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LIMIT THEOREMS FOR PURE DEATH PROCESSES COMING DOWN FROM INFINITY

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Abstract

In this paper we treat a pure death process coming down from infinity as a natural generalization of the death process associated with the Kingman coalescent. We establish a number of limit theorems including a strong law of large numbers and a large deviation theorem.

Keywords: Almost sure convergence; large deviation; Kingman's coalescent

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1. Introduction

The number of lineages in the Kingman coalescent [5] instantaneously comes down from infinity by jumps $n \to n-1$ at rate $\lambda_n = \binom{n}{2}$. As a natural extension of the Kingman setting, we consider a pure death process $(Z(t),\ t \ge 0)$ absorbing at state $n = 1$ and having death rates $(\lambda_n,\ n \ge 2)$ such that $\sum_{n=2}^{\infty}\lambda_n^{-1} < \infty$. Assume that $Z(0) = \infty$ and denote, for $n \ge 2$,

$$T_n = \inf\{t\colon Z(t) = n-1\} = X_n + X_{n+1} + \cdots,$$

where $X_2, X_3, \dots$ are independent exponentially distributed holding times with $\mathbb{E}X_i = \lambda_i^{-1}$. Since the mean values $A_n = \mathbb{E}T_n = \sum_{i=n}^{\infty}\lambda_i^{-1}$ are finite, we have $A_n \to 0$ as $n \to \infty$, which implies that the process instantaneously comes down from infinity, in that $\mathbb{P}(Z(t) < \infty \mid Z(0) = \infty) = 1$ for any $t > 0$.

In this paper we are interested in the asymptotic properties of $Z(t)$ as $t \to 0$. In view of the relation $\{Z(t) \ge n\} = \{T_n > t\}$, the step function

$$v(t) = \sum_{n=2}^{\infty} n\,\mathbf{1}_{[A_{n+1},A_n)}(t) + \mathbf{1}_{[A_2,\infty)}(t),$$

being a generalized inverse of the sequence (A_n), gives the *speed of coming down from infinity* for the process $Z(t)$; see [2]. Recall that for the Kingman coalescent, $A_n = 2/n$ and $v(t) \sim 2/t$ as $t \to 0$.

Our main results, Theorems 1–3, are presented in Section 2. In Theorem 1 we neatly summarize limit theorems for T_n that, with some additional effort, can be deduced from more general results recently obtained in [1] for birth–death processes. We give direct concise proofs.

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In Theorem 2 we deal with $Z(t)$. In particular, we show weak and strong laws of large numbers

$$\frac{Z(t)}{v(t)} \to 1, \qquad t \to 0, \tag{1}$$

which improve their counterparts from [1]. Theorem 3 is an explicit large deviation theorem generalizing a recent result in [4] obtained for the Kingman coalescent.

Our results can be also interpreted in terms of an explosive pure birth process $N(u) = Z(T_2 - u)$ obtained from the pure death process $(Z(t),\ 0 < t \leq T_2)$ by time reversing. The time-reversed process $N(u)$ can be viewed as a model for the number of neutrons at time u in a nuclear chain reaction exploding at a finite random time T_2; see [7] and [9]. Knowing the speed of explosion $v(t)$ and the current population size $N(u)$, one can hope to predict the time $t = T_2 - u$ left to the explosion event; see [8].

2. Results

Consider a pure death process with finite $A_n = \mathbb{E}T_n$, and set $B_n^2 = \sum_{i=n}^{\infty} \lambda_i^{-2}$ and $C_n^3 = \sum_{i=n}^{\infty} \lambda_i^{-3}$. In this paper we use a natural notational agreement of the type $A_{nx} := A_{\lfloor nx \rfloor}$. We start by mapping the connections among various conditions on the death rates appearing in our limit theorems. Besides condition $B_n = o(A_n)$, we will mention a group of related conditions requiring the existence of a limit for λ_n/λ_{n+1}, or existence of a (possibly infinite) limit for $\lambda_n A_n$, or

$$A_n \inf_{k>nx} \lambda_k \to \infty, \qquad n \to \infty \text{ for all } x > 1. \tag{2}$$

Also, the following two pairs of conditions are important:

$$\sum_{i=2}^{\infty} \lambda_i^{-2} A_i^{-2} < \infty, \tag{3}$$

$$\sum_{i=2}^{\infty} \lambda_i^{-2} A_{ix}^{-2} < \infty \quad \text{for all } x \in (0, 1), \tag{4}$$

$$A_{nx} = o(A_n) \quad \text{for all } x > 1, \tag{5}$$

$$\limsup_{n\to\infty} \frac{A_{nx}}{A_n} < 1 \quad \text{for all } x > 1. \tag{6}$$

Condition $\lambda_n/\lambda_{n+1} \to \alpha \in [0, 1)$ is equivalent to $\lambda_n A_n \to 1/(1-\alpha)$, due to the recursion $\lambda_n A_n = 1 + (\lambda_n/\lambda_{n+1})\lambda_{n+1} A_{n+1}$. Under this condition, we have $A_{n+1}/A_n \to \alpha$, which implies (5), which implies (6).

Condition $\lambda_n A_n \to \infty$ is equivalent to $B_n = o(A_n)$ and holds, in particular, if $\lambda_n/\lambda_{n+1} \to 1$. Indeed, for an arbitrary small $\varepsilon \in (0, 1)$, if $B_n = o(A_n)$ then, for sufficiently large n, $\lambda_n^{-2} \leq B_n^2 \leq \varepsilon^2 A_n^2$, implying that $\lambda_n^{-1} \leq \varepsilon(\lambda_n^{-1} + A_{n+1})$. Thus, $\lambda_n^{-1} \leq \varepsilon(1-\varepsilon)^{-1} A_{n+1}$ and $\lambda_n A_n \to \infty$. On the other hand, given $\lambda_n A_n \to \infty$,

$$A_n^2 - B_n^2 = 2 \sum_{i=n}^{\infty} \sum_{j=i+1}^{\infty} \lambda_i^{-1} \lambda_j^{-1} = 2 \sum_{i=n}^{\infty} \lambda_i^{-1} A_{i+1} \geq \varepsilon^{-1} B_n^2$$

for all sufficiently large n, which yields $B_n = o(A_n)$.

Condition $\lambda_n A_n \to \infty$ implies (2), because $\inf_{k>nx} \lambda_k A_k \leq A_n \inf_{k>nx} \lambda_k$ for $x > 1$.

Condition (3) implies $B_n = o(A_n)$ and (4), since $A_n^2 \sum_{i=n}^{\infty} \lambda_i^{-2} A_i^{-2} \geq B_n^2$.

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Theorem 1. (i) *If* $\lambda_n/\lambda_{n+1} \to \alpha \in [0, 1)$ *then, for all* $x \geq 0$,

$$\mathbb{P}(A_n^{-1}T_n \leq x) \to F_\alpha(x), \quad n \to \infty, \qquad \int_0^\infty \mathrm{e}^{-ux}\,\mathrm{d}F_\alpha(x) = \prod_{i\geq 0} \frac{1}{u\alpha^i(1-\alpha)+1}.$$

(ii) *If* $B_n = o(A_n)$ *then* $A_n^{-1}T_n \to 1$ *in probability as* $n \to \infty$.

(iii) *If (3) holds then* $A_n^{-1}T_n \to 1$ *almost surely as* $n \to \infty$.

(iv) *If* $B_n = o(A_n)$ *and* $C_n = o(B_n)$ *then, for* $x \in (-\infty, \infty)$,

$$\mathbb{P}(T_n \leq A_n + B_n x) \to \Phi(x), \qquad n \to \infty,$$

where $\Phi(\cdot)$ *is the standard normal distribution function.*

Theorem 2. (i) *If either (5) holds, or (2) and (6) hold together, then (1) holds in probability.*

(ii) *If (4) and (6) hold then (1) holds almost surely.*

(iii) *If* $\lambda_n/\lambda_{n+1} \to \alpha \in [0, 1)$ *then, for each* $k = 0, \pm 1, \pm 2, \ldots$,

$$\mathbb{P}(Z(A_n) < n + k) \to F_\alpha(\alpha^{-k}), \qquad n \to \infty.$$

(iv) *Let* $B_n = o(A_n)$ *and* $C_n = o(B_n)$*. If* $b_n = o(n)$ *is such that*

$$\frac{A_n - A_{n+xb_n}}{B_{n+xb_n}} \to h(x), \qquad n \to \infty \text{ for all } x \in (-\infty, \infty),$$

then

$$\mathbb{P}\bigg(\frac{Z(t) - v(t)}{b_{v(t)}} \leq x\bigg) \to \Phi(h(x)) \quad \text{as } t \to 0 \text{ for all } x \in (-\infty, \infty).$$

Parts (i) and (ii) of Theorem 2 improve the pure death case results of [1], which can be stated in our notation as follows:

> If either $\lambda_n A_n$ has a finite limit, or $\lambda_n A_n \to \infty$ holds together with (6), then (1) holds in probability. If $\lambda_n A_n \to \infty$ together with (3) and (6) then (1) holds almost surely.

Part (iii) has no counterpart in [1]. Part (iv) should be compared to the pure death case of [1, Proposition 4.6].

An important class of pure death processes coming down from infinity is set out by the constraint

$$\lambda_n = n^\beta L(n), \qquad \beta > 1, \tag{7}$$

where the function $L\colon [1, \infty) \to (0, \infty)$ is assumed to slowly vary at ∞. For the Kingman coalescent, this condition holds with $\beta = 2$. By the properties of regularly varying functions (see [3]), condition (7) entails

$$A_n = n^{1-\beta}L_1(n), \qquad L_1(n) \sim (\beta-1)^{-1}L^{-1}(n), \qquad n \to \infty,$$

implying that $v(t)$ regularly varies at 0 with index $1/(1-\beta)$. In this case, condition (6) holds but not (5). By Theorem 2, if (7) holds then (1) is valid almost surely and the limit distribution of $(Z(t) - v(t))/\sqrt{v(t)}$ is normal with mean 0 and variance $1/(2\beta - 1)$.

Examples. We give five simple examples illustrating the wide range of regimes covered by Theorem 2. For all our examples, the key condition $\sum_{n=2}^{\infty}\lambda_n^{-1} < \infty$ is easily verified. We see that the faster the decay of A_n as $n \to \infty$, the slower is the speed of coming down from infinity.

1. Let $A_n = (\log n)^{-a}$ for some $a > 0$. Then, as $n \to \infty$,

$$\lambda_n \sim a^{-1}n(\log n)^{1+a}, \qquad B_n \sim a^{-1}n^{-1/2}(\log n)^{-1-a},$$
$$C_n \sim a^{-1}n^{-2/3}(\log n)^{-1-a}.$$

In this case, $v(t) \sim \mathrm{e}^{t^{-1/a}}$ as $t \to 0$. Since (6) is not valid, Theorem 2 does not apply.

2. If $A_n \sim cn^{1-\beta}$ for some $\beta > 1$ and $c > 0$, then condition (7) is valid and we obtain $v(t) \sim c^{-1/(\beta-1)}t^{1/(\beta-1)}$ as $t \to 0$. A special case with $\beta = 2$ is obtained when $\lambda_n = \binom{2n}{3}$, so that the process $2Z(t)$ describes a triplewise coalescent (in contrast to the pairwise Kingman coalescent).

3. If $A_n = \mathrm{e}^{-n^{\rho}}$ for some $\rho \in (0, 1)$ then $v(t) \sim (\log(t^{-1}))^{1/\rho}$ as $t \to 0$. This yields an example when both $\lambda_n/\lambda_{n+1} \to 1$ and (5) take place. Observe that, for $\rho \in [\frac{1}{2}, 1)$, condition (4) holds, while (3) is not satisfied.

4. Set $A_n = \mathrm{e}^{-n/\log n}$. In [1] it was shown that in this case, $A_n^{-1}T_n \to 1$ in probability, but not almost surely. Here $v(t) \sim \log(t^{-1})\log\log(t^{-1})$ as $t \to 0$. Since conditions (4) and (5) are satisfied, we conclude that (1) holds almost surely.

5. If $A_n = \mathrm{e}^{-n}$ then the fast decay of A_n ensures that $\lambda_n/\lambda_{n+1} \to 1/\mathrm{e}$, and (1) holds almost surely with $v(t) \sim \log(t^{-1})$ as $t \to 0$. For this example, condition $C_n = o(B_n)$ fails and the central limit theorem does not apply.

In the next results we focus on pure death processes satisfying condition (7). For a given $x > 0$, define $\tau = \tau(x)$ as the solution of

$$\int_1^{\infty} \frac{\mathrm{d}y}{(\beta-1)^{-1}y^{\beta} - \tau} = x.$$

Observe that $\tau(x)$ is a strongly increasing function with

$$\lim_{x\to 0}\tau(x) = -\infty, \qquad \tau(1) = 0, \qquad \lim_{x\to\infty}\tau(x) = (\beta-1)^{-1}.$$

Define two families of functions by $I(x) = -(\beta-1)x\tau(x) - \log(1-(\beta-1)\tau(x))$ and $J(x) = xI(x^{\beta-1})$, which are illustrated in Figure 1.

Lemma 1. *The above defined functions $I(x)$ and $J(x)$ are both nonnegative and strictly convex over $x \in (0, \infty)$ with $I(1) = J(1) = 0$. They satisfy the following asymptotical relations:*

$$I(x) \sim (\beta-1)^{-1}x, \qquad J(x) \sim (\beta-1)^{-1}x^{\beta}, \qquad x \to \infty,$$
$$I(x) = c(\beta)x^{-1/(\beta-1)} - \beta(\beta-1)^{-1}\log(x^{-1}) - \log c(\beta) - \beta + o(1), \qquad x \to 0,$$
$$J(x) = c(\beta) - (\beta\log x + \log c(\beta) + \beta)x + o(x), \qquad x \to 0,$$

where $c(\beta) = \{(1-1/)\pi/\sin(\pi/\beta)\}^{\beta/(\beta-1)}$.

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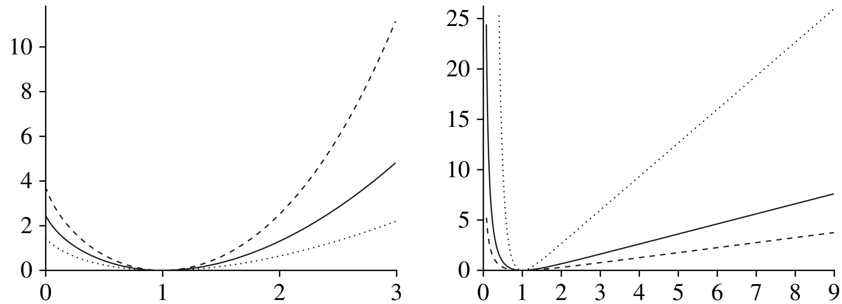

FIGURE 1: Three pairs of profiles for the rate functions $I(x)$ (*left*) and $J(x)$ (*right*). Parameter values $\beta = 1.3$ (*dotted lines*), $\beta = 2$ (*solid lines*), and $\beta = 3$ (*dashed lines*).

Theorem 3. *Consider a death process satisfying (7) with* $\beta > 1$. *If* $x \geq 1$ *then*

$$n^{-1} \log \mathbb{P}(T_n > xA_n) \to -I(x), \qquad n \to \infty, \tag{8}$$

$$v(t)^{-1} \log \mathbb{P}(Z(t) > xv(t)) \to -J(x), \qquad t \to 0. \tag{9}$$

On the other hand, if $0 < x \leq 1$ *then*

$$n^{-1} \log \mathbb{P}(T_n < xA_n) \to -I(x), \qquad n \to \infty,$$

$$v(t)^{-1} \log \mathbb{P}(Z(t) < xv(t)) \to -J(x), \qquad t \to 0.$$

3. Proofs

We start with two lemmas. Lemma 2 is a version of Kolmogorov's inequality used in the proof of Lemma 3.

Lemma 2. *If an infinite sum* $\xi_1+\xi_2+\cdots$ *of independent zero-mean random variables converges almost surely, and* $\zeta_n := \xi_n + \xi_{n+1} + \cdots$, *then, for each* $\varepsilon > 0$,

$$\mathbb{P}\Big(\sup_{k\geq n} |\zeta_k| \geq \varepsilon\Big) \leq \varepsilon^{-2}\mathbb{E}\zeta_n^2, \qquad n \geq 1.$$

Proof. It is easy to check that the sequence ζ_n forms a backward martingale. Setting $B_k = \{|\zeta_k| \geq \varepsilon, |\zeta_{k+1}| < \varepsilon, |\zeta_{k+2}| < \varepsilon, \dots\}$, we obtain

$$\mathbb{E}(\zeta_n^2) \geq \sum_{k=n}^{\infty} \mathbb{E}(\zeta_n^2 \mathbf{1}_{B_k}) \geq \sum_{k=n}^{\infty} \mathbb{E}(\zeta_k^2 \mathbf{1}_{B_k}) \geq \varepsilon^2 \sum_{k=n}^{\infty} \mathbb{P}(B_k) = \varepsilon^2 \mathbb{P}\Big(\sup_{k\geq n} |\zeta_k| \geq \varepsilon\Big),$$

using the submartingale property of ζ_n^2. □

Lemma 3. *If* $\sum \lambda_i^{-2} A_{i(1-\varepsilon)}^{-2} < \infty$ *for some* $\varepsilon \in [0, 1)$ *then, for any* $\delta > 0$,

$$\mathbb{P}\bigg(\sup_{k\geq n} \frac{|T_k - A_k|}{A_{k(1-\varepsilon)}} > \delta\bigg) \to 0, \qquad n \to \infty.$$

Proof. The following proof is an adaptation of the proof of [6, Proposition 1]. For a given n, let u_n be the unique natural number satisfying $2^{-u_n-1} < A_{n(1-\varepsilon)} \leq 2^{-u_n}$. Clearly, $u_n \leq u_{n+1}$

and $u_n \to \infty$. Setting $v_j = \min\{k\colon u_k = j\}$, we obtain

$$\begin{aligned}\mathbb{P}\Big(\sup_{k\geq n} A_{k(1-\varepsilon)}^{-1}|T_k - A_k| \geq \varepsilon\Big) &\leq \sum_{j\geq u_n} \mathbb{P}\Big(\max_{k\colon u_k=j} A_{k(1-\varepsilon)}^{-1}|T_k - A_k| \geq \varepsilon\Big)\\ &\leq \sum_{j\geq u_n} \mathbb{P}\Big(\max_{k\colon u_k=j} |T_k - A_k| \geq \varepsilon 2^{-j-1}\Big)\\ &\leq \sum_{j\geq u_n} \mathbb{P}\Big(\sup_{k\geq v_j} |T_k - A_k| \geq \varepsilon 2^{-j-1}\Big).\end{aligned}$$

Note that for some j the set of indices $\{k\colon u_k = j\}$ might be empty—in such a case the corresponding maximum is assumed to be 0.

By Lemma 2 applied to $\xi_i = X_i - \lambda_i^{-1}$ having centered exponential distributions, we see that there is a positive constant c such that

$$\begin{aligned}\sum_{j\geq u_n} \mathbb{P}\Big(\sup_{k\geq v_j} |T_k - A_k| \geq \varepsilon 2^{-j-1}\Big) &\leq \sum_{j\geq u_n} c\varepsilon^{-2} 4^{j+1} \sum_{k\geq v_j} \lambda_k^{-2}\\ &= c\varepsilon^{-2} \sum_{j\geq u_n} 4^{j+1} \sum_{l\geq j} 4^{-l} \sum_{k\colon u_k=l} (\lambda_k 2^{-l})^{-2}\\ &\leq c\varepsilon^{-2} \sum_{l\geq u_n} \sum_{j=u_n}^{l} 4^{j-l+1} \sum_{k\colon u_k=l} (\lambda_k A_{k(1-\varepsilon)})^{-2}.\end{aligned}$$

Thus,

$$\mathbb{P}\Big(\sup_{k\geq n} A_{k(1-\varepsilon)}^{-1}|T_k - A_k| \geq \varepsilon\Big) \leq 4c\varepsilon^{-2} \sum_{l\geq u_n} \sum_{k\colon u_k=l} \lambda_k^{-2} A_{k(1-\varepsilon)}^{-2} = 4c\varepsilon^{-2} \sum_{k\geq K_n} \lambda_k^{-2} A_{k(1-\varepsilon)}^{-2},$$

where $K_n = \min\{k\colon u_k = u_n\}$ is v_j for $j = u_n$. By the monotonicity of A_n, we have $K_n \to \infty$ as $n \to \infty$, and the statement of Lemma 3 follows. □

Proof of Theorem 1. (i) Observe first that, for any given $u_0 > 0$, the moment generating function

$$\mathbb{E}e^{uT_n} = \prod_{i=n}^{\infty} \frac{\lambda_i}{\lambda_i - u} = \exp\Big\{-\sum_{i=n}^{\infty} \log(1 - u\lambda_i^{-1})\Big\}, \qquad u \in (-\infty, u_0], \tag{10}$$

is well defined for all sufficiently large n. Since $(\lambda_{n+i} A_n)^{-1} \to \alpha^i(1-\alpha)$ for all $i \geq 0$, a bounded convergence argument yields that, for each $u \geq 0$,

$$\mathbb{E}e^{-uT_n/A_n} = \prod_{k\geq n} \frac{1}{u(\lambda_k A_n)^{-1} + 1} \to \prod_{i\geq 0} \frac{1}{u\alpha^i(1-\alpha) + 1}, \qquad n \to \infty.$$

(ii) Convergence in probability is easily derived using the Chebyshev inequality.

(iii) Almost sure convergence is a straightforward corollary of Lemma 3 with $\varepsilon = 0$.

(iv) Using (10) and the notation $\xi_i = X_i - \lambda_i^{-1}$, we find that, for each $u \geq 0$,

$$\mathbb{E}\exp\{-u(\xi_n + \xi_{n+1} + \cdots)\} = \exp\Big\{-\sum_{i=n}^{\infty} (\log(1 + u\lambda_i^{-1}) - u\lambda_i^{-1})\Big\}.$$

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Using the inequalities $x - x^2/2 \le \log(1+x) \le x - x^2/2 + x^3/3$ available for $x \ge 0$, we conclude that under condition $C_n = o(B_n)$,

$$\mathbb{E}\exp\{-uB_n^{-1}(\xi_{n+1}+\xi_{n+2}+\cdots)\} \sim \exp\left\{\sum_{i=n}^{\infty}\frac{(uB_n^{-1}\lambda_i^{-1})^2}{2}\right\} = \exp\left\{\frac{u^2}{2}\right\}. \qquad \square$$

Proof of Theorem 2. Observe that (1) is equivalent to $Z(A_n)/n \to 1$ as $n \to \infty$, since

$$\frac{Z(A_{n-1})}{n} \le \frac{Z(t)}{v(t)} \le \frac{Z(A_n)}{n}, \qquad n = v(t).$$

(i) We first show that

$$\mathbb{P}(T_{n(1+\varepsilon)} > A_n) \to 0, \qquad n \to \infty,\ \varepsilon \in (0,1). \tag{11}$$

If (5) holds then (11) easily follows from the Markov inequality.

Given (2) and (6), we fix arbitrary $\varepsilon \in (0,1)$ and $u \in (0,\infty)$, and observe that there exist a $\delta \in (0,1)$ and an $n_0 = n_0(\varepsilon, u)$ such that, for all $n \ge n_0$,

$$\frac{A_{n(1+\varepsilon)}}{A_n} < \delta, \qquad A_n\lambda_k > (2-\delta)(1-\delta)^{-1}u, \qquad k > n(1+\varepsilon),$$

so that the moment generating function

$$\mathbb{E}\exp\left\{\frac{uT_{n(1+\varepsilon)}}{A_n}\right\} = \prod_{k>n(1+\varepsilon)}\frac{1}{1-(A_n\lambda_k)^{-1}u}$$

is well defined. By an exponential version of Markov's inequality,

$$\begin{aligned}\mathbb{P}(T_{n(1+\varepsilon)} > A_n) &\le \exp\{-u\}\mathbb{E}\exp\left\{\frac{uT_{n(1+\varepsilon)}}{A_n}\right\}\\ &= \exp\{-u\}\exp\left\{-\sum_{k>n(1+\varepsilon)}\log(1-(A_n\lambda_k)^{-1}u)\right\}.\end{aligned}$$

This together with $(1-x)\log((1-x)^{-1}) \le x$ for $x \in (0,1)$, yields

$$\begin{aligned}\mathbb{P}(T_{n(1+\varepsilon)} > A_n) &\le \exp\{-u\}\exp\left\{\sum_{k>n(1+\varepsilon)}\frac{(A_n\lambda_k)^{-1}u}{1-(A_n\lambda_k)^{-1}u}\right\}\\ &\le \exp\{-u\}\exp\left\{(2-\delta)A_n^{-1}u\sum_{k>n(1+\varepsilon)}\lambda_k^{-1}\right\}\\ &\le \exp\{-u(1-\delta)^2\}.\end{aligned}$$

Letting $u \to \infty$, we see that (11) holds also under conditions (2) and (6). Since

$$\mathbb{P}(Z(A_n) \ge n(1+\varepsilon)) = \mathbb{P}(T_{n(1+\varepsilon)} > A_n),$$

we derive from (11) that $\mathbb{P}(Z(A_n) \ge n(1+\varepsilon)) \to 0$. Similarly, $\mathbb{P}(Z(A_n) \le n(1-\varepsilon)) \to 0$ as $n \to \infty$.

(ii) It suffices to prove $Z(A_n)/n \to 1$ almost surely or, in other terms,

$$\mathbb{P}\bigg(\sup_{k\geq n}\frac{Z(A_k)-k}{k}\geq\varepsilon\bigg)\to 0,\qquad \mathbb{P}\bigg(\inf_{k\geq n}\frac{Z(A_k)-k}{k}\leq-\varepsilon\bigg)\to 0,\qquad n\to\infty.$$

To check the first convergence, observe that

$$\begin{aligned}\mathbb{P}\bigg(\sup_{k\geq n}\frac{Z(A_k)-k}{k}\geq\varepsilon\bigg)&=\mathbb{P}(\text{there exist } k\geq n\colon Z(A_k)\geq(1+\varepsilon)k)\\&=\mathbb{P}\Big(\text{there exist } k\geq n\colon T_{(1+\varepsilon)k}>A_k\Big)\\&=\mathbb{P}\bigg(\text{there exist } k\geq n\colon \frac{T_{(1+\varepsilon)k}-A_{(1+\varepsilon)k}}{A_k}>-\frac{A_{(1+\varepsilon)k}}{A_k}\bigg).\end{aligned}$$

By condition (6), it follows that, for some $\delta\in(0,1)$ and all $n\geq n_0(\varepsilon)$,

$$\begin{aligned}\mathbb{P}\bigg(\sup_{k\geq n}\frac{Z(A_k)-k}{k}\geq\varepsilon\bigg)&\leq\mathbb{P}\bigg(\text{there exist } k\geq n\colon \frac{T_{(1+\varepsilon)k}-A_{(1+\varepsilon)k}}{A_k}>\delta\bigg)\\&\leq\mathbb{P}\bigg(\text{there exist } k\geq n(1+\varepsilon)\colon \frac{T_k-A_k}{A_{k/(1+\varepsilon)}}>\delta\bigg).\end{aligned}$$

Given (4), we can apply Lemma 3 and obtain the first required convergence. The second convergence is verified similarly.

(iii) This follows from Theorem 1(i) in view of the relations

$$\mathbb{P}(Z(A_n)\geq n+k)=\mathbb{P}\bigg(\frac{T_{n+k}}{A_{n+k}}>\frac{A_n}{A_{n+k}}\bigg),\qquad A_{n+k}/A_n\to\alpha^k.$$

(iv) This is obtained from Theorem 1(iv) using the equality

$$\mathbb{P}\bigg(\frac{Z(A_n)-n}{b(n)}\geq x\bigg)=\mathbb{P}(T_{n+xb(n)}>A_n)=\mathbb{P}\bigg(\frac{T_{n+xb(n)}-A_{n+xb(n)}}{B_{n+xb(n)}}>\frac{A_n-A_{n+xb(n)}}{B_{n+xb(n)}}\bigg).$$

This completes the proof. □

Proof of Lemma 1. The function $\tau(x)$ satisfies $\Lambda'(\tau(x))=x$, where

$$\Lambda(u)=-\int_1^\infty\log(1-(\beta-1)uy^{-\beta})\,\mathrm{d}y,\qquad u\leq 1/(\beta-1).$$

This yields $\tau'(x)=1/\Lambda''(\tau(x))$. Using integration by parts, we have

$$\Lambda(\tau(x))=-\int_1^\infty\log(1-(\beta-1)\tau(x)y^{-\beta})\,\mathrm{d}y=\log(1-(\beta-1)\tau(x))+\beta x\tau(x).$$

Thus, the defining expression for $I(x)$ can be written as $I(x)=x\tau(x)-\Lambda(\tau(x))$ for $x>0$. It follows that $I'(x)=\tau(x)$ and $I''(x)=\tau'(x)=1/\Lambda''(\tau(x))$. In view of

$$\Lambda''(u)=\int_1^\infty\frac{\mathrm{d}y}{((\beta-1)^{-1}y^\beta-u)^2}>0,$$

we conclude that $I(x)$ is a convex function with a minimal value $I(1)=0$.

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On the other hand, $J(x) = xI(x^{\beta-1})$ is also a convex function with minimal value $J(1) = 0$. Indeed,

$$J'(x) = (\beta - 1)x^{\beta-1}\tau(x^{\beta-1}) + I(x^{\beta-1}) = R(x^{\beta-1}),$$

where $R(x) = (\beta-1)x\tau(x)+I(x)$. In particular, $J'(1) = R(1) = 0$. To verify that $R'(x) > 0$, observe that

$$R'(x) = (\beta - 1)\tau(x) + (\beta - 1)x\tau'(x) + \tau(x) = \beta\tau(x) + (\beta - 1)x\tau'(x).$$

We have $R'(x) = \tau'(x)r(\tau(x))$, where $\tau'(x) > 0$ and

$$\begin{aligned} r(u) &= \beta u\Lambda''(u) + (\beta - 1)\Lambda'(u) \\ &= \int_1^\infty \frac{\beta u\,\mathrm{d}y}{((\beta-1)^{-1}y^\beta - u)^2} + \int_1^\infty \frac{(\beta-1)\,\mathrm{d}y}{(\beta-1)^{-1}y^\beta - u} \\ &= \int_1^\infty \frac{(y^\beta + u)\,\mathrm{d}y}{((\beta-1)^{-1}y^\beta - u)^2}. \end{aligned}$$

Clearly, $r(u) > 0$ for $u \geq -1$, and it remains to show that $r(-u) > 0$ for $u > 1$. To see this, observe that in view of

$$r(-u) = \int_1^\infty \frac{(y^\beta - u)\,\mathrm{d}y}{((\beta-1)^{-1}y^\beta + u)^2} = u^{1/\beta-1}\int_{u^{-1/\beta}}^\infty \frac{(y^\beta - 1)\,\mathrm{d}y}{((\beta-1)^{-1}y^\beta + 1)^2},$$

we find that, using MATHEMATICA® software, for $\beta > 1$,

$$r(-u) > u^{1/\beta-1}(\beta-1)^2\int_0^\infty \frac{(y^\beta - 1)\,\mathrm{d}y}{(y^\beta + \beta - 1)^2} = 0.$$

Turning to the stated asymptotics as $x \to \infty$, observe that

$$\begin{aligned} \Lambda''(u) &= \int_1^\infty \frac{\mathrm{d}y}{((\beta-1)^{-1}y^\beta - u)^2} \\ &= \frac{(\beta-1)^2}{h^2}\int_1^\infty \frac{\mathrm{d}y}{((y^\beta - 1)/h + 1)^2} \\ &= \frac{(\beta-1)^2}{\beta h}\int_0^\infty \frac{(1+zh)^{1-1/\beta}}{(z+1)^2}\,\mathrm{d}z, \end{aligned}$$

where $h = 1 - (\beta - 1)u$ and $z = (y^\beta - 1)/h$. This yields

$$\Lambda''(u) \sim \frac{(\beta-1)^2}{\beta(1-(\beta-1)u)}, \qquad u \to (1-\beta)^{-1}.$$

Therefore, using L'Hospital's rule, we find that, as $x \to \infty$,

$$x^{-1}\log(1-(\beta-1)\tau(x))^{-1} \sim \frac{(\beta-1)\tau'(x)}{1-(\beta-1)\tau(x)} = \frac{\beta-1}{\Lambda''(\tau(x))(1-(\beta-1)\tau(x))} \to \frac{\beta}{\beta-1}.$$

This implies that

$$x^{-1}I(x) = -(\beta-1)\tau(x) - x^{-1}\log(1-(\beta-1)\tau(x)) \to -1 + \frac{\beta}{\beta-1} = \frac{1}{\beta-1}.$$

The last assertion of the lemma yields an asymptotic as $x \to 0$. We prove it by first noting that (again using MATHEMATICA)

$$\begin{aligned}\Lambda'(-u) &= u^{1/\beta-1}\int_{u^{-1/\beta}}^{\infty}\frac{\mathrm{d}y}{(\beta-1)^{-1}y^{\beta}+1}\\ &= u^{1/\beta-1}\int_{0}^{\infty}\frac{\mathrm{d}y}{(\beta-1)^{-1}y^{\beta}+1} - u^{1/\beta-1}\int_{0}^{u^{-1/\beta}}\frac{\mathrm{d}y}{(\beta-1)^{-1}y^{\beta}+1}\\ &= u^{1/\beta-1}(\beta-1)^{1/\beta}\frac{\pi/\beta}{\sin(\pi/\beta)} - u^{-1} + o(u^{-1}) \quad \text{as } u\to\infty,\end{aligned}$$

and, therefore, as $x \to 0$,

$$x = |\tau(x)|^{1/\beta-1}(\beta-1)^{1/\beta}\frac{\pi/\beta}{\sin(\pi/\beta)} - |\tau(x)|^{-1} + o(|\tau(x)|^{-1}).$$

Solving the last equation, we obtain, as a first approximation,

$$\tau(x) \sim -b(\beta)x^{-\beta/(\beta-1)}, \qquad b(\beta) := (\beta-1)^{1/(\beta-1)}\bigg(\frac{\pi/\beta}{\sin(\pi/\beta)}\bigg)^{\beta/(\beta-1)} = \frac{c(\beta)}{\beta-1},$$

and then more exactly $\tau(x) = x^{-1} - b(\beta)x^{-\beta/(\beta-1)} + o(x^{-1})$ as $x \to 0$. Thus,

$$\begin{aligned}I(x) &= -(\beta-1)x\tau(x) - \log(1-(\beta-1)\tau(x))\\ &= c(\beta)x^{-1/(\beta-1)} - \frac{\beta}{\beta-1}\log(x^{-1}) - \log c(\beta) - \beta + o(1),\end{aligned}$$

and $J(x) = c(\beta) - (\beta\log x + \log c(\beta) + \beta)x + o(x)$ as $x \to 0$. □

Proof of Theorem 3. Here we prove only the first half of Theorem 3 since the second half is proved similarly. Our proof is more direct than that of [4] and uses the classical Cramer's device of 'tilted distributions'.

Let $x > 1$. The required upper bound for (8) is obtained from

$$\mathbb{P}(T_n > xA_n) = \mathbb{P}(\mathrm{e}^{\tau(x)nA_n^{-1}T_n} > \mathrm{e}^{x\tau(x)n}) \le \mathbb{E}\mathrm{e}^{\tau(x)nA_n^{-1}T_n}\mathrm{e}^{-x\tau(x)n}.$$

Indeed, using (10) we have

$$n^{-1}\log\mathbb{P}(T_n > xA_n) \le -\frac{1}{n}\sum_{i=n}^{\infty}\log\bigg(1-\frac{\tau(x)}{\lambda_i A_n n^{-1}}\bigg) - x\tau(x),$$

and it remains to see that, by the dominated convergence theorem,

$$-\frac{1}{n}\sum_{i=n}^{\infty}\log\bigg(1-\frac{u}{\lambda_i A_n n^{-1}}\bigg) = \int_1^{\infty}\log\bigg(1-\frac{u}{\lambda_{yn}A_n n^{-1}}\bigg)\,\mathrm{d}y \to \Lambda(u).$$

Here the dominating function is found from the uniform bounds

$$(1-\varepsilon)y^{\beta-\varepsilon} \le \frac{\lambda_{yn}}{\lambda_n} \le (1+\varepsilon)y^{\beta+\varepsilon}, \qquad n \ge n_0(\varepsilon),\ y\in[1,\infty),$$

where $n_0(\varepsilon)$ does not depend on y.

The lower bound for (8) is derived using the so-called tilted distributions. □

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Lemma 4. *Let (7) hold with $\beta > 1$, and set $\tilde{\lambda}_{i,n} = \lambda_i - \tau(x_n) n A_n^{-1}$. If $x_n \to x$ then $\tilde{\lambda}_{i,n} > 0$, $i \geq n$, for all sufficiently large n. Moreover, as $n \to \infty$,*

$$\frac{\widetilde{A}_n}{A_n} \to x, \qquad n\left(\frac{\widetilde{B}_n}{A_n}\right)^2 \to \Lambda''(\tau(x)), \qquad n^2\left(\frac{\widetilde{C}_n}{A_n}\right)^3 \to \frac{1}{2}\Lambda'''(\tau(x)),$$

where $\widetilde{A}_n$, $\widetilde{B}_n^2$, and $\widetilde{C}_n^3$ denote the sums $\sum_{i=n}^{\infty}(\tilde{\lambda}_{i,n})^{-j}$ with $j = 1, 2, 3$, respectively. There exists a sequence $x_n \to x$ such that $\widetilde{A}_n = x A_n$ for all sufficiently large n.

Proof. We suppose that $x_n \to x$. We have $\lambda_{ny} \sim y^{\beta}(\beta - 1)^{-1} A_n^{-1}$ for $y \geq 1$. Since $\tau(x) < (\beta - 1)^{-1}$, it follows that $\tilde{\lambda}_{i,n} > 0$ for all $i \geq n$ and sufficiently large n. Furthermore, by the dominated convergence theorem,

$$\begin{aligned}
\frac{\widetilde{A}_n}{A_n} &= n^{-1} \sum_{i=n}^{\infty} \frac{1}{\lambda_i A_n n^{-1} - \tau(x_n)} \\
&= \int_1^{\infty} \frac{\mathrm{d}y}{\lambda_{yn} A_n n^{-1} - \tau(x_n)} \\
&\to \int_1^{\infty} \frac{\mathrm{d}y}{y^{\beta}(\beta - 1)^{-1} - \tau(x)} \\
&= \Lambda'(\tau(x)) \\
&= x, \\
n\left(\frac{\widetilde{B}_n}{A_n}\right)^2 &\to \int_1^{\infty} \frac{1}{(y^{\beta}(\beta - 1)^{-1} - \tau(x))^2}\,\mathrm{d}y = \Lambda''(\tau(x)), \\
n^2\left(\frac{\widetilde{C}_n}{A_n}\right)^3 &\to \int_1^{\infty} \frac{1}{(y^{\beta}(\beta - 1)^{-1} - \tau(x))^3}\,\mathrm{d}y = \frac{1}{2}\Lambda'''(\tau(x)), \qquad n \to \infty.
\end{aligned}$$

To prove the last statement of Lemma 4, take $x_n \equiv u$ and consider the sequence $a_n(u) = \widetilde{A}_n / A_n = \int_1^{\infty} \mathrm{d}y / (\lambda_{yn} A_n n^{-1} - \tau(u))$. We know that each function a_n is continuous and strictly monotone, and that $a_n(u) \to u$. Therefore, for the given $x > 1$ and a small $\varepsilon > 0$, if n is sufficiently large, we have

$$x - 2\varepsilon < a_n(x - \varepsilon) < x - \frac{\varepsilon}{2} < x + \frac{\varepsilon}{2} < a_n(x + \varepsilon) < x + 2\varepsilon.$$

We conclude that there exists an x_n such that $a_n(x_n) = x$ and $x_n \to x$. □

We return to the proof of Theorem 3. Besides the random variables X_i with exponential distributions $\operatorname{Exp}(\lambda_i)$, we introduce their tilted versions $\tilde{X}_{i,n}$ having exponential distributions $\operatorname{Exp}(\tilde{\lambda}_{i,n})$, where $\tilde{\lambda}_{i,n}$ are defined according to Lemma 4 in such a way that $\widetilde{A}_n = x A_n$. If $F_n(y)$ and $\tilde{F}_n(y)$ are the distribution functions for $T_n = \sum_{i=n}^{\infty} X_i$ and $\tilde{T}_n = \sum_{i=n}^{\infty} \tilde{X}_{i,n}$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{e}^{uy}\,\mathrm{d}\widetilde{F}_n(y) &= \mathbb{E}\mathrm{e}^{u\tilde{T}_n} \\
&= \prod_{i=n}^{\infty} \frac{\tilde{\lambda}_{i,n}}{\tilde{\lambda}_{i,n} - u}
\end{aligned}$$

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$$= \frac{\mathbb{E}e^{(u+\tau(x)n/A_n)T_n}}{\mathbb{E}e^{(\tau(x)n/A_n)T_n}}$$
$$= \frac{1}{\mathbb{E}e^{\tau(x)nT_n/A_n}} \int_{-\infty}^{\infty} e^{(u+\tau(x)n/A_n)y}\, dF_n(y),$$

implying that

$$d\widetilde{F}_n(y) = \frac{e^{\tau(x)ny/A_n}}{\mathbb{E}e^{\tau(x)nT_n/A_n}}\, dF_n(y).$$

Thus, for any $b > x$, we obtain

$$\mathbb{P}(T_n > xA_n) = \int_{xA_n}^{\infty} dF_n(y) \ge \mathbb{E}[e^{\tau(x)nT_n/A_n}]e^{-\tau(x)nb} \int_{xA_n}^{bA_n} d\widetilde{F}_n(y).$$

Lemma 4 yields an analog of Theorem 1(iv) stating that $\mathbb{E}(\widetilde{T}_n \le \widetilde{A}_n + u\widetilde{B}_n) \to \Phi(u)$. Since $\widetilde{A}_n = xA_n$, this implies that $\int_{xA_n}^{bA_n} d\widetilde{F}_n(y) \to \frac{1}{2}$, so that

$$\liminf_{n\to\infty} n^{-1} \log \mathbb{P}(T_n > xA_n) \ge \Lambda(\tau(x)) - b\tau(x).$$

To complete the proof of (8), send $b \to x$. To prove (9), observe that

$$n^{-1} \log \mathbb{P}(Z(A_n) > nx) \sim x(nx)^{-1} \log \mathbb{P}(T_{nx} > x^{\beta-1}A_{nx}) \to xI(x^{\beta-1}), \qquad n \to \infty. \quad \square$$

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