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ON A PLANAR SCHRÖDINGER–POISSON SYSTEM INVOLVING A NON-SYMMETRIC POTENTIAL

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Abstract We prove the existence of a ground state positive solution of Schrödinger–Poisson systems in the plane of the form

$$-\Delta u + V(x)u + \frac{\gamma}{2\pi}\left(\log|\cdot| * u^2\right) u = b|u|^{p-2}u \quad \text{in } \mathbb{R}^2,$$

where $p \geq 4$, $\gamma, b > 0$ and the potential V is assumed to be positive and unbounded at infinity. On the potential we do not require any symmetry or periodicity assumption, and it is not supposed it has a limit at infinity. We approach the problem by variational methods, using a variant of the mountain pass theorem and the Cerami compactness condition.

Keywords: nonlinear Schrödinger–Poisson systems; planar case; positive solutions

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1. Introduction

This work deals with the existence of positive solutions of Schrödinger–Poisson systems of the form

$$\begin{cases} -\Delta u + V(x)u + \gamma\phi(x)u = b\,|u|^{p-2}u & \text{in } \mathbb{R}^2 \\ \Delta\phi = u^2 & \text{in } \mathbb{R}^2, \end{cases} \tag{S}$$

where we assume $b > 0$, the nonlinearity $p \geq 4$, the coupling coefficient $\gamma > 0$ and the potential V is a positive measurable function.

For their relevance in physics, Schrödinger–Poisson systems have been extensively studied in the three-dimensional case, see, for example, the survey paper [1], the recent paper [15] or also [2, 3, 8, 9, 25] and the references therein. On the contrary, in dimension two they are much less studied, also due to some difficulties intrinsic to the planar case (see [4, 10, 11, 13, 14, 20, 21, 27] and references therein).

Concerning the semiclassical analysis related to (S), very little is known. The autonomous case has been treated first in [20], while the case where a potential is involved has been studied only recently, in [4], where the problem is fronted by perturbation methods.

We observe that in the two-dimensional Choquard problem with logarithmic kernel, analogous difficulties come out. We refer the reader to the recent paper [7] for more details and references on this subject. In [7], the authors considered non homogeneous nonlinearities and the positive potential is assumed to be continuous and one-periodic.

Since the Poisson equation in (S) gives

$$\phi = \phi_u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, u^2(y)\,\mathrm{d}y, \tag{1.1}$$

the system (S) can be reduced to the single nonlocal equation

$$-\Delta u + V(x)u + \frac{\gamma}{2\pi}(\log(|\cdot|) * u^2)u = b|u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{E}$$

It is natural to expect that the solutions of equation (E) correspond to the critical points of the following action functional, where we have normalized the constants:

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \\ & + \frac{1}{4}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x. \end{aligned}$$

However, a first difficulty is that I is not well defined on the usual Sobolev space $H^1(\mathbb{R}^2)$, for the presence of the sign-changing and unbounded logarithmic kernel. Moreover, in the framework considered in this paper, another point to be fixed is related to the potential, that is assumed to be unbounded and possibly not coercive at infinity. To overcome the difficulty related to the Newtonian kernel, Stubbe in [27] introduced the following Hilbert space, to study (S) in the autonomous case $V \equiv 0$, $b = 0$,

$$\widetilde{X} := \left\{ u \in H^1(\mathbb{R}^2) \ : \ \int_{\mathbb{R}^2}[\log(1+|x|)]u^2(x)\,\mathrm{d}x < \infty \right\}. \tag{1.2}$$

The variational framework introduced in [27] was then developed in detail by Cingolani and Weth in [10] to front non-autonomous problems of the form (S), with $b > 0$, when the potential is assumed to be $\mathbb{Z}^2$-periodic or constant. The Cerami condition turns out to be the more convenient compactness condition to tackle the two-dimensional case in a variational way, indeed it is not evident how to show the boundedness of the Palais–Smale sequences. However, since the logarithmic convolution term does not have a definite sign on $\widetilde{X}$, also the boundedness of Cerami sequences becomes a major difficulty. The idea, developed in [27], to handle the different signs of the kernel is to take advantage of the relation $\log r = \log(1+r) - \log(1+\frac{1}{r})$, for all $r > 0$, to work in $\widetilde{X}$ and decompose the action of the nonlocal part in its positive and negative contribution. Once obtained the boundedness of the Cerami sequences, another difficulty is to prove that the weak limit actually is a strong limit for the Cerami sequences. To this aim, some more assumptions on the potential has been done; for example, its $\mathbb{Z}^2$ periodicity considered in [10] (see also [6] and references therein), or its axial symmetry, assumed for example in [11, 12].

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To the best of our knowledge, up to the semiclassical analysis performed in [4], there are not works concerning problem (S) without any periodicity or symmetry assumption on the potential. Our contribution in this paper is to consider a class of problems of the form (S), where V does not enjoy those assumptions. On V, we assume the following conditions

$$\begin{array}{ll} (a) & V \in L^1_{\rm loc}(\mathbb{R}^2) \\ (b) & \inf_{\mathbb{R}^2} V > 0 \\ (c) & \left|\{x \in \mathbb{R}^2 : V(x) \leq M\}\right| < +\infty, \ \forall\, M > 0. \end{array} \tag{V}$$

If the potential verifies (V), the functional I could be not well defined on $\widetilde{X}$, so that its natural domain turns out to be the weighted Hilbert space

$$X := \left\{ u \in \widetilde{X} \ : \ \int_{\mathbb{R}^2} V(x) u^2 \,\mathrm{d}x < \infty \right\} \tag{1.3}$$

endowed with the norm

$$\|u\|_X^2 := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2 + \log(1+|x|)u^2\right) \mathrm{d}x \quad u \in X.$$

We assume (V)(a) to avoid pathological potentials such that $X = \emptyset$ and to guarantee that $\mathcal{C}_c^\infty(\mathbb{R}^2) \subseteq X$, so that the weak formulation of problem (E) is meaningful.

In Proposition 3.2, we employ assumption (V)(c) to get the compactness condition. In this regard, it is worth to observe that even if the coercivity of the logarithmic weight in the definition of $\widetilde{X}$ guarantees the compactness of the embedding of $\widetilde{X}$ in the Lebesgue spaces L^p, for $p \in [2, +\infty)$, the logarithmic weight in the functional I does not work in the compactness of the functional, because its contribution is invariant by translation.

We prove the following result.

Theorem 1.1. *If $p \geq 4$, and V verifies (V), then equation (E) has at least a non-trivial weak solution $\bar{u} \in X$, with $\bar{u} \geq 0$. Moreover, if $p > 4$ then the solution $\bar{u}$ is a ground state solution:*

$$I(\bar{u}) = \inf\{I(u) \ : \ u \in X \setminus \{0\}, \ u \text{ is a solution of } (\mathrm{E})\}.$$

If $V \in L^q_{\rm loc}(\mathbb{R}^2)$ for some $q > 1$, then $\bar{u} \in \mathcal{C}^{0,\alpha}_{\rm loc}(\mathbb{R}^2)$ and $\bar{u}(x) > 0$, for all $x \in \mathbb{R}^2$.
If $V \in L^q_{\rm loc}(\mathbb{R}^2)$ for some $q > 2$, then $\bar{u} \in \mathcal{C}^{1,\alpha}_{\rm loc}(\mathbb{R}^2)$.
If $V \in \mathcal{C}^{0,\alpha}_{\rm loc}(\mathbb{R}^2)$, then $\bar{u} \in \mathcal{C}^2(\mathbb{R}^2)$ is a classical solution.

To prove Theorem 1.1, we use a variant of the mountain pass Theorem (see Theorem 2.6). Indeed, it is not convenient to try a direct application of the mountain pass Theorem because it is not straightforward to see that I is a positive quadratic form near 0 with respect to the X-norm. On the contrary, it is easy to see that I is quadratic in zero with respect to the usual H^1-norm. Let us remark that for $p = 4$, the analysis of I on the fiber $t \mapsto tu(\cdot)$ is not sufficient to guarantee the mountain pass structure, so for $p = 4$ a different fiber $t \mapsto u_t$ has to be considered. Moreover, the assumption $p > 4$ is necessary to apply the fibering techniques that guarantee that the mountain pass solution is a ground state. To apply Theorem 2.6, a basic tool is the Cerami compactness condition, that can be verified because of (V) and $p \geq 4$.

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The paper is organized as follows: in § 2, we introduce the main notation and some preliminary results, in § 3, we prove the compactness condition and in § 4, we prove Theorem 1.1.

2. Variational framework and preliminary facts

Since in the present work we focus on (E) in the case $\gamma, b, \inf V > 0$, to simplify the notation, we assume

$$\gamma = 2\pi, \quad b = 1, \ \inf V = 1.$$

We use the following notation:

- $\int = \int_{\mathbb{R}^2}$.
- $B_r(y)$ denotes the open ball of radius $r > 0$ and centre $y \in \mathbb{R}^2$.
- L^p, $1 \leq p < +\infty$, and H^1 are the usual Lebesgue and Sobolev spaces in $\mathbb{R}^2$, with norm $|u|_p = (\int |u|^p \mathrm{d}x)^{1/p}$ and $\|u\| = (\int (|\nabla u|^2 + u^2)\,\mathrm{d}x)^{1/2}$, respectively.
- We denote by $C, c_1, c_2, \ldots$ various constants that can also vary from one line to another.

First, in Lemma 2.1, we recall from [10, 27] that the nonlocal term is well defined and regular in $\widetilde{X}$. To state this result, let us introduce the following bilinear forms:

$$(u,v) \mapsto B_1(u,v) := \iint \log(1 + |x-y|) u(x) v(y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$(u,v) \mapsto B_2(u,v) := \iint \log\left(1 + \frac{1}{|x-y|}\right) u(x) v(y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$(u,v) \mapsto B_0(u,v) := B_1(u,v) - B_2(u,v) = \iint \log(|x-y|) u(x) v(y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are measurable functions such that the integrals are well defined. Correspondingly, let

$$V_1(u) := B_1(u^2, u^2), \quad V_2(u) := B_2(u^2, u^2), \quad V_0(u) := B_0(u^2, u^2) = V_1(u) - V_2(u).$$

With this notation, the functional related to problem (E) takes the form

$$I(u) = \frac{1}{2} \int (|\nabla u|^2 + V(x) u^2)\,\mathrm{d}x + \frac{1}{4} V_0(u) - \frac{1}{p} |u|_p^p.$$

Lemma 2.1. *The functional I is a well-defined C^2 functional on the Hilbert space X. Moreover, V_2 is of class C^1 on $L^{\frac{8}{3}}(\mathbb{R}^2)$ and the following formulas hold*

$$V_i'(u)[v] = 4B_i(u^2, uv), \quad \forall u, v \in X, \ i = 0, 1, 2. \tag{2.1}$$

In particular,

$$V_i'(u)[u] = 4V_i(u), \quad \forall u \in X, \ i = 0, 1, 2.$$

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For the proof, see [10, Lemma 2.2] and [26, Lemma 3.1.5].

Moreover, let us recall the Hardy–Littlewood–Sobolev inequality (see [18, Theorem 4.3]):

$$\left|\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\lambda}\,\mathrm{d}x\mathrm{d}y\right| \le C(\lambda,p)|f|_p|h|_r, \quad \forall f\in L^p(\mathbb{R}^N),\ \forall g\in L^r(\mathbb{R}^N), \tag{2.2}$$

where $p,r>1$ and $0<\lambda<N$ are such that $\frac{1}{p}+\frac{\lambda}{N}+\frac{1}{r}=2$. Then, by (2.2) and log $(1+r)\le r$, $\forall r>0$, it is readily seen that

$$V_2(u)\le \iint \frac{u^2(x)u^2(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \le c\,|u^2|_{4/3}|u^2|_{4/3} = c|u|^4_{8/3}. \tag{2.3}$$

Next, we introduce the Cerami compactness condition.

Definition 2.2. Let X be a Banach space, $I\in C^1(X)$ and $c\in\mathbb{R}$. A sequence $\{x_n\}_n$ is called a (C) sequence for I at the level c if

$$I(x_n)\to c \text{ as } n\to\infty \text{ and } \|I'(x_n)\|_{\mathcal{L}}(1+\|x_n\|_X)\to 0,\ \text{ as } n\to\infty. \tag{C}$$

We say that I satisfies the Palais–Smale–Cerami condition (at the level c), (C) condition for short, if any (C) sequence (at the level c) possesses a converging subsequence.

To analyse the compactness condition, we report some known facts: Lemma 2.3 is Lemma 2.6 in [10] and states a continuity property of the bilinear form B_1, Lemma 2.4 provides the compactness of the embedding of X in the Lebesgue spaces.

Lemma 2.3. *Let $\{u_n\}_n,\{v_n\}_n,\{w_n\}_n$ be bounded sequences in X such that $u_n\to u$ weakly in X. Then, for every $z\in X$, we have $B_1(v_nw_n, z(u_n-u))\to 0$ as $n\to\infty$.*

Lemma 2.4. *The space X is compactly embedded in $L^q(\mathbb{R}^2)$ for all $q\in[2,+\infty)$.*

Since bounded sets in X are also bounded sets in $\widetilde{X}$ (see (1.2), (1.3)), to verify Lemma 2.4 it is sufficient to exploit the coercivity of the map $|x|\mapsto \log(1+|x|)$ and apply, for example, Theorem XIII.65 in [24] (see also [23]).

Finally, let us state the topological tools we will use in the proof of Theorem 1.1.

Lemma 2.5 (see [17, Lemma 2.6])**.** *Let X be a Banach space, $I\in C^1(X)$, $c\in\mathbb{R}$, $\varepsilon>0$ and suppose that there exists $\alpha>0$ such that*

$$(1+\|x\|_X)\|I'(x)\|_{\mathcal{L}}\ge\alpha>0 \quad \forall x\in I^{c+2\epsilon}_{c-2\epsilon}. \tag{2.4}$$

Then there exists a continuous map $\eta: X\to X$, such that:

1. *$\eta(I^{c+\epsilon})\subset(I^{c-\epsilon})$;*
2. *$\eta(x)=x$, for all $x\notin I^{c+2\epsilon}_{c-2\epsilon}$,*

where $I^a=\{x\in X\ :\ I(x)\le a\}$ and $I^b_a=\{x\in X\ :\ a\le I(x)\le b\}$, for all $a<b$.

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From Lemma 2.5 there follows the following variant of the Mountain Pass Theorem.

Theorem 2.6. *Let $I \in C^1(X)$, X a Banach space. We suppose that $I(0) = 0$ and there exists a closed subspace $S \subset X$ that disconnects X in two path-connected components X_1 and X_2. We assume that $0 \in X_1$ and there exists $A > 0$, $\bar{u} \in X_2$ such that*

- $I\big|_S \geq A$;
- $I(\bar{u}) \leq 0$.

Set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{2.5}$$

where $\Gamma := \{\gamma : [0,1] \to X : \gamma \text{ is continuous}, \ \gamma(0) = 0, \ \gamma(1) \in X_2, \ I(\gamma(1)) \leq 0\}$. If I verifies the (C) condition at c, then c is a critical value for I.

The proof of Theorem 2.6 is standard so we give here only a sketch of it, for the sake of completeness.

If, by contradiction, c is not a critical value for I, then (2.4) has to be verified for suitable $\varepsilon, \alpha > 0$, because I verifies the (C) condition at the level c. We can choose $\varepsilon < A$. Now, let $\gamma_\varepsilon \in \Gamma$ be such that $\max_{[0,1]} I(\gamma_\varepsilon(t)) \leq c + \varepsilon$. If we consider $\tilde{\gamma}_\varepsilon := \eta \circ \gamma_\varepsilon$, where η is the deformation provided by Lemma 2.5, it turns out that

$$\tilde{\gamma}_\varepsilon \in \Gamma, \quad \max_{[0,1]} I(\tilde{\gamma}_\varepsilon(t)) \leq c - \varepsilon,$$

contrary to the definition of c.

3. Compactness

This section is devoted to prove that the (C) condition holds at positive values, that is the range where we are looking for critical levels:

Remark 3.1. Let u be a nontrivial critical point of I, then $I(u) > 0$.

Indeed, by $p \geq 4$ and assumption (V)(b),

$$I(u) = I(u) - \frac{1}{4} I'(u)[u] = \frac{1}{4} \int (|\nabla u|^2 \mathrm{d}x + V(x) u^2) \, \mathrm{d}x + \left(\frac{1}{4} - \frac{1}{p} \right) |u|_p^p$$
$$\geq \frac{1}{4} \int (|\nabla u|^2 + V(x) u^2) \, \mathrm{d}x \geq C \|u\|^2 > 0.$$

Proposition 3.2. *Assume that $p \geq 4$ and V satisfies (V), and let $\{u_n\}_n$ be a (C) sequence at the level c. If $c > 0$ then $\{u_n\}_n$ is relatively compact.*

A key point to prove the proposition is the following lemma that states the boundedness of the (C) sequences at the positive levels.

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Lemma 3.3. *Under the assumptions of Proposition 3.2, the sequence* $\{u_n\}_n$ *is bounded in* X. *Moreover, there exist a ball* $B_2(\bar{x})$ *and a constant* $d > 0$ *such that, up to a subsequence,*

$$\int_{B_2(\bar{x})} u_n^2(x)\,\mathrm{d}x \geq d \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Proof. Since $\{u_n\}_n$ is a (C) sequence, it turns out that

$$I'(u_n)[u_n] = o(1) \tag{3.2}$$

and so we have

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{4} I'(u_n)[u_n] \\ &= \frac{1}{4}\int (|\nabla u_n|^2 + V(x)u_n^2)\,\mathrm{d}x + \left(\frac{1}{4} - \frac{1}{p}\right)|u_n|_p^p \\ &\geq C\|u_n\|^2. \end{aligned}$$

Therefore, $\{u_n\}_n$ is bounded in $H^1(\mathbb{R}^2)$, hence in $L^q(\mathbb{R}^2)$ for all $q \in [2, +\infty)$, and

$$\int (|\nabla u_n|^2 + V(x)u_n^2)\,\mathrm{d}x \leq C \quad \forall n \in \mathbb{N}. \tag{3.3}$$

So, we are left to prove that

$$|u_n|_*^2 := \int \log(1 + |y|)u_n^2(x)\,\mathrm{d}x < C, \quad \forall n \in \mathbb{N}.$$

Let us define

$$\delta := \limsup_{n\to\infty} \left(\max_{y\in\mathbb{R}^2} \int_{B_1(y)} |u_n|^2 \mathrm{d}x \right).$$

We claim that $\delta > 0$. Assume, by contradiction, that $\delta = 0$. Then by [19, Lemma I.1] there follows

$$u_n \to 0 \text{ in } \ L^q(\mathbb{R}^2), \quad \forall q \in (2, +\infty) \tag{3.4}$$

and so, by (2.3), we get

$$V_2(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.5}$$

From (3.2), (3.4) and (3.5), we infer

$$\int \left(|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x + V_1(u_n) = I'(u_n)[u_n] + V_2(u_n) + |u_n|_p^p = o(1),$$

that, by the positivity of V_1, implies

$$\int (|\nabla u_n|^2 + V(x)u_n^2)\,\mathrm{d}x = o(1) \quad \text{and} \quad V_1(u_n) = o(1). \tag{3.6}$$

Summarizing, from (3.4), (3.5) and (3.6), we infer

$$I(u_n) = \frac{1}{2}\int (|\nabla u_n|^2 + V(x)u_n^2)\,\mathrm{d}x + \frac{1}{4}V_1(u_n) - \frac{1}{4}V_2(u_n) - \frac{1}{p}|u_n|_p^p = o(1),$$

contrary to $I(u_n) \to c > 0$, and the claim follows.

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Then, $\delta > 0$ and there exists a sequence $\{x_n\}_n \subset \mathbb{R}^2$ such that

$$\int_{B_1(x_n)} u_n^2(x)\,\mathrm{d}x > \frac{\delta}{2}, \tag{3.7}$$

for n large enough.

The task is now to prove that $\{x_n\}$ is bounded. Assume by contradiction that $|x_n| \to \infty$, up to a subsequence.

Let us fix an arbitrary $M > 0$ and call

$$A_n := \{x \in B_1(x_n) \ : \ V(x) \leq M\}, \quad B_n := \{x \in B_1(x_n) \ : \ V(x) > M\}.$$

By (V) (c) and $|x_n| \to \infty$, we obtain

$$|A_n| \leq \left|\{x \in \mathbb{R}^2 \ : \ |x| \geq |x_n| - 2,\ V(x) \leq M\}\right| \longrightarrow 0, \qquad \text{as } n \to \infty. \tag{3.8}$$

Observe that from (3.8) we infer, for any fixed $q > 2$,

$$\int_{A_n} u_n^2(x)\,\mathrm{d}x \leq |u_n|_q^2 |A_n|^{1-\frac{2}{q}} = o(1).$$

As a consequence

$$\int_{B_n} u_n^2(x)\,\mathrm{d}x = \int_{B_1(x_n)} u_n^2(x)\,\mathrm{d}x - \int_{A_n} u_n^2(x)\,\mathrm{d}x \geq \frac{\delta}{2} + o(1).$$

Then, taking into account (2.3), we have

$$\begin{aligned} c + o(1) = I(u_n) &\geq \frac{1}{2}\int_{B_n} V(x) u_n^2 \mathrm{d}x - \frac{1}{4}V_2(u_n) - C_1 \frac{1}{p}\|u_n\|^p \\ &\geq \frac{\delta}{4} M - C_2, \end{aligned} \tag{3.9}$$

where C_2 is a constant independent of M. Letting $M \to \infty$ in (3.9), we get a contradiction, hence $\{x_n\}$ has to be bounded.

Therefore, there exists $\bar{x} \in \mathbb{R}^2$ such that $x_n \to \bar{x}$, up to a subsequence. Taking into account (3.7), we can also assume that

$$\int_{B_2(\bar{x})} u_n^2 \mathrm{d}x \geq \frac{\delta}{2} > 0, \tag{3.10}$$

that proves (3.1).

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Now, observe that for every $R > 0$

$$1 + |x-y| \geq 1 + \frac{|y|}{2} \geq \sqrt{1+|y|} \quad \forall y \in \mathbb{R}^2 \setminus B_{2R}(0),\ \forall x \in B_R(0).$$

Then, fixing $\bar{R} > |\bar{x}| + 2$ and taking into account (3.10), we get

$$\begin{aligned} V_1(u_n) = B_1(u_n^2, u_n^2) &\geq \int_{\mathbb{R}^2\setminus B_{2\bar{R}}(0)} \left(\int_{B_{\bar{R}}(0)} \log(1+|x-y|) u_n^2(x)\,\mathrm{d}x \right) u_n^2(y)\,\mathrm{d}y \\ &\geq \left(\int_{B_2(\bar{x})} u_n^2(x)\mathrm{d}x \right) \cdot \left(\int_{\mathbb{R}^2\setminus B_{2\bar{R}}(0)} \log\left(1+\frac{|y|}{2}\right) u_n^2(y)\,\mathrm{d}y \right) \\ &\geq \frac{\delta}{4} \int_{\mathbb{R}^2\setminus B_{2\bar{R}}(0)} \log(1+|y|) u_n^2(y)\,\mathrm{d}y \\ &= \frac{\delta}{4} \left(|u_n|_*^2 - \int_{B_{2\bar{R}}(0)} \log(1+|y|)\, u_n^2(y)\,\mathrm{d}y \right) \\ &\geq \frac{\delta}{4} (|u_n|_*^2 - \log(1+2\bar{R})|u_n|_2^2). \end{aligned} \tag{3.11}$$

Thus, we conclude that

$$|u_n|_*^2 \leq \frac{4}{\delta} V_1(u_n) + C|u_n|_2^2. \tag{3.12}$$

Since $\{|u_n|_2\}_n$ is bounded, if we prove that $\{V_1(u_n)\}_n$ is bounded, we are done.

Observe that

$$-\frac{1}{4}V_1(u_n) + \frac{1}{4}V_2(u_n) + \left(\frac{1}{2} - \frac{1}{p}\right)|u|_p^p = I(u_n) - \frac{1}{2}I'(u_n)[u_n] = c + o(1), \tag{3.13}$$

from which we infer, taking into account (2.3),

$$\frac{1}{4}V_1(u_n) = \frac{1}{4}V_2(u_n) + \left(\frac{1}{2} - \frac{1}{p}\right)|u_n|_p^p - c + o(1) \leq C|u_n|_{8/3}^4 + C|u_n|_p^p - c + o(1) \leq C, \tag{3.14}$$

that is the desired result. □

Proof of Proposition 3.2. By Lemma 3.3, $\{u_n\}_n$ is bounded in the Hilbert space X, so, taking also into account Lemma 2.4, there exists $\bar{u} \in X$ such that, up to a subsequence,

$$u_n \to \bar{u} \text{ weakly in } X, \text{ in } L^s(\mathbb{R}^2), \text{ for every } s \in [2, +\infty), \text{ and a.e. in } \mathbb{R}^2. \tag{3.15}$$

Clearly, the sequence $\{u_n\}$ is bounded also in the weighted Hilbert space

$$H_V := \left\{ u \in H^1(\mathbb{R}^2) \;:\; \int V(x)u^2(x)\,\mathrm{d}x < \infty \right\}, \quad \|u\|_V^2 := \int (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x.$$

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Hence, we can assume that $\{u_n\}$ weakly converges in H_V to a function $\tilde{u} \in H_V$. Observe that $\tilde{u} = \bar{u}$. Indeed for every fixed $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ the map

$$u \mapsto \int u\varphi \, \mathrm{d}x$$

is a continuous linear form on X and on H_V, so that

$$\lim_{n\to\infty} \int u_n\varphi \, \mathrm{d}x = \int \bar{u}\varphi \, \mathrm{d}x \quad \text{and} \quad \lim_{n\to\infty} \int u_n\varphi \, \mathrm{d}x = \int \tilde{u}\varphi \, \mathrm{d}x.$$

Then we can conclude that

$$\int (\bar{u} - \tilde{u})\varphi \, \mathrm{d}x = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$$

and the assertion follows by the fundamental lemma of Calculus of Variations (see e.g. [5, Corollary 4.24]).

Now, from the boundedness in X of the (C) sequence $\{u_n\}_n$, we get

$$\begin{aligned} o(1) = I'(u_n)[u_n - \bar{u}] &= \int (|\nabla u_n|^2 + V(x)u_n^2) \, \mathrm{d}x - \int (\nabla u_n \cdot \nabla \bar{u} + V(x)u_n\bar{u}) \, \mathrm{d}x \\ &\quad + V_1'(u_n)(u_n - \bar{u}) - V_2'(u_n)(u_n - \bar{u}) - \int |u_n|^{p-2}u_n(u_n - \bar{u}) \, \mathrm{d}x. \end{aligned} \tag{3.16}$$

The weak convergence in H_V yields

$$\liminf_{n\to\infty} \int (|\nabla u_n|^2 + V(x)u_n^2) \, \mathrm{d}x \geq \int (|\nabla \bar{u}|^2 + V(x)\bar{u}^2) \, \mathrm{d}x. \tag{3.17}$$

By Lemmas 2.1 and 2.3,

$$\begin{aligned} V_1'(u_n)[u_n - \bar{u}] &= 4B_1(u_n^2, u_n(u_n - \bar{u})) = 4B_1(u_n^2, (u_n - \bar{u})^2) + 4B_1(u_n^2, \bar{u}(u_n - \bar{u})) \\ &= 4B_1(u_n^2, (u_n - \bar{u})^2) + o(1). \end{aligned} \tag{3.18}$$

By (2.2) and (3.15)

$$\begin{aligned} |V_2'(u_n)[u_n - \bar{u}]| &= |4B_2(u_n^2, u_n(u_n - \bar{u}))| \\ &= 4\left|\iint \log\left(1 + \frac{1}{|x-y|}\right) u_n^2(x)u_n(y)(u_n - \bar{u})(y) \, \mathrm{d}x \, \mathrm{d}y\right| \\ &\leq 4 \iint \frac{u_n^2(x)\, |u_n(y)|\, |(u_n - \bar{u})(y)|}{|x-y|} \, \mathrm{d}x \, \mathrm{d}y \\ &\leq 4|u_n|_{\frac{8}{3}}^3 |u_n - \bar{u}|_{\frac{8}{3}} = o(1). \end{aligned} \tag{3.19}$$

By (3.15),

$$\left|\int |u_n|^{p-2}u_n(u_n - \bar{u}) \, \mathrm{d}x\right| \leq |u_n|_p^{p-1}|u_n - u|_p = o(1). \tag{3.20}$$

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From (3.16), taking into account (3.17), (3.18), (3.19) and (3.20), we infer

$$o(1) = \int \left(|\nabla u_n|^2 + V(x)u_n^2\right) \mathrm{d}x - \int \left(|\nabla \bar{u}|^2 + V(x)\bar{u}^2\right) \mathrm{d}x + 4B_1(u_n^2, (u_n - \bar{u})^2) + o(1)$$
$$\geq o(1).$$

Thus, it follows that

$$\int (|\nabla u_n|^2 + V(x)u_n^2)\, \mathrm{d}x \to \int (|\nabla \bar{u}|^2 + V(x)\bar{u}^2)\, \mathrm{d}x, \tag{3.21}$$

and

$$B_1(u_n^2, (u_n - \bar{u})^2) \to 0, \text{ as } n \to +\infty. \tag{3.22}$$

Taking into account (3.1) and arguing exactly as in (3.11), we get

$$B_1(u_n^2, (u_n - \bar{u})^2) \geq \frac{d}{2}(|u_n - \bar{u}|_*^2 - C|u_n - \bar{u}|_2^2). \tag{3.23}$$

From (3.23), (3.22) and $u_n \to \bar{u}$ in L^2, there follows

$$|u_n - \bar{u}|_* \to 0,$$

that, together with (3.21), gives $\|u_n - \bar{u}\|_X \to 0$. □

4. Proof of Theorem 1.1

To prove Theorem 1.1, we are going to apply Theorem 2.6. Since condition (C) holds on $(0, +\infty)$ by Proposition 3.2, we are left to show that also the geometric conditions hold.

Observe that, by (2.3) and by the Sobolev inequality, for every $u \in X \setminus \{0\}$

$$\begin{aligned} I(u) &\geq \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\, \mathrm{d}x - \frac{1}{4}\iint \log\left(1 + \frac{1}{|x-y|}\right) u^2(x)u^2(y)\, \mathrm{d}x\, \mathrm{d}y - \frac{1}{p}|u|_p^p \\ &\geq c_1\|u\|^2 - c_2|u|_{\frac{8}{3}}^4 - c_3\|u\|^p \\ &\geq c_1\|u\|^2 - c_4\|u\|^4 - c_3\|u\|^p. \end{aligned} \tag{4.1}$$

Since $p \geq 4$, there exist $A > 0$ and $\bar{\rho} > 0$ such that $I_{|_S} \geq A$, where $S := \{u \in X : \|u\| = \bar{\rho}\}$. The closed set S disconnects X in the two path-connected components

$$X_1 := \{u \in X : \|u\| < \bar{\rho}\} \quad \text{and} \quad X_2 := \{u \in X : \|u\| > \bar{\rho}\}.$$

Clearly, $0 \in X_1$ and $I(0) = 0$.

Now, we are going to show that there exists $\bar{u} \in X_2$ such that $I(\bar{u}) < 0$. If $p > 4$, let us fix $\tilde{u} \in X \setminus \{0\}$ and observe that $\lim_{t\to\infty} I(t\tilde{u}) = -\infty$, so the claim follows.

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If $p = 4$, then let us fix $\tilde{u} \in \mathcal{C}(\mathbb{R}^2)$ with compact support and define $u_t := t^2\tilde{u}(t\,\cdot)$, for $t > 0$. We can assume that 0 is a Lebesgue point for V, then, by standard computations,

$$\begin{aligned} I(u_t) = {} & \left[\frac{1}{2}|\nabla \tilde{u}|_2^2\right] t^4 + \left[\frac{V(0)+o(1)}{2}|\tilde{u}|_2^2\right] t^2 \\ & + \left[\frac{1}{4}\iint \log(|x-y|)\tilde{u}^2(x)\tilde{u}^2(y)\,\mathrm{d}x\,\mathrm{d}y\right] t^4 \\ & - \left[\frac{1}{4}|\tilde{u}|_2^4\right] t^4 \log t - \left[\frac{1}{4}|\tilde{u}|_4^4\right] t^6, \end{aligned} \tag{4.2}$$

where $o(1) \to 0$ as $t \to \infty$. Since (4.2) implies $\lim_{t\to\infty} I(u_t) = -\infty$, and $\|u_t\| \to \infty$ as $t \to \infty$, the existence of $\bar{u} \in X_2$ such that $I(\bar{u}) < 0$ follows.

Then, the value

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{4.3}$$

where $\Gamma := \{\gamma : [0,1] \to X : \gamma \text{ is continuous}, \gamma(0) = 0, \gamma(1) \in X_2, I(\gamma(1)) \le 0\}$, is a critical value for I, according to Theorem 2.6, with $c \ge A > 0$.

Now, let $p > 4$. To verify that c is a ground state level, let us fix any nontrivial solution $\tilde{u}$ and consider the path $t \mapsto t\tilde{u}$. A direct computation shows that the real function $t \mapsto I(t\tilde{u})$ has a unique critical point $t_{\tilde{u}}$, that corresponds to a maximum. From $\frac{d}{dt}I(t\tilde{u}) = I'(t\tilde{u})[\tilde{u}]$ we infer $t_{\tilde{u}} = 1$. Then, taking into account that $\lim_{t\to\infty} I(t\tilde{u}) = -\infty$ and (4.3), it is readily seen that

$$c \le \max_{t\ge 0} I(t\tilde{u}) = I(\tilde{u}),$$

that is our claim.

Our next goal is to show that there exists a constant sign solution. Let u be a critical point for I at the level c and consider the path $t \mapsto t|u|$. Since the functional is even, $c = \max_{t\ge0} I(t|u|) = I(|u|)$ from which we infer that $\bar{u} := |u|$ is a critical point for I, too. So, $\bar{u}$ is the weak solution we are looking for.

Now, we consider the regularity issue. For classical regularity results, we refer the reader to [18, §10] (in particular Theorems 10.2 and 10.3), or to [16, §8].

Observe that

$$-\Delta \bar{u} = \varphi(x) \quad \text{for } \varphi(x) = -V(x)\bar{u} + \phi_u(x)\bar{u} + \bar{u}^{p-1}, \tag{4.4}$$

(see (1.1)).

Assume $V \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for some $q > 1$. Since $\bar{u} \in L^r(\mathbb{R}^2)$ for every $r \in [2,\infty)$, and $\phi_u \in \mathcal{C}^3(\mathbb{R}^2)$ (see [10, Proposition 2.3]), we get $\varphi \in L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for every $s \in [1,q)$. So we are in position to apply classical regularity results and obtain $\bar{u} \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$. Moreover, we can also write

$$-\Delta \bar{u} = \psi(x)\bar{u} \quad \text{for } \psi(x) = -V(x) + \phi_u(x) + \bar{u}^{p-2}$$

and then, since $\psi \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$, by the Harnack inequality, we can conclude $\bar{u} > 0$ (see [22, Theorem 7.2.1]).

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If $V \in L^q_{\text{loc}}(\mathbb{R}^2)$ for some $q > 2$, then we can argue as before and conclude $\varphi \in L^s_{\text{loc}}(\mathbb{R}^2)$ for every $s \in [1, q)$, so that $\bar{u} \in \mathcal{C}^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$.

If $V \in \mathcal{C}^{0,\alpha}_{\text{loc}}(\mathbb{R}^2)$, then the RHS in (4.4) is locally Hölder continuous, so $u \in \mathcal{C}^2(\mathbb{R}^2)$ by standard elliptic regularity (see, for example, [18, §10]).

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