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A group-theoretic generalization of the p-adic local monodromy theorem

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Abstract. Let G be a connected reductive group over a p-adic number field F. We propose and study the notions of G-φ-modules and G-(φ,∇)-modules over the Robba ring, which are exact faithful F-linear tensor functors from the category of G-representations on finite-dimensional F-vector spaces to the categories of φ-modules and (φ,∇)-modules over the Robba ring, respectively, commuting with the respective fiber functors. We study Kedlaya's slope filtration theorem in this context, and show that G-(φ,∇)-modules over the Robba ring are "G-quasi-unipotent," which is a generalization of the p-adic local monodromy theorem proved independently by Y. André, K. S. Kedlaya, and Z. Mebkhout.

1 Introduction

Let p be a prime number and q a power of p. Let K be a complete nonarchimedean discretely valued field of characteristic 0 equipped with an automorphism φ, the *Frobenius*, inducing the q-power map on the residue field $\kappa \supseteq \mathbb{F}_q$. We also require K to be unramified over the fixed subfield F under φ. See Hypothesis 2.1 for a concrete example.

The *Robba ring* $\mathcal{R} = \mathcal{R}(K,t)$ is the ring of bidirectional power series $\sum_{i\in\mathbb{Z}} c_i t^i$ in one variable t with coefficients in K which converge in an annulus $[\alpha,1)$ for some series-dependent $0<\alpha<1$. The Robba ring $\mathcal{R}$ is endowed with an absolute Frobenius lift φ which extends the Frobenius on K and lifts the q-power map on $\kappa((t))$, and with the derivation $\partial = d/dt$.

A (φ,∇) *-module* over $\mathcal{R}$ is a triple (M,Φ,∇), where M is a finite free $\mathcal{R}$-module, Φ is a *Frobenius*, i.e., a φ-linear endomorphism of M whose image spans M over $\mathcal{R}$, and $\nabla\colon M \to M\otimes_{\mathcal{R}} \mathcal{R}dt$ is a connection. Moreover, Φ and ∇ should satisfy the *gauge compatibility condition*, which says that, after choosing an $\mathcal{R}$-basis for M, the actions Φ and ∇ are given by matrices A and N, respectively, and these matrices should satisfy $N = \boldsymbol{\mu}\cdot A(\varphi(N))A^{-1} - \partial(A)A^{-1}$, where $\boldsymbol{\mu} := \partial(\varphi(t))$.

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The (φ, ∇)-modules, also known as the *overconvergent* F *-isocrystals* in the literature, are closely related to p-adic local systems on $\operatorname{Spec} \kappa((t))$ (for a summary, we refer to [13]), for which the correct monodromy theorem is the *p-adic local monodromy theorem* (pLMT), conjectured by Crew [5], and proved independently by André [1], Kedlaya [9], and Mebkhout [17]. It states that every (φ, ∇)-module over $\mathcal{R}$ is quasi-unipotent. Concretely, a (φ, ∇)-module M over $\mathcal{R}$, after an étale extension to $\mathcal{R}_L$ (the Robba ring canonically associated to some finite separable extension L of $\kappa((t))$), admits a filtration by (φ, ∇)-submodules such that the connections induced on the gradiation are trivial. A matricial description of the theorem is given as follows. Let d be the rank of M over $\mathcal{R}$, and let $A \in \mathrm{GL}_d(\mathcal{R})$ (resp. $N \in \mathrm{Mat}_{d,d}(\mathcal{R})$) be the matrix of Φ (resp. ∇) in some basis. Then, there exists $B \in \mathrm{GL}_d(\mathcal{R}_L)$ such that $BNB^{-1} - \partial(B)B^{-1}$ is an upper-triangular block matrix with zero blocks in the diagonal.

We mention two applications of the pLMT in p-adic Hodge theory.

- In [3], Berger associated to every p-adic de Rham representation V a (φ, ∇)-module $\mathrm{N}_{\mathrm{dR}}(V)$ over a Robba ring. He showed that V is potentially semistable if and only if $\mathrm{N}_{\mathrm{dR}}(V)$ is quasi-unipotent. Using the pLMT, he could prove the p-adic monodromy theorem (previously a conjecture of Fontaine): every p-adic de Rham representation is potentially semistable.
- In [16], Marmora used the pLMT to construct a functor from the category of (φ, ∇)-modules over $\mathcal{R}$ to that of K^{nr}-valued Weil–Deligne representations of the Weil group $\mathcal{W}_{\kappa((t))}$, where K^{nr} is the maximal unramified extension of K in a fixed algebraic closure of K.

Rather than the general linear group, a Galois representation may take value in some connected reductive group G, such as a special linear group or a symplectic group. In order to have appropriate formulations of the above results in this context, it is helpful to establish a G-version of the pLMT, which is the main motivation of our present paper.

In this paper, we introduce the notion of *G-φ -modules over* $\mathcal{R}$ (resp. *G-(φ, ∇) -modules over* $\mathcal{R}$), which are exact faithful F-linear tensor functors from the category $\mathbf{Rep}_F(G)$ of G-representations on finite-dimensional F-vector spaces to the category $\mathbf{Mod}_{\mathcal{R}}^{\varphi}$ of φ-modules over $\mathcal{R}$ (resp. to the category $\mathbf{Mod}_{\mathcal{R}}^{\varphi,\nabla}$ of (φ, ∇)-modules over $\mathcal{R}$), commuting with the respective fiber functors. These constructions are inspired by that of *G-isocrystals* introduced in [6, Section IX.1].

Before coming to the main theorem, we first explain the group-theoretic gauge compatibility condition (Definition 4.6). Let G be an affine algebraic F-group, and let $\mathfrak{g}$ be its Lie algebra. For any $y \in G(\mathcal{R})$ and $Y \in \mathfrak{g} \otimes_F \mathcal{R}$, we define $\Gamma_y(Y) := \mathrm{Ad}(y)(Y) - \mathrm{dlog}(y)$, where $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ is the adjoint representation, and $\mathrm{dlog}\colon G(\mathcal{R}) \to \mathfrak{g} \otimes_F \mathcal{R}$ is defined in Construction 4.4. We say $g \in G(\mathcal{R})$ and $X \in \mathfrak{g} \otimes_F \mathcal{R}$ satisfy the *gauge compatibility condition* if $X = \Gamma_g\big(\boldsymbol{\mu}\, \varphi(X)\big)$. When $G = \mathrm{GL}_d$, we have $\mathrm{Ad}(y)(Y) = yYy^{-1}$ and $\mathrm{dlog}(y) = \partial(y)y^{-1}$. In this context, the group-theoretic gauge compatibility condition coincides with the aforementioned matrical one.

Our main theorem is the following G-version of the pLMT.

Theorem 1.1 (Theorem 4.21) *Let G be a connected reductive F-group, and let $\mathfrak{g}$ be its Lie algebra. If $g \in G(\mathcal{R})$ and $X \in \mathfrak{g} \otimes_F \mathcal{R}$ satisfy $X = \Gamma_g\big(\boldsymbol{\mu}\,\varphi(X)\big)$, then there exists a finite separable extension L over $\kappa((t))$ and an element $b \in G(\mathcal{R}_L)$ such that $\Gamma_b(X) \in$* $\operatorname{Lie}\big(U_{G_{\mathcal{R}}}(-\lambda_g)\big) \otimes_{\mathcal{R}} \mathcal{R}_L$.

Here, $\lambda_g\colon \mathbb{G}_{m,\mathcal{R}} \to G_{\mathcal{R}}$ is a cocharacter associated to g whose reciprocal is denoted by $-\lambda_g$, and $U_{G_{\mathcal{R}}}(-\lambda_g)$ denotes the unipotent radical of the parabolic subgroup of $G_{\mathcal{R}}$ associated to $-\lambda_g$. When $G = \mathrm{GL}_d$, g (resp. X) should be thought as the matrix of the Frobenius (resp. the matrix of the connection), and $\Gamma_b(\_)$ as the matrix of a connection under the change-of-basis via b^{-1}. Moreover, $\operatorname{Lie}\big(U_{G_{\mathcal{R}}}(-\lambda_g)\big) \otimes_{\mathcal{R}} \mathcal{R}_L$ consists of upper-triangular matrices over $\mathcal{R}_L$ with zero blocks (of certain sizes) in the diagonal. As such, Theorem 1.1 recovers the matricial pLMT described above.

In Proposition 4.9, we show that G-(φ,∇)-modules over $\mathcal{R}$ are indeed pairs (g, X) subject to the gauge compatibility condition in the theorem. In this sense, the theorem can be interpreted as saying that G-(φ,∇)-modules over $\mathcal{R}$ are "G-quasi-unipotent." In Examples 4.10 and 4.11, we give examples of the existence of such pairs for G a special linear group and a symplectic group, respectively.

More examples of G-(φ,∇)-modules are expected from Berger's functor N_{dR} mentioned previously. Explicitly, we hope to show in a future work that if a p-adic de Rham representation V takes value in a connected reductive group G, then $\mathrm{N}_{\mathrm{dR}}(V)$ is a G-(φ,∇)-module. As another future work, we intend to use Theorem 1.1 to formulate a G-version of Marmora's functor, namely, to construct a functor from the category of G-(φ,∇)-modules over $\mathcal{R}$ to that of Weil–Deligne representations of the Weil group $\mathcal{W}_{\kappa((t))}$ taking value in $G(K^{\mathrm{nr}})$.

Our approach to the theorem closely follows that of the pLMT in [9] for absolute Frobenius lifts, wherein the author used his slope filtration theorem (along with applying the pushforward functor and twisting to each quotient of the filtration) to reduce the problem to the unit-root case, and then apply the unit-root pLMT attributed to Tsuzuki [23] to finish. More precisely, we use Kedlaya's slope filtration theorem to construct a $\mathbb{Q}$-filtered fiber functor HN_g from $\mathbf{Rep}_F(G)$ to $\mathbb{Q}$-$\mathbf{Fil}_{\mathcal{R}}$, the category of $\mathbb{Q}$-filtered modules over $\mathcal{R}$ (see Theorem 3.4). We then reduce HN_g to a $\mathbb{Z}$-filtered fiber functor $\mathrm{HN}_g^{\mathbb{Z}}$ from $\mathbf{Rep}_F(G)$ to $\mathbb{Z}$-$\mathbf{Fil}_{\mathcal{R}}$, the category of $\mathbb{Z}$-filtered modules over $\mathcal{R}$ (see Lemma 3.10). Then, a result of Ziegler (Theorem 2.12) immediately implies that $\mathrm{HN}_g^{\mathbb{Z}}$ is *splittable*, i.e., factors through a $\mathbb{Z}$-graded fiber functor (see Proposition 3.11). In particular, for any splitting of $\mathrm{HN}_g^{\mathbb{Z}}$, we construct a morphism $\lambda_g\colon \mathbb{G}_{m,\mathcal{R}} \to G_{\mathcal{R}}$ of $\mathcal{R}$-groups in Section 3.4, which is called the $\mathbb{Z}$ *-slope morphism* of g. With this, we can reduce the G-(φ,∇)-module (g, X) over $\mathcal{R}$, involving the (generalized) pushforward functor and twisting, to a unit-root one (see Corollary 4.20). Theorem 1.1 then follows from the unit-root pLMT and a Tannakian argument.

The paper is organized as follows. In Section 2, we set up basic notation and conventions, and then recall some necessary background on the theory of slopes and Tannakian formalism. In Section 3, we study G-φ-modules over the Robba ring, and construct slope morphisms. In Section 4, we consider G-(φ,∇)-modules over the Robba ring, and prove our main result, Theorem 1.1, in the last subsection.

2 Preliminaries

2.1 Notation and conventions

When k is a field, we denote by $\mathbf{Vec}_k$ the category of finite-dimensional k-vector spaces. When R is a k-algebra,[1] we denote by $\mathbf{Mod}_R$ the category of R-modules, and by $\mathbf{Alg}_R$ the category of R-algebras. When $V, W \in \mathbf{Vec}_k$, we write V_R for $V \otimes_k R$, and write $\alpha_R := \alpha \otimes R$, the R-linear extension of α, for all k-linear maps $\alpha\colon V \to W$. When G is an affine algebraic k-group, we denote by $k[G]$ the Hopf algebra of G, by $G_R := G \times_{\operatorname{Spec} k} \operatorname{Spec} R$ the base extension, by $H^1(k, G) := H^1\big(\operatorname{Gal}(k^{\mathrm{sep}}/k), G(k^{\mathrm{sep}})\big)$ the first Galois cohomology set, and by $\mathbf{Rep}_k(G)$ the category of representations of G on finite-dimensional k-vector spaces.

By a reductive k-group, we mean a (not necessarily connected) affine algebraic k-group G such that every smooth connected unipotent normal subgroup of $G_{\bar{k}}$ is trivial, where $\bar{k}$ is an algebraic closure of k.

For the rest of this paper, we work under the following hypothesis.

Hypothesis 2.1 Let p be a prime number and $q = p^f$ an integral power of p. Let F be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_F$, a fixed uniformizer ϖ_F and the residue field $\mathbb{F}_q$ of q elements. Let κ be a perfect field containing $\mathbb{F}_q$. Let $\mathcal{O}_K = \mathcal{O}_F \otimes_{W(\mathbb{F}_q)} W(\kappa)$, where $W(\mathbb{F}_q)$ (resp. $W(\kappa)$) denotes the ring of Witt vectors with coefficients in $\mathbb{F}_q$ (resp. in κ). Then, $K := \operatorname{Frac}(\mathcal{O}_K) \cong F \otimes_{W(\mathbb{F}_q)} W(\kappa)$ is a complete discretely valued field with ring of integers $\mathcal{O}_K$, a uniformizer $\varpi := \varpi_F \otimes 1$ and residue field κ. Let Frob be the ring endomorphism of $W(\kappa)$ induced by the p-power map on κ, and let

$$\varphi := \mathrm{Id}_F \otimes \mathrm{Frob}^f \colon K \longrightarrow K$$

be the *Frobenius automorphism* on K relative to F. Then, φ reduces to the q-power map on κ, and the fixed field of φ on K is $F \otimes_{W(\mathbb{F}_q)} W(\mathbb{F}_q) \cong F$.

2.2 The Robba ring and its variants

For $\alpha \in (0,1)$, we put

$$\mathcal{R}_\alpha := \Big\{ \sum_{i\in\mathbb{Z}} c_i t^i \;\Big|\; c_i \in K,\ \lim_{i\to\pm\infty} |c_i|\rho^i = 0,\ \forall \rho \in [\alpha,1) \Big\}.$$

For any $\rho \in [\alpha,1)$, we define the ρ-Gauss norm on $\mathcal{R}_\alpha$ by setting $\big|\sum_i c_i t^i\big|_\rho := \sup_i\{|c_i|\rho^i\}$. The *Robba ring* is defined to be the union $\mathcal{R} := \mathcal{R}(K,t) := \bigcup_{\alpha\in(0,1)} \mathcal{R}_\alpha$. For any $\sum_i c_i t^i \in \mathcal{R}$, we define $\big|\sum_i c_i t^i\big|_1 := \sup_i\{|c_i|\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the 1-Gauss norm.

The *bounded Robba ring* $\mathcal{E}^\dagger = \mathcal{E}^\dagger(K,t)$ is the subring of $\mathcal{R}$ consisting of bounded elements (i.e., elements with finite 1-Gauss norm), which is actually a Henselian discretely valued field w.r.t. the 1-Gauss norm with residue field $\kappa((t))$.

[1] By an algebra, we always mean a commutative algebra with 1.

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Let $R \in \{\mathcal{R}, \mathcal{E}^\dagger\}$. An *absolute q-power Frobenius lift* on R is a ring endomorphism $\varphi: R \to R$ given by $\sum_{i\in\mathbb{Z}} c_i t^i \longmapsto \sum_{i\in\mathbb{Z}} \varphi(c_i) u^i$ for $u = \varphi(t) \in R$ such that $|u - t^q|_1 < 1$.

For any $\alpha \in (0,1)$, we define $\tilde{\mathcal{R}}_\alpha$ to be the ring of formal sums $\sum_{i\in\mathbb{Q}} c_i t^i$ with $c_i \in K$, subject to the following properties.

- For any $c > 0$, the set $\{i \in \mathbb{Q} \mid |c_i| \geq c\}$ is well-ordered.
- For any $\rho \in [\alpha, 1)$, we have $\lim_{i\to\pm\infty} |c_i|\rho^i = 0$.

For any $\rho \in [\alpha, 1)$, we define the ρ-Gauss norm on $\tilde{\mathcal{R}}_\alpha$ by setting

$$\Big|\sum_i c_i t^i\Big|_\rho = \sup_{i\in\mathbb{Q}}\{|c_i|\rho^i\}.$$

We define $\tilde{\mathcal{R}} := \tilde{\mathcal{R}}(K,t) := \bigcup_{\alpha\in(0,1)} \tilde{\mathcal{R}}_\alpha$, the *extended Robba ring*. The *absolute Frobenius lift* on $\tilde{\mathcal{R}}$ is the ring automorphism on $\tilde{\mathcal{R}}$ given by $\sum_{i\in\mathbb{Q}} c_i t^i \mapsto \sum_{i\in\mathbb{Q}} \varphi(c_i) t^{iq}$. We denote by $\tilde{\mathcal{E}}^\dagger$ the subring of $\tilde{\mathcal{R}}$ consisting of bounded elements. By [11, Proposition 2.2.6], we have a φ-equivariant embedding $\psi : \mathcal{R} \to \tilde{\mathcal{R}}$ such that $|\psi(x)|_\rho = |x|_\rho$ for ρ sufficiently close to 1.

2.3 The slope filtration theorem

We recall Kedlaya's theory of slopes. Let $R \in \{\mathcal{E}^\dagger, \mathcal{R}, \tilde{\mathcal{E}}^\dagger, \tilde{\mathcal{R}}\}$ equipped with a Frobenius lift φ. For the notions of φ-modules and (φ, ∇)-modules over R, we refer to [9, Section 2.5]. We denote by $\mathbf{Mod}_{\mathrm{R}}^{\varphi}$ (resp. $\mathbf{Mod}_{\mathrm{R}}^{\varphi,\nabla}$) the category of φ-modules (resp. (φ, ∇)-modules) over R.

Let $(M, \varphi) \in \mathbf{Mod}_{\mathrm{R}}^{\varphi}$, and let n be a positive integer. Then, (M, φ^n) is a φ^n-module over (R, φ^n). The *n-pushforward functor* is given by

$$[n]_*: \mathbf{Mod}_{\mathrm{R}}^{\varphi} \longrightarrow \mathbf{Mod}_{R}^{\varphi^n}, \qquad (M, \varphi) \longmapsto (M, \varphi^n).$$

For any $s \in \mathbb{Z}$, we define the *twist* $M(s)$ of (M, φ) by s to be the φ-module $(M, \varpi^s \Phi)$.

Now, let M be a φ-module over R of rank d.

(i) We say that M is a *unit-root* φ-module if there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_d$ of M over R in which φ acts via an invertible matrix in $\mathrm{GL}_d(\mathcal{O}_{\mathcal{E}^\dagger})$ if $R \in \{\mathcal{E}^\dagger, \mathcal{R}\}$, or $\mathrm{GL}_d(\mathcal{O}_{\tilde{\mathcal{E}}^\dagger})$ if $R \in \{\tilde{\mathcal{E}}^\dagger, \tilde{\mathcal{R}}\}$.

(ii) Let $\mu = s/r \in \mathbb{Q}$ with $r > 0$ and $(s, r) = 1$. We say that M is *pure of slope* μ if $([r]_* M)(-s)$ is unit-root.

Let $M \in \mathbf{Mod}_{\mathrm{R}}^{\varphi}$. We have a canonical filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$ of φ-submodules over R such that each quotient M_i/M_{i-1} is pure of some slope μ_i with $\mu_1 < \cdots < \mu_l$, by [11, Theorem 1.7.1] if $R = \mathcal{R}$ or [11, Proposition 1.4.15 and Theorem 2.1.8]) if $R = \tilde{\mathcal{R}}$. This is called the *slope filtration* of M. We call $\mu_1, \ldots, \mu_l$ the *jumps* of the slope filtration. The (uniquely determined, not necessarily strictly) increasing sequence $(\mu_1, \ldots, \mu_1, \ldots, \mu_l, \ldots, \mu_l)$, with each μ_i appearing $\mathrm{rk}_R(M_i/M_{i-1})$ times, is said to be the *slope sequence* for M. We call $\mathrm{rk}_R(M_i/M_{i-1})$ the *multiplicity* of μ_i for all $1 \leq i \leq l$.

Moreover, if M is a (φ, ∇)-module over $\mathcal{R}$, then the slope filtration can be refined to a filtration of (φ, ∇)-submodules. This is [9, Theorem 6.12], and is referred to as the *slope filtration theorem for* (φ, ∇) *-modules* over $\mathcal{R}$.

To continue, we need to recall some notions introduced in [12, Section 14]. A *difference ring* (resp. *difference field*) is a ring (resp. field) R equipped with an endomorphism ϕ. A *difference module* over R is an R-module M equipped with a ϕ-linear endomorphism Φ. A finite free difference module M over R is said to be *dualizable* (resp. *trivial*) if M admits a basis over R such that Φ acts via an invertible matrix (resp. the identity matrix). For example, a φ-module over R is a dualizable difference module over R where R is any of the rings constructed in Section 2.2. A dualizable difference module M over R is said to be *standard* if it admits a basis $e_1, \dots, e_d$ such that $e_i = \Phi(e_{i-1})$ for $2 \le i \le d$ and $\Phi(e_d) = \lambda e_1$ for some $\lambda \in R^\times$. A difference field (k, ϕ_k) is called *strongly difference-closed* if ϕ_k is an automorphism and any dualizable difference module over k is trivial.

Let k be a complete nonarchimedean valued field and (k, ϕ_k) is a difference field in which ϕ_k is bijective. An *admissible extension* of (k, ϕ_k) is a difference field (ℓ, ϕ_ℓ), where ℓ is a field extension of k complete for the valuation extending the one on k with the same value group, and ϕ_ℓ is an automorphism of ℓ extending ϕ_k. (See [11, Definition 3.2.1].)

Lemma 2.2 *[15, Lemma 1.5.3] The field K admits an admissible extension E such that the residue field κ_E of E is strongly difference-closed.*

The following lemma is a recollection of some results which will be used in the sequel.

Lemma 2.3 *Let E be an admissible extension of K such that κ_E is strongly difference-closed.*

(i) *Let $M \in \mathbf{Mod}^{\varphi}_{\mathcal{R}}$. Then, tensoring the slope filtration of M with $\tilde{\mathcal{R}}(E, t)$ gives the slope filtration of $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t)$.*
(ii) *Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence of φ-modules over $\tilde{\mathcal{R}}(E, t)$ such that the slopes of M_1 are all less than the smallest slope of M_2. Then, the sequence splits.*
(iii) *Every φ-module over $\tilde{\mathcal{R}}(E, t)$ admits a Dieudonné–Manin decomposition, i.e., it is a direct sum of standard φ-submodules.*
(iv) *Let M and N be φ-modules over $\tilde{\mathcal{R}}(E, t)$. If the slopes of M are all less than the smallest slope of N, then no nonzero morphism from M to N exists.*

Assertion (i) is [15, Proposition 1.5.6]. Assertion (ii) is [15, Proposition 1.5.11], and assertion (iii) is Proposition 1.5.12 in loc. cit. Assertion (iv) is [11, Proposition 1.4.18].

2.4 The Tannakian duality

In this subsection, k denotes a field unless otherwise specified. We follow the definitions and notations in [7]. We denote by ω^G the forgetful functor $\mathbf{Rep}_k(G) \to \mathbf{Vec}_k$, which is called the *fiber functor*.

The following *Tannakian duality* will be repeatedly used in this paper, whose proof can be found, e.g., in [18, Theorem 9.2].

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Theorem 2.4 *Let G be an affine algebraic k-group, and let $R \in \mathbf{Alg}_k$. Suppose that for any $(V, \rho_V) \in \mathbf{Rep}_k(G)$, we are given an R-linear map $\lambda_V \colon V_R \to V_R$. If the family $\{\lambda_V \mid (V, \rho_V) \in \mathbf{Rep}_k(G)\}$ satisfies*

(i) *$\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$ for all $V, W \in \mathbf{Rep}_k(G)$;*
(ii) *$\lambda_{\mathbb{1}}$ is the identity map where $\mathbb{1}$ is the trivial representation on k;*
(iii) *for all G-equivariant maps $\alpha \colon V \to W$, we have $\lambda_W \circ \alpha_R = \alpha_R \circ \lambda_V$.*

Then, there exists a unique $g \in G(R)$ such that $\lambda_V = \rho_V(g)$ for all $(V, \rho_V) \in \mathbf{Rep}_k(G)$.

Corollary 2.5 *Let G be an affine algebraic k-group. We have an isomorphism $G \cong \underline{\mathrm{Aut}}^{\otimes}(\omega^G)$ of affine algebraic k-groups.*

Corollary 2.6 *Let G be a smooth affine algebraic k-group. Let ℓ/k be a field extension, and let $\eta \colon \mathbf{Rep}_\ell(G) \to \mathbf{Vec}_\ell$ be a fiber functor over ℓ. Then, $\underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta)$ is a G-torsor over ℓ. In particular, if $H^1(\ell, G) = \{1\}$ and $G(\ell) \neq \varnothing$, then ω^G is isomorphic to η over ℓ.*

Proof Notice that we have an action

$$\underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta) \times \underline{\mathrm{Aut}}^{\otimes}(\omega^G) \longrightarrow \underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta)$$

by precomposition. By [7, Theorem 3.2(i)], $\underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta)$ is an $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)$-torsor. In particular, it is a G-torsor over ℓ by Corollary 2.5.

Because G is a ℓ-group variety, G-torsors over ℓ are ℓ-varieties by [18, Proposition 2.69], whose isomorphism classes are classified by $H^1(\ell, G)$. It follows from the triviality of $H^1(\ell, G)$ that $\underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta)(\ell) \cong G(\ell)$; hence, $\underline{\mathrm{Hom}}^{\otimes}(\omega^G, \eta)(\ell) \neq \varnothing$. [7, Proposition 1.13] then implies the second assertion. ∎

To end this subsection, we give a Lie algebra version of Theorem 2.4. We start with recalling the notion of the Lie algebra of a k-group functor. (See [8, Chapitre II, Section 4] for more details. Notice that k denotes a ring in loc. cit.)

For any $R \in \mathbf{Alg}_k$, we define the *R-algebra of dual numbers* $R[\varepsilon] := R[X]/(X^2)$. Put $\varepsilon := X + (X^2)$; we then have the canonical projection $\pi_R \colon R[\varepsilon] \to R$, $\varepsilon \mapsto 0$. Let G be a k-group functor. We define

$$\mathrm{Lie}(G)(R) := \mathrm{Ker}\, G(\pi_R).$$

Let $f \colon G \to H$ be a morphism of k-group functors. The commutative diagram

$$\begin{array}{ccc} \mathrm{Lie}(G)(R) = \mathrm{Ker}(G(\pi_R)) & & \mathrm{Lie}(H)(R) = \mathrm{Ker}(H(\pi_R)) \\ \downarrow{\scriptstyle \iota_G} & & \downarrow{\scriptstyle \iota_H} \\ G(R[\varepsilon]) & \xrightarrow{f(R[\varepsilon])} & H(R[\varepsilon]) \\ \downarrow{\scriptstyle G(\pi_R)} & & \downarrow{\scriptstyle H(\pi_R)} \\ G(R) & \xrightarrow{f(R)} & H(R) \end{array} \tag{1}$$

implies that $f(R[\varepsilon]) \circ \iota_G(X) \in \mathrm{Lie}(H)(R)$ for all $X \in \mathrm{Lie}(G)(R)$. We define $\mathrm{Lie}(f) := f(R[\varepsilon]) \circ \iota_G \colon \mathrm{Lie}(G)(R) \to \mathrm{Lie}(H)(R)$. Hence, $\mathrm{Lie}(\_)(R)$ is a functor from the category of k-group functors to that of abelian groups.

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For an affine algebraic k-group G, we write I for the kernel of the counit $\varepsilon_G\colon k[G] \to k$. Because $k[G]$ is Noetherian, I/I^2 is a finite-dimensional vector space over $k \cong k[G]/I$. We then have $\mathrm{Hom}_k(I/I^2, R) \cong \mathrm{Hom}_k(I/I^2, k) \otimes_k R$. By [8, Corollaire II.3.3], we have canonical group isomorphisms $\mathrm{Lie}(G)(R) \cong \mathrm{Hom}_k(I/I^2, R)$ and $\mathfrak{g} = \mathrm{Lie}(G)(k) \cong \mathrm{Hom}_k(I/I^2, k)$, whence $\mathrm{Lie}(G)(R) \cong \mathfrak{g}_R$. The Lie structure on $\mathfrak{g}$ then canonically gives a Lie structure on $\mathfrak{g}_R$ and hence on $\mathrm{Lie}(G)(R)$. We call $\mathrm{Lie}(G)(R)$ the *Lie algebra* of G over R, and will identify it with $\mathfrak{g}_R$. Moreover, $\mathrm{Lie}(\_)(R)$ is a functor from the category of affine algebraic k-groups to that of Lie algebras over R.

Remark 2.7 More generally, let k be a commutative ring with 1 and let G be a smooth k-group scheme. For any k-algebra R, we can similarly define $\mathrm{Lie}(G)(R)$ as above. Because the $\mathscr{O}_G$-module $\Omega^1_{G/k}$ is finite locally free, we have $\mathrm{Lie}(G)(R) \cong \mathrm{Lie}(G)(k) \otimes_k R$ by [8, Proposition II.4.8].

Remark 2.8 For any d-dimensional G-representation (V, ρ_V), we write $\mathfrak{gl}_V := \mathrm{Lie}(\mathrm{GL}_V)(k)$. We then have $\mathfrak{gl}_{V,R} = \{I_d + \varepsilon B \mid B \in \mathrm{Mat}_{d,d}(R)\}$, after choosing a k-basis for V. Then, $I_d + \varepsilon B \mapsto B$ gives a group isomorphism from $\mathfrak{gl}_{V,R}$ to $\mathrm{End}_R(V_R)$. Henceforth, we will identify $\mathrm{Lie}(\rho_V)(X)$ as an endomorphism of V_R, for all $X \in \mathfrak{g}_R$.

Replacing H with GL_V and f with ρ_V in diagram (1), we obtain a morphism $\mathrm{Lie}(\rho_V) = \rho_V(R[\varepsilon]) \circ \iota_G\colon \mathfrak{g}_R \to \mathfrak{gl}_{V,R}$ of Lie algebras over R. Let $(W, \rho_W) \in \mathbf{Rep}_k(G)$, and let $\alpha \in \mathrm{Hom}_G(V, W)$. We then have $\alpha_R \circ \mathrm{Lie}(\rho_V)(X) = \mathrm{Lie}(\rho_W)(X) \circ \alpha_R$ for all $X \in \mathfrak{g}_R$.

Applying the functor $\mathrm{Lie}(\_)(R)$ on both sides of the isomorphism in Corollary 2.5 gives us an isomorphism $\mathfrak{g}_R \cong \mathrm{Lie}(\underline{\mathrm{Aut}}^{\otimes}(\omega^G))(R)$ of Lie algebras over R. The following corollary indicates that elements in $\mathrm{Lie}(\underline{\mathrm{Aut}}^{\otimes}(\omega^G))(R)$ are exactly the derivatives (in the sense of taking derivations of conditions (i–iii) in Theorem 2.4) of elements in $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R)$.

Corollary 2.9 *Let G be an affine algebraic k-group, and let R be a k-algebra. Suppose that for any $(V, \rho_V) \in \mathbf{Rep}_k(G)$, we are given an R-linear endomorphism θ_V of V_R subject to the conditions:*

(i) $\theta_{V \otimes W} = \theta_V \otimes \mathrm{Id}_{W_R} + \mathrm{Id}_{V_R} \otimes \theta_W$ *for all* $V, W \in \mathbf{Rep}_k(G)$*;*
(ii) $\theta_{\mathbb{1}} = 0$*, where* $\mathbb{1} = k$ *is the trivial* G*-representation;*
(iii) $\theta_W \circ \alpha_R = \alpha_R \circ \theta_V$ *for all* $\alpha \in \mathrm{Hom}_G(V, W)$.

Then, there exists a unique element $X \in \mathfrak{g}_R$ *such that* $\theta_V = \mathrm{Lie}(\rho_V)(X)$ *for all* $(V, \rho_V) \in \mathbf{Rep}_k(G)$.

Proof For any $(V, \rho_V) \in \mathbf{Rep}_k(G)$ and $\theta_V\colon V_R \to V_R$, we define an $R[\varepsilon]$-linear map

$$\varepsilon\theta_V\colon V_{R[\varepsilon]} \longrightarrow V_{R[\varepsilon]}, \qquad v \otimes (x + y\varepsilon) \longmapsto \theta_V(v \otimes x) \otimes \varepsilon.$$

We define an $R[\varepsilon]$-linear endomorphism

$$\tilde{\theta}_V := \mathrm{Id}_{V_{R[\varepsilon]}} + \varepsilon\theta_V\colon V_{R[\varepsilon]} \longrightarrow V_{R[\varepsilon]}.$$

Then, $\tilde{\theta}_V \in \mathrm{Lie}(\mathrm{GL}_V)(R) \subseteq \mathrm{GL}_V(R[\varepsilon])$, because $\pi_R(\tilde{\theta}_V) = \mathrm{Id}_{V_R}$.

We claim that the family

$$\left\{\tilde{\theta}_V\colon V_{R[\varepsilon]} \to V_{R[\varepsilon]} \mid (V, \rho_V) \in \mathbf{Rep}_k(G)\right\} \tag{2}$$

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of $R[\varepsilon]$-linear endomorphisms satisfies conditions (i–iii) in Theorem 2.4. Granting this claim for a moment, we conclude that $\tilde{\theta} \in \underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R[\varepsilon])$. In particular, there exists a unique element $X \in G(R[\varepsilon])$ such that $\tilde{\theta}_V = \rho_V(X)$ for all $(V, \rho_V) \in \mathbf{Rep}_k(G)$. Because $\pi_R(\tilde{\theta}) = \mathrm{Id} \in \underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R)$, we have $\tilde{\theta} \in \mathrm{Lie}(\underline{\mathrm{Aut}}^{\otimes}(\omega^G))(R)$. The isomorphism $\mathfrak{g}_R \cong \mathrm{Lie}(\underline{\mathrm{Aut}}^{\otimes}(\omega^G))(R)$ then implies that $X \in \mathfrak{g}_R$. Furthermore, it follows from the construction that $\theta_V = \mathrm{Lie}(\rho_V)(X)$ for all $(V, \rho_V) \in \mathbf{Rep}_k(G)$, and the corollary follows.

It remains to prove the claim. Condition (ii) is clear from the construction. Given $(W, \rho_W) \in \mathbf{Rep}_k(G)$, we compute

$$\begin{aligned}\tilde{\theta}_{V \otimes W} &= \mathrm{Id}_{(V \otimes W)_R} + \varepsilon \theta_{V \otimes W} \\ &= \mathrm{Id}_{(V \otimes W)_R} + \varepsilon(\theta_V \otimes \mathrm{Id}_{W_R} + \mathrm{Id}_{V_R} \otimes \theta_W) \\ &= (\mathrm{Id}_{V_R} + \varepsilon \theta_V) \otimes (\mathrm{Id}_{W_R} + \varepsilon \theta_W) \\ &= \tilde{\theta}_V \otimes \tilde{\theta}_W.\end{aligned}$$

Hence, (2) satisfies condition (i). It remains to show that Theorem 2.4 satisfies condition (iii). Let $\alpha \in \mathrm{Hom}_G(V, W)$. For any $v \otimes (x + y\varepsilon) \in V_{R[\varepsilon]}$, we compute

$$\begin{aligned}\alpha_{R[\varepsilon]} \circ \varepsilon \theta_V(v \otimes (x + y\varepsilon)) &= \alpha_{R[\varepsilon]}(\theta_V(v \otimes x) \otimes \varepsilon) = (\alpha_R \circ \theta_V)(v \otimes x) \otimes \varepsilon \\ &= (\theta_W \circ \alpha_R)(v \otimes x) \otimes \varepsilon = \theta_W(\alpha(v) \otimes x) \otimes \varepsilon \\ &= \varepsilon \theta_W(\alpha(v) \otimes (x + y\varepsilon)) = \varepsilon \theta_W \circ \alpha_{R[\varepsilon]}(v \otimes (x + y\varepsilon)).\end{aligned}$$

It follows that

$$\begin{aligned}\alpha_{R[\varepsilon]} \circ \tilde{\theta}_V &= \alpha_{R[\varepsilon]} \circ (\mathrm{Id}_{V_{R[\varepsilon]}} + \varepsilon \theta_V) = \alpha_{R[\varepsilon]} + \alpha_{R[\varepsilon]} \circ \varepsilon \theta_V \\ &= \alpha_{R[\varepsilon]} + \varepsilon \theta_W \circ \alpha_{R[\varepsilon]} = (\mathrm{Id}_{W_{R[\varepsilon]}} + \varepsilon \theta_W) \circ \alpha_{R[\varepsilon]} \\ &= \tilde{\theta}_W \circ \alpha_{R[\varepsilon]},\end{aligned}$$

as desired. ■

2.5 Filtered and graded fiber functors

We recall the notions of filtered and graded fiber functors on Tannakian categories following [25]. Let Γ be a totally ordered abelian group (written additively), and let $R \in \mathbf{Alg}_k$. A Γ *-graded* R-module is an R-module M together with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$. A morphism between two Γ-graded R modules M and N is an R-linear map $f: M \to N$ such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$. We denote by Γ-$\mathbf{Grad}_R$ the category of Γ-graded modules over R. For $M, N \in \Gamma$-$\mathbf{Grad}_R$, we define the tensor product $(M \otimes_R N)_\gamma = \bigoplus_{\gamma' + \gamma'' = \gamma} (M_{\gamma'} \otimes_R N_{\gamma''})$.

Let M be an R-module. A Γ *-filtration* on M is an increasing map

$$\mathcal{F}: \Gamma \longrightarrow \{R\text{-submodules of } M\}, \quad \gamma \longmapsto \mathcal{F}^\gamma M,$$

such that $\mathcal{F}^\gamma M = 0$ for $\gamma \ll 0$ and $\mathcal{F}^\gamma M = M$ for $\gamma \gg 0$, which is *increasing* in the sense that $\mathcal{F}^\gamma M \subseteq \mathcal{F}^{\gamma'} M$ whenever $\gamma \leq \gamma'$. A Γ *-filtered R-module* is an R-module M with a Γ-filtration. To abbreviate notations, we sometimes denote $\mathcal{F}^\gamma M$ by M^γ if no confusion

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shall arise. A morphism between two Γ-filtered R-modules M and N is an R-linear map $f: M \to N$ such that $f(M^\gamma) \subseteq N^\gamma$ for all $\gamma \in \Gamma$. We denote by Γ-$\mathbf{Fil}_R$ the category of Γ-filtered modules over R.

Let M be a Γ-filtered module over R. For any $\gamma \in \Gamma$, we put $\mathcal{F}^{\gamma-}M := \sum_{\gamma'<\gamma} \mathcal{F}^{\gamma'}M$. We define

$$\mathrm{gr}^\gamma_{\mathcal{F}} M := \mathcal{F}^\gamma M / \mathcal{F}^{\gamma-} M.$$

Then, $\mathrm{gr}_{\mathcal{F}} M := \bigoplus_{\gamma\in\Gamma} \mathrm{gr}^\gamma_{\mathcal{F}} M$ is a Γ-graded R module, and is called the *Γ-graded R-module associated to $\mathcal{F}$*. We thus have a functor

$$\mathrm{gr}\colon \Gamma\text{-}\mathbf{Fil}_R \longrightarrow \Gamma\text{-}\mathbf{Grad}_R .$$

Elements $\gamma \in \Gamma$ such that $\mathrm{gr}^\gamma_{\mathcal{F}} M \neq 0$ are said to be the Γ-*jumps* (or simply jumps) of $\mathcal{F}$.

The tensor product structure in Γ-$\mathbf{Fil}_R$ is defined by

$$\mathcal{F}^\gamma(M \bigotimes_R N) = \sum_{\gamma'+\gamma''=\gamma} \mathcal{F}^{\gamma'}M \bigotimes_R \mathcal{F}^{\gamma''}N,$$

for all Γ-filtered modules M and N over R.

A morphism $f: M \to N$ in Γ-$\mathbf{Fil}_R$ is said to be *admissible* (or *strict*) if

$$f(M^\gamma) = f(M) \cap N^\gamma, \quad \forall \gamma \in \Gamma.$$

Following [25, Section 4.1], we say that a short sequence $0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0$ in Γ-$\mathbf{Fil}_R$ is *exact* if both of f' and f'' are admissible, and the underlying short sequence in $\mathbf{Mod}_R$ is exact.

Let $\mathcal{T}$ be a Tannakian category over k, and let R be a k-algebra.

(i) A *Γ-graded fiber functor* on $\mathcal{T}$ over R is an exact faithful k-linear tensor functor $\tau: \mathcal{T} \to \Gamma$-$\mathbf{Grad}_R$.

(ii) A *Γ-filtered fiber functor* on $\mathcal{T}$ over R is an exact faithful k-linear tensor functor $\eta: \mathcal{T} \to \Gamma$-$\mathbf{Fil}_R$.

(iii) Given an object $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ in Γ-$\mathbf{Grad}_R$, we put $\mathcal{F}^\gamma(M) := \bigoplus_{\gamma'\leq\gamma} M_{\gamma'}$. This gives rise to a functor $\mathrm{fil}\colon \Gamma$-$\mathbf{Grad}_R \to \Gamma$-$\mathbf{Fil}_R$.

(iv) A Γ-filtered fiber functor η is called *splittable* if there exists a Γ-graded fiber functor τ such that $\eta = \mathrm{fil} \circ \tau$, and τ is called a *splitting* of η.

Remark 2.10 More concretely, a Γ-filtered fiber functor is a k-linear functor $\eta: \mathcal{T} \to \Gamma$-$\mathbf{Fil}_R$ satisfying the following properties (cf. [6, Definition 4.2.6 and Remark 4.2.7]).

(i) It is *admissibly* (or *strictly*) functorial, i.e., for any morphism $\alpha: X \to Y$ in $\mathcal{T}$, we have $\eta(\alpha)\big(\mathcal{F}^\gamma\eta(X)\big) = \eta(\alpha)(\eta(X)) \cap \mathcal{F}^\gamma\eta(Y)$ for all $\gamma \in \Gamma$.

(ii) It is compatible with tensor products, i.e., we have

$$\mathcal{F}^\gamma\big(\eta(X \otimes Y)\big) = \sum_{\gamma'+\gamma''=\gamma} \mathcal{F}^{\gamma'}\big(\eta(X)\big) \otimes \mathcal{F}^{\gamma''}\big(\eta(Y)\big),$$

for all $X, Y \in \mathrm{Ob}(\mathcal{T})$ and $\gamma \in \Gamma$.

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(iii)

$$\mathcal{F}^{\gamma}\eta(\mathbb{1}) = \begin{cases} R & \text{for } \gamma \geq 0, \\ 0 & \text{for } \gamma < 0, \end{cases}$$

where $\mathbb{1}$ is the identity object in $\mathcal{T}$. Note that $\big(\eta(\mathbb{1}), \gamma \mapsto \mathcal{F}^{\gamma}\eta(\mathbb{1})\big)$ is the identity object in Γ-$\mathbf{Fil}_R$.

Construction 2.11 Let $(M, \mathcal{F}) \in \mathbb{Z}$-$\mathbf{Fil}_R$ be a $\mathbb{Z}$-filtered module with $\mathbb{Z}$-jumps $\jmath_1 < \cdots < \jmath_n$. For any $\gamma \in \mathbb{Q}_{>0}$, we define a $\mathbb{Q}$-filtered module $(M, [\gamma]_*\mathcal{F})$ by

$$([\gamma]_*\mathcal{F})^x M := \begin{cases} 0 & \text{for } x < \jmath_1\gamma, \\ M^{\jmath_i} & \text{for } \jmath_i\gamma \leq x < \jmath_{i+1}\gamma,\ 1 \leq i \leq n-1, \\ M & \text{for } x \geq \jmath_n\gamma. \end{cases}$$

We then have a fully faithful embedding $[\gamma]_*\colon \mathbb{Z}$-$\mathbf{Fil}_R \to \mathbb{Q}$-$\mathbf{Fil}_R$. Similarly, we have a fully faithful embedding $[\gamma]_*\colon \mathbb{Z}$-$\mathbf{Grad}_R \to \mathbb{Q}$-$\mathbf{Grad}_R$ by defining $[\gamma]_* := \mathrm{gr} \circ [\gamma]_* \circ \mathrm{fil}$.

To end this subsection, we exhibit the following theorem for latter use. (Be aware that in [25], the author only considers Γ-gradings and Γ-filtrations for $\Gamma = \mathbb{Z}$.)

Theorem 2.12 *[25, Theorem 4.15] Let $\mathcal{T}$ be a Tannakian category over a field k, and let R be a k-algebra. Let $\eta\colon \mathcal{T} \to \mathbb{Z}$-$\mathbf{Fil}_R$ be a $\mathbb{Z}$-filtered fiber functor. If $\underline{\mathrm{Aut}}^{\otimes}_R(\mathrm{forg} \circ \eta)$ is prosmooth (i.e., a limit of smooth algebraic group schemes) over R, where* forg: *$\mathbb{Z}$-$\mathbf{Fil}_R \to \mathbf{Mod}_R$ is the forgetful functor, then η is splittable.*

3 G-φ-modules over the Robba ring

We fix an affine algebraic F-group G in this section.

3.1 Definition

Let $R \in \{\mathcal{E}^{\dagger}, \mathcal{R}, \tilde{\mathcal{E}}^{\dagger}, \tilde{\mathcal{R}}\}$ equipped with an absolute Frobenius lift φ. The following definition is motivated by that of *G-isocrystals* introduced in [6, Section IX.1].

Definition 3.1 A *G-φ -module* over R is an exact faithful F-linear tensor functor

$$\mathrm{I}\colon \mathbf{Rep}_F(G) \longrightarrow \mathbf{Mod}^{\varphi}_{\mathrm{R}},$$

which satisfies $\mathrm{forg} \circ \mathrm{I} = \omega^G \otimes R$, where $\mathrm{forg}\colon \mathbf{Mod}^{\varphi}_{\mathrm{R}} \to \mathbf{Mod}_R$ is the forgetful functor. The category of G-φ-modules over R is denoted by G-$\mathbf{Mod}^{\varphi}_{\mathrm{R}}$, whose morphisms are morphisms of tensor functors.

Let $(V, \rho) \in \mathbf{Rep}_F(G)$, and let $g \in G(R)$. We define $\mathrm{I}(g)(V) := (V_R, g\varphi)$, where

$$g\varphi\colon V_R \longrightarrow V_R, \qquad v \otimes f \longmapsto \rho(g)(v \otimes 1)\varphi(f).$$

Let $V, W \in \mathbf{Rep}_F(G)$. We have a canonical isomorphism $(V \otimes W)_{\mathcal{R}} \cong V_{\mathcal{R}} \otimes_{\mathcal{R}} W_{\mathcal{R}}$, and we will henceforth identify them. Given any $\alpha \in \mathrm{Hom}_G(V, W)$, we define

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$\mathrm{I}(g)(\alpha) := \alpha_R$. We thus have the following G-φ-module over R (associated to g).

$$\mathrm{I}(g)\colon \mathbf{Rep}_F(G) \longrightarrow \mathbf{Mod}_{\mathrm{R}}^{\varphi}, \qquad V \longmapsto (V_R, g\varphi).$$

We call $\mathrm{I}(g)(V) = (V_R, g\varphi)$ the *G-φ -module over R associated to g.*

For any $g \in G(R)$, we sometimes write $\Phi_g = \Phi_{g,V}$ for the φ-linear action $g\varphi$ on V_R. Both notations have their own advantages in practice.

Remark 3.2 For any $g \in G(R)$, we define $\Phi(g) := G(\varphi)(g)$. For any $(V, \rho) \in \mathbf{Rep}_F(G)$, we have a commutative diagram

$$\begin{array}{ccc} G(R) & \xrightarrow{\rho(R)} & \mathrm{GL}_V(R) \\ {\scriptstyle G(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{GL}_V(\varphi)} \\ G(R) & \xrightarrow[\rho(R)]{} & \mathrm{GL}_V(R) \end{array}$$

Hence, $\rho(\varphi(g)) = \varphi(\rho(g))$. For any $h \in G(R)$ and $n, m \geq 0$, we have the following formula in $G(R) \rtimes \langle \varphi \rangle$:

$$(h\varphi^n) \circ (g\varphi^m) = \big(h\varphi^n(g)\big)\varphi^{n+m}.$$

3.2 The $\mathbb{Q}$-filtered fiber functor HN_g

We fix an element $g \in G(\mathcal{R})$.

Construction 3.3 For any $V \in \mathbf{Rep}_F(G)$, we have a φ-module $(V_{\mathcal{R}}, g\varphi)$ over $\mathcal{R}$. Kedlaya's slope filtration theorem [9, Theorem 6.10] then provides a filtration

$$0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_l} = V_{\mathcal{R}},$$

satisfying

- $V_{\mathcal{R}}^{\mu_1}$ is pure of some slope $\mu_1 \in \mathbb{Q}$ and each $V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}$ is pure of some slope $\mu_i \in \mathbb{Q}$ for $2 \leq i \leq l$;
- $\mu_1 < \cdots < \mu_l$.

We thus have an increasing map

$$\begin{aligned} \mathcal{HN}_g\colon \mathbb{Q} &\longrightarrow \{\mathcal{R}\text{-modules of } V_{\mathcal{R}}\} \\ x &\longmapsto \mathcal{HN}_g^x(V_{\mathcal{R}}), \end{aligned}$$

where

$$\mathcal{HN}_g^x(V_{\mathcal{R}}) = \begin{cases} 0 & \text{for } x < \mu_1, \\ V_{\mathcal{R}}^{\mu_i} & \text{for } \mu_i \leq x < \mu_{i+1}, 1 \leq i \leq l-1, \\ V_{\mathcal{R}} & \text{for } x \geq \mu_l. \end{cases}$$

Then, $(V_{\mathcal{R}}, \mathcal{HN}_g)$ is a $\mathbb{Q}$-filtered module over $\mathcal{R}$ with $\mathbb{Q}$-jumps $\mu_1 < \cdots < \mu_l$. We will denote $\mathcal{HN}_g^x(V_{\mathcal{R}})$ by $V_{\mathcal{R}}^x$ when $\mathcal{HN}_g$ is clear in the context.

Theorem 3.4 *The assignments*

$$V \longmapsto (V_{\mathcal{R}}, \mathcal{HN}_g) \quad \text{and} \quad \alpha \longmapsto \alpha_{\mathcal{R}},$$

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for all $\alpha \in \mathrm{Hom}_G(V, W)$, *define a* $\mathbb{Q}$*-filtered fiber functor*

$$\mathrm{HN}_g: \boldsymbol{Rep}_F(G) \longrightarrow \mathbb{Q}\text{-}\boldsymbol{Fil}_{\mathcal{R}}.$$

Proof This is Propositions 3.5 and 3.6 below. ∎

For any admissible extension E of K, we first remark that the φ-equivariant embedding $\psi: \mathcal{R} \to \tilde{\mathcal{R}}(E,t)$ is faithfully flat (see [11, Remark 3.5.3]). We also remark that, if M_1 and M_2 are pure φ-modules over $\mathcal{R}$ of slopes μ_1 and μ_2, respectively, then $M_1 \otimes_{\mathcal{R}} M_2$ is pure of slope $\mu_1 + \mu_2$ (cf. [11, Corollary 1.6.4]). These facts will be repeatedly used in the sequel.

Proposition 3.5 *The assignments in Theorem 3.4 yield a faithful F-linear tensor functor* $\mathrm{HN}_g: \boldsymbol{Rep}_F(G) \to \mathbb{Q}\text{-}\boldsymbol{Fil}_{\mathcal{R}}$.

Proof Let $\mathbb{1} = F$ be the trivial G-representation. Then, $\mathbb{1} \otimes_F \mathcal{R} = \mathcal{R}$ is of rank 1 with slope 0, proving that HN_g preserves identity objects.

We claim that HN_g is functorial. Let $\alpha \in \mathrm{Hom}_G(V, W)$ be a morphism of finite-dimensional G-modules. We need to show that $\alpha_{\mathcal{R}}(V_{\mathcal{R}}^x) \subseteq W_{\mathcal{R}}^x$ for all $x \in \mathbb{Q}$. Choose by Lemma 2.2 an admissible extension E of K such that κ_E is strongly difference-closed. For any fixed $x \in \mathbb{Q}$, we set $V_{\tilde{\mathcal{R}}(E,t)}^x := V_{\mathcal{R}}^x \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t)$, and $W_{\tilde{\mathcal{R}}(E,t)}^x := W_{\mathcal{R}}^x \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t)$. By Lemma 2.3(iii), we have a decomposition $W_{\tilde{\mathcal{R}}(E,t)} = W_{\tilde{\mathcal{R}}(E,t)}^x \oplus W_{\tilde{\mathcal{R}}(E,t)}'$ of φ-modules over $\tilde{\mathcal{R}}(E,t)$, where $W_{\tilde{\mathcal{R}}(E,t)}^x$ (resp. $W_{\tilde{\mathcal{R}}(E,t)}'$) has slopes less or equal to x (resp. greater than x). By Lemma 2.3(iv), the induced morphism $V_{\tilde{\mathcal{R}}(E,t)}^x \to W_{\tilde{\mathcal{R}}(E,t)}'$ of φ-modules is zero. We thus have $\alpha_{\tilde{\mathcal{R}}(E,t)}\big(V_{\tilde{\mathcal{R}}(E,t)}^x\big) \subseteq W_{\tilde{\mathcal{R}}(E,t)}^x$. Given any $\mathbf{v} \in V_{\mathcal{R}}^x$, we may write $\alpha_{\tilde{\mathcal{R}}(E,t)}(\mathbf{v} \otimes 1) = \alpha_{\mathcal{R}}(\mathbf{v}) \otimes 1 = \sum_{i \in I} \mathbf{w}_i \otimes s_i$ for some finite set I, with $\mathbf{w}_i \in W_{\mathcal{R}}^x$ and $s_i \in \tilde{\mathcal{R}}(E,t)$ for all $i \in I$. Let M be the $\mathcal{R}$-submodule of $W_{\mathcal{R}}$ generated by $\alpha_{\mathcal{R}}(\mathbf{v})$ and the $\mathbf{w}_i$, and let N be the $\mathcal{R}$-submodule of $W_{\mathcal{R}}^x$ generated by the $\mathbf{w}_i$. We then have $(M/N) \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t) \cong (M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t))/(N \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t)) = 0$. It follows that $M/N = 0$ as $\mathcal{R} \to \tilde{\mathcal{R}}(E,t)$ is faithfully flat. We thus have $\alpha_{\mathcal{R}}(\mathbf{v}) \in N \subseteq W_{\mathcal{R}}^x$, as desired.

It remains to show that HN_g preserves tensor products (in the sense of Remark 2.10(ii)). Let V and W be two finite-dimensional G-modules, and suppose that the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ (resp. $(W_{\mathcal{R}}, g\varphi)$) has jumps $\mu_1 < \cdots < \mu_{l_V}$ (resp. $\nu_1 < \cdots < \nu_{l_W}$). By [12, Lemma 16.4.3], $\big((V \otimes_F W)_{\mathcal{R}}, g\varphi\big)$ has jumps $\{\mu_i + \nu_j \mid 1 \le i \le l_V, 1 \le j \le l_W\}$. Fix any $1 \le l \le l_V$ and $1 \le s \le l_W$; we need to show

$$(V \bigotimes_F W)_{\mathcal{R}}^{\mu_l + \nu_s} = \sum_{\substack{x, y \in \mathbb{Q} \\ x + y = \mu_l + \nu_s}} V_{\mathcal{R}}^x \bigotimes_{\mathcal{R}} W_{\mathcal{R}}^y, \tag{3}$$

and we will do so in the remainder of the proof.

We claim that

$$\sum_{\substack{x, y \in \mathbb{Q} \\ x + y = \mu_l + \nu_s}} V_{\mathcal{R}}^x \bigotimes_{\mathcal{R}} W_{\mathcal{R}}^y = \sum_{\substack{\mu_i + \nu_j \le \mu_l + \nu_s \\ 1 \le i \le l_V, 1 \le j \le l_W}} V_{\mathcal{R}}^{\mu_i} \bigotimes_{\mathcal{R}} W_{\mathcal{R}}^{\nu_j}.$$

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It is clear that the RHS is contained in the LHS; we now show the reverse inclusion. Let $x, y \in \mathbb{Q}$ such that $x + y = \mu_l + \nu_s$. If $x < \mu_1$ or $y < \nu_1$, then $V^x_{\mathcal{R}} \otimes_{\mathcal{R}} W^y_{\mathcal{R}} = 0$ which is contained in the RHS. Otherwise, there exists the largest integer $1 \le i \le l_V$ (resp. $1 \le j \le l_W$) with the property that $\mu_i \le x$ (resp. $\nu_j \le y$). We then have $V^x_{\mathcal{R}} \otimes_{\mathcal{R}} W^y_{\mathcal{R}} = V^{\mu_i}_{\mathcal{R}} \otimes_{\mathcal{R}} W^{\nu_j}_{\mathcal{R}}$ and $\mu_i + \nu_j \le \mu_l + \nu_s$. The claim is thus proved.

From Lemma 2.3(iii), we see that

$$\Big(V \bigotimes_F W\Big)^{\mu_l+\nu_s}_{\tilde{\mathcal{R}}(E,t)} = \Big(\sum_{\substack{\mu_i + \nu_j \le \mu_l + \nu_s \\ 1 \le i \le l_V, 1 \le j \le l_W}} V^{\mu_i}_{\mathcal{R}} \bigotimes_{\mathcal{R}} W^{\nu_j}_{\mathcal{R}} \Big) \bigotimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t).$$

Therefore, we have

$$\Big(V \bigotimes W\Big)^{\mu_l+\nu_s}_{\mathcal{R}} = \sum_{\substack{\mu_i + \nu_j \le \mu_l + \nu_s \\ 1 \le i \le l_V, 1 \le j \le l_W}} V^{\mu_i}_{\mathcal{R}} \bigotimes_{\mathcal{R}} W^{\nu_j}_{\mathcal{R}}$$

by Lemma 2.3(i) and the fact that $\mathcal{R} \to \tilde{\mathcal{R}}(E,t)$ is faithfully flat. The desired equality (1) then follows from the preceding claim. ∎

Let (M, φ) be a φ-module over $\tilde{\mathcal{R}}$ of rank d. Then, Φ is invertible, because the Frobenius lift on $\tilde{\mathcal{R}}$ is bijective, and (M, φ^{-1}) is a φ^{-1}-module over $\tilde{\mathcal{R}}$. More explicitly, let $A \in \mathrm{GL}_d(\tilde{\mathcal{R}})$ be the matrix of action of φ in some basis for M over $\tilde{\mathcal{R}}$. Then, in the same basis, the matrix of action of φ^{-1} is $\varphi^{-1}(A^{-1})$. For example, if $M = V_{\tilde{\mathcal{R}}}$ for some $V \in \mathbf{Rep}_F(G)$, and $\Phi = \psi(g)\varphi$ where ψ denotes (by abuse of notation) the group morphism $G(\mathcal{R}) \to G(\tilde{\mathcal{R}})$ induced by the embedding $\psi\colon \mathcal{R} \to \tilde{\mathcal{R}}$ recalled above Proposition 3.5, then

$$\big(\psi(g)\varphi\big) \cdot \big(\varphi^{-1}(\psi(g^{-1}))\varphi^{-1}\big) = 1$$

in $G(\tilde{\mathcal{R}}) \rtimes \langle \varphi \rangle$ (cf. Remark 3.2), which implies that $\varphi^{-1} = \varphi^{-1}(\psi(g^{-1}))\varphi^{-1}$.

Let M be a standard φ-module over $\tilde{\mathcal{R}}$ of slope $\mu = s/r$ with $r > 0$ and $(s, r) = 1$. Namely, we have a standard basis $e_1, \dots, e_r$ in which φ acts via

$$A = \begin{pmatrix} 0 & & & \varpi^s \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Then,

$$\varphi^{-1}(A^{-1}) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \varpi^{-s} & & & 0 \end{pmatrix},$$

which implies that (M, φ^{-1}) is a standard φ^{-1}-module pure of slope $-\mu$.

Proposition 3.6 *The functor* $\mathrm{HN}_g\colon \boldsymbol{Rep}_F(G) \to \mathbb{Q}\text{-}\boldsymbol{Fil}_{\mathcal{R}}$ *is exact.*

Proof Let $\alpha \in \mathrm{Hom}_G(V, W)$ be a morphism of finite-dimensional G-modules. We need to show that $\alpha_{\mathcal{R}}(V^x_{\mathcal{R}}) = \alpha_{\mathcal{R}}(V_{\mathcal{R}}) \cap W^x_{\mathcal{R}}$ for all $x \in \mathbb{Q}$. For any fixed $x \in \mathbb{Q}$, the functoriality in Proposition 3.5 already implies that $\alpha_{\mathcal{R}}(V^x_{\mathcal{R}}) \subseteq \alpha_{\mathcal{R}}(V_{\mathcal{R}}) \cap W^x_{\mathcal{R}}$. Thus, it suffices to show that for any nonzero element $\mathbf{v} \in V_{\mathcal{R}}$ such that $\alpha_{\mathcal{R}}(\mathbf{v}) \in W^x_{\mathcal{R}}$, there exists $\mathbf{v}' \in V^x_{\mathcal{R}}$ with $\alpha_{\mathcal{R}}(\mathbf{v}) = \alpha_{\mathcal{R}}(\mathbf{v}')$.

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By Lemma 2.3(iii), we have decompositions

$$V_{\tilde{\mathcal{R}}(E,t)} = V^x_{\tilde{\mathcal{R}}(E,t)} \bigoplus V'_{\tilde{\mathcal{R}}(E,t)} \quad \text{and} \quad W_{\tilde{\mathcal{R}}(E,t)} = W^x_{\tilde{\mathcal{R}}(E,t)} \bigoplus W'_{\tilde{\mathcal{R}}(E,t)} \tag{4}$$

of φ-modules over $\tilde{\mathcal{R}}(E,t)$, in which $V^x_{\tilde{\mathcal{R}}(E,t)}$ and $W^x_{\tilde{\mathcal{R}}(E,t)}$ have slopes less or equal to x, while $V'_{\tilde{\mathcal{R}}(E,t)}$ and $W'_{\tilde{\mathcal{R}}(E,t)}$ have slopes greater than x. Notice that the composition

$$\xi\colon V'_{\tilde{\mathcal{R}}(E,t)} \longrightarrow V^x_{\tilde{\mathcal{R}}(E,t)} \oplus V'_{\tilde{\mathcal{R}}(E,t)} \xrightarrow{\alpha_{\tilde{\mathcal{R}}(E,t)}} W^x_{\tilde{\mathcal{R}}(E,t)} \oplus W'_{\tilde{\mathcal{R}}(E,t)} \longrightarrow W^x_{\tilde{\mathcal{R}}(E,t)}$$

is a morphism of φ-modules. We claim that $\xi = 0$. We write $\varphi = \psi(g)\varphi$, then $\varphi^{-1} = \varphi^{-1}(\psi(g^{-1}))\varphi^{-1}$. Because α is G-equivariant and $\varphi^{-1}(\psi(g^{-1})) \in G(\tilde{\mathcal{R}}(E,t))$, we have that $\alpha_{\tilde{\mathcal{R}}}\colon (V_{\tilde{\mathcal{R}}(E,t)}, \varphi^{-1}) \to (W_{\tilde{\mathcal{R}}(E,t)}, \varphi^{-1})$ is a morphism of φ^{-1}-modules. On the other hand, we also have decompositions of φ^{-1}-modules as in (2), together with the induced morphism $\xi\colon V'_{\tilde{\mathcal{R}}(E,t)} \to W^x_{\tilde{\mathcal{R}}(E,t)}$ of φ^{-1}-modules. But in this case, $V'_{\tilde{\mathcal{R}}(E,t)}$ has slopes less than x, while $W^x_{\tilde{\mathcal{R}}(E,t)}$ has slopes greater or equal to x. It then follows from Lemma 2.3(iv) that $\xi = 0$, as claimed.

Therefore, we find $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V^x_{\mathcal{R}}$ and $s_1, \ldots, s_n \in \tilde{\mathcal{R}}(E,t)$ such that

$$\alpha_{\tilde{\mathcal{R}}(E,t)}(\mathbf{v} \otimes 1) = \alpha_{\mathcal{R}}(\mathbf{v}) \otimes 1 = \sum_{i=1}^{n} \alpha_{\mathcal{R}}(\mathbf{v}_i) \otimes s_i.$$

Let M be the submodule of $W_{\mathcal{R}}$ generated by $\alpha_{\mathcal{R}}(\mathbf{v})$ and the $\alpha_{\mathcal{R}}(\mathbf{v}_i)$, and let N be the submodule generated by the $\alpha_{\mathcal{R}}(\mathbf{v}_i)$. We then have

$$(M/N) \bigotimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t) \cong (M \bigotimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t))/(N \bigotimes_{\mathcal{R}} \tilde{\mathcal{R}}(E,t)) = 0.$$

It follows that $M/N = 0$ as $\mathcal{R} \to \tilde{\mathcal{R}}(E,t)$ is faithfully flat, and hence, $\alpha_{\mathcal{R}}(\mathbf{v}) = \sum_{i=1}^{n} r_i \alpha_{\mathcal{R}}(\mathbf{v}_i) \in W^x_{\mathcal{R}}$ for some $r_i \in \mathcal{R}$. Put $\mathbf{v}' := \sum_{i=1}^{n} r_i \mathbf{v}_i \in V^x_{\mathcal{R}}$, we then have $\alpha_{\mathcal{R}}(\mathbf{v}') = \alpha_{\mathcal{R}}(\mathbf{v})$, as desired. ■

3.3 Splittings of HN_g

As before, we fix an element $g \in G(\mathcal{R})$. In Section 3.2, we have constructed a $\mathbb{Q}$-filtered fiber functor $\mathrm{HN}_g\colon \mathbf{Rep}_F(G) \to \mathbb{Q}\text{-}\mathbf{Fil}_{\mathcal{R}}$. In this subsection, we show that HN_g is splittable whenever G is smooth. Our strategy goes as follows. We first use Lemma 3.10 reducing HN_g to a $\mathbb{Z}$-filtered fiber functor $\mathrm{HN}_g^{\mathbb{Z}}$ to which Theorem 2.12 is applicable. This $\mathrm{HN}_g^{\mathbb{Z}}$ then admits a $\mathbb{Z}$-splitting. Finally, in Theorem 3.12, we lift such a $\mathbb{Z}$-splitting to a $\mathbb{Q}$-splitting of HN_g.

Definition 3.7 We define the *support* of HN_g by

$$\mathrm{Supp}(\mathrm{HN}_g) := \{x \in \mathbb{Q} \mid \mathrm{gr}^x_{\mathrm{HN}_g}(V) \neq 0 \text{ for some } V \in \mathbf{Rep}_F(G)\}.$$

Notice that $\mathrm{Supp}(\mathrm{HN}_g)$ is the set of jumps of the slope filtrations of $(V_{\mathcal{R}}, g\varphi)$ for all $V \in \mathbf{Rep}_F(G)$.

The general idea of the following construction was addressed in [2], after Definition 2.5 in loc. cit.; we will make it more explicit in our case.

Construction 3.8 Let $W \in \mathbf{Rep}_F(G)$ be a faithful representation. Because G is algebraic, W is a tensor generator for $\mathbf{Rep}_F(G)$, i.e., any representation V of G is a subquotient of a direct sum of representations $\otimes^m(W \oplus W^\vee)$ for various $m \in \mathbb{N}$. (See [18, Theorem 4.14].) Therefore, $\mathrm{Supp}(\mathrm{HN}_g)$ is the additive subgroup of $\mathbb{Q}$ finitely generated by the $\mathbb{Q}$-jumps $v_1, \ldots, v_n$ of $(W_{\mathcal{R}}, g\varphi)$. We write $v_i = s_i/d_i$ with $d_i > 0$ and $(s_i, d_i) = 1$ for $1 \le i \le n$. Let $d_g \in \mathbb{N}$ be the least common multiple of the d_i. We then have $d_g v_i \in \mathbb{Z}$ for $1 \le i \le n$. In particular, we have

$$d_g = \min\{m \in \mathbb{N} \mid mx \in \mathbb{Z}, \forall x \in \mathrm{Supp}(\mathrm{HN}_g)\}.$$

Therefore, d_g is uniquely determined by g. We call d_g the *least common denominator* of g.

Remark 3.9 We conclude from Construction 3.8 that $\mathrm{Supp}(\mathrm{HN}_g)$ is isomorphic to $\mathbb{Z}$ or 0. In fact, if $(W_{\mathcal{R}}, g\varphi)$ has only one jump 0, then $\mathrm{Supp}(\mathrm{HN}_g) = 0$. Otherwise, the choice of d_g implies that $\gcd(d_g v_1, \ldots, d_g v_n) = 1$. We then have $d_g \,\mathrm{Supp}(\mathrm{HN}_g) = \mathbb{Z}$, because the $d_g v_i$ generate $\mathbb{Z}$ as a $\mathbb{Z}$-module. Therefore, $x \mapsto d_g x$ gives an isomorphism $\mathrm{Supp}(\mathrm{HN}_g) \cong \mathbb{Z}$.

Lemma 3.10 *HN_g factors through a $\mathbb{Z}$-filtered fiber functor $\mathrm{HN}_g^{\mathbb{Z}}\colon \boldsymbol{Rep}_F(G) \to \mathbb{Z}$-$\boldsymbol{Fil}_{\mathcal{R}}$ which makes the diagram*

$$\begin{array}{ccc} \boldsymbol{Rep}_F(G) & \xrightarrow{\ \mathrm{HN}_g\ } & \mathbb{Q}\text{-}\boldsymbol{Fil}_{\mathcal{R}} \\ {\scriptstyle \mathrm{HN}_g^{\mathbb{Z}}}\big\downarrow & \nearrow{\scriptstyle [d_g^{-1}]_*} & \\ \mathbb{Z}\text{-}\boldsymbol{Fil}_{\mathcal{R}} & & \end{array}$$

commute.

We remark that the functor $[d_g^{-1}]_*$ (see Construction 2.11) is nothing but relabeling the jumps by multiplying all jumps with d_g^{-1}. In particular, this lemma implies that $\mathrm{gr}^x_{\mathrm{HN}_g}(V) = \mathrm{gr}^{d_g^{-1}x}_{\mathrm{HN}_g^{\mathbb{Z}}}(V)$ for all $x \in \mathbb{Q}$ and $V \in \mathbf{Rep}_F(G)$.

Proof of Lemma 3.10 Let $V \in \mathbf{Rep}_F(G)$, and let $\mu_1, \ldots, \mu_l$ be the $\mathbb{Q}$-jumps of $(V_{\mathcal{R}}, g\varphi)$. We then have $d_g \mu_i \in \mathbb{Z}$ for all i. We have an increasing map

$$\begin{aligned} \mathcal{F}_g\colon \mathbb{Z} &\longrightarrow \{\mathcal{R}\text{-submodules of } V_{\mathcal{R}}\}, \\ x &\longmapsto \mathcal{F}_g^x(V_{\mathcal{R}}), \end{aligned}$$

where

$$\mathcal{F}_g^x(V_{\mathcal{R}}) := \begin{cases} 0 & \text{for } x < d_g\mu_1, \\ \mathcal{HN}_g^{\mu_i}(V_{\mathcal{R}}) & \text{for } d_g\mu_i \le x < d_g\mu_{i+1}, 1 \le i \le l-1, \\ V_{\mathcal{R}} & \text{for } x \ge d_g\mu_l. \end{cases}$$

Then, $(V_{\mathcal{R}}, \mathcal{F}_g)$ is a $\mathbb{Z}$-filtered module over $\mathcal{R}$ with $\mathbb{Z}$-jumps $d_g\mu_1 < \cdots < d_g\mu_l$. We thus have a $\mathbb{Z}$-filtered fiber functor

$$\begin{aligned} \mathrm{HN}_g^{\mathbb{Z}}\colon \mathbf{Rep}_F(G) &\longrightarrow \mathbb{Z}\text{-}\mathbf{Fil}_{\mathcal{R}}, \\ V &\longmapsto (V_{\mathcal{R}}, \mathcal{F}_g), \end{aligned}$$

satisfying $\mathrm{HN}_g = [d_g^{-1}]_* \circ \mathrm{HN}_g^{\mathbb{Z}}$. ■

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By the definition of $\underline{\mathrm{Aut}}^{\otimes}$ and Corollary 2.5, we have $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R) = \mathrm{Aut}^{\otimes}(\omega_R^G) \cong G(R)$ for all $R \in \mathbf{Alg}_k$. For any R-algebra S, we then have

$$\underline{\mathrm{Aut}}^{\otimes}(\omega_R^G)(S) = \mathrm{Aut}^{\otimes}(\omega_R^G \otimes S) = \mathrm{Aut}^{\otimes}(\omega_S^G) \cong G_R(S).$$

Proposition 3.11 *Let G be a smooth F-group. Then, $\mathrm{HN}_g^{\mathbb{Z}}$ is splittable.*

Proof Because $\mathrm{forg} \circ \mathrm{HN}_g^{\mathbb{Z}} = \omega^G \otimes \mathcal{R}$, we have

$$\underline{\mathrm{Aut}}_{\mathcal{R}}^{\otimes}(\mathrm{forg} \circ \mathrm{HN}_g^{\mathbb{Z}}) = \underline{\mathrm{Aut}}_{\mathcal{R}}^{\otimes}(\omega_{\mathcal{R}}^G) \cong G_{\mathcal{R}}.$$

Notice that $G_{\mathcal{R}}$ is smooth over $\mathcal{R}$; the proposition then follows from Theorem 2.12. ∎

Theorem 3.12 *Let G be a smooth F-group. Then, the $\mathbb{Q}$-filtered fiber functor HN_g is splittable.*

Proof Choose a splitting $\tau_g : \mathbf{Rep}_F(G) \to \mathbb{Z}\text{-}\mathbf{Grad}_{\mathcal{R}}$ of $\mathrm{HN}_g^{\mathbb{Z}}$ by Proposition 3.11, we then have a $\mathbb{Q}$-graded fiber functor $[d_g^{-1}]_* \circ \tau_g : \mathbf{Rep}_F(G) \to \mathbb{Q}\text{-}\mathbf{Grad}_{\mathcal{R}}$. On the other hand, we have the diagram

$$
\begin{array}{ccccc}
 & & \mathbf{Rep}_F(G) & & \\
 & \swarrow^{\tau_g} & \downarrow{\scriptstyle \mathrm{HN}_g^{\mathbb{Z}}} & \searrow^{\mathrm{HN}_g} & \\
\mathbb{Z}\text{-}\mathbf{Grad}_{\mathcal{R}} & \xrightarrow{\mathrm{fil}} & \mathbb{Z}\text{-}\mathbf{Fil}_{\mathcal{R}} & \xrightarrow{[d_g^{-1}]_*} & \mathbb{Q}\text{-}\mathbf{Fil}_{\mathcal{R}} \\
 & \searrow_{[d_g^{-1}]_*} & & \nearrow_{\mathrm{fil}} & \\
 & & \mathbb{Q}\text{-}\mathbf{Grad}_{\mathcal{R}} & &
\end{array} \tag{5}
$$

with the upper-left, the upper-right, and the bottom triangles commutative. Here, the commutativity of the upper-left (resp. the upper-right) triangle follows from Proposition 3.11 (resp. Lemma 3.10); for the bottom one, we note that $[d_g^{-1}]_* \circ \mathrm{fil} = \mathrm{fil} \circ [d_g^{-1}]_*$. Hence, the outer diagram also commutes, which implies that HN_g factors through the $\mathbb{Q}$-graded fiber functor $[d_g^{-1}]_* \circ \tau_g$, as desired. ∎

3.4 The slope morphism

Let R be a commutative ring with 1, and let Γ be an abelian group (not necessarily finitely generated). We first continue the discussions in Section 2.5 to see how Γ-gradings over R are related to $D_R(\Gamma)$-modules, for some affine group scheme $D_R(\Gamma)$ which will be defined as follows.

The group algebra $R[\Gamma] := \bigoplus_{\gamma \in \Gamma} Re_\gamma$ carries a Hopf algebra structure, where the comultiplication is given by $\Delta(e_\gamma) = e_\gamma \otimes e_\gamma$, the counit is given by $\varepsilon(e_\gamma) = 1$, and the antipode is given by $S(e_\gamma) = e_{-\gamma}$, for all $\gamma \in \Gamma$. We denote by $D_R(\Gamma)$ the affine R-group scheme represented by $R[\Gamma]$. For any $\gamma \in \Gamma$, the Hopf algebra morphism $R[\mathbb{Z}] \to R[\Gamma]$, $e_1 \mapsto e_\gamma$ gives rise to a character $\chi_\gamma : D_R(\Gamma) \to \mathbb{G}_{m,R}$ of $D_R(\Gamma)$. For the remainder of this paper, we denote by $\mathbb{D}_R$ the R-group scheme $D_R(\mathbb{Q})$.

Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a Γ-graded module over R. Then, M becomes a $D_R(\Gamma)$-module where $D_R(\Gamma)$ acts on each M_γ via χ_γ. The following lemma shows that this assignment gives an equivalence of categories.

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Lemma 3.13 *[8, Proposition II.2.5]* Γ-$\mathbf{Grad}_R$ *is equivalent to the category of* $D_R(\Gamma)$-*modules.*

Corollary 3.14 *For any* $\gamma \in \mathbb{Q}_{>0}$, *the functor* $[\gamma]_*: \mathbb{Z}\text{-}\mathbf{Grad}_R \to \mathbb{Q}\text{-}\mathbf{Grad}_R$ *corresponds to the character* $\chi_\gamma: \mathbb{D}_R \to \mathbb{G}_{m,R}$.

Proof Let $M \in \mathbb{Z}\text{-}\mathbf{Grad}_R$. By Lemma 3.13, we may write $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as a direct sum of eigenmodules. By construction, we have $[\gamma]_*(M) = \bigoplus_{n\in\mathbb{Z}} ([\gamma]_*(M))_{\gamma n}$ with $([\gamma]_*(M))_{\gamma n} = M_n$ for all n, which is also a decomposition into eigenmodules. Therefore, giving $[\gamma]_*$ is equivalent to giving the commutative diagram

$$\begin{array}{ccc} M_n & = & ([\gamma]_*(M))_{\gamma n} \\ \downarrow & & \downarrow \\ M_n \otimes_R R[\mathbb{Z}] & \longrightarrow & ([\gamma]_*(M))_{\gamma n} \otimes_R R[\mathbb{Q}] \end{array}$$

of R-modules for all $n \in \mathbb{Z}$ such that $M_n \neq 0$. Here, the left (resp. the right) vertical arrow is given by $m \mapsto m \otimes e_n$ (resp. $m \mapsto m \otimes e_{\gamma n}$). The diagram then corresponds to $R[\mathbb{Z}] \to R[\mathbb{Q}]$, $e_1 \mapsto e_\gamma$, as desired. ∎

We now apply the preceding discussions to the functors constructed in Section 3.3, following [14, Section 4].

Construction 3.15 Let $g \in G(\mathcal{R})$; we fix a splitting τ_g of $\mathrm{HN}_g^{\mathbb{Z}}$ given by Proposition 3.11. For any $(V,\rho) \in \mathbf{Rep}_F(G)$, τ_g gives a decomposition of $V_{\mathcal{R}}$, which induces a morphism $\lambda_{\rho,g}: \mathbb{G}_{m,\mathcal{R}} \to \mathrm{GL}_{V,\mathcal{R}}$ by Lemma 3.13. Let S be an $\mathcal{R}$-algebra, and let $a \in \mathbb{G}_{m,\mathcal{R}}(S)$. We then have a family

$$\{\lambda_{\rho,g}(a): V_S \to V_S \mid (V,\rho) \in \mathbf{Rep}_F(G)\}$$

of S-linear maps. Because τ_g is a tensor functor, this family satisfies conditions (i–iii) in Theorem 2.4. We thus find a unique element $b \in G_{\mathcal{R}}(S)$ such that $\lambda_{\rho,g}(a) = \rho(b)$ for all $(V,\rho) \in \mathbf{Rep}_F(G)$. The assignment $a \mapsto b$ is functorial in S, because both $\lambda_{\rho,g}$ and ρ are functorial. We then have a morphism of $\mathcal{R}$-groups

$$\lambda_g: \mathbb{G}_{m,\mathcal{R}} \longrightarrow G_{\mathcal{R}},$$

which is said to be the $\mathbb{Z}$ -*slope morphism* of g.

By Corollary 3.14, $[d_g^{-1}]_*$ gives a unique morphism $\chi_{d_g^{-1}}: \mathbb{D}_{\mathcal{R}} \to \mathbb{G}_{m,\mathcal{R}}$. We define

$$\nu_g := \lambda_g \circ \chi_{d_g^{-1}}: \mathbb{D}_{\mathcal{R}} \longrightarrow G_{\mathcal{R}},$$

which is said to be the $\mathbb{Q}$ -*slope morphism* of g.

The following example demonstrates explicitly how λ_g and ν_g are related to the splittings constructed in Section 3.3 (see Diagram 3).

Example 3.16 Let $(V,\rho) \in \mathbf{Rep}_F(G)$ and suppose that the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ is

$$0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_l} = V_{\mathcal{R}}$$

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with jumps $\mu_1 < \cdots < \mu_l$. By Theorem 3.12, the functor $[d_g^{-1}]_* \circ \tau_g : \mathbf{Rep}_F(G) \to \mathbb{Q}\text{-}\mathbf{Grad}_{\mathcal{R}}$ gives a splitting

$$V_{\mathcal{R}} = V_{\mathcal{R},\mu_1} \oplus \cdots \oplus V_{\mathcal{R},\mu_l} \tag{6}$$

of $\mathrm{HN}_g(V)$, i.e., we have $\bigoplus_{i=1}^{j} V_{\mathcal{R},\mu_i} = V_{\mathcal{R}}^{\mu_j}$ for all $1 \le j \le l$.

First, we fix $1 \le i \le l$. Let $S \in \mathbf{Alg}_{\mathcal{R}}$ and $a \in \mathbb{D}_{\mathcal{R}}(S)$, then $\rho \circ \nu_g(a)$ acts on $V_{\mathcal{R},\mu_i} \otimes_{\mathcal{R}} S$ via multiplication by $\chi_{\mu_i}(a)$. By Lemma 3.10, $\rho \circ \lambda_g(b)$ acts on $V_{\mathcal{R},\mu_i}$ via multiplication by $b^{d_g \mu_i}$, for all $b \in \mathbb{G}_{m,\mathcal{R}}(S)$. Notice that for any $\frac{m}{n} \in \mathbb{Q}$, we have $e_{\frac{m}{n}} = (e_{\frac{1}{n}})^m \in \mathcal{R}[\mathbb{Q}]$, and hence, $\chi_{\frac{m}{n}} = (\chi_{\frac{1}{n}})^m$. In particular, we have $\chi_{\mu_i} = \chi_{\frac{d_g \mu_i}{d_g}} = (\chi_{d_g^{-1}})^{d_g \mu_i}$. Then, on $V_{\mathcal{R},\mu_i} \otimes_{\mathcal{R}} S$, we have

$$\rho \circ \nu_g(a) = \chi_{\mu_i}(a) = (\chi_{d_g^{-1}}(a))^{d_g \mu_i} = \rho \circ \lambda_g\big(\chi_{d_g^{-1}}(a)\big) = \rho \circ \lambda_g \circ \chi_{d_g^{-1}}(a).$$

We next apply this result to all $1 \le i \le l$. Because $V_{\mathcal{R}} = \bigoplus_{i=1}^{l} V_{\mathcal{R},\mu_i}$, we conclude that $\rho \circ \nu_g = \rho \circ \lambda_g \circ \chi_{d_g^{-1}}$. It follows that $\nu_g = \lambda_g \circ \chi_{d_g^{-1}}$ once we choose a faithful representation, as is expected from the definition of ν_g.

If $G = \mathrm{GL}_V$ for some $V \in \mathbf{Vec}_F$, we consider the standard representation (V, ρ) of G where ρ is the identity. The discussion in the above example then indicates that the image of λ_g is contained in a split maximal torus in $G_{\mathcal{R}}$; we conjecture that this property holds true for an arbitrary split reductive F-group G, and we shall give one more evidence as follows.

Example 3.17 Fix a d-dimensional F-vector space V. For any $R \in \mathbf{Alg}_F$, we define

$$\mathrm{SL}_V(R) := \{g \in \mathrm{GL}_V(R) \mid \det(g) = 1\}.$$

The affine algebraic F-group SL_V is called the *special linear group* (associated to V).

Fix an arbitrary $g \in \mathrm{SL}_V(\mathcal{R})$. With the notation as in Construction 4.14, we suppose the jumps of the slope filtration of $(V_{\mathcal{R}}, \Phi_g)$ are $\mu_1, \ldots, \mu_l$ and $\xi_g(V) = \bigoplus_{i=1}^{l} V_{\mathcal{R},\mu_i}$. For each i, we write $r_i = \mathrm{rk}_{\mathcal{R}}(V_{\mathcal{R},\mu_i})$, then the r_i-th exterior product $\Lambda^{r_i}(V_{\mathcal{R},\mu_i})$ is of rank 1. We choose a generator m_i, then $\Lambda^{r_i}(\Phi_{g,\mu_i})(m_i) = f_i m_i$ for some $f_i \in \mathcal{R}^\times = (\mathcal{E}^\dagger)^\times$. Let v be the valuation of the 1-Gauss norm on $\mathcal{E}^\dagger$. We then have $\mu_i = \frac{v(f_i)}{r_i}$ by [11, Definition 1.4.4].

Let $e_1, \ldots, e_d$ be a basis for V over F, and let $A \in \mathrm{SL}_d(\mathcal{R})$ be the matrix of action of Φ_g in $e_1 \otimes 1, \ldots, e_d \otimes 1$. Let $B \in \mathrm{GL}_d(\mathcal{R})$ represent a change-of-basis over $\mathcal{R}$. Then, in the new basis, the matrix of action of Φ_g is $B^{-1} A \varphi(B)$. Notice that $\det(B) \in (\mathcal{E}^\dagger)^\times$ and φ preserves v, we then have

$$v\big(\det(B^{-1} A \varphi(B))\big) = v\big(\det(B^{-1}) \det(A) \varphi(\det(B))\big) = v(\det(A)),$$

which implies that the valuation of the determinant of the matrix of action of Φ_g is invariant under change-of-basis. We denote by $v(\det(\Phi_g))$ this invariant. In particular, we have $v(\det(\Phi_g)) = 0$, because $\det(A) = 1$ by assumption. Put $\Phi'_g := \bigoplus_{i=1}^{l} \Phi_{g,\mu_i}$,

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where each Φ_{g,μ_i} is the projection of Φ_g to the μ_i-th graded piece of $\xi_g(V)$ (cf. Construction 4.14 below). We thus have

$$0 = v(\det(\Phi_g)) = v(\det(\Phi'_g)) = v(f_1) + \cdots + v(f_l) = r_1\mu_1 + \cdots + r_l\mu_l.$$

Let $S \in \mathbf{Alg}_{\mathcal{R}}$ and $t \in \mathbb{G}_{m,\mathcal{R}}(S)$. Because $\lambda_g(t)$ acts on each $V_{\mathcal{R},\mu_i} \otimes_{\mathcal{R}} S$ via multiplication by $t^{d_g\mu_i}$ where d_g is the least common denominator of g, we then have

$$\det(\lambda_g(t)) = t^{d_g(r_1\mu_1+\cdots+r_l\mu_l)} = 1.$$

Therefore, the image of λ_g is contained in a split maximal torus in $\mathrm{SL}_{V,\mathcal{R}}$.

4 G-(φ,∇)-modules over the Robba ring

In this section, we fix an affine algebraic group F-group G.

4.1 Definition and an identification

Let $R \in \{\mathcal{E}^\dagger, \mathcal{R}\}$ equipped with an absolute Frobenius lift φ and the usual derivation $\partial = \partial_t = d/dt$ on R.

Definition 4.1 A G-(φ,∇) *-module* over R is an exact faithful F-linear tensor functor

$$\mathrm{I}\colon \mathbf{Rep}_F(G) \longrightarrow \mathbf{Mod}_R^{\varphi,\nabla},$$

which satisfies $\mathrm{forg}\circ \mathrm{I} = \omega^G \otimes R$, where $\mathrm{forg}\colon \mathbf{Mod}_{\mathrm{R}}^{\varphi} \to \mathbf{Mod}_R$ is the forgetful functor. The category of G-(φ,∇)-modules over R is denoted by G-$\mathbf{Mod}_{\mathrm{R}}^{\varphi,\nabla}$, whose morphisms are morphisms of tensor functors. A G-(φ,∇)-module I over R is called *unit-root* if $\mathrm{I}(V,\rho)$ is a unit-root (φ,∇)-module over R for all $(V,\rho) \in \mathbf{Rep}_F(G)$.

Remark 4.2 We remark that G-$\mathbf{Mod}_{\mathrm{R}}^{\varphi,\nabla}$ is a groupoid, because both $\mathbf{Rep}_F(G)$ and $\mathbf{Mod}_R^{\varphi,\nabla}$ are rigid tensor categories over F, and any morphism of tensor functors between rigid tensor categories is an isomorphism by [7, Proposition 1.13]. Note that tensor products and duals in $\mathbf{Mod}_R^{\varphi,\nabla}$ are defined as in [22, Section 3.1], and the identity object is (R,φ,∂).

We put

$$\boldsymbol{\mu} := \boldsymbol{\mu}(\varphi,t) := \partial(\varphi(t)).$$

Let $\Omega_R^1 := \Omega_{R/K}^1$ be the free R-module generated by the symbol dt, with the K-linear derivation $d\colon R \to \Omega_R^1, f \mapsto \partial(f)dt$. We also define a φ-linear endomorphism

$$d\varphi\colon \Omega_R^1 \longrightarrow \Omega_R^1, \quad f\,dt \longmapsto \boldsymbol{\mu}\,\varphi(f)dt.$$

Given a finite-dimensional representation $\rho\colon G \to \mathrm{GL}_V$, we have a morphism $\mathrm{Lie}(\rho)\colon \mathfrak{g} \to \mathfrak{gl}_V$ of Lie algebras, and hence a morphism $\mathfrak{g}_R \to \mathfrak{gl}_V \otimes R \cong \mathrm{End}_R(V_R)$ of Lie algebras over R (which is injective if ρ is a closed embedding). For any $X \in \mathfrak{g}_R$, we denote by X the action of $\mathrm{Lie}(\rho)(X)$ on V_R (see Remark 2.8). We define the *connection* ∇_X of V_R associated to X by

$$\nabla_X := \nabla_{X,V}\colon V_R \longrightarrow V_R \bigotimes_R \Omega_R^1,$$
$$v\otimes f \longmapsto (v\otimes 1)\otimes d(f) + X(v\otimes f)\otimes dt.$$

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Because $f dt \mapsto f$ gives an isomorphism $\Omega^1_R \cong R$, we have an isomorphism $\iota\colon V_R \otimes_R \Omega^1_R \to V_R$. Let $\Theta_X := \Theta_{X,V}$ be the *differential operator associated to* ∇_X given by the following composition:

$$V_R \xrightarrow{\nabla_X} V_R \otimes_R \Omega^1_R \xrightarrow{\iota} V_R.$$

We have that $\Theta_X(v \otimes f) = v \otimes \partial(f) + X(v \otimes f)$ for all $v \otimes f \in V_R$.

When $G = \mathrm{GL}_V$ for some $V \in \mathbf{Vec}_F$, we may canonically associate to any G-(φ, ∇)-module I over R a (φ, ∇)-module (V_R, Φ, ∇) over R, where $(V_R, \Phi, \nabla) := \mathrm{I}(V, \rho)$ and $\rho\colon G \to G$ is the identity. Choose a basis $e_1, \ldots, e_d$ of V, we define elements $g \in G(R)$ and $X \in \mathfrak{g}_R$ by setting $g(e_i \otimes 1) := \Phi(e_i \otimes 1)$ and $X(e_i \otimes 1) := \iota \circ \nabla(e_i \otimes 1)$, respectively. We then have $\Phi = g\varphi$ and $\nabla = \nabla_X$.

Lemma 4.3 *Let $V, W \in \boldsymbol{Rep}_F(G)$, and let $\alpha \in \mathrm{Hom}_G(V, W)$. We then have*

$$\alpha_R \circ \Theta_{X,V} = \Theta_{X,W} \circ \alpha_R, \quad \textit{and} \quad \Theta_{X,V\otimes W} = \Theta_{X,V} \otimes \mathrm{Id}_{W_{\mathcal{R}}} + \mathrm{Id}_{V_{\mathcal{R}}} \otimes \Theta_{X,W}.$$

Proof The first equality holds, because α_R commutes with X (see Remark 2.8), and the second one follows from a direct computation. ■

Construction 4.4 We consider the R-algebra morphism

$$\hat{\partial}\colon R \longrightarrow R[\varepsilon], \quad r \longmapsto r + \partial(r)\varepsilon,$$

which induces a morphism $G(\hat{\partial})\colon G(R) \to G(R[\varepsilon])$. Notice that $\pi_R \circ \hat{\partial} = \mathrm{Id}_R$; we then have $G(\pi_R) \circ G(\hat{\partial}) = \mathrm{Id}_{G(R)}$, in particular, $G(\pi_R)\big(G(\hat{\partial})(g)\big) = g$. Identifying g with its image in $G(R[\varepsilon])$ induced by the inclusion $R \to R[\varepsilon]$, $r \mapsto r$, we then have

$$G(\hat{\partial})(g)g^{-1} \in \operatorname{Ker} G(\pi_R) = \mathfrak{g}_R.$$

For $g \in G(R)$, we define $\partial(g) := G(\hat{\partial})(g) \in G(R[\varepsilon])$, and put

$$\mathrm{dlog}(g) := \partial(g)g^{-1} \in \mathfrak{g}_R.$$

Example 4.5 Let $G = \mathrm{GL}_d$ for some $d \in \mathbb{N}$, and let $B \in G(R)$. We have that $\mathrm{dlog}(B) = I_d + \varepsilon\partial(B)B^{-1}$, where I_d is the $d \times d$ identity matrix and ∂ acts on B entrywise. Using the isomorphism $\mathrm{Lie}(G)(R) = \{I_d + \varepsilon B \mid B \in \mathrm{Mat}_{d,d}(R)\} \cong \{B \mid B \in \mathrm{Mat}_{d,d}(R)\}$, we may identify $\mathrm{dlog}(B)$ with $\partial(B)B^{-1}$.

Definition 4.6

(i) We define the *gauge transformation*

$$\Gamma_g\colon \mathfrak{g}_R \longrightarrow \mathfrak{g}_R, \qquad X \longmapsto \mathrm{Ad}(g)(X) - \mathrm{dlog}(g),$$

where $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ is the adjoint representation.

(ii) We define $\mathbf{B}^{\varphi,\nabla}(G, R)$ to be the groupoid whose objects are $(g, X) \in G(R) \times \mathfrak{g}_R$ satisfying $X = \Gamma_g(\boldsymbol{\mu}\,\varphi(X))$, and whose morphisms $(g, X) \to (g', X')$ are elements $x \in G(R)$ such that $g' = xg\varphi(x^{-1})$ and $X' = \Gamma_x(X)$.

Lemma 4.7 *Let $(g, X) \in \boldsymbol{B}^{\varphi,\nabla}(G, R)$. Then, $(V_R, g\varphi, \nabla_X)$ is a (φ, ∇)-module over R for all $V \in \boldsymbol{Rep}_F(G)$.*

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Proof Choose a basis $e_1, \dots, e_d$ for V over F where $d = \dim_F V$. Let $A = (a_{ij})_{i,j} \in \mathrm{GL}_d(R)$ (resp. $N = (n_{ij})_{i,j} \in \mathrm{Mat}_{n,n}(R)$) be the representing matrix of $\rho(g)$ (resp. X). For any $\mathbf{v} = \sum_{i=1}^{d} e_i \otimes f_i \in V_{\mathcal{R}}$, we compute

$$\begin{aligned} g\varphi(\Theta_X(\mathbf{v})) &= g\varphi\Big(\sum_{i=1}^{d} e_i \otimes \partial(f_i) + \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} n_{ji} f_i\Big) \\ &= \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji}\varphi(\partial(f_i)) + \sum_{k=1}^{d} e_k \otimes \sum_{i=1}^{d}\sum_{j=1}^{d} a_{kj}\varphi(n_{ji} f_i), \end{aligned}$$

and

$$\begin{aligned} \Theta_X(g\varphi(\mathbf{v})) &= \Theta_X\Big(\sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji}\varphi(f_i)\Big) \\ &= \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} \partial(a_{ji})\varphi(f_i) + \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji}\partial(\varphi(f_i)) \\ &\quad + \sum_{k=1}^{d} e_k \otimes \sum_{i=1}^{d}\sum_{j=1}^{d} n_{kj} a_{ji}\varphi(f_i). \end{aligned}$$

Because $\boldsymbol{\mu} \cdot \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji}\varphi(\partial(f_i)) = \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji}\partial(\varphi(f_i))$, we have that $\boldsymbol{\mu} \cdot g\varphi \circ \Theta_X = \Theta_X \circ g\varphi$ if and only if $\mu A\varphi(N) = \partial(A) + NA$, i.e., $N = \boldsymbol{\mu} A\varphi(N)A^{-1} - \partial(A)A^{-1}$. The last equality holds because of the assumption $X = \Gamma_g\big(\boldsymbol{\mu}\,\varphi(X)\big)$, which completes the proof. ■

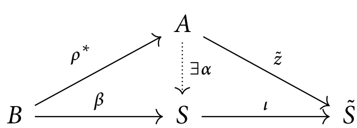
As a consequence, we may define a functor

$$\mathbf{B}^{\varphi,\nabla}(G, R) \longrightarrow G\text{-}\mathbf{Mod}_{\mathrm{R}}^{\varphi,\nabla}, \qquad (g, X) \longmapsto \mathrm{I}(g, X), \tag{7}$$

where $\mathrm{I}(g, X)(V) := (V_R, g\varphi, \nabla_X)$. We next show that this functor is an isomorphism. To do this, we need the following elementary descent result.

Lemma 4.8 *Fix a field k, and let A and B be finitely generated k-algebras. Let* $\rho\colon X \to Y$ *be a closed embedding of affine algebraic k-schemes for* $X = \operatorname{Spec} A$ *and* $Y = \operatorname{Spec} B$. *Let* $\iota\colon S \hookrightarrow \tilde{S}$ *be an embedding in* $\mathbf{Alg}_k$. *Suppose that we are given an element* $\tilde{z} \in X(\tilde{S})$ *such that* $\rho(\tilde{z}) \in Y(\iota(S))$, *then there exists a unique element* $z \in X(S)$ *such that* $\tilde{z} = X(\iota)(z)$.

Proof We have a diagram



with the outer triangle commutative in which ρ^* is surjective. We prove the lemma by constructing a unique k-algebra morphism $\alpha\colon A \to S$ such that $\tilde{z} = \iota \circ \alpha$, as follows. For any $a \in A$, the surjectivity of ρ^* gives us some $b \in B$ such that $\rho^*(b) = a$. We define $\alpha(a) := \beta(b)$. Because ι is injective, we have $\operatorname{Ker}\rho^* \subseteq \operatorname{Ker}\beta$, which implies that α is well-defined. We then have $\tilde{z} \circ \rho^* = \iota \circ \beta = \iota \circ \alpha \circ \rho^*$, which implies that $\tilde{z} = \iota \circ \alpha$,

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because ρ^* is surjective. Moreover, α is a k-algebra morphism, because ι is injective and both ι and $\tilde{z} = \iota \circ \alpha$ are k-algebra morphisms. Finally, we see that α is unique, again because ι is injective. ∎

Proposition 4.9 *The functor* $\boldsymbol{B}^{\varphi,\nabla}(G,R) \to \text{G-}\boldsymbol{Mod}_{\mathrm{R}}^{\varphi,\nabla}$ *defined in (7) is an isomorphism of categories.*

Proof The proof is similar to that of [6, Lemma 9.1.4]. We first show that the functor is fully faithful. Let $(g,X),(g',X') \in \mathbf{B}^{\varphi,\nabla}(G,R)$. Then, any morphism $\eta\colon \mathrm{I}(g,x) \to \mathrm{I}(g',X')$ is an isomorphism according to [7, Proposition 1.13] (see also Remark 4.2). By composing η with the forgetful functor, we then have an automorphism of the fiber functor $\omega^G \otimes R$. By Corollary 2.5, this automorphism is given by a unique element $x \in G(R)$, which then gives an isomorphism between (g,X) and (g',X'), as desired.

It remains to show that, for any $\mathrm{I} \in \text{G-}\mathbf{Mod}_{\mathrm{R}}^{\varphi,\nabla}$, there exists a unique $(g,X) \in \mathbf{B}^{\varphi,\nabla}(G,R)$ such that $\mathrm{I} = \mathrm{I}(g,X)$. For any $(V,\rho) \in \mathbf{Rep}_F(G)$, we write $\mathrm{I}(V,\rho_V) = (V_R,\Phi_V,\nabla_V)$ for a φ-linear map Φ_V and a connection ∇_V on V_R. The proof consists of two steps.

Step 1: There exists a unique $X \in \mathfrak{g}_R$ such that $\nabla_V = \nabla_X$. We write Θ_V for the composition of

$$V_R \xrightarrow{\nabla_V} V_R \otimes \Omega^1_R \xrightarrow{\iota} V_R,$$

where ι is induced by $f dt \mapsto f$, and put $\theta_V := \Theta_V - \mathrm{Id}_V \otimes \partial$. It is clear that $\theta_{\mathbb{1}} = 0$, where $\mathbb{1}$ denotes the trivial representation. Lemma 4.3 then implies that the family

$$\{\theta_V\colon V_R \to V_R \mid (V,\rho_V) \in \mathbf{Rep}_F(G)\}$$

of R-linear endomorphisms satisfies conditions (i–iii) in Corollary 2.9. We thus find a unique $X \in \mathfrak{g}_R$ such that $\theta_V = \mathrm{Lie}(\rho_V)(X)$ for all $(V,\rho_V) \in \mathbf{Rep}_F(G)$, which implies that $\nabla_V = \nabla_X$.

Step 2: There exists a unique $g \in G(R)$ such that $\Phi_V = g\varphi$. We put $\tilde{\Phi}_V := \Phi_V \otimes \varphi$, where φ is the Frobenius lift on $\tilde{\mathcal{R}}$ (in particular, $\tilde{\mathcal{R}}$ is viewed as an $\mathcal{R}$-module via the φ-equivariant embedding ψ described in Section 2.3). The family

$$\{\lambda_V := \tilde{\Phi}_V \circ (\mathrm{Id}_V \otimes \varphi^{-1})\colon V_{\tilde{\mathcal{R}}} \to V_{\tilde{\mathcal{R}}} \mid V \in \mathbf{Rep}_F(G)\}$$

of $\tilde{\mathcal{R}}$-linear endomorphisms satisfies conditions (i–) in Theorem 2.4, which provides a unique element $\tilde{g} \in G(\tilde{\mathcal{R}})$ such that $\lambda_V = \rho_V(\tilde{g})$ for all $(V,\rho_V) \in \mathbf{Rep}_F(G)$. We next reduce $\tilde{g}$ to an element in $G(\mathcal{R})$. We compute

$$\tilde{\Phi}_V \circ (\mathrm{Id}_V \otimes \varphi^{-1})(v \otimes f) = \tilde{\Phi}_V(v \otimes \varphi^{-1}(f)) = \rho_V(\tilde{g})(v \otimes f),$$

which implies that $\tilde{\Phi}_V(v \otimes f) = \rho_V(\tilde{g})(v \otimes \varphi(f))$, and hence, $\tilde{\Phi}_V = \tilde{g}\varphi$. We now fix a d-dimensional faithful representation (V,ρ_V), and an F-basis $e_1,\dots,e_d$ for V. Suppose that $\Phi_V(e_i) = \sum_{j=1}^{d} a_{ji}e_j$, where $a_{ij} \in R$ for all $1 \le i,j \le d$. Put $A = (a_{ij})_{i,j} \in \mathrm{GL}_d(R)$. Then, $\psi(A) = (\psi(a_{ij}))_{i,j} \in \mathrm{GL}_d(\tilde{\mathcal{R}})$ describes the φ-linear action of $\tilde{\Phi}_V$ as well as the $\tilde{\mathcal{R}}$-linear action $\rho(\tilde{g})$ in the basis $e_1 \otimes 1,\dots,e_d \otimes 1$. By replacing X with G, Y with GL_d, S with R, $\tilde{S}$ with $\tilde{\mathcal{R}}$, and ι with ψ in Lemma 4.8, we find a unique element $g \in G(R)$ such that $\psi(g) = \tilde{g}$. It follows that $\Phi_V = g\varphi$, as desired. ∎

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Example 4.10 Let $d \in \mathbb{N}$. The affine algebraic F-group SL_d is defined by

$$\mathrm{SL}_d(S) = \{A \in \mathrm{GL}_d(S) \mid \det(A) = 1\}$$

for all $S \in \mathbf{Alg}_F$, whose Lie algebra $\mathfrak{sl}_d$ consists of $d \times d$ matrices with entries in F and with trace zero.

(i) We claim that any pair $(A, N) \in \mathrm{SL}_d(\mathcal{R}) \times \mathrm{Mat}_{d,d}(\mathcal{E}^\dagger)$ satisfying $N = \boldsymbol{\mu} A\varphi(N)A^{-1} - \partial(A)A^{-1}$ is already an object in $\mathbf{B}^{\varphi,\nabla}(\mathrm{SL}_d, \mathcal{R})$. It is equivalent to showing that the trace $\mathrm{Tr}(N)$ of N is zero. Recall that the Frobenius lift φ on $\mathcal{E}^\dagger$ is given by $\varphi\big(\sum_{i\in\mathbb{Z}} c_i t^i\big) = \sum_{i\in\mathbb{Z}} \varphi(c_i)u^i$, where $u = \varphi(t)$ satisfies $|u - t^q|_1 < 1$. If we write $u = \sum_{i\in\mathbb{Z}} u_i t^i, u_i \in K$, we then have $|u_j| < 1$ for all $j \neq q$ and $|u_q| = 1$. It follows that $|\boldsymbol{\mu}|_1 = |\partial(u)|_1 = |\sum_{i\in\mathbb{Z}} iu_i t^{i-1}|_1 < 1$. On the other hand, we have $\mathrm{Tr}\big(\partial(A)A^{-1}\big) = 0$, because $\partial(A)A^{-1} \in \mathfrak{sl}_{d,\mathcal{R}}$ (see Construction 4.4). Assume, to the contrary, that $\mathrm{Tr}(N) \neq 0$, we have

$$|\mathrm{Tr}(N)|_1 = |\boldsymbol{\mu}\,\mathrm{Tr}(\varphi(N))|_1 = |\boldsymbol{\mu}\,\varphi(\mathrm{Tr}(N))|_1 < |\varphi(\mathrm{Tr}(N))|_1 = |\mathrm{Tr}(N)|_1,$$

a contradiction (we have the last equality, because φ preserves the 1-Gauss norm on $\mathcal{E}^\dagger$).

(ii) We use the *Bessel isocrystal* as described in [12, Example 20.2.1] (see also [9, Section 1.5] and [24, Example 6.2.6]) to construct an object in $\mathbf{B}^{\varphi,\nabla}(\mathrm{SL}_2, \mathcal{R})$. We first briefly recall the Bessel isocrystal. In Hypothesis 2.1, we let $q = p$ be an odd prime, $\kappa = \mathbb{F}_p$, and $F = \mathbb{Q}_p(\pi)$, where π is a $(p-1)$st root of $-p$ in $\bar{\mathbb{Q}}_p$. Then, the (p-power) Frobenius on $K = F$ is the identity. Let φ be the Frobenius lift on $\mathcal{R}$ given by $\varphi(t) = t^p$. Then, [12, Example 20.2.1] gives rise to a pair $(A_0, N_0) \in \mathrm{GL}_2(\mathcal{R}) \times \mathrm{Mat}_{2,2}(\mathcal{E}^\dagger)$ with $\det(A_0) = p$ satisfying the gauge compatibility condition, in which $N_0 = \begin{pmatrix} 0 & t^{-1} \\ \pi^2 t^{-2} & 0 \end{pmatrix} \in \mathfrak{sl}_{2,\mathcal{E}^\dagger}$ .We now assume that $p \equiv 1 (\mathrm{mod}\, 4)$, and $\mathbf{i}$ is a square root of -1 in $\mathbb{Q}_p$. Because $p-1$ is even, we may set $\alpha := \frac{\mathbf{i}}{\pi^{(p-1)/2}} \in F^\times$. We then have $\alpha^2 = p^{-1} = \det(A_0)^{-1}$. Put $D_0 = \left(\begin{smallmatrix} 0 & 1 \\ \alpha & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$. Then, $D_0 A_0 D_0 \in \mathrm{SL}_2(\mathcal{R})$. Moreover, we see that $D_0 N_0 D_0^{-1} = D_0^{-1} N_0 D_0 \in \mathfrak{sl}_{2,\mathcal{E}^\dagger}$. Put $A := D_0 A_0 D_0$ and $N := D_0 N_0 D_0^{-1}$. Then, a straightforward verification shows $N = \boldsymbol{\mu} A\varphi(N)A^{-1} - \partial(A)A^{-1}$ (noting that $\varphi(D_0) = D_0$ and $\partial(D_0) = 0$). We thus have $(A, N) \in \mathbf{B}^{\varphi,\nabla}(\mathrm{SL}_2, \mathcal{R})$, as desired.

(iii) Let $(A, N) \in \mathbf{B}^{\varphi,\nabla}(\mathrm{SL}_d, \mathcal{R})$. We show that (A, N) is“SL_d-quasi-unipotent” (as described in the introduction) by modifying the classical monodromy as follows. By the classical pLMT, we find a finite separable extension L of $\kappa((t))$ and $B \in \mathrm{GL}_d(\mathcal{R}_L)$ such that $BNB^{-1} - \partial(B)B^{-1}$ has trace zero being an upper-triangular block matrix with zero blocks in the diagonal. We wish to replace B with an element in $\mathrm{SL}_d(\mathcal{R}_L)$. To this end, we deduce first that $\mathrm{Tr}(\partial(B)B^{-1}) = \mathrm{Tr}(BNB^{-1}) = \mathrm{Tr}(N) = 0$. It then follows from Jacobi's formula that $\partial(\det(B)) = \det(B) \cdot \mathrm{Tr}(B^{-1}\partial(B)) = 0$. Put $D := \mathrm{Diag}(\det(B)^{-1}, 1, \ldots, 1)$. Then, $DB \in \mathrm{SL}_d(\mathcal{R}_L)$ and $\partial(D) = 0$. We then have

$$(DB)N(DB)^{-1} - \partial(DB)(DB)^{-1} = D\big(BNB^{-1} - \partial(B)B^{-1}\big)D^{-1},$$

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which is an upper-triangular block matrix with zero blocks, and the sizes of the blocks are the same as those in $BNB^{-1} - \partial(B)B^{-1}$ (the said properties are preserved under conjugation by a diagonal matrix). Hence, DB is a desired replacement of B and we are done.

Example 4.11 For any matrix X, we denote by X^{T} its transpose, and by $X^{-\mathrm{T}}$ the inverse of transpose if X is invertible.

We fix the skew-symmetric matrix $J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$. The affine algebraic F-group Sp_4 is defined by

$$\mathrm{Sp}_4(S) := \{A \in \mathrm{GL}_4(S) \mid A^{-1} = J^{-1}A^{\mathrm{T}}J\},$$

for all $S \in \mathbf{Alg}_F$. We denote by $\mathfrak{sp}_4$ the Lie algebra of Sp_4. For any $S \in \mathbf{Alg}_F$, we then have $\mathfrak{sp}_{4,S} = \{X \in \mathrm{Mat}_{4,4}(S) \mid X = JX^{\mathrm{T}}J\}$. We remark that the specific choice of J preserves Borel subgroups under conjugation, which will be useful in the monodromy considered below.

Given any (φ, ∇)-module over $\mathcal{R}$ of rank 2, e.g., the Bessel isocrystal described above, we obtain a pair $(A_0, N_0) \in \mathrm{GL}_2(\mathcal{R}) \times \mathrm{Mat}_{2,2}(\mathcal{R})$ satisfying $N_0 = \boldsymbol{\mu} A_0 \varphi(N_0) A_0^{-1} - \partial(A_0)A_0^{-1}$. Put

$$A := \begin{pmatrix} A_0 & 0 \\ 0 & \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)^{-1} A_0^{-\mathrm{T}} \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} N_0 & 0 \\ 0 & \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) N_0^{\mathrm{T}} \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \end{pmatrix}.$$

A straightforward verification shows that $A \in \mathrm{Sp}_4(\mathcal{R}), N \in \mathfrak{sp}_{4,\mathcal{R}}$, and, moreover, $N = \boldsymbol{\mu} A\varphi(N)A^{-1} - \partial(A)A^{-1}$ (noting that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{-1} = -\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$), which implies that $(A, N) \in \mathbf{B}^{\varphi,\nabla}(\mathrm{Sp}_4, \mathcal{R})$.

We next show that (A, N) is "Sp_4-quasi-unipotent." By the classical pLMT, we find a finite separable extension L of $\kappa((t))$ and $B_0 \in \mathrm{GL}_2(\mathcal{R}_L)$ such that

$$B_0 N_0 B_0^{-1} - \partial(B_0)B_0^{-1} = \left(\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\right),$$

for some $n \in \mathcal{R}_L$ (n could be 0). Put

$$B := \begin{pmatrix} B_0 & 0 \\ 0 & \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)^{-1} B_0^{-\mathrm{T}} \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \end{pmatrix}.$$

We then have $B \in \mathrm{Sp}_4(\mathcal{R}_L)$, and

$$BNB^{-1} - \partial(B)B^{-1} = \left(\begin{smallmatrix} 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right),$$

again by straightforward computations.

4.2 The pushforward functor

Let $R \in \{\mathcal{E}^\dagger, \mathcal{R}\}$. For any $g \in G(R)$ and $n \in \mathbb{N}$, we define

$$[n]_*(g) := g\varphi(g)\cdots\varphi^{n-1}(g) \in G(R),$$

the *n-pushforward* of g. Notice that $[n]_*(g)\varphi^n = (g\varphi)^n \in G(R) \rtimes \langle \varphi \rangle$ for all $n \in \mathbb{N}$.

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We define the *n-pushforward functor* by

$$[n]_*: \mathbf{B}^{\varphi,\nabla}(G,R) \longrightarrow \mathbf{B}^{\varphi^n,\nabla}(G,R), \qquad (g,X) \longmapsto \big([n]_*(g), X\big),$$

and $[n]_*(x) = x$ for all morphisms $x \in \mathbf{B}^{\varphi,\nabla}(G,R)$. The following lemma shows that this functor is well-defined (in particular, faithful).

Lemma 4.12 *Let $(g,X) \in \boldsymbol{B}^{\varphi,\nabla}(G,R)$. We then have $\big([n]_*(g), X\big) \in \boldsymbol{B}^{\varphi^n,\nabla}(G,R)$ for all $n \in \mathbb{N}$.*

Proof We show by induction on n that

$$X + \operatorname{dlog}\big([n]_*(g)\big) = \boldsymbol{\mu}(\varphi^n, t)\operatorname{Ad}\big([n]_*(g)\big)\big(\varphi^n(X)\big).$$

There is nothing to show when $n = 1$. We now assume by the induction hypothesis that

$$X + \operatorname{dlog}\big([n-1]_*(g)\big) = \boldsymbol{\mu}(\varphi^{n-1}, t)\operatorname{Ad}\big([n-1]_*(g)\big)\big(\varphi^{n-1}(X)\big),$$

We notice that $\boldsymbol{\mu}(\varphi^{n-1}, t) = \boldsymbol{\mu}\,\varphi(\boldsymbol{\mu})\cdots\varphi^{n-2}(\boldsymbol{\mu})$, and hence,

$$\partial(\varphi^{n-1}(f)) = \boldsymbol{\mu}\,\varphi(\boldsymbol{\mu})\cdots\varphi^{n-2}(\boldsymbol{\mu})\varphi^{n-1}(\partial(f)) = \boldsymbol{\mu}(\varphi^{n-1}, t)\varphi^{n-1}(\partial(f)), \qquad \forall f \in R,$$

which implies that

$$\operatorname{dlog}(\varphi^{n-1}(g)) = \boldsymbol{\mu}(\varphi^{n-1}, t)\varphi^{n-1}(\operatorname{dlog}(g)).$$

On the other hand, because $X + \operatorname{dlog}(g) = \boldsymbol{\mu}\operatorname{Ad}(g)(\varphi(X))$, we have

$$\varphi^{n-1}(X) + \varphi^{n-1}(\operatorname{dlog}(g)) = \varphi^{n-1}(\boldsymbol{\mu})\operatorname{Ad}\big(\varphi^{n-1}(g)\big)\big(\varphi^n(X)\big).$$

We now compute

$$\begin{aligned} X + \operatorname{dlog}\big([n]_*(g)\big) &= X + \operatorname{dlog}\big([n-1]_*(g)\big) + \operatorname{Ad}\big([n-1]_*(g)\big)\big(\operatorname{dlog}(\varphi^{n-1}(g))\big) \\ &= \boldsymbol{\mu}(\varphi^{n-1}, t)\operatorname{Ad}\big([n-1]_*(g)\big)\big(\varphi^{n-1}(X)\big) \\ &\quad + \boldsymbol{\mu}(\varphi^{n-1}, t)\operatorname{Ad}\big([n-1]_*(g)\big)\big(\varphi^{n-1}(\operatorname{dlog}(g))\big) \\ &= \boldsymbol{\mu}(\varphi^{n-1}, t)\operatorname{Ad}\big([n-1]_*(g)\big)\big(\varphi^{n-1}(X) + \varphi^{n-1}(\operatorname{dlog}(g))\big) \\ &= \boldsymbol{\mu}(\varphi^{n-1}, t)\operatorname{Ad}\big([n-1]_*(g)\big)\big(\varphi^{n-1}(\boldsymbol{\mu})\operatorname{Ad}\big(\varphi^{n-1}(g)\big)\big(\varphi^n(X)\big)\big) \\ &= \boldsymbol{\mu}(\varphi^n, t)\operatorname{Ad}\big([n]_*(g)\big)\big(\varphi^n(X)\big), \end{aligned}$$

which proves the lemma. ∎

In connection with the pushforward functor on φ-modules as recalled in Section 2.3, we state the following lemma resulting from [11, Lemma 1.6.3 and Remark 1.7.2], which will not be explicitly used in the sequel.

Lemma 4.13 *Let $g \in G(R)$. Then, $(V_R, g\varphi)$ is pure of slope μ if and only if $(V_R, [n]_*(g)\varphi^n)$ is pure of slope $n\mu$ for all $n \in \mathbb{N}$. Moreover, if $(V_R, g\varphi)$ has jumps $\mu_1, \ldots, \mu_l$, then $(V_R, [n]_*(g)\varphi^n)$ has jumps $n\mu_1, \ldots, n\mu_l$.*

4.3 *G*-φ-modules attached to splittings

Let $g \in G(\mathcal{R})$. We fix a splitting ξ_g of HN_g by Theorem 3.12.

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Construction 4.14 Let $(V_{\mathcal{R}}, g\varphi, \nabla_X)$ be a (φ, ∇)-module over $\mathcal{R}$ with the slope filtration

$$0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_l} = V_{\mathcal{R}},$$

with jumps $\mu_1 < \cdots < \mu_l$. Then, $\xi_g(V)$ is the decomposition

$$V_{\mathcal{R}} = \bigoplus_{i=1}^{l} V_{\mathcal{R},\mu_i}$$

of $\mathcal{R}$-modules such that $\bigoplus_{i=1}^{j} V_{\mathcal{R},\mu_i} = V_{\mathcal{R}}^{\mu_j}$ for $j = 1, \ldots, l$.

(i) For any $1 \le j \le l$ and $\mathbf{v} \in V_{\mathcal{R},\mu_j}$, we have $\Phi_g(\mathbf{v}) \in V_{\mathcal{R}}^{\mu_j}$, whence a unique expression $\Phi_g(\mathbf{v}) = \sum_{i=1}^{j} \mathbf{v}_i$ with $\mathbf{v}_i \in V_{\mathcal{R},\mu_i}$. We thus have a φ-linear map

$$\Phi_{g,\mu_j}: V_{\mathcal{R},\mu_j} \longrightarrow V_{\mathcal{R},\mu_j}, \quad \mathbf{v} \longmapsto \mathbf{v}_j.$$

We then define $\Phi'_g := \bigoplus_{i=1}^{l} \Phi_{g,\mu_i}$. We define

$$\mathrm{I}'(g)(V) := (V_{\mathcal{R}}, \Phi'_g).$$

For a morphism $\alpha: V \to W$ of finite-dimensional G-modules, we define $\mathrm{I}'(g)(\alpha) := \alpha_{\mathcal{R}}$.

(ii) Similarly, for any $1 \le j \le l$ and $\mathbf{v} \in V_{\mathcal{R},\mu_j}$, we have $\Theta_X(\mathbf{v}) \in V_{\mathcal{R}}^{\mu_j}$, whence a unique expression $\Theta_X(\mathbf{v}) = \sum_{i=1}^{j} \mathbf{v}_i$ with $\mathbf{v}_i \in V_{\mathcal{R},\mu_i}$. We thus have a K-linear differential operator

$$\Theta_{X,\mu_j}: V_{\mathcal{R},\mu_j} \longrightarrow V_{\mathcal{R},\mu_j}, \quad \mathbf{v} \longmapsto \mathbf{v}_j.$$

We then define $\Theta'_X := \bigoplus_{i=1}^{l} \Theta_{X,\mu_i}$.

Notice that $(V_{\mathcal{R},\mu_1}, \Phi_{g,\mu_1}) = (V_{\mathcal{R}}^{\mu_1}, \Phi_g|_{V_{\mathcal{R}}^{\mu_1}})$, and $(V_{\mathcal{R},\mu_i}, \Phi_{g,\mu_i})$ is isomorphic to $V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}$ as φ-modules for $2 \le i \le l$. Similarly, we have $(V_{\mathcal{R},\mu_1}, \Theta_{X,\mu_1}) = (V_{\mathcal{R}}^{\mu_1}, \Theta_X|_{V_{\mathcal{R}}^{\mu_1}})$, and $(V_{\mathcal{R},\mu_i}, \Theta_{X,\mu_i})$ is isomorphic to $V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}$ as a differential module for $2 \le i \le l$.

The remainder of this subsection is devoted to the consequences of Construction 4.14 (i). We will continue to discuss (ii) in Section 4.4; we will show, in particular, that both constructions assemble to give a G-(φ, ∇)-module over $\mathcal{R}$.

Lemma 4.15 *$\mathrm{I}'(g)$: $\mathbf{Rep}_F(G) \to \varphi$-$\mathbf{Mod}_{\mathcal{R}}$ is a G-φ-module over $\mathcal{R}$.*

Proof By Definition 3.1, it amounts to show that $\mathrm{I}'(g)$ is an exact faithful F-linear tensor functor. In this proof, we fix $V, W \in \mathbf{Rep}_F(G)$, and suppose the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ (resp. of $(W_{\mathcal{R}}, g\varphi)$) has jumps $\mu_1 < \cdots < \mu_{l_V}$ (resp. $\nu_1 < \cdots < \nu_{l_W}$).

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We first check the functoriality of $\mathrm{I}'(g)$ (the exactness, faithfulness, and F-linearity will follow immediately). Given $\alpha \in \mathrm{Hom}_G(V, W)$, we need to show that

$$\alpha_{\mathcal{R}} \circ \Phi'_g = \Phi'_g \circ \alpha_{\mathcal{R}}.$$

For any fixed $1 \le l \le l_V$, we have that $\alpha_{\mathcal{R}}(V_{\mathcal{R},\mu_l}) \subseteq W_{\mathcal{R},\mu_l}$ by Theorem 3.12. Notice that $W_{\mathcal{R},\mu_l} = W_{\mathcal{R},\nu_s}$ if $\mu_l = \nu_s$ for some $1 \le s \le l_W$, and $W_{\mathcal{R},\mu_l} = 0$ otherwise. In the latter case, it is clear that $\alpha_{\mathcal{R}} \circ \Phi'_g = \Phi'_g \circ \alpha_{\mathcal{R}} = 0$ on $V_{\mathcal{R},\mu_l}$, and we are done. Suppose now we are in the former case. Let $\mathbf{v}$ be a nonzero element in $V_{\mathcal{R},\mu_l}$. We then have $\Phi_g(\mathbf{v}) \in V_{\mathcal{R}}^{\mu_l}$ and $\alpha_{\mathcal{R}}(\mathbf{v}) \in W_{\mathcal{R},\nu_s}$. We have unique expressions

$$\Phi_g(\mathbf{v}) = \sum_{i=1}^{l} \mathbf{v}_i, \qquad \mathbf{v}_i \in V_{\mathcal{R},\mu_i},$$

and

$$\alpha_{\mathcal{R}} \circ \Phi_g(\mathbf{v}) = \sum_{i=1}^{s} \mathbf{w}_i, \qquad \mathbf{w}_i \in W_{\mathcal{R},\nu_i};$$

therefore $\alpha_{\mathcal{R}}(\mathbf{v}_l) = \mathbf{w}_s$. We also write

$$\Phi_g \circ \alpha_{\mathcal{R}}(\mathbf{v}) = \sum_{i=1}^{s} \mathbf{w}'_i, \quad \mathbf{w}'_i \in W_{\mathcal{R},\nu_i};$$

we then have $\mathbf{w}_i = \mathbf{w}'_i$ for $i = 1, \dots, s$, as $\alpha_{\mathcal{R}} \circ \Phi_g = \Phi_g \circ \alpha_{\mathcal{R}}$. We thus have $\alpha_{\mathcal{R}} \circ \Phi_{g,\mu_l}(\mathbf{v}) = \alpha_{\mathcal{R}}(\mathbf{v}_l) = \mathbf{w}_s$ and $\Phi_{g,\nu_s} \circ \alpha_{\mathcal{R}}(\mathbf{v}) = \mathbf{w}'_s = \mathbf{w}_s$, which implies that $\alpha_{\mathcal{R}} \circ \Phi_{g,\mu_l} = \Phi_{g,\nu_s} \circ \alpha_{\mathcal{R}}$, as desired.

It remains to show that $\mathrm{I}'(g)$ preserves tensor products. Because τ_g is a tensor functor, the $(\mu_l + \nu_s)$th graded piece of $\tau_g(V \otimes W)$ is then

$$\Big(V \bigotimes_F W\Big)_{\mathcal{R},\mu_l+\nu_s} = \bigoplus_{\substack{\mu_i + \nu_j = \mu_l + \nu_s \\ 1 \le i \le l_V, 1 \le j \le l_W}} \Big(V_{\mathcal{R},\mu_i} \bigotimes_{\mathcal{R}} W_{\mathcal{R},\nu_j}\Big),$$

for all $1 \le l \le l_V$ and $1 \le s \le l_W$. It then follows from Construction 4.14(i) that

$$\Phi'_{g,\mu_l+\nu_s} = \bigoplus_{\substack{\mu_i + \nu_j = \mu_l + \nu_s \\ 1 \le i \le l_V, 1 \le j \le l_W}} \big(\Phi'_{g,\mu_i} \otimes \Phi'_{g,\nu_j}\big),$$

which implies that $\mathrm{I}'(g)(V \otimes W)$ coincides with $\mathrm{I}'(g)(V) \otimes \mathrm{I}'(g)(W)$ on all $(V \otimes W)_{\mathcal{R},\mu_l+\nu_s}$, whence on $(V \otimes W)_{\mathcal{R}}$. This completes the proof. ■

With Lemma 4.15, we imitate *Step 2* in the proof of Proposition 4.9 and have the following proposition.

Proposition 4.16 *There exists a unique element $z \in G(\mathcal{R})$ such that $\mathrm{I}'(g) = \mathrm{I}(z)$.*

4.4 G-(φ, ∇)-modules attached to splittings

We fix $(g, X) \in \mathbf{B}^{\varphi,\nabla}(G, \mathcal{R})$. We also fix a splitting ξ_g of HN_g given by Theorem 3.12.

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We now look back at Construction 4.14(ii). We claim that $\Theta'_X - \mathrm{Id}_V \otimes\partial\colon V_R \to V_R$ is R-linear for all $(V,\rho_V) \in \mathbf{Rep}_F(G)$. Let $1 \le j \le l$ and let $v \otimes f \in V_{\mathcal{R},\mu_j}$. Suppose that $\Theta_X(v \otimes f) = \sum_{i=1}^{j} \mathbf{v}_i$ with $\mathbf{v}_i \in V_{\mathcal{R},\mu_i}$. Then, $\Theta'_X(v \otimes f) = \mathbf{v}_j$ by construction. Let $f' \in R$. We compute

$$\begin{aligned}\Theta'_X(v \otimes ff') &= v \otimes \partial(f)f' + v \otimes f\partial(f') + X(v \otimes ff')\\ &= \big(v \otimes \partial(f) + X(v \otimes f)\big)f' + v \otimes f\partial(f')\\ &= \Theta'_X(v \otimes f)f' + v \otimes f\partial(f')\\ &= f'\sum_{i=1}^{j}\mathbf{v}_i + v \otimes f\partial(f'),\end{aligned}$$

which implies that $\Theta'_X(v \otimes ff') = f'\mathbf{v}_j + v \otimes f\partial(f')$. We thus have

$$\begin{aligned}(\Theta'_X - \mathrm{Id}_V \otimes\partial)(v \otimes ff') &= f'\mathbf{v}_j + v \otimes f\partial(f') - v \otimes \partial(ff')\\ &= f'\mathbf{v}_j + v \otimes f\partial(f') - v \otimes \partial(f)f' - v \otimes f\partial(f')\\ &= f'(\mathbf{v}_j - v \otimes \partial(f))\\ &= f'(\Theta'_X - \mathrm{Id}_V \otimes\partial)(v \otimes f),\end{aligned}$$

as desired.

The following proposition (and its proof) is analogous to Lemma 4.15.

Proposition 4.17 *There exists a unique element $X_0 \in \mathfrak{g}_{\mathcal{R}}$ such that $\Theta'_X = \Theta_{X_0}$.*

Proof For any $(V,\rho_V) \in \mathbf{Rep}_F(G)$, we define $\theta_V := \Theta'_X - \mathrm{Id}_V \otimes\partial$. We claim that the family

$$\{\theta_V\colon V_{\mathcal{R}} \to V_{\mathcal{R}} \mid (V,\rho_V) \in \mathbf{Rep}_F(G)\}$$

of R-linear endomorphisms satisfies conditions (i–iii) in Corollary 2.9. The lemma will follow immediately.

It is clear that $\theta_V = 0$ if $V = F$ is the trivial G-representation. For the remainder of the proof, we fix $(V,\rho_V),(W,\rho_W) \in \mathbf{Rep}_F(G)$, and suppose the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ (resp. of $(W_{\mathcal{R}}, g\varphi)$) has jumps $\mu_1 < \cdots < \mu_{l_V}$ (resp. $\nu_1 < \cdots < \nu_{l_W}$). Let $\alpha \in \mathrm{Hom}_G(V,W)$. To show that $\theta_V \circ \alpha_{\mathcal{R}} = \alpha_{\mathcal{R}} \circ \theta_W$, it suffices to show that $\Theta'_X \circ \alpha_{\mathcal{R}} = \alpha_{\mathcal{R}} \circ \Theta'_X$. Notice that $\alpha_{\mathcal{R}}$ respects gradings. Replacing Φ_g with Θ_X (possibly with proper decorations) in the second paragraph of the proof of Lemma 4.15, we have the desired result.

It remains to show that

$$\theta_{V\otimes W} = \theta_V \otimes \mathrm{Id}_{W_{\mathcal{R}}} + \mathrm{Id}_{V_{\mathcal{R}}} \otimes\theta_W.$$

Because τ_g is a tensor functor, the $(\mu_l + \nu_s)$th graded piece of $\tau_g(V \otimes W)$ is then

$$(V \bigotimes W)_{\mathcal{R},\mu_l+\nu_s} = \bigoplus_{\substack{\mu_i + \nu_j = \mu_l + \nu_s\\ 1 \le i \le l_V, 1 \le j \le l_W}} (V_{\mathcal{R},\mu_i} \bigotimes_{\mathcal{R}} W_{\mathcal{R},\nu_j}),$$

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for all $1 \le l \le l_V$ and $1 \le s \le l_W$. It follows from Lemma 4.3 and Construction 4.14 that

$$\Theta'_{X,\mu_l+\nu_s} = \bigoplus_{\substack{\mu_i+\nu_j=\mu_l+\nu_s \\ 1\le i\le l_V, 1\le j\le l_W}} \left(\Theta'_{X,\mu_i} \otimes \mathrm{Id}_{W_{\mathcal{R},\nu_j}} + \mathrm{Id}_{V_{\mathcal{R},\mu_i}} \otimes \Theta'_{X,\nu_j}\right).$$

Let $v \otimes f \otimes w \otimes f' \in V_{\mathcal{R},\mu_i} \otimes_{\mathcal{R}} W_{\mathcal{R},\nu_j}$. We compute

$$\begin{aligned}
&\left(\theta_V \otimes \mathrm{Id}_{W_{\mathcal{R}}} + \mathrm{Id}_{V_{\mathcal{R}}} \otimes \theta_W\right)(v \otimes f \otimes w \otimes f') \\
=&\left(\Theta'_{X,\mu_i}(v \otimes f) - v \otimes \partial(f)\right) \otimes w \otimes f' + v \otimes f \otimes \left(\Theta'_{X,\nu_j}(w \otimes f') - w \otimes \partial(f')\right) \\
=&\left(\Theta'_{X,\mu_i} \otimes \mathrm{Id}_{W_{\mathcal{R},\nu_j}} + \mathrm{Id}_{V_{\mathcal{R},\mu_i}} \otimes \Theta'_{X,\nu_j}\right)(v \otimes f \otimes w \otimes f') - v \otimes 1 \otimes w \otimes \partial(ff') \\
=&\left(\Theta'_{X,\mu_l+\nu_s} - \mathrm{Id}_{V\otimes W} \otimes \partial\right)(v \otimes w \otimes ff') \\
=&\theta_{V\otimes W}(v \otimes w \otimes ff'),
\end{aligned}$$

which completes the proof. ■

We now summarize what we have shown thus far. The splitting ξ_g of HN_g gives a unique element $z \in G(\mathcal{R})$ such that $\mathrm{I}'(g) = \mathrm{I}(z)$ by Proposition 4.16, and a unique element $X_0 \in \mathfrak{g}_{\mathcal{R}}$ such that $\Theta'_X = \Theta_{X_0}$ by Proposition 4.17. These two elements are related as in Proposition 4.19 below.

We next recall some notions from [4, Section 2.1]. Let k be a commutative ring with 1, and let $\mathfrak{G}$ be a reductive k-group. Hereupon, we denote by $\kappa(s)$ the residue field of s and $\bar{\kappa}(s)$ an algebraic closure of $\kappa(s)$, for all $s \in \operatorname{Spec} k$. A subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a *parabolic* (resp. *Borel*) subgroup if $\mathfrak{P}$ is smooth and $\mathfrak{P}_{\bar{\kappa}(s)}$ is a parabolic (resp. Borel) subgroup of $\mathfrak{G}_{\bar{\kappa}(s)}$, for all $s \in \operatorname{Spec} k$.

Suppose we have a cocharacter $\lambda\colon \mathbb{G}_m \to \mathfrak{G}$ over k. For any k-algebra R, we let $\mathbb{G}_{m,R}$ act on $\mathfrak{G}_R$ via the conjugation

$$\mathbb{G}_{m,R}(S) \times \mathfrak{G}_R(S) \longrightarrow \mathfrak{G}_R(S), \quad (t,x) \longmapsto t.x := \lambda(t)x\lambda(t)^{-1}$$

for all R-algebra S. For any $x \in \mathfrak{G}(R)$, we have an *orbit map* $\alpha_x\colon \mathbb{G}_{m,R} \to \mathfrak{G}_R$ given by

$$\alpha_x\colon \mathbb{G}_{m,R}(S) \longrightarrow \mathfrak{G}_R(S), \quad t \longmapsto t.x$$

for all R-algebras S. Let $\mathbb{A}^1$ be the affine k-line. We say that the *limit* $\lim_{t\to 0} t.x$ exists if α_x extends (necessarily uniquely) to a morphism $\tilde{\alpha}_x\colon \mathbb{A}^1_R \to \mathfrak{G}_R$ of affine R-schemes, and put $\lim_{t\to 0} t.x := \tilde{\alpha}_x(0) \in \mathfrak{G}_R(R)$. We define

$$P_{\mathfrak{G}}(\lambda)(R) := \left\{x \in \mathfrak{G}(R) \mid \lim_{t\to 0} t.x \text{ exists}\right\},$$

$$U_{\mathfrak{G}}(\lambda)(R) := \left\{x \in \mathfrak{G}(R) \mid \lim_{t\to 0} t.x = 1\right\},$$

and

$$Z_{\mathfrak{G}}(\lambda)(R) := P_{\mathfrak{G}}(\lambda)(R) \cap P_{\mathfrak{G}}(-\lambda)(R),$$

where $-\lambda$ is the reciprocal of λ. Then, $P_{\mathfrak{G}}(\lambda)$ is a closed k-subgroup of $\mathfrak{G}$ [4, Lemma 2.1.4], $U_{\mathfrak{G}}(\lambda)$ is an affine algebraic k-normal subgroup of $P_{\mathfrak{G}}(\lambda)$, and $Z_{\mathfrak{G}}(\lambda)$ is the

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centralizer of the $\mathbb{G}_m$-action in $\mathfrak{G}$ [4, Lemma 2.1.5]. By [4, Proposition 2.1.8(3)], these subgroups are smooth, because $\mathfrak{G}$ is smooth.

It follows from the definitions that the formations of $P_{\mathfrak{G}}(\lambda)$, $U_{\mathfrak{G}}(\lambda)$, and $Z_{\mathfrak{G}}(\lambda)$ commute with any base extension on k. In particular, for every $s \in \operatorname{Spec} k$, we have $P_{\mathfrak{G}}(\lambda)_{\bar{\kappa}(s)} = P_{\mathfrak{G}_{\bar{\kappa}(s)}}(\lambda_{\bar{\kappa}(s)})$, which is a parabolic subgroup of $\mathfrak{G}_{\bar{\kappa}(s)}$ by [20, Proposition 8.4.5]. Hence, $P_{\mathfrak{G}}(\lambda)$ is a parabolic k-group.

By [4, Proposition 2.1.8(2)], the multiplication map gives an isomorphism

$$U_{\mathfrak{G}}(\lambda) \rtimes Z_{\mathfrak{G}}(\lambda) \longrightarrow P_{\mathfrak{G}}(\lambda)$$

of affine algebraic k-groups.

Now, let $\mathbb{G}_m$ act on $\mathfrak{g} = \operatorname{Lie}(\mathfrak{G})(k)$ through the adjoint representation. We then have $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{X \in \mathfrak{g} \mid t.X = t^n X, \forall t \in \mathbb{G}_m\}$ for all $n \in \mathbb{Z}$. We have $\operatorname{Lie}\left(Z_{\mathfrak{G}}(\lambda)\right) = \mathfrak{g}_0$ (which is the centralizer of the $\mathbb{G}_m$-action on $\mathfrak{g}$), $\operatorname{Lie}\left(U_{\mathfrak{G}}(\lambda)\right) = \bigoplus_{n>0} \mathfrak{g}_n$, and $\operatorname{Lie}\left(P_{\mathfrak{G}}(\lambda)\right) = \bigoplus_{n\geq 0} \mathfrak{g}_n$. In particular, we have the following decomposition:

$$\operatorname{Lie}\left(P_{\mathfrak{G}}(\lambda)\right) = \operatorname{Lie}\left(Z_{\mathfrak{G}}(\lambda)\right) \oplus \operatorname{Lie}\left(U_{\mathfrak{G}}(\lambda)\right). \tag{8}$$

Lemma 4.18 *With the notion above, we have*

$$Z - \operatorname{Ad}(u)(Z) \in \operatorname{Lie}\left(U_{\mathfrak{G}}(\lambda)\right),$$

for all $u \in U_{\mathfrak{G}}(\lambda)(k)$ *and* $Z \in \operatorname{Lie}\left(Z_{\mathfrak{G}}(\lambda)\right)$.

Proof Recall that $Z \in Z_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ by definition; we may also view u as an element in $U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ via the inclusion $\iota: k \hookrightarrow k[\varepsilon]$. By the definition of the adjoint representation, we have

$$Z - \operatorname{Ad}(u)(Z) = Z(uZu^{-1})^{-1} = ZuZ^{-1}u^{-1} \in P_{\mathfrak{G}}(\lambda)(k[\varepsilon]).$$

Because $U_{\mathfrak{G}}(\lambda)$ is normal in $P_{\mathfrak{G}}(\lambda)$, we have that $ZuZ^{-1} \in U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$, and so is $ZuZ^{-1}u^{-1}$. Consider the following commutative diagram:

$$\begin{array}{ccc} U_{\mathfrak{G}}(\lambda)(k[\varepsilon]) & \hookrightarrow & P_{\mathfrak{G}}(\lambda)(k[\varepsilon]) \\ \downarrow & & \downarrow \\ U_{\mathfrak{G}}(\lambda)(k) & \hookrightarrow & P_{\mathfrak{G}}(\lambda)(k) \end{array}$$

Because both Z and $uZ^{-1}u^{-1}$ lie in the kernel of the right vertical map, so does their product $ZuZ^{-1}u^{-1}$. Hence, $ZuZ^{-1}u^{-1} \in U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ lies in the kernel of the left vertical map. The lemma then follows. ∎

Proposition 4.19 *Let* $z \in G(\mathcal{R})$ *and* $X_0 \in \mathfrak{g}_{\mathcal{R}}$ *be the unique elements given by Propositions 4.16 and 4.17, respectively. We have* $X_0 = \Gamma_z\left(\boldsymbol{\mu}\, \varphi(X_0)\right)$. *In particular,* $\mathrm{I}(z, X_0)$ *is a* G-(φ, ∇)*-module over* $\mathcal{R}$.

Proof The second assertion follows from the first assertion and Lemma 4.7. For the first assertion, we need to show

$$X_0 = \boldsymbol{\mu} \cdot \operatorname{Ad}(z)\left(\varphi(X_0)\right) - \operatorname{dlog}(z). \tag{9}$$

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It suffices to show (3) with both sides understood as elements in $\mathrm{End}_{\mathcal{R}}(V_{\mathcal{R}})$ for some faithful representation $(V,\rho)\in\mathbf{Rep}_F(G)$. Suppose that $\dim_F V=d$, and suppose that $v_g(V)$ is the decomposition $V_{\mathcal{R}}=\bigoplus_{i=1}^{l}V_{\mathcal{R},\mu_i}$. We choose for each graded-piece $V_{\mathcal{R},\mu_i}$ a basis. They altogether give a basis $\mathbf{v}_1,\dots,\mathbf{v}_d$ of $V_{\mathcal{R}}$, in which Φ_g acts via a block-upper-triangular matrix

$$A=\begin{pmatrix}A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_l\end{pmatrix}\in \mathrm{GL}_d(\mathcal{R}),$$

where each A_i is an m_i by m_i invertible matrix with m_i the multiplicity of μ_i. Then, Φ_z acts in this basis via $Z:=\mathrm{Diag}(A_1,\dots,A_l)$. Likewise, Θ_X acts in the basis $\mathbf{v}_1,\dots,\mathbf{v}_d$ via a block-upper-triangular matrix

$$N=\begin{pmatrix}N_1 & & & * \\ & N_2 & & \\ & & \ddots & \\ & & & N_l\end{pmatrix}\in \mathrm{Mat}_{d,d}(\mathcal{R}),$$

where each N_i is an m_i by m_i matrix, and Θ_{X_0} acts via $\overline{N}:=\mathrm{Diag}(N_1,\dots,N_l)$. Write $A=ZU$ for $U\in\mathrm{GL}_d(\mathcal{R})$, and $N=\overline{N}+N_+$ for $N_+\in\mathrm{Mat}_{d,d}(\mathcal{R})$. Because $X=\Gamma_g(\boldsymbol{\mu}\,\varphi(X))$, we have $N=\boldsymbol{\mu}\cdot A\varphi(N)A^{-1}-\partial(A)A^{-1}$, and then

$$\begin{aligned}\overline{N}+N_+ &=\boldsymbol{\mu}\cdot(UZ)(\varphi(\overline{N}+N_+))(UZ)^{-1}-\partial(UZ)(UZ)^{-1}\\ &=\boldsymbol{\mu}\cdot(UZ)\varphi(\overline{N})Z^{-1}U^{-1}+\boldsymbol{\mu}\cdot(UZ)\varphi(N_+)Z^{-1}U^{-1}-\partial(U)U^{-1}-U\partial(Z)Z^{-1}U^{-1}.\end{aligned}$$

Applying $\mathrm{Ad}(U^{-1})$ on both sides, we then have

$$\begin{aligned}&\boldsymbol{\mu}\cdot Z\varphi(\overline{N})Z^{-1}-\partial(Z)Z^{-1}+\boldsymbol{\mu}\cdot Z\varphi(N_+)Z^{-1}-U^{-1}\partial(U)\\ =&U^{-1}\overline{N}U+U^{-1}N_+U=\overline{N}-(\overline{N}-U^{-1}\overline{N}U-U^{-1}N_+U).\end{aligned}$$

We claim that $\boldsymbol{\mu}\cdot Z\varphi(\overline{N})Z^{-1}-\partial(Z)Z^{-1}=\overline{N}$. Put $\lambda_{\rho,g}:=\rho\circ\lambda_g\colon\mathbb{G}_{m,\mathcal{R}}\to\mathrm{GL}_{V,\mathcal{R}}$, where $\lambda_g\colon\mathbb{G}_{m,\mathcal{R}}\to G_{\mathcal{R}}$ is the slope morphism defined in Construction 3.15. Identifying $\mathrm{GL}_{V,\mathcal{R}}$ with $\mathrm{GL}_{d,\mathcal{R}}$ via the basis $\mathbf{v}_1,\dots,\mathbf{v}_d$ given in the preceding paragraph, and letting $\mathfrak{G}=\mathrm{GL}_{d,\mathcal{R}}$, we then have an isomorphism

$$U_{\mathfrak{G}}(-\lambda_{\rho,g})\rtimes Z_{\mathfrak{G}}(-\lambda_{\rho,g})\cong P_{\mathfrak{G}}(-\lambda_{\rho,g})$$

of affine algebraic $\mathcal{R}$-groups. Because $\mu_1<\cdots<\mu_l$, we have

$$A\in P_{\mathfrak{G}}(-\lambda_{\rho,g})(\mathcal{R}),\ U\in U_{\mathfrak{G}}(-\lambda_{\rho,g})(\mathcal{R}),\ Z\in Z_{\mathfrak{G}}(-\lambda_{\rho,g})(\mathcal{R});$$

$$N\in\mathrm{Lie}\big(P_{\mathfrak{G}}(-\lambda_{\rho,g})\big),\ N_+\in\mathrm{Lie}\big(U_{\mathfrak{G}}(-\lambda_{\rho,g})\big),\ \overline{N}\in\mathrm{Lie}\big(Z_{\mathfrak{G}}(-\lambda_{\rho,g})\big).$$

It follows from Lemma 4.18 that $\overline{N}-U^{-1}\overline{N}U\in\mathrm{Lie}\big(U_{\mathfrak{G}}(-\lambda_{\rho,g})\big)$. In particular, we have $\overline{N}-U^{-1}\overline{N}U-U^{-1}N_+U\in\mathrm{Lie}\big(U_{\mathfrak{G}}(-\lambda_{\rho,g})\big)$. On the other hand, it is clear that $\boldsymbol{\mu}\cdot Z\varphi(\overline{N})Z^{-1}-\partial(Z)Z^{-1}\in\mathrm{Lie}\big(Z_{\mathfrak{G}}(-\lambda_{\rho,g})\big)$ and $\boldsymbol{\mu}\cdot Z\varphi(N_+)Z^{-1}-U^{-1}\partial(U)\in\mathrm{Lie}\big(U_{\mathfrak{G}}(-\lambda_{\rho,g})\big)$. By decomposition (2), we have $\boldsymbol{\mu}\cdot Z\varphi(\overline{N})Z^{-1}-\partial(Z)Z^{-1}=\overline{N}$, and the desired equality (3) follows. ■

Recall that the least common denominator d_g of g is constructed in Construction 3.8, and $\lambda_g\colon\mathbb{G}_{m,\mathcal{R}}\to G_{\mathcal{R}}$ is the slope morphism (see Construction 3.15). We next reduce the G-(φ,∇)-module (z,X_0) over $\mathcal{R}$ to a unit-root one by applying the pushforward functor $[d_g]_*$ and *twisting* by $\lambda_g(\varpi^{-1})$.

Corollary 4.20 $\mathrm{I}\big(\lambda_g(\varpi^{-1})[d_g]_*(z),X_0\big)$ *is a unit-root* G-(φ^{d_g},∇)*-module over* $\mathcal{R}$.

Proof For any $V\in\mathbf{Rep}_F(G)$, it suffices to show that $(V_{\mathcal{R}},[d_g]_*(z)\varphi^{d_g},\nabla_{X_0})$ is unit-root. By Proposition 4.19 and Lemma 4.12, $(V_{\mathcal{R}},[d_g]_*(z)\varphi^{d_g},\nabla_{X_0})$ is a (φ^{d_g},∇)-module over $\mathcal{R}$. Equivalently, we have $\Theta_{X_0}\circ\Phi_z^{d_g}=\boldsymbol{\mu}\cdot\Phi_z^{d_g}\circ\Theta_{X_0}$. Suppose that $(V_{\mathcal{R}},g\varphi)$ has jumps $\mu_1,\ldots,\mu_l$, then $\big(V_{\mathcal{R}},[d_g]_*(z)\varphi^{d_g}\big)$ has jumps $d_g\mu_1,\ldots,d_g\mu_l$ by Lemma 4.13. For any $1\le i\le l$, $\rho(\lambda_g(\varpi^{-1}))$ acts via multiplication by $\varpi^{-d_g\mu_i}\in K$ on the graded-piece $V_{\mathcal{R},\mu_i}$, which implies that $\big(V_{\mathcal{R},\mu_i},\lambda_g(\varpi^{-1})[d_g]_*(z)\varphi^{d_g}\big)$ is unit-root. It follows from [10, Proposition 4.6.3(a)] that $\big(V_{\mathcal{R}},\lambda_g(\varpi^{-1})[d_g]_*(z)\varphi^{d_g}\big)$ is unit-root. Moreover, because Θ_{X_0} is K-linear, we have

$$\Theta_{X_0}\circ\rho(\lambda_g(\varpi^{-1}))\circ\Phi_z^{d_g}=\rho(\lambda_g(\varpi^{-1}))\circ\Theta_{X_0}\circ\Phi_z^{d_g}=\boldsymbol{\mu}\cdot\rho(\lambda_g(\varpi^{-1}))\circ\Phi_z^{d_g}\circ\Theta_{X_0},$$

which completes the proof. ∎

4.5 A *G*-version of the *p*-adic local monodromy theorem

Let L be a finite separable extension of $\kappa((t))$, and let $\mathcal{E}_L^\dagger$ be the unique unramified extension of $\mathcal{E}^\dagger$ with residue field L. We put $\mathcal{R}_L:=\mathcal{R}\otimes_{\mathcal{E}^\dagger}\mathcal{E}_L^\dagger$.

We put

$$\mathcal{E}^{\dagger,\mathrm{nr}}:=\varinjlim_L\mathcal{E}_L^\dagger,\quad\text{and}\quad\mathcal{B}_0:=\varinjlim_L\mathcal{R}_L\cong\mathcal{R}\bigotimes_{\mathcal{E}^\dagger}\mathcal{E}^{\dagger,\mathrm{nr}},$$

where L runs through all finite separable extensions of $\kappa((t))$. In fact, $\mathcal{E}^{\dagger,\mathrm{nr}}$ is the maximal unramified extension of $\mathcal{E}^\dagger$ with residue field $\kappa((t))^{\mathrm{sep}}$, the separable closure of $\kappa((t))$.

The main result of this paper is the following theorem.

Theorem 4.21 *Let G be a connected reductive F-group, and let* $(g,X)\in\boldsymbol{B}^{\varphi,\nabla}(G,\mathcal{R})$. *Then, there exist a finite separable extension L of $\kappa((t))$ and an element $b\in G(\mathcal{R}_L)$ such that* $\Gamma_b(X)\in\mathrm{Lie}\big(U_{G_{\mathcal{R}}}(-\lambda_g)\big)_{\mathcal{R}_L}$.

We will make use of the following lemma, which is often mentioned as Steinberg's theorem. The theory of fields of cohomological dimension ≤ 1 can be found in, e.g., [19, Chapter II, Section 3]; for us, the most important example will be a Henselian discretely valued field of characteristic 0 with algebraically closed residue field (see [19, Chapter II, Section 3.3]).

Lemma 4.22 ([21, Theorem 1.9]) *Suppose that k is a field of cohomological dimension* ≤ 1 *and $\mathfrak{G}$ is a connected reductive k-group, then* $H^1(k,\mathfrak{G})=1$.

Proof of Theorem 4.21 Let $z\in G(\mathcal{R})$ and $X_0\in\mathfrak{g}_{\mathcal{R}}$ be the unique elements given by Propositions 4.16 and 4.17, respectively.

Let (V,ρ) be a d-dimensional G-representation (not necessarily faithful). Suppose the slope filtration of $(V_{\mathcal{R}},g\varphi)$ has jumps $\mu_1,\ldots,\mu_l$. Suppose that $\xi_g(V)=\bigoplus_{i=1}^{l}V_{\mathcal{R},\mu_i}$, we put $d_i:=\mathrm{rk}_{\mathcal{R}}(V_{\mathcal{R},\mu_i})$ for all i. In the proof of Corollary 4.20, we see

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that $\left(V_{\mathcal{R},\mu_i}, \lambda_g(\varpi^{-1})[d_g]_*(z)\varphi^{d_g}, \nabla_{X_0}\right)$ is a unit-root (φ,∇)-module over $\mathcal{R}$ for all $1 \le i \le l$. Let $\Phi_z = z\varphi$, and let $\Theta_{X_0}\colon V_{\mathcal{R}} \to V_{\mathcal{R}}$ be the differential operator associated to ∇_{X_0}. Then, Φ_z (resp. Θ_{X_0}) may be extended to $V \otimes_F \mathcal{B}_0$, which is still denoted by Φ_z (resp. Θ_{X_0}). By the unit-root pLMT [9, Theorem 6.11], we find:

(i) a finite separable extension $L(V)$ of $\kappa((t))$;

(ii) for each $1 \le i \le l$, a basis $\mathbf{w}_1^{(i)}, \dots, \mathbf{w}_{d_i}^{(i)}$ for $V_{\mathcal{R},\mu_i} \otimes_{\mathcal{R}} \mathcal{R}_{L(V)}$ over $\mathcal{R}_{L(V)}$ such that $\Theta_{X_0}(\mathbf{w}_j^{(i)}) = 0$, for all $1 \le j \le d_i$.

Then, for each $1 \le i \le l$, we have that

$$W_i := (V_{\mathcal{R},\mu_i} \bigotimes_{\mathcal{R}} \mathcal{B}_0)^{\Theta_{X_0}=0} = \left\{x \in V_{\mathcal{R},\mu_i} \bigotimes_{\mathcal{R}} \mathcal{B}_0 \mid \Theta_{X_0}(x) = 0\right\}$$

is a d_i-dimensional K^{nr}-vector space spanned by $\mathbf{w}_1^{(i)}, \dots, \mathbf{w}_{d_i}^{(i)}$. In particular, we have

$$(V_{\mathcal{B}_0})^{\Theta_{X_0}=0} = \left\{x \in V_{\mathcal{B}_0} \mid \Theta_{X_0}(x) = 0\right\} = \bigoplus_{i=1}^{l} W_i,$$

which is a d_i-dimensional K^{nr}-vector space.

We now have two K^{nr}-valued fiber functors

$$\omega_1 = \omega^G \otimes K^{\mathrm{nr}}\colon \mathbf{Rep}_F(G) \longrightarrow \mathbf{Vec}_{K^{\mathrm{nr}}}, \quad V \longmapsto V \otimes K^{\mathrm{nr}},$$

and

$$\omega_2\colon \mathbf{Rep}_F(G) \longrightarrow \mathbf{Vec}_{K^{\mathrm{nr}}}, \quad V \longmapsto (V_{\mathcal{B}_0})^{\Theta_{X_0}=0}.$$

Moreover, we have an action

$$\underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2) \times \underline{\mathrm{Aut}}^{\otimes}(\omega_1) \longrightarrow \underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)$$

of $\underline{\mathrm{Aut}}^{\otimes}(\omega_1)$ on $\underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)$, given by precomposition. We note that $\underline{\mathrm{Aut}}^{\otimes}(\omega_1) = \underline{\mathrm{Aut}}^{\otimes}(\omega^G \otimes K^{\mathrm{nr}}) \cong G_{K^{\mathrm{nr}}}$,[2] so $\underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)$ may be viewed as a $G_{K^{\mathrm{nr}}}$-torsor over K^{nr}. By Lemma 4.22, we have $H^1(K^{\mathrm{nr}}, G_{K^{\mathrm{nr}}}) = 1$. Thus, $\underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)$ is isomorphic to the trivial $G_{K^{\mathrm{nr}}}$-torsor over K^{nr}, i.e., we have $\underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)_{K^{\mathrm{nr}}} \cong G_{K^{\mathrm{nr}}}$.

On the other hand, we have an isomorphism $\gamma\colon \omega_2 \otimes \mathcal{B}_0 \to \omega_1 \otimes \mathcal{B}_0$ of tensor functors, induced by the $\mathcal{B}_0$-linear extension of the inclusion

$$(V_{\mathcal{B}_0})^{\Theta_{X_0}=0} \hookrightarrow V_{\mathcal{B}_0}$$

for all $(V,\rho) \in \mathbf{Rep}_F(G)$. We now fix $\beta \in \underline{\mathrm{Isom}}^{\otimes}(\omega_1,\omega_2)(K^{\mathrm{nr}})$; we then have an element $\tilde{\beta} := \gamma \circ \beta_{\mathcal{B}_0} \in \underline{\mathrm{Aut}}^{\otimes}(\omega_1 \otimes \mathcal{B}_0)(\mathcal{B}_0) = G(\mathcal{B}_0)$. Let $b \in G(\mathcal{B}_0)$ be the inverse of the image of $\tilde{\beta}$ under the following isomorphism:

$$\underline{\mathrm{Aut}}^{\otimes}(\omega_1 \otimes \mathcal{B}_0)(\mathcal{B}_0) \longrightarrow G(\mathcal{B}_0).$$

Because $F[G]$ is finitely presented over F, the functor $\mathrm{Hom}_{\mathbf{Alg}_F}(F[G], \_)$ commutes with colimits. We have

$$G(\mathcal{B}_0) = G(\varinjlim_L \mathcal{R}_L) = \varinjlim_L G(\mathcal{R}_L),$$

[2] For this isomorphism, we refer to the discussion above Proposition 3.11.

where L runs over all finite separable extensions of $\kappa((t))$; we thus find a finite separable extension L of $\kappa((t))$ such that $b \in G(\mathcal{R}_L)$.

For any $(V,\rho) \in \mathbf{Rep}_F(G)$, it follows from the construction of b that the automorphism $\rho(b^{-1})\colon V_{\mathcal{B}_0} \to V_{\mathcal{B}_0}$ factors through $(V_{\mathcal{B}_0})^{\Theta_{X_0}=0} \otimes \mathcal{B}_0$. Notice that Θ_{X_0} and X_0 agree on $\omega_1(V) = V_{K^{\mathrm{nr}}}$. Therefore, we have

$$\rho(b)X_0\rho(b^{-1}) - \partial(\rho(b))\rho(b^{-1}) = 0. \tag{10}$$

We now fix a faithful representation (V,ρ). The equality (4) then implies

$$\Gamma_b(X_0) = 0.$$

Put $X_1 := X - X_0 \in \mathfrak{g}_{\mathcal{R}}$; we then have

$$\begin{aligned}\Gamma_b(X) &= \mathrm{Ad}(b)(X_0 + X_1) - \mathrm{dlog}(b)\\ &= \mathrm{Ad}(b)(X_0) - \mathrm{dlog}(b) + \mathrm{Ad}(b)(X_1)\\ &= \Gamma_b(X_0) + \mathrm{Ad}(b)(X_1)\\ &= \mathrm{Ad}(b)(X_1).\end{aligned}$$

Conserving the notation as in the second paragraph, $\Theta_X = \rho(b)X_1\rho(b^{-1})$ acts in the basis $\mathbf{w}_1^{(1)},\dots,\mathbf{w}_{d_1}^{(1)},\dots,\mathbf{w}_1^{(l)},\dots,\mathbf{w}_{d_l}^{(l)}$ via a matrix of the form

$$\begin{pmatrix} 0 & & * \\ & 0 \ddots & \\ & & 0 \end{pmatrix} \in \mathrm{Mat}_{d,d}(\mathcal{R}_L).$$

Here, the ith 0 in the diagonal denotes the zero matrix of size $d_i \times d_i$. Hence, $\Gamma_b(X) \in \mathrm{Lie}\left(U_{G_{\mathcal{R}_L}}(-\lambda_{g,\mathcal{R}_L})\right) = \mathrm{Lie}\left(U_{G_{\mathcal{R}}}(-\lambda_g)_{\mathcal{R}_L}\right) = \mathrm{Lie}\left(U_{G_{\mathcal{R}}}(-\lambda_g)\right)_{\mathcal{R}_L}$, as desired. ∎

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