) : |z| < R\}$, which is a perfect set [**4**, Theorem 3]. So there is a non-exceptional point $z_2 \in O^-(z_1)$ with $|z_2| < R$. The set of repelling periodic points is a dense subset of $J(f)$ [**4**, Theorem 4]. Since $I(f)$ contains no periodic point, we have $\overline{J(f) \setminus I(f)} = J(f)$. For each $k < \omega$, we also note that $f^{-k}\{z_2\}$ is closed and $f^{-k}\{z_2\} \subseteq I(f)$. So each pre-image $f^{-k}\{z_2\}$ is nowhere dense in $J(f)$. Therefore $V := U \setminus \bigcup\{f^{-k}\{z_2\} : 0 \leq k < n\}$ is a non-empty open subset of $J(f)$. By [**4**, Lemma 4] there exists $k < \omega$ such that $f^{-k}\{z_2\} \cap V \neq \varnothing$. Then $k \geq n$ and $f^{-k}\{z_2\} \cap U \neq \varnothing$. Let $z_3 \in f^{-k}\{z_2\} \cap U$. Then $|f^k(z_3)| = |z_2| < R$, so $z_3 \notin X_n$. Since X is backward-invariant, $z_3 \in (U \cap X) \setminus X_n$ as desired. Clearly X_n is a relatively closed subset of X. We conclude that $(U \cap X) \setminus X_n$ is a non-empty X-open subset of $U \cap X$ missing X_n. Recall U was an arbitrary open subset of $J(f)$ intersecting X, so this proves X_n is nowhere dense in X. □

COROLLARY 3.2. *$I(f) \cap J(f)$ is first category.*

Proof. Theorem 3.1 applies since $I(f) \cap J(f)$ is completely invariant under f. □

COROLLARY 3.3. *$J(f) \setminus I(f)$ is not first category.*

Proof. $J(f)$ is a closed subset of $\mathbb{C}$, and is therefore not the union of two first category sets. Since $I(f) \cap J(f)$ is first category (Corollary 3.2), $J(f) \setminus I(f)$ is not. □

4. *Applications to complex exponentials f_a*

THEOREM 4.1. *$I(f_a)$, $A(f_a)$, $\dot{E}(f_a)$, and $\ddot{E}(f_a)$ are first category.*

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Proof. These are completely invariant subsets of $I(f_a)$. And $I(f_a) \subseteq J(f_a)$; this actually holds for all $a \in \mathbb{C}$ [**11**, §2]. So Theorem 3.1 applies to each set. □

Remark 4.1. Complete invariance of $\dot{E}(f_a)$ was also applied in [**2**, p. 68] to generalize the main result in [**13**].

THEOREM 4.2. *Neither $\dot{E}(f_a)$ nor $\ddot{E}(f_a)$ is homeomorphic to $E(f_a)$.*

Proof. $E(f_a)$ is completely metrizable (recall $E(f_a) \simeq \mathfrak{E}_c$, which is a G_δ-subset of ℓ^2), so by the Baire category theorem $E(f_a)$ is not first category. The result now follows from Theorem 4.1. □

THEOREM 4.3. $\ddot{E}(f_a) \not\simeq \mathfrak{E}$.

Proof. $\ddot{E}(f_a)$ is an absolute $G_{\delta\sigma}$-space because $A(f_a)$ and $E(f_a)$ are F_σ and G_δ subsets of $\mathbb{C}$, respectively. On the other hand, $\mathfrak{E}$ is not absolute $G_{\delta\sigma}$ because it has a closed subspace homeomorphic to $\mathbb{Q}^\omega$; see [**9**, p. 23]. □

QUESTION 1. *Is $\dot{E}(f_a)$ homeomorphic to $\mathfrak{E}$?*

QUESTION 2. *Is $\ddot{E}(f_a)$ homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$?*

Acknowledgements. Lasse Rempe-Gillen supplied the graphics for Figure 1. Philip Rippon indicated a strengthening of Corollary 3.3. Namely, $J(f) \setminus I(f)$ contains a dense G_δ-subset of $J(f_a)$ by [**3**, Lemma 1].

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