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A generalised approach for the modelling of articulated open chain planar linkages

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SUMMARY
In this paper a model to cover all possible topologies of robot manipulators composed of prismatic and revolute joints is presented. For simplicity, only planar systems are considered, hence to provide plane positioning, systems handled are of three degrees of freedom. The physical model assumes three moving rigid links in articulation with one revolute and one prismatic joint between each link pair, forming a six degrees of freedom open chain linkage. Among each joint pair, one is real and the other fictitious. The real joint is arbitrarily actuated by an externally applied force or torque while the fictitious one is acted upon by an appropriately controlled force or torque as to keep that joint velocity zero, keeping fixed at its initial position. The physical model is accompanied by a mathematical model obtained by Lagrange formualtion. This approach is called 'The method of Fictititous Degrees of Freedom'.

KEYWORDS: Robot dynamics; Equations of motion; Articulated linkages.

1. INTRODUCTION
In robotics, extensive research is carried out to explore and understand the dynamic behaviour of open kinematic chains composed of rigid links articulated through various types of joints and actuators to drive them. Once the dynamic behaviour of a manipulator is understood, it can be controlled to move smoothly, free of jerk and vibrations even if it is a cheap machine having dynamically disadvantageous features. Problems of this kind have been studied at various works. Jones for example has shown how the residual vibrations on the translational motion of a cam follower can be eliminated by a velocity shock given by the cam at the beginning of the rise period in his paper published in 1977.[1] The effect of velocity shocking is practically obtainable from hydraulic servo systems which are widely used to power heavy duty robot manipulators. Alici has extended this basic idea to a two degrees of freedom planar manipulator carrying a dynamic load and shown that residual vibrations of the load can be eliminated by velocity shocks superimposed onto the harmonic motion profile of each degree of freedom in a simulation work reported in his paper published in 1993.[2] A realistic digital simulation of the system provides the designer a versatile tool, with which ideas on motion design can be tested. Many simulation programs have been developed capable of handling a variety of mechanical networks, such as DAMN, DRAM, ADAMS, IMP, VECNET, KIDYAN, CATIA[3] and Working Model, some of which are commercially available. Most of these programs can simulate robotic manipulators but since they are multi-purpose and large, their use is time consuming and costly. Computation efficiency can be of great concern as discussed by Walker and Orin in their 1982 paper[4] based on the dynamic modelling of multi degrees of freedom open kinematic chains with Newton-Euler equations. Further, some of them will not generate equations of motion to work on in closed form. A variety of approaches in the description of the linkage and mathematical modelling exists. Gorur, for example locates the joints first and describes the links by assigning various stiffnesses between joints.[5] Variable stiffnesses enable statically determine and indeterminate structures, mechanisms and open chains composed of binary links with revolute elements to be modelled by Newton-Euler formulation. Robotic manipulators are relatively simple and the model presented in this article is bringing a neat approach, easily understandable both physically and mathematically.

2. MODEL REQUIREMENTS
The basic task of a robot manipulator is to bring an object to a specific location in space at the required angular orientation. In general the manipulator must have 6 degrees of freedom, at least 3 of which are rotational. Gross spatial displacement is obtained by a 3 degrees of freedom arm, which is composed of a 2 degrees of freedom, planar dyad, movable either around a revolute axis in, or along a prismatic axis perpendicular to, the dyad plane. The wrist generally provides 3 or less rotational degrees of freedom which facilitate the angular orientation. In robot manipulators multi degrees of freedom joints are unfavourable due to the requirement for multiple power transmission elements to be packed in a small space and the greater bearing loads in comparison to that of single degree of freedom joints. Therefore, universally only single degree of freedom, single input-single output joints are utilised. These the *revolute* and *prismatic* joints. In most manipulators at least 3 links are co-planar and so, in this article 3 degrees of freedom, planar articulated open chains composed of

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revolute and prismatic joints are aimed at. Planar manipulators have been the subject of numerous studies on robotics like that of Horn,[6] Backhouse,[7] and Alici[2] as the initial step towards a thorough understanding of their dynamic behaviour. The mathematical model should provide compatibility with the real system in the following aspects:

1. Kinematic compatibility
The motion of all the bodies in the model must be the same as that of the real system. To obtain kinematic compatibility, the joint types, their degrees of freedom and ranges of motion and structural dimensions of the links must be in consistency with the real system. This ensures the joints to be positioned relative to each other correctly, leading to a properly modelled relative motion of the moving planes and points on these planes like mass centers, points of application of forces etc. kinematic compatibility is defined by constraint equations in the form:

$$f_j(x_i) = 0 \quad \text{for} \quad \begin{aligned} i &= 1 \cdots m \\ j &= 1 \cdots (m-n) \end{aligned} \tag{1}$$

for an n degree of freedom system. The total number of such equations is $(m-n)$ if m many coordinates are defined in x.

Kinematic compatibility can be sufficient for systems where forces acting are negligibly small.

2. Static compatibility
The forces external to the system and the bearing forces interacting at the joints must be consistent with that of the real system when the system is static. This can be expressed by a set of algebraic equations as:

$$\sum_k Q_k \frac{\partial x_k}{\partial x_i} = 0 \tag{2}$$

where Q_k are k many externally applied forces or torques and x_k are coordinates on which these forces directly act. Static compatibility can be sufficient for systems moving with negligible accelerations.

3. Dynamic compatibility
All the static and dynamic forces external to the system and interacting at the joints must be consistent with that of the real system. Such forces may come from actuators, viscous dampers, balancing springs or friction. To achieve this, apart from kinematic compatibility, all the inertial properties in the model must be assigned the same numerical values as of the real system. Dynamic compatibility can be represented by *Lagrange Formulation* as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \sum_k Q_k \frac{\partial x_k}{\partial x_i} \tag{3}$$

where L is the Lagrangian and t is time. A dynamically compatible simulation will generate the same motion profile as the real system upon the application of the same magnitudes of external forces.

These three criteria are closely related to each other. Kinematic compatibility defines the structure of the manipulator while static and dynamic compatibilities define the response of the manipulator to external forces. Static compatibility is a subset of dynamic compatibility and both stand on correctly formulated constraint equations and are taken care of in the mathematical model.

3. THE PHYSICAL MODEL

The physical model describes the structure of the manipulator. The manipulator to be modelled will have three moving links connected to each other in articulation and to the ground to form a 3 degrees of freedom open chain. Normally two of the movabilities will provide the gross point positioning in plane while the third will provide the angular orientation. So at least one degree of freedom must be revolute. The joint between any link pair can either be a prismatic (P) or a revolute (R) joint, hence a total of 7 chain configurations are possible as: RRR, RRP, RPR, PRR, RPP, PRP and PPR. As it can not facilitate angular orientation, a PPP configuration is not suitable. A link can be defined by the type of joint it starts and ends with, hence possible link types are RR, RP, PR and PP links. To define the position of a link plane, 3 independent variables are required. These can be the x and y coordinates of a characteristic point and the angle of a characteristic straight line in the plane. For convenience, the characteristic straight line is selected to pass through a characteristic point. The location of a revolute joint will be represented by the coordinates of the point where a joint axis pierces the plane of motion. Similarly, a prismatic joint is represented by a vector in the plane of motion, coinciding with the direction of the sliding axis and with magnitude equal to the instantaneous displacement of that joint. A link starts with the characteristic point, which always coincides with the point where the preceding link ends.

Joints between any link pair can be revolute or prismatic. The model must facilitate both and constrain whichever is not existing or fictitious. Three moving links in articulation, each having the ability to rotate or translate with respect to the precedling link form a 6 degrees of freedom system. Among the two joints between a link pair, one is real and the other is fictitious according to the type of the manipulator to be modelled. Distinction between the functions of the joints is done by appropriate *Existence Factors.* Existence factors are binary information bits one for each joint, as EF_1 and EF_2 for the prismatic and revolute joints respectively between the fixed link and the first moving link. Similarly EF_3 and EF_4 are for the prismatic and revolute joints, respectively, between the first and second moving links and EF_5 and EF_6 are for the prismatic and revolute joints respectively between the second and third moving links

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to describe whether they are real or fictitious. If an EF is 1, that particular joint is real and if 0, it is fictitious. Therefore, the following statements are always true:

$$(EF_1 = 1 \cdot AND \cdot EF_2 = 0) \cdot OR \cdot (EF_1 = 0 \cdot AND \cdot EF_2 = 1)$$
$$(EF_3 = 1 \cdot AND \cdot EF_4 = 0) \cdot OR \cdot (EF_3 = 0 \cdot AND \cdot EF_4 = 1)$$
$$(EF_5 = 1 \cdot AND \cdot EF_6 = 0) \cdot OR \cdot (EF_5 = 0 \cdot AND \cdot EF_6 = 1) \qquad (4)$$

In the mathematical model, topology of the actual manipulator to be modelled will be defined and adaptation of the equations to describe the motion of that particular topology is achieved by making use of magnitudes of the existence factors.

All 6 joints in the model can be thought of having motors of their own, each one being controllable independently. If for example first joint is a revolute, EF_1 will be 0 and EF_2 will be 1. The prismatic axis there will be appropriately controlled to produce zero joint velocity and acceleration. Prismatic action will freeze at its initial position while the revolute joint will be actuated arbitrarily. If on the other hand this joint is prismatic, the rotary motor actuating the revolute joint will be appropriately controlled to produce zero joint velocity and acceleration. Revolute action will freeze at its initial position while the prismatic joint will be actuated arbitrarily. The force or torque generated to constrain a movability is called the *Generalised Constraint Force or Torque*. It can be thought that two different command inputs are made available to the servo amplifier of a motor, one being the variable command signal from a controller and the other a constant voltage corresponding to the initial joint position. Switching of the amplifier input to the variable command signal is provided by the corresponding EF, and to the constant voltage by $\overline{EF}$ or $(1 - EF)$. Normally the Generalised Constraint Forces or Torques acting on a joint defined as fictitious can be obtained as the product of the positional error of that joint with respect to its initial value and a high gain. Normally a high frequency noise is imposed onto the profile of this force and the solution can get out of control.[3] Noise can be dampened by an appropriate derivative component. Similar techniques have been developed at which simultaneous solutions of algebraic,[8] differential[9] or mixtures of algebraic and differential equations are solved, the solution modifying itself in proportion to the error and to the gradient of the error and descending towards its correct value. This kind of an approach is easy to grasp and formulate,[3] but contains undetermined expressions. A closed form solution should always be preferred for faster and more efficient digital computation. Further, closed form equations of motion may yield clues on how to shape the profile of the actuation forces, which then can presumably be obtained from computer controlled servo actuators. The generalised constraint forces or torques can be extracted from closed form equations of motion, as to make the relative acceleration of the succeeding link with respect to the preceding zero throughout the time. Within this context, construction of each link must facilitate a relative rotation and a relative translation, hence a link plane will contain two kinematic elements, one a revolute at the characteristic point and a prismatic.

A vector of constant magnitude, named A is placed between the revolute and prismatic elements on a link, to identify their spacing. It starts from the characteristic point and stretches up to the prismatic element, perpendicular to it and constitutes the characteristic line. Figure 1 shows The generalised 6 degrees of freedom model on which the forecoming formulation is done, showing all the related nomenclature. Figure 2 shows different link types in this model. It must be noted that:

(*i*) Dimension of an RR link, that is magnitude of its A vector can not be zero, otherwise two revolute joints coincide and one of them becomes redundant reducing the total degrees of freedom by 1.
(*ii*) Two successive prismatic axes can not be parallel, otherwise one of them becomes redundant reducing the total degrees of freedom by 1.
(*iii*) Magnitude of vector A of a PR or RP link can be zero. This combines a rotation and translation into a *turn-slide* joint.
(*iv*) Vector A of a PP link can be of zero magnitude.
Location of points of interest like mass centers, actuator connections etc. are defined by a pair of relative

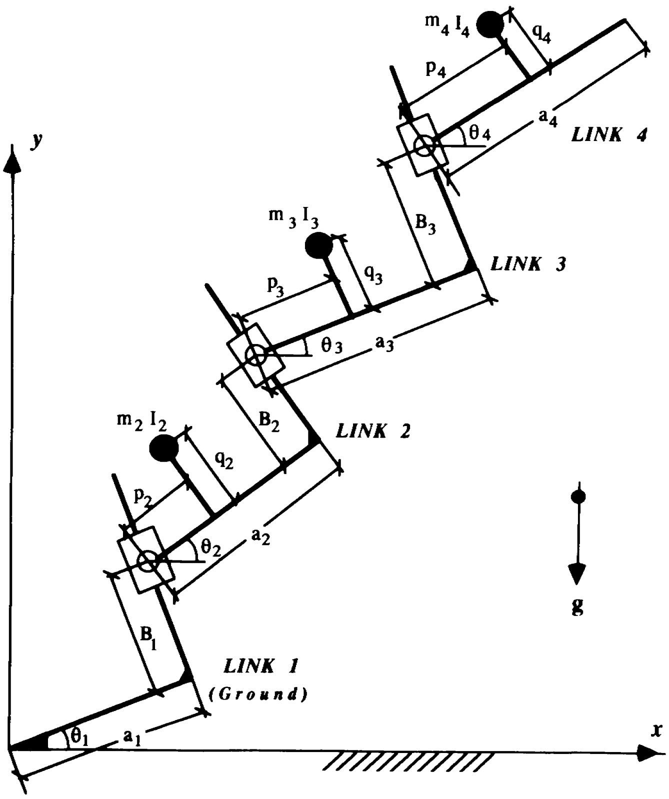

Fig. 1. 6 degrees of freedom PRPRPR open chain linkage. This chain can model any configuration of 3 degrees of freedom planar manipulators composed of revolute and prismatic joints by the appropriate control of joint motors.

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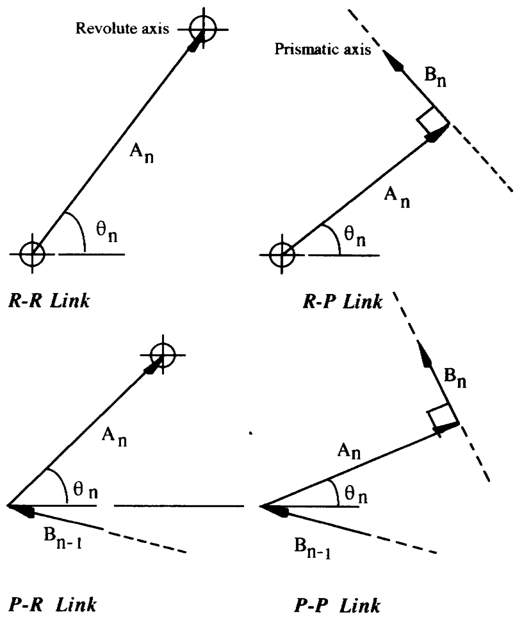

Fig. 2. Possible link types to form planar manipulators composed of Prismatic and revolute joints.

coordinates p and q, p along vector A and q perpendicular to it. This physical model provides kinematic, static and dynamic compatibilities with a real system excepting forces of Coulomb friction. Coulomb friction is dependent on actual joint configuration and dimensions.

4. THE MATHEMATICAL MODEL

The differential equations of motion of the system shown in Figure 1, working in the vertical plane can be derived using the *Lagrange* formulation of equation (3). Lagrangian of the system is;

$$L = T - V \tag{5}$$

where T is the total kinetic energy as:

$$T = \sum_{n=2}^{4} \left(\tfrac{1}{2} m_n v_{g_n}^2 + \tfrac{1}{2} I_n \dot\theta_n^2\right) \tag{6}$$

and V is the total potential energy as:

$$V = \sum_{n=2}^{4} (m_n y_{g_n} g) \tag{7}$$

m_n is the mass and I_n is the mass moment of inertia of the n'th link, y_{g_n} is the elevation and v_{g_n} is the velocity of its mass center and $\dot\theta_n$ is its absolute angular velocity. Counters of equations (6) and (7) start from 2 as the first link is ground, which is not movable. For each one of six degrees of freedom, one equation has to be derived. Equation defining the motion of the generalised coordinate B_1 comes up as:

$$\begin{aligned}
&\ddot B_1[m_2+m_3+m_4] + \ddot\theta_2[\{m_2q_2+B_2(m_3+m_4)\}\\
&\quad \times \sin(\theta_1-\theta_2) + \{m_2p_2+A_2(m_3+m_4)\}\cos(\theta_1-\theta_2)]\\
&\quad + \ddot B_2[(m_3+m_4)\cos(\theta_1-\theta_2)] + \ddot\theta_3[\{m_3q_3+m_4B_3\}\\
&\quad \times \sin(\theta_1-\theta_3) + \{m_3p_3+m_4A_3\}\cos(\theta_1-\theta_3)]\\
&\quad + \ddot B_3[m_4\cos(\theta_1-\theta_3)] + \ddot\theta_4[m_4\{q_4\sin(\theta_1-\theta_4)\\
&\quad + p_4\cos(\theta_1-\theta_4)\}] = F_{B_l} - (m_2+m_3+m_4)g\cos\theta_1\\
&\quad - [m_2p_2\dot\theta_2^2 + \dot\theta_2(A_2\dot\theta_2+2\dot B_2)(m_3+m_4)]\sin(\theta_1-\theta_2)\\
&\quad + [m_2q_2\dot\theta_2^2 + B_2\dot\theta_2^2(m_3+m_4)]\cos(\theta_1-\theta_2)\\
&\quad - [m_3p_3\dot\theta_3^2 + m_4\dot\theta_3(A_3\dot\theta_3+2\dot B_3)]\sin(\theta_1-\theta_3)\\
&\quad + \dot\theta_3^2(m_3q_3+m_4B_3)\cos(\theta_1-\theta_3)\\
&\quad - m_4p_4\dot\theta_4^2\sin(\theta_1-\theta_4) + m_4q_4\dot\theta_4^2\cos(\theta_1-\theta_4)
\end{aligned} \tag{8}$$

Similarly equation of motion of the coordinate θ_2 is:

$$\begin{aligned}
&\ddot B_1[\{m_2q_2+B_2(m_3+m_4)\}\sin(\theta_1-\theta_2) + \{m_2p_2\\
&\quad + A_2(m_3+m_4)\}\cos(\theta_1-\theta_2)] + \ddot\theta_2[I_2 + m_2(p_2^2+q_2^2)\\
&\quad + (m_3+m_4)(A_2^2+B_2^2)] + \ddot B_2[A_2(m_3+m_4)] + \ddot\theta_3[\{m_3\\
&\quad \times (A_2q_3 - B_2p_3) + m_4(A_2B_3 - B_2A_3)\}\sin(\theta_2-\theta_3)\\
&\quad + \{m_3(A_2p_3+B_2q_3) + m_4(A_2A_3+B_2B_3)\}\cos(\theta_2-\theta_3)]\\
&\quad + \ddot B_3[m_4\{A_2\cos(\theta_2-\theta_3) - B_2\sin(\theta_2-\theta_3)\}]\\
&\quad + \ddot\theta_4[m_4\{(A_2q_4 - B_2p_4)\sin(\theta_2-\theta_4) + (A_2p_4+B_2q_4)\\
&\quad \times \cos(\theta_2-\theta_4)\}] = \tau_{\theta_2} - (m_3+m_4)[2B_2\dot B_2\dot\theta_2\\
&\quad + g(A_2\cos\theta_2 - B_2\sin\theta_2)] - m_2g(p_2\cos\theta_2 - q_2\sin\theta_2)\\
&\quad - [m_3\dot\theta_3^2(A_2p_3+B_2q_3) + m_4\{2A_2\dot B_3\dot\theta_3 + \dot\theta_3^2(A_2A_3\\
&\quad + B_2B_3)\}]\sin(\theta_2-\theta_3) - [-m_3\dot\theta_3^2(A_2q_3 - B_2p_3)\\
&\quad + m_4\{2B_2\dot B_3\dot\theta_3 - \dot\theta_3^2(A_2B_3 - B_2A_3)\}]\\
&\quad \times \cos(\theta_2-\theta_3) - \dot\theta_4^2 m_4(A_2p_4+B_2q_4)\sin(\theta_2-\theta_4)\\
&\quad + \dot\theta_4^2 m_4(A_2q_4 - B_2p_4)\cos(\theta_2-\theta_4)
\end{aligned} \tag{9}$$

and equation of motion of the coordinate B_2 is:

$$\begin{aligned}
&\ddot B_1[(m_3+m_4)\cos(\theta_1-\theta_2)] + \ddot\theta_2[A_2(m_3+m_4)]\\
&\quad + \ddot B_2[m_3+m_4] + \ddot\theta_3[(m_3q_3+m_4B_3)\sin(\theta_2-\theta_3)\\
&\quad + (m_3p_3+m_4A_3)\cos(\theta_2-\theta_3)] + \ddot B_3[m_4\cos(\theta_2-\theta_3)]\\
&\quad + \ddot\theta_4[m_4\{q_4\sin(\theta_2-\theta_4) + p_4\cos(\theta_2-\theta_4)\}]\\
&= F_{B_2} + (m_3+m_4)(B_2\dot\theta_2^2 - g\cos\theta_2)\\
&\quad + [-m_3p_3\dot\theta_3^2 - m_4\dot\theta_3(A_2\dot\theta_3 + 2\dot B_3)]\\
&\quad \times \sin(\theta_2-\theta_3) + \dot\theta_3^2(m_3q_3+m_4B_3)\cos(\theta_2-\theta_3)\\
&\quad - m_4p_4\dot\theta_4^2\sin(\theta_2-\theta_4) + m_4q_4\dot\theta_4^2\cos(\theta_2-\theta_4)
\end{aligned} \tag{10}$$

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Equation of motion of the coordinate θ_3 is:

$$\begin{aligned}
&\ddot{B}_1[(m_3q_3+m_4B_3)\sin(\theta_1-\theta_3)+(m_3p_3+m_4A_3)\\
&\quad\times\cos(\theta_1-\theta_3)]+\ddot{\theta}_2[\{m_3(A_2q_3-B_2p_3)\\
&\quad+m_4(A_2B_3-B_2A_3)\}\sin(\theta_2-\theta_3)+\{m_3(A_2p_3+B_2q_3)\\
&\quad+m_4(A_2A_3+B_2B_3)\}\cos(\theta_2-\theta_3)]\\
&\quad+\ddot{B}_2[(m_3q_3+m_4B_3)\sin(\theta_2-\theta_3)\\
&\quad+(m_3p_3+m_4A_3)\cos(\theta_2-\theta_3)]+\ddot{\theta}_3[I_3+m_3(p_3^2+q_3^2)\\
&\quad+m_4(A_3^2+B_3^2)]+\ddot{B}_3[m_4A_3]+\ddot{\theta}_4[m_4\{(A_3q_4-B_3p_4)\\
&\quad\times\sin(\theta_3-\theta_4)+(A_3p_4+B_3q_4)\cos(\theta_3-\theta_4)\}]\\
&=\tau_{\theta_3}-m_3g(p_3\cos\theta_3-q_3\sin\theta_3)\\
&\quad-m_4[2B_3\dot{B}_3\dot{\theta}_3+g(A_3\cos\theta_3-B_3\sin\theta_3)]\\
&\quad+[m_3\{2p_3\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2p_3+B_2q_3)\}\\
&\quad+m_4\{2A_3\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2A_3+B_2B_3)\}]\sin(\theta_2-\theta_3)\\
&\quad-[m_3\{2q_3\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2q_3-B_2p_3)\}\\
&\quad+m_4\{2B_3\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2B_3-B_2A_3)\}]\cos(\theta_2-\theta_3)\\
&\quad-m_4\dot{\theta}_4^2(A_3p_4+B_3q_4)\sin(\theta_3-\theta_4)\\
&\quad+m_4\dot{\theta}_4^2(A_3q_4-B_3p_4)\cos(\theta_3-\theta_4)
\end{aligned}\tag{11}$$

Equation of motion of the coordinate B_3 is:

$$\begin{aligned}
&\ddot{B}_1[m_4\cos(\theta_1-\theta_3)]+\ddot{\theta}_2[m_4\{A_2\cos(\theta_2-\theta_3)\\
&\quad-B_2\sin(\theta_2-\theta_3)\}]+\ddot{B}_2[m_4\cos(\theta_2-\theta_3)]+\ddot{\theta}_3[m_4A_3]\\
&\quad+\ddot{B}_3[m_4]+\ddot{\theta}_4[m_4\{q_4\sin(\theta_3-\theta_4)+p_4\cos(\theta_3-\theta_4)\}]\\
&=F_{B_3}+m_4(B_3\dot{\theta}_3^2-g\cos\theta_3)+m_4\dot{\theta}_2(A_2\dot{\theta}_2+2\dot{B}_2)\\
&\quad\times\sin(\theta_2-\theta_3)+m_4B_2\dot{\theta}_2^2\cos(\theta_2-\theta_3)\\
&\quad-m_4p_4\dot{\theta}_4^2\sin(\theta_3-\theta_4)+m_4q_4\dot{\theta}_4^2\cos(\theta_3-\theta_4)
\end{aligned}\tag{12}$$

and the equation of motion of coordinate θ_4 is:

$$\begin{aligned}
&\ddot{B}_1[m_4\{q_4\sin(\theta_1-\theta_4)+p_4\cos(\theta_1-\theta_4)\}]\\
&\quad+\ddot{\theta}_2[m_4\{(A_2q_4-B_2p_4)\sin(\theta_2-\theta_4)\\
&\quad+(A_2p_4+B_2q_4)\cos(\theta_2-\theta_4)\}]\\
&\quad+\ddot{B}_2[m_4\{q_4\sin(\theta_2-\theta_4)+p_4\cos(\theta_2-\theta_4)\}]\\
&\quad+\ddot{\theta}_3[m_4\{(A_3q_4-B_3p_4)\sin(\theta_3-\theta_4)\\
&\quad+(A_3p_4+B_3q_4)\cos(\theta_3-\theta_4)\}]\\
&\quad+\ddot{B}_3[m_4\{q_4\sin(\theta_3-\theta_4)+p_4\cos(\theta_3-\theta_4)\}]\\
&\quad+\ddot{\theta}_4[I_4+m_4(p_4^2+q_4^2)]\\
&=\tau_{\theta_4}-m_4g(p_4\cos\theta_4-q_4\sin\theta_4)\\
&\quad+[m_4\{2p_4\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2p_4+B_2q_4)\}]\sin(\theta_2-\theta_4)\\
&\quad-[m_4\{2q_4\dot{B}_2\dot{\theta}_2+\dot{\theta}_2^2(A_2q_4-B_2p_4)\}]\cos(\theta_2-\theta_4)\\
&\quad+[m_4\{2p_4\dot{B}_3\dot{\theta}_3+\dot{\theta}_3^2(A_3p_4+B_3q_4)\}]\sin(\theta_3-\theta_4)\\
&\quad-[m_4\{2q_4\dot{B}_3\dot{\theta}_3+\dot{\theta}_3^2(A_3q_4B_3p_4)\}]\cos(\theta_3-\theta_4)
\end{aligned}\tag{13}$$

Nomenclature is as shown in Figure 1. Equations (7–13) can be represented in matrix form as:

$$|M|\cdot\begin{vmatrix}\ddot{B}_1\\\ddot{\theta}_2\\\ddot{B}_2\\\ddot{\theta}_3\\\ddot{B}_3\\\ddot{\theta}_4\end{vmatrix}=\begin{vmatrix}F_{B_1}+\phi_{B_1}\\\tau_{\theta_2}+\phi_{\theta_2}\\F_{B_2}+\phi_{B_2}\\\tau_{\theta_3}+\phi_{\theta_3}\\F_{B_3}+\phi_{B_3}\\\tau_{\theta_4}+\phi_{\theta_4}\end{vmatrix}\tag{14}$$

M is the symmetric *mass matrix*. F_{xi} and τ_{xi}'s are externally applied driving forces and torques whose values are arbitrary and ϕ_{xi}'s are the sum of respective components of all velocity dependent forces, namely *Coriolis* and *Centrifugal* and the effects of *gravity*.

Generalised Constraint Forces, which are activated by appropriate *Existence Factors* should be added on the motion equation of each degree of freedom in the same format as the external driving forces. Also, if there is a rotary actuator to drive a rotational degree of freedom, its reaction torque acts on the preceding rotational movability. With their inclusion, equation (14) becomes:

$$|M|\cdot\begin{vmatrix}\ddot{B}_1\\\ddot{\theta}_2\\\ddot{B}_2\\\ddot{\theta}_3\\\ddot{B}_3\\\ddot{\theta}_4\end{vmatrix}=\begin{vmatrix}F_{B_l}+F_{\text{const}_{B_l}}\cdot EF_2+\phi_{B_1}\\\tau_{\theta_2}+\tau_{\text{const}_{\theta_2}}\cdot EF_1-\tau_{\theta_3}-\tau_{\text{const}_{\theta_3}}\cdot EF_3+\phi_{\theta_2}\\F_{B_2}+F_{\text{const}_{B_2}}\cdot EF_4+\phi_{B_2}\\\tau_{\theta_3}+\tau_{\text{const}_{\theta_3}}\cdot EF_3-\tau_{\theta_4}-\tau_{\text{const}_{\theta_4}}\cdot EF_5+\phi_{\theta_3}\\F_{B_3}+F_{\text{const}_{B_3}}\cdot EF_6+\phi_{B_3}\\\tau_{\theta_4}+\tau_{\text{const}_{\theta_4}}\cdot EF_5+\phi_{\theta_4}\end{vmatrix}\tag{15}$$

where $F_{\text{const}_{x_i}}$ and $\tau_{\text{const}_{x_i}}$ are corresponding generalised constraint forces and torques.

If the tipmost or third joint is prismatic, then $EF_5=1$ and $EF_6=0$. Generalised constraint torque on coordinate 6, i.e. $\tau_{\text{const}_{\theta_4}}$ will be active, constraining any rotation of the tipmost link with respect to the preceding. This torque is a variable taking its instantaneous value from the dynamics of the rest of the system as to make the acceleration of the coordinate θ_4 the same as the acceleration of the preceding link, that is, $\ddot{\theta}_3=\ddot{\theta}_4$. This particular constraint torque hence can be formulated as:

$$\begin{aligned}
\tau_{\text{const}_{\theta_4}}&=M(6,1)\ddot{B}_1+M(6,2)\ddot{\theta}_2+M(6,3)\ddot{B}_2\\
&\quad+M(6,4)\ddot{\theta}_3+M(6,5)\ddot{B}_3+M(6,6)\ddot{\theta}_3-\tau_{\theta_4}-\phi_{\theta_4}
\end{aligned}\tag{16}$$

Substitution of equation (16) into equation (15) produces the final form of the motion equation for the coordinate θ_4 as:

$$\begin{aligned}
&\ddot{B}_1[M(6,1)\cdot EF_6]+\ddot{\theta}_2[M(6,2)\cdot EF_6]+\ddot{B}_2[M(6,3)\cdot EF_6]\\
&\quad+\ddot{\theta}_3[M(6,4)\cdot EF_6-M(6,6)\cdot EF_5]+\ddot{B}_3[M(6,5)\cdot EF_6]\\
&\quad+\ddot{\theta}_4[(M(6,6)]=[\tau_{\theta_4}+\phi_{\theta_4}]\cdot EF_6
\end{aligned}\tag{17}$$

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Instead, if the tipmost joint is revolute, then $EF_5 = 0$ and $EF_6 = 1$. Generalised constraint force on coordinate B_3 is active constraining any translation of the tipmost link with respect to the preceding. Instantaneous value of this constraint force is calculable from equation (15) as to make the acceleration of B_3 zero as:

$$F_{\mathrm{const}_{B_3}} = M(5,1)\ddot{B}_1 + M(5,2)\ddot{\theta}_2 + M(5,3)\ddot{B}_2 + M(5,4)\ddot{\theta}_3 + M(5,6)\ddot{\theta}_4 - F_{B_3} - \phi_{B_3} \quad (18)$$

Substitution of equation (18) into (15) yields the final form of the motion equation for the coordinate B_3 as;

$$\ddot{B}_1[M(5,1)\cdot EF_5] + \ddot{\theta}_2[M(5,2)\cdot EF_5] + \ddot{B}_2[M(5,3)\cdot EF_5] + \ddot{\theta}_3[M(5,4)\cdot EF_5] - \ddot{B}_3[M(5,5)] + \ddot{\theta}_4[M(5,6)\cdot EF_5] = [F_{B_3} + \phi_{B_3}]\cdot EF_5 \quad (19)$$

Similarly if the second joint is prismatic, $EF_3 = 1$ and $EF_4 = 0$ hence 4'th coordinate, that is θ_3 becomes fictitious with the action of the Generalised Constraint Torque $\tau_{\mathrm{const}_{\theta_3}}$ to make the angular accelerations of links 2 and 3 the same. The expression for $\tau_{\mathrm{const}_{\theta_3}}$ can be extracted from equation (15) with the substitution of $\ddot{\theta}_2 = \ddot{\theta}_3$ as:

$$\begin{aligned}\tau_{\mathrm{const}_{\theta_3}} = {} & \ddot{B}_1[M(4,1) + M(6,1)\cdot EF_5] + \ddot{\theta}_2[M(4,2) \\ & + M(4,4) + \{M(6,2) + M(6,4) + M(6,6)\}\cdot EF_5] \\ & + \ddot{B}_2[M(4,3) + M(6,3)\cdot EF_5] \\ & + \ddot{B}_3[M(4,5) + M(6,5)\cdot EF_5] + \ddot{\theta}_4[M(4,6)] \\ & + [-\tau_{\theta_3} + \tau_{\theta_4} - \phi_{\theta_3} - \{\tau_{\theta_4} + \phi_{\theta_4}\}\cdot EF_5]\end{aligned} \quad (20)$$

Substitution of equation (20) into equation (15) yields the final form of the motion equation for the coordinate θ_3 as:

$$\begin{aligned}& \ddot{B}_1[\{M(4,1) + M(6,1)\cdot EF_5\}\cdot EF_4] \\ & \quad + \ddot{\theta}_2[\{M(4,2) + M(6,2)\cdot EF_5\}\cdot EF_4 - M(4,4)\cdot EF_3 \\ & \quad - \{M(6,4) + M(6,6)\}\cdot EF_5\cdot EF_3] \\ & \quad + \ddot{B}_2[\{M(4,3) + M(6,3)\cdot EF_5\}\cdot EF_4] \\ & \quad + \ddot{\theta}_3[M(4,4) + \{M(6,4) + M(6,6)\}\cdot EF_5] \\ & \quad + \ddot{B}_3[\{M(4,5) + M(6,5)\cdot EF_5\}\cdot EF_4] \\ & \quad + \ddot{\theta}_4[M(4,6)\cdot EF_4] \\ & = [\tau_{\theta_3} - \tau_{\theta_4}\cdot EF_6 + \phi_{\theta_3} + \phi_{\theta_4}\cdot EF_5]\cdot EF_4\end{aligned} \quad (21)$$

Instead, if the second joint is revolute, then $EF_3 = 0$ and $EF_4 = 1$, hence the third coordinate becomes fictitious with the action of the associated Generalised Constraint Force to make that particular joint acceleration zero. Instantaneous value of $F_{\mathrm{const}_{B_2}}$ is calculable from equation (15) by substituting $\ddot{B}_2 = 0$ as:

$$F_{\mathrm{const}_{B_2}} = M(3,1)\ddot{B}_1 + M(3,2)\ddot{\theta}_2 + M(3,4)\ddot{\theta}_3 + M(3,5)\ddot{B}_3 + M(3,6)\ddot{\theta}_4 - F_{B_2} - \phi_{B_2} \quad (22)$$

Substitution of equation (22) into equation (15) yields the final form of the motion equation for the coordinate B_2 as:

$$\begin{aligned}& \ddot{B}_1[M(3,1)\cdot EF_3] + \ddot{\theta}_2[M(3,2)\cdot EF_3] + \ddot{B}_2[M(3,3)] \\ & \quad + \ddot{\theta}_3[M(3,4)\cdot EF_3] + \ddot{B}_3[M(3,5)\cdot EF_3] \\ & \quad + \ddot{\theta}_4[M(3,6)\cdot EF_3] = [F_{B_2} + \phi_{B_2}]\cdot EF_3\end{aligned} \quad (23)$$

If the first joint, that is, the one connecting the first moving link to ground is prismatic, then $EF_1 = 1$ and $EF_2 = 0$. Generalised Constraint Torque on coordinate 2, i.e. $\tau_{\mathrm{const}_{\theta_2}}$ will be active constraining any rotation of the first moving link. This torque is a variable whose instantaneous value can be extracted from equation (15) to satisfy $\ddot{\theta}_2 = 0$ as:

$$\begin{aligned}\tau_{\mathrm{const}_{\theta_2}} = {} & \ddot{B}_1[M(2,1) + \{M(4,1) + M(6,1)\cdot EF_5\}\cdot EF_3] \\ & + \ddot{B}_2[M(2,3) + \{M(4,3) + M(6,3)\cdot EF_5\}\cdot EF_3] \\ & + \ddot{\theta}_3[M(2,4)] + \ddot{B}_3[M(2,5) \\ & + \{M(4,5) + M(6,5)\cdot EF_5\}\cdot EF_3] \\ & + \ddot{\theta}_4[M(2,6) + M(4,6)\cdot EF_3] \\ = {} & [-\tau_{\theta_2} + \tau_{\theta_3} - \phi_{\theta_2} - \{\tau_{\theta_3} + \phi_{\theta_3} - \tau_{\theta_4} \\ & + \tau_{\theta_4}\cdot EF_5 + \phi_{\theta_4}\cdot EF_5\}\cdot EF_3]\end{aligned} \quad (24)$$

Substitution of equation (24) into equation (15) yields the final form of the motion equation for the coordinate θ_2 as:

$$\begin{aligned}& \ddot{B}_1[\{M(2,1) + (M(4,1) + M(6,1)\cdot EF_5)\cdot EF_3\}\cdot EF_2] \\ & \quad + \ddot{\theta}_2[M(2,2) + \{M(4,2) + M(4,4) + (M(6,2) \\ & \quad + M(6,4) + M(6,6))\cdot EF_5\}\cdot EF_3] + \ddot{B}_2[\{M(2,3) \\ & \quad + (M(4,3) + M(6,3)\cdot EF_5)\cdot EF_3\}\cdot EF_2] \\ & \quad + \ddot{\theta}_3[M(2,4)\cdot EF_2] \\ & \quad + \ddot{B}_3[\{M(2,5) + (M(4,5) + M(6,5)\cdot EF_5)\cdot EF_3\}\cdot EF_2] \\ & \quad + \ddot{\theta}_4[\{M(2,6) + M(4,6)\cdot EF_3\}\cdot EF_2] \\ & = [\tau_{\theta_2} - \tau_{\theta_3}\cdot EF_4 - \tau_{\theta_4}\cdot EF_3\cdot EF_6 \\ & \quad + \phi_{\theta_2} + \phi_{\theta_3}\cdot EF_3 + \phi_{\theta_4}\cdot EF_3\cdot EF_5]\cdot EF_2\end{aligned} \quad (25)$$

Instead, if the first joint is revolute, then $EF_1 = 0$ and $EF_2 = 1$, hence the first coordinate becomes fictitious with the action of the Generalised Constraint Force to make that particular joint acceleration zero. Instantaneous value of $F_{\mathrm{const}_{B_1}}$ is derivable from equation (15), by substituting $\ddot{B}_1 = 0$ as:

$$F_{\mathrm{const}_{B_1}} = M(1,2)\ddot{\theta}_2 + M(1,3)\ddot{B}_2 + M(1,4)\ddot{\theta}_3 + M(1,5)\ddot{B}_3 + M(1,6)\ddot{\theta}_4 - F_{B_1} - \phi_{B_1} \quad (26)$$

Substitution of equation (26) into equation (15) yields the final form of the motion equation for the coordinate

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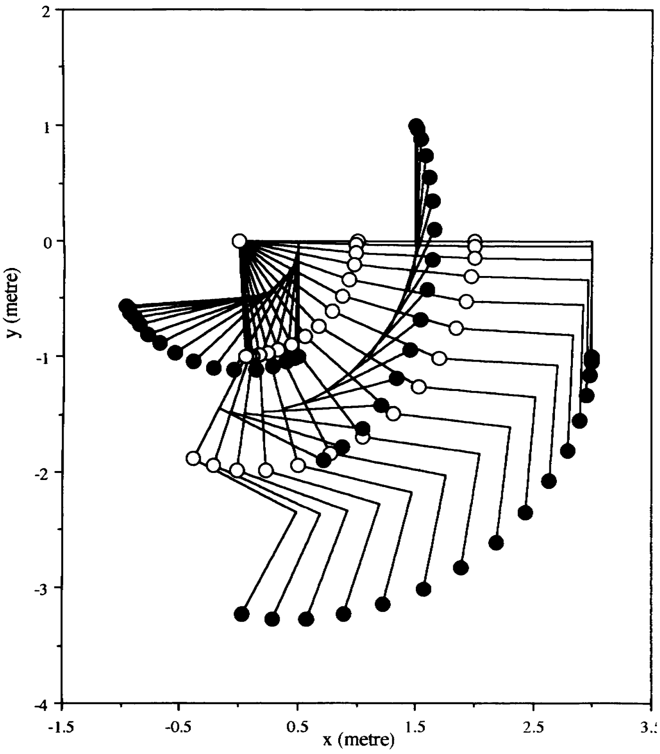
B_1 as:

$$\ddot{B}_1[M(1,1) + \ddot{\theta}_2[M(1,2)\cdot EF_1] + \ddot{B}_2[M(1,3)\cdot EF_1]$$
$$+ \ddot{\theta}_3[M(1,4)\cdot EF_1] + \ddot{B}_3[M(1,5)\cdot EF_1]$$
$$+ \ddot{\theta}_4[M(1,6)\cdot EF_1] = [F_{B_1} + \phi_{B_1}]\cdot EF_1 \qquad (27)$$

Equations (17), (19), (21), (23), (25), (27) together, when used with proper Existence Factors define the motion of any 3-degrees of freedom, open loop link chain composed of prismatic and revolute joints.



LINKAGE PARAMETERS: INITIAL CONDITIONS:
A1= 0.0 m. M2=5.0 kg B1=0.0 m.
A2= 1.0 m. M3=2.0 kg. B2=0.0 m.
A3= 1.0 m. M4=1.0 kg. B3=0.0 m.
A4= 1.0 m. I2=0.0 kg-m^2 THETA 2=0.0 deg.
P2= 0.5 m. I3=0.0 kg-m^2 THETA 3=0.0 deg.
Q2=-1.0 m I4=0.0 kg.m^2 THETA 4=0.0 deg.
P3= 0.5 m.
Q3= 1.0 m. B1 DOT=0.0 m/sec.
P4= 1.0 m B2 DOT=0.0 m/sec.
Q4=-1.0 m B3 DOT=0.0 m/sec.
THETA 1=0.0 deg. THETA 2 DOT=0.0 rad/sec.
THETA 3 DOT=0.0 rad/sec.
GEN.FORCES: THETA 4 DOT=0.0 rad/sec.
F1=0.0 N
F2=0.0 N. R JOINT DAMPING COEFF.:
F3=0.0 N. 20 N-m-sec/rad.
T2=0.0 N-m.
T3=0.0 N-m. EXISTENCE FACTORS:
T4=0.0 N-m. EF1=0 EF3=0 EF5=0
EF2=1 EF4=1 EF6=1

Fig. 3. Chrono-cyclograph of a triple pendulum with offset mass centers, motion starting from rest when all the revolute joints are alligned on a horizontal line. Time increment is 0.1 second. Linkage parameters, initial values of generalised coordinates and velocities and the forcing functions active are listed below the figure.

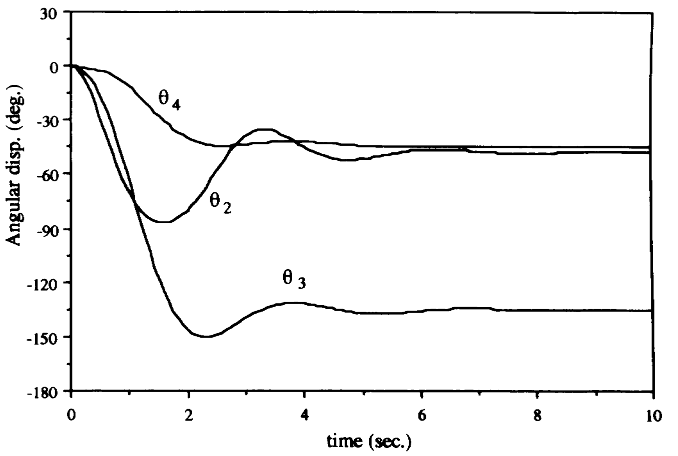
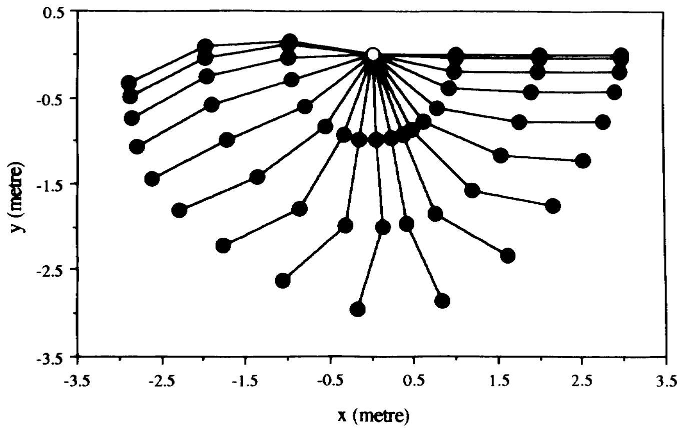


Fig. 4. Motion profile of the triple pendulum shown in Figure 3. Due to damping, system comes to rest after some time. While at rest, overall mass center is on the vertical line passing through the ground pivot.



LINKAGE PARAMETERS: INITIAL CONDITIONS:
A1=0.0 m. M2=1.0 kg. B1=0.0 m.
A2=1.0 m. M3=1.0 kg. B2=0.0 m.
A3=1.0 m. M4=1.0 kg. B3=0.0 m.
A4=1.0 m. I2=0.0 kg-m^2 THETA 2=0.0 deg.
P2=1.0 m. I3=0.0 kg-m^2 THETA 3=0.0 deg.
Q2=0.0 m I4=0.0 kg.m^2 THETA 4=0.0 deg.
P3=1.0 m.
Q3=0.0 m. B1 DOT=0.0 m/sec.
P4=1.0 m. B2 DOT=0.0 m/sec.
Q4=0.0 m. B3 DOT=0.0 m/sec.
THETA 1=0.0 deg. THETA 2 DOT=0.0 rad/sec.
GEN.FORCES: THETA 3 DOT=0.0 rad/sec.
THETA 4 DOT=0.0 rad/sec.
F1=0.0 N.
F2=0.0 N.
F3=0.0 N. EXISTENCE FACTORS:
T2=0.0 N-m. EF1=0 EF3=0 EF5=0
T3=0.0 N-m. EF2=1 EF4=1 EF6=1
T4=0.0 N-m.

Fig. 5. Chrono-cyclograph of a triple pendulum, motion starting from rest when all the links are horizontal. Time increment is 0.1 seconds. Linkage parameters, initial values of generalised coordinates and velocities and tbe forcing functions active are shown below the picture.

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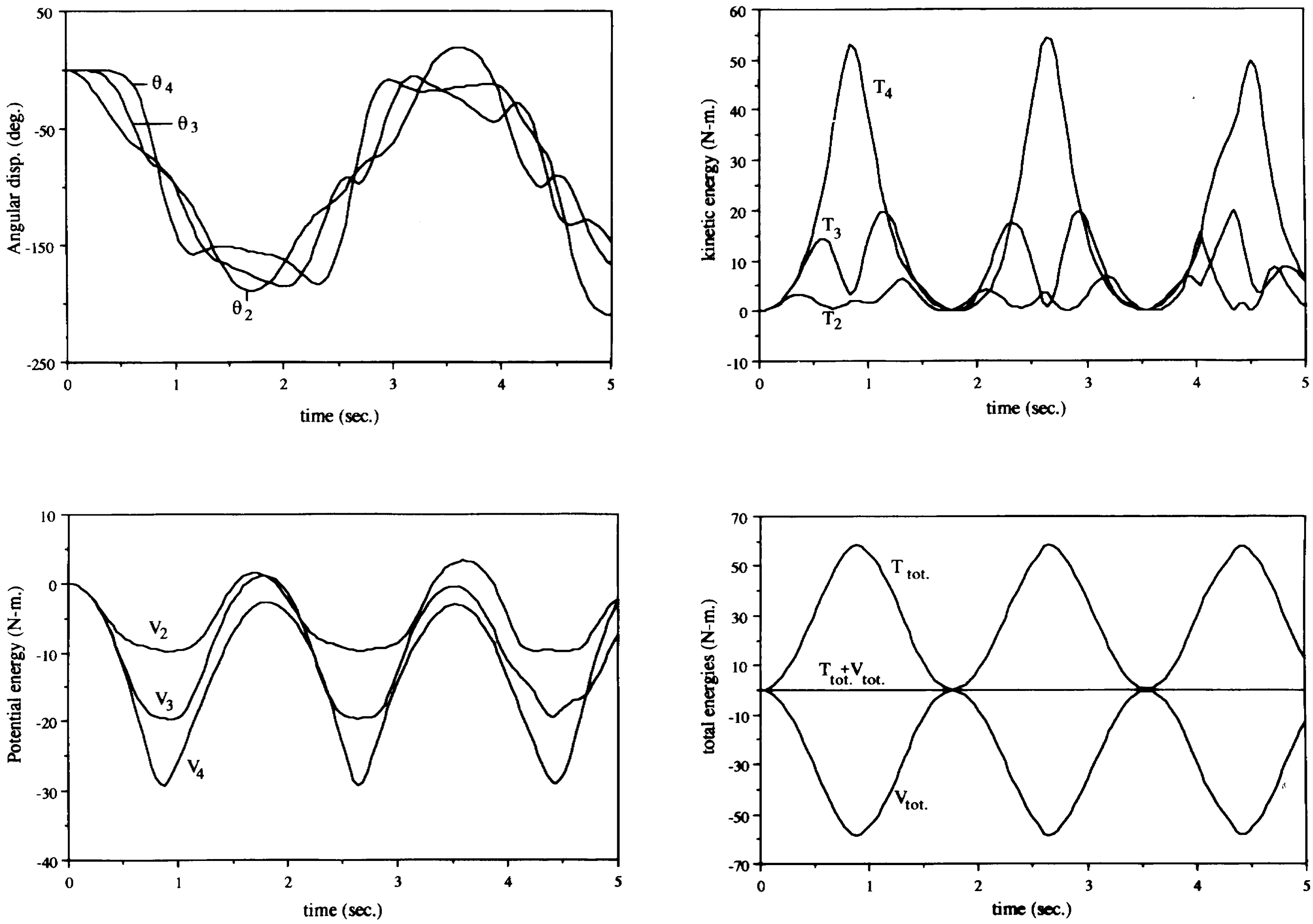

Fig. 6. Profiles of angular positions and energy levels of the links of the triple pendulum shown in Figure 5. As the system is conservative, total energy stays constant.

5. EXAMPLES

First example to demonstrate the applicability of the mathematical model presented with equations (17), (19), (21), (23), (25) and (27) is the motion of a triple pendulum with offset mass centers. The linkage parameters, initial values of generalised coordinates and velocities and the forcing functions active are shown in Figure 3. The equations of motion can be integrated by any commercially available numerical integration routline like DRKGS of IBM or D02BAF of NAG libraries. A chrono-cyclograph of the RRR chain under consideration is shown in Figure 3 for the first 1.5 seconds of the motion at 0.1 second intervals. Revolute joints are dampened and system comes to rest after some time as shown in the motion profile of Figure 4. The final angular position of the links are such that the system mass center is on the vertical line passing through the ground pivot, hence demonstrating the static compatibility.

The second example is the motion of a triple pendulum whose linkage parameters, initial values of generalised coordinates and velocities and the forcing functions active are shown in Figure 5. All the active and dissipative forces are zero, hence the system is conservative. Motion develops with the profile shown in Figure 6. Total kinetic and potential energy levels vary in opposite polarity to keep the total energy constant, demonstrating a dynamic compatibility of the model with a triple pendulum.

The third example is the motion of an RPR linkage under PD control to trace a circular trajectory. Figure 7 shows the first few discrete positions of the manipulator at the beginning of the motion and the path traced by the tip point of the manipulator. At the beginning of the motion, manipulator is at rest with gravity being the only active force. As seen, joint motions display some overshoots and vibrations due to the initial offsets in actual and commanded positions which decay and manipulator follows the commanded trajectory smoothly.

6. CONCLUSION

6 second order differential equations defining the system are simultaneously integrated. Results for joint positions

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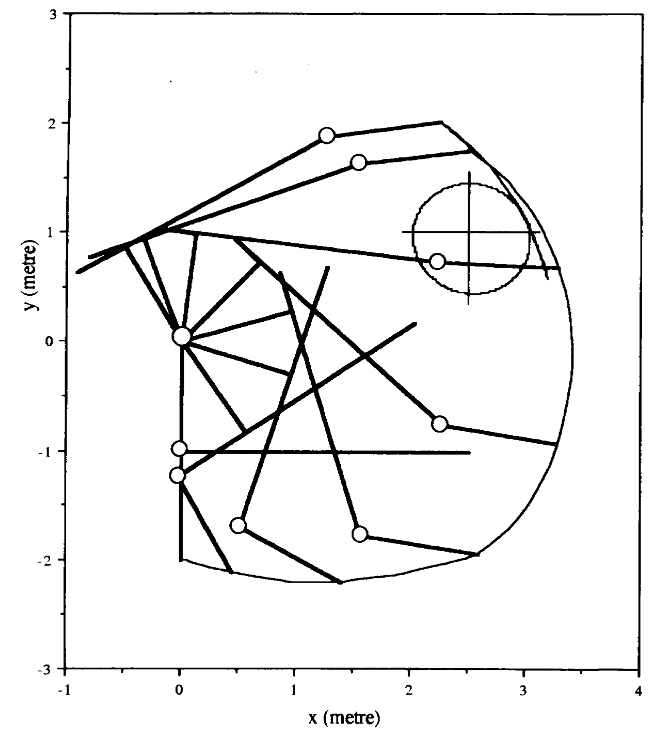

LINKAGE PARAMETERS:

A1=0.0 m.	M2=1.0 kg.	
A2=1.0 m.	M3=1.0 kg.	
A3=2.5 m.	M4=1.0 kg.	
A4=1.0 m.	I2=1.0 kg-m^2	
P2=1.0 m.	I3=1.0 kg-m^2	
Q2=0.0 m	I4=1.0 kg.m^2	
P3=2.5 m.		
Q3=0.0 m.		
P4=1.0 m.		
Q4=0.0 m.		
THETA 4=0.0 deg.		

INITIAL CONDITIONS:

B1=0.0 m.
B2=2.5 m.
B3=0.0 m.
THETA 2=-90 deg.
THETA 3=180 deg.
THETA 4=-90 deg.

B1 DOT=0.0 m/sec.
B2 DOT=0.0 m/sec
B3 DOT=0.0 m/sec.
THETA 2 DOT=0.0 rad/sec.
THETA 3 DOT=0.0 rad/sec.
THETA 4 DOT=0.0 rad/sec.

GEN.FORCES:

F1=0.0 N.
F3=0.0 N.
T3=0.0 N-m.

F2 P GAIN=20000 N/m.
F2 D GAIN=50 N-sec/m.
F2 SATURATING AT:300 N

T2 P GAIN=1500 N-m/rad.
T2 D GAIN=80 N-m-sec/rad.
T2 SATURATING AT:1000 N-m.

T4 P GAIN=800 N-m/rad.
T4 D GAIN=50 N-m-sec/rad.
T4 SATURATING AT: 400 N-m.

EXISTENCE FACTORS:

EF1=0
EF2=1
EF3=1
EF4=0
EF5=0
EF6=1

Fig. 7. An RPR manipulator, initially at rest under the action of gravity, commanded to trace a circular path. Joints are actuated by classical PD control whose characteristics are listed below the figure with other system parameters and initial values of generalised coordinates and velocities.

velocities and accelerations at incremental time intervals are printed out and a neat zero acceleration and velocity on the relative motion of two successive links connected at a fictitious joint are observed. As there is no noise or dither on the constrained motion, integration process becomes fast and accurate. The 6 degrees of freedom system behaves like having only 3 degrees of freedom during the solution. For a complete robot simulation the mathematical models for the servo drives and the profiles of any external forces which might be acting on the end effector should be included in the equations of motion. The original simulation work a part of which is presented in this paper is capable of attaching intermediary links between any link pair, exerting some sort of active forces like balancing springs, dashpots and linear actuators. Their masses and variable inertias are included in the equations of motion. a graphics package produces a pictorial representation of the manipulator under consideration in animated form. The package as a whole and the method of fictitious degrees of freedom have proved to be an easy and neat way of providing flexibility in system definition and modelling.

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