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ON STARLIKE FUNCTIONS

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Let S denote the class of functions f analytic and univalent in the open disc $\{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$, and $S^*(\alpha)$ denote the set of starlike functions of order α $(0 \le \alpha \le 1)$ in S. In this paper, the results of William M. Causey and William L. White [*J. Math. Anal. Appl.* **64** (1978), 458-466] are generalized by the following:

THEOREM 1. *Let f, g and h belong to $S^*(\alpha)$, $S^*(\lambda)$ and $S^*(\gamma)$, respectively and let F be defined by*

$$** \qquad F(z)^a = \frac{a+c}{g(z)^c} \int_0^z h(t)^{c-1} f(t)^a \, dt$$

where $a, c \in N$. Then F belongs to $S^*(\beta)$ for $|z|$ less than a suitably chosen number.

THEOREM 2. *Let F, g and h belong to $S^*(\alpha)$ $S^*(\lambda)$ and $S^*(\gamma)$, respectively and f be given by **. Then f belongs to $S^*(\beta)$, for $|z|$ less than a suitably chosen number.*

1. Introduction

Let S denote the class of functions f analytic and univalent in the open disc $E\{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. Let the set of convex functions in S be denoted by K and let $S^*(\alpha)$ denote the set of starlike functions of order α $(0 \le \alpha \le 1)$ in S. If

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$\alpha = 1$ then $f(z) = z$. Let $P(\delta)$ denote the class of analytic functions $p(z)$ satisfying $\mathrm{Re}\{p(z)\} > \delta$ in E and normalized by $p(0) = 1$.

Libera [6] proved that if $f(z)$ is in $S^*(0)$, then the function $F(z) = \frac{2}{z}\int_0^z f(t)dt$, is likewise in $S^*(0)$. Subsequently, Livingston [7] obtained the converse of this result. Many generalizations and refinements of these two results were obtained by several authors in [1] to [9].

In 1978, Causey and White [3] obtained certain results concerning the univalence and starlikeness of normalized analytic functions in E defined by integral representations of the form

$$F(z)^a = \frac{a+c}{z^c}\int_0^z g(t)^{c-1}f(t)^a dt \quad (a,\ c \in N) ,$$

and

$$F(z) = \frac{2}{z}\int_0^z f(t)\left[\frac{g(t)}{h(t)}\right]^\eta p(t)dt , \quad \eta \geq 0 .$$

The purpose of this paper is to generalize the results obtained by Causey and White [3] and the results obtained in [4], [5]. Section 2 deals with statements of the results, whereas the last section is devoted to the proofs of the main results.

2. Statements of the results

THEOREM 1. *Let $f \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\gamma)$ and F be defined by*

$$(2.1) \qquad F(z)^a = \frac{a+c}{g(z)^c}\int_0^z h(t)^{c-1}f(t)^a dt \quad (a,\ c \in N) .$$

Then F belongs to $S^*(\beta)$ for $|z| < r_0$, where r_0 is given by

$$r_0 = \begin{cases} \dfrac{(1+c-\delta-c\lambda)+\{(\delta-1-c+c\lambda)^2-(2-c-a\beta)(a\beta-2\delta+2c\lambda-c)\}^{\frac{1}{2}}}{a\beta-2\delta+2c\lambda-c} & \text{when } a\beta-2\delta+2c\lambda-c \neq 0 , \\[2ex] \dfrac{(2-c-a\beta)}{2(1+c-c\lambda-\delta)} , & \text{when } a\beta-2\delta+2c\lambda-c = 0 , \end{cases}$$

where $\delta = a\alpha + (c-1)\gamma$.

COROLLARY [3]. *Let* $f, h \in K$. *Then* F *defined by*

$$F(z)^a = \frac{a+c}{z^c} \int_0^z h(t)^{c-1} f(t)^a dt$$

belongs to $S^*(0)$.

THEOREM 2. *Let* $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\gamma)$ *and* f *be defined by the relation* (2.1). *Then* f *belongs to* $S^*(\beta)$ *for* $|z| < r_1$ *where* r_1 *is the smallest positive root of the equation*

$$a(1-\beta)bDr^2 + [1-b-a(1-\beta)(b-D)]r - a(1-\beta) = 0$$

with

$$b = \frac{[a(2\alpha-1)+c(2\gamma-1)]}{(a+c)},$$

$$D = \frac{|d-\beta|}{(1-\beta)} = |d'|,$$

$$d = 2\alpha - 1 + \left\{\frac{c(2\lambda-1)+(1-c)(2\gamma-1)-1}{a}\right\}.$$

THEOREM 3. *Let* $F \in S^*(\alpha)$, $g \in S^*(\lambda)$, $h \in S^*(\gamma)$, $p \in P(0)$ *and* f *be given by*

$$F(z)^a = \frac{a+c}{z^c} \int_0^z f(t)^a \left[\frac{g(t)}{h(t)}\right]^\eta p(t)dt, \quad \eta \geq 0.$$

Then $f(z) \in S^*(\beta)$ *for* $|z| < r_2$ *where* r_2 *is the smallest positive root of the equation*

$$Ar^3 + Br^2 + C'r + a' = 0$$

where

$$A = -b'[a(2\alpha-1)+2\eta(\gamma-\lambda)+c-1-a\beta],$$

$$B = b'[2a(\alpha-1)+2\eta(\gamma+\lambda-2)+1] - a(2\alpha-1) - 2\eta(\gamma-\lambda) - c + a\beta + 2,$$

$$C' = a(2\alpha-2+b') + 2\eta(\gamma+\lambda-2) + b(c-2-a\beta) - 3,$$

$$b' = \frac{c-a+2a\alpha}{a+c},$$

$$a' = a(1-\beta) + c - 1.$$

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COROLLARY [3]. *Suppose* $F \in S^*(0)$, $g, h \in K$, $p \in P(0)$ *and* f *is given by*

$$F(z) = \frac{2}{z}\int_0^z f(t)\left[\frac{g(t)}{h(t)}\right]^{\eta} p(t)dt\ , \quad \eta \geq 0\ .$$

Then f *maps* $\{z : |z| < r(\eta)\}$ *one-to-one onto a starlike domain where*

$$r_0(\eta) = \frac{2\eta+5-\left(4\eta^2+20\eta+17\right)^{\frac{1}{2}}}{4}\ .$$

REMARK. Theorems 1 and 2 are respectively generalizations of Theorems 1 and 2 of Gupta and Jain [4] which correspond to $\alpha = \beta$, $\gamma = \lambda = 1$ and of Theorems 1 and 3 of Karunakaran [5] which correspond to $a = c = 1$.

3. Proofs of theorems

We first state certain lemmas which will be needed in proving the theorems stated in Section 2. The proofs of these lemmas are omitted for conciseness as they are analogous to the lemmas in [1] and the technique of their proofs is almost the same.

LEMMA 1. *If* $f \in S^*(\alpha)$ *and* $h \in S^*(\gamma)$, *then* H *defined by* $H(z) = h(z)^{c-1}f(z)^a$ *is* $(a+c-1)$*-valently starlike function of order* δ *where* $\delta = a\alpha + (c-1)\gamma$ *in* E .

LEMMA 2. *If* N *and* D *are regular functions in* E , $N(0) = D(0) = 0$, D *maps* E *onto a many sheeted region and* $N'/D' \in P(\delta)$, *then* $N/D \in P(\delta)$.

LEMMA 3. *If* J *is defined as* $J(z) = \int_0^z H(t)dt$, *then* J *is* $(a+c)$*-valently starlike function.*

Now we give the proofs of the theorems.

From (2.1), we have

$$\frac{azF'(z)}{F(z)} = 1 + \frac{zJ'(z)-J(z)}{J(z)} - c\,\frac{zg'(z)}{g(z)} \tag{3.1}$$

where $J(z)$ is as given in Lemma 3. Choosing $N(z) = zJ'(z) - J(z)$ and $D(z) = J(z)$, we have

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$$\frac{N'(z)}{D'(z)} = z\,\frac{H'(z)}{H(z)} .$$

Now by using Lemma 1, we get $N'/D' \in P(\delta)$. Further N and D satisfy the conditions of Lemma 2, therefore, $N/D \in P(\delta)$ so

$$\frac{N(z)}{D(z)} = \frac{zJ'(z)-J(z)}{J(z)} = \frac{1+(2\delta-1)r}{1+r} .$$

Further $g \in S^*(\lambda)$ and so relation (3.1) gives

$$a\ \mathrm{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geq 1 + \frac{1+(2\delta-1)r}{1+r} - \frac{1+(1-2\lambda)r}{1-r} .$$

Now the function $F \in S^*(\beta)$ for $|z| < r$ only when

$$1 + \frac{1+(2\delta-1)r}{1+r} - c\,\frac{1+(1-2\lambda)r}{1-r} - a\beta \geq 0 .$$

This condition easily gives $r < r_0$ where r_0 is as given in Theorem 1.

This result together with the fact that convex functions are starlike functions of order $\frac{1}{2}$ implies the corollary on taking $\beta = 0$ and $\lambda = 1$.

We now turn to the proof of Theorem 2. From (2.1) we have

$$f(z)^a = \frac{F(z)^a g(z)^c h(z)^{1-c}}{(a+c)z}\left[a\,\frac{zF'(z)}{F(z)} + c\,\frac{zg'(z)}{g(z)}\right] . \tag{3.2}$$

Now

$$\frac{zF'(z)}{F(z)} \in P(\alpha) , \quad \frac{zg'(z)}{g(z)} \in P(\lambda) ,$$

therefore

$$\frac{1}{a+c}\left[a\,\frac{zF'(z)}{F(z)} + c\,\frac{zg'(z)}{g(z)}\right] \in P\left(\frac{a\alpha+c\lambda}{a+c}\right)$$

and hence by a well known representation formula, we have

$$\frac{1}{a+c}\left[a\,\frac{zF'(z)}{F(z)} + c\,\frac{zg'(z)}{g(z)}\right] = \frac{1+bw(z)}{1+w(z)} \tag{3.3}$$

where

$$b = \frac{a(2\alpha-1)+c(2\lambda-1)}{a+c} .$$

Equating (3.2) and (3.3), we have

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$$f(z)^a = \frac{F(z)^a \cdot g(z)^c \cdot h(z)^{1-c}}{z} \left[\frac{1+bw(z)}{1+w(z)}\right] .$$

On differentiating logarithmically, we get

$$\frac{azf'(z)}{f(z)} = a\,\frac{zF'(z)}{F(z)} + c\,\frac{zg'(z)}{g(z)} + (1-c)\,\frac{zh'(z)}{h(z)} - 1 - \frac{(1-b)zw'(z)}{\{1+w(z)\}\{1+bw(z)\}}$$
$$= a\left[\frac{1+(2\alpha-1)w(z)}{1+w(z)}\right] + c\left[\frac{1+(2\lambda-1)w(z)}{1+w(z)}\right] + (1-c)\left[\frac{1+(2\gamma-1)w(z)}{1+w(z)}\right] - 1 - \frac{(1-b)zw'(z)}{\{1+w(z)\}\{1+bw(z)\}}$$

or

$$\frac{zf'(z)}{f(z)} - \beta = (1-\beta)\left[\frac{1+d'w(z)}{1+w(z)}\right] - \frac{(1-b)}{a}\,\frac{zw'(z)}{\{1+bw(z)\}\{1+w(z)\}}$$

where $d' = (d-\beta)/(1-\beta)$ and

$$d = 2\alpha - 1 + \left\{\frac{c(2\lambda-1)+(1-c)(2\gamma-1)-1}{a}\right\} .$$

Now $f \in S^*(\beta)$ only if $\operatorname{Re}[(zf'/f)-\beta] \geq 0$; this result after a little computation gives that $f \in S^*(\beta)$ if

$$(1-b)\,|zw'(z)| \leq a(1-\beta)\,|1+d'w(z)|\,|1+bw(z)| .$$

Since $|w'(z)| \leq \left(1-|w(z)|^2\right)/\left(1-r^2\right)$, therefore the above inequality will be satisfied if

$$r(1-b) \leq a(1-\beta)(1-Dr)(1+br)$$

with $D = |d'|$. On simplification this yields

$$a(1-\beta) - r[1-b-a(1-\beta)(b-D)] - a(1-\beta)bDr^2 \geq 0 .$$

Let $Q(r) = a(1-\beta) - r[1-b-a(1-\beta)(b-D)] - a(1-\beta)bDr^2$. Since $Q(0) = a(1-\beta) > 0$ and $Q'(r) < 0$, the positive root for which $Q(r) > 0$ must be less than the root of the polynomial $Q(r) = 0$. This gives the required root r_1 of Theorem 2 and hence the proof of Theorem 2 is complete.

The proof of Theorem 3 can be obtained on the lines of the proof of Theorem 2 and hence is omitted.

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