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Global well-posedness and decay estimates for three-dimensional compressible Navier–Stokes–Allen–Cahn systems

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We study the small data global well-posedness and time-decay rates of solutions to the Cauchy problem for three-dimensional compressible Navier–Stokes–Allen–Cahn equations via a refined pure energy method. In particular, the optimal decay rates of the higher-order spatial derivatives of the solution are obtained, the $\dot{H}^{-s}$ $(0 \leqslant s < \frac{3}{2})$ negative Sobolev norms is shown to be preserved along time evolution and enhance the decay rates.

Keywords: Compressible Navier–Stokes–Allen–Cahn equations; decay estimates; global well-posedness; pure energy method

2020 *Mathematics subject classification:* 35Q30; 76N10

1. Introduction

A fluid-mechanical theory for two-phase mixtures of fluids faces a well-known mathematical difficulty: the movement of the interfaces is naturally amenable to a Lagrangian description, while the bulk fluid flow is usually considered in the Eulerian framework [**10**]. Recently, the phase-field methods, or sometimes called the diffuse interface approaches, has been introduced to overcome this difficulty by postulating the existence of a 'diffuse' interface spread over a possibly narrow region covering the 'real' sharp interface boundary. Now, these models become one of the major tools to study a variety of interfacial phenomena. As the underlying physical problem still conceptually consists of sharp interfaces, the dynamics of the phase variable remains to a considerable extent purely fictitious [**10**]. Typically, different variants of Allen–Cahn, Cahn–Hilliard or other types of dynamics were used to describe the models, see previous studies [**1**, **14**, **20**, **21**] for example.

One of the well-known diffuse interface model for two-phase flow is the coupled Navier–Stokes/Allen–Cahn system. In this model, the interfaces between the phases are assumed to be of 'diffuse' nature, that is, sharp interfaces are replaced by narrow transition layers. These regions are located by a phase field variable χ governed by the Allen–Cahn equation, while the dynamics are described by the Navier–Stokes equations. In [2], Blesgen proposed this model and developed a thermodynamically and mechanically consistent set of partial differential equations extending the Navier–Stokes equations to a compressible binary Allen–Cahn mixture. As Blesgen [2] has pointed out before, the Navier–Stokes–Allen–Cahn model can be seen as a first step towards incorporating transport mechanism into the description of phase-formation processes.

During the past few years, many authors studied the properties of solutions for the incompressible Navier–Stokes system with matched density. For example, Favre and Schimperna [9] considered the existence of weak solution in three-dimensional (3D) and well-posedness of strong solution in two-dimensional (2D), Xu *et al.* [29] studied the existence of axisymmetric solutions, Zhao *et al.* [31] studied the vanishing viscosity limit, Gal and Grasselli [11, 12] considered the asymptotic behaviour and attractors. For the system based on the incompressible Navier–Stokes system with different densities, Li *et al.* [18, 19] studied the local well-posedness and blow-up criterion of strong solutions. Moreover, by using an energetic variational approach, Jiang *et al.* [15] derived a different model of Navier–Stokes–Allen–Cahn, then proved the existence of weak solutions in 3D, the well-posedness of strong solutions in 2D, and studied the long-time behaviour of the strong solutions.

As far as we know, there are also some classical results are available for the initial-boundary value problem of compressible Navier–Stokes–Allen–Cahn system. In [10], for the initial-boundary value problem of 3D compressible model, by using Faedo–Galerkin approximation, Feireisl *et al.* proved the existence of global-in-time weak solutions without any restriction on the size of initial data for the exponent of pressure $\gamma > 6$. This result was extended to $\gamma > 2$ in Chen *et al.* [4]. Moreover, supposed that the boundary conditions of the model are of mixed type (Neumann–Dirichlet) and may be nonhomogeneous, the density is Hölder continuous for instance, Kotschote [17] established the local well-posedness of solutions for sufficiently smooth data, in a general C^2 bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$. Ding *et al.* [5, 6] proved the existence and uniqueness of global classical solution, the existence of weak solutions and the existence of unique global strong solution of the initial-boundary value problem of one-dimensional (1D) compressible Navier–Stokes–Allen–Cahn systems for the initial data without vacuum states. Recently, Zhu and his authors [22, 30] paid their attention to the large time behaviour of solutions for the Cauchy problem and inflow problem of 1D compressible Navier–Stokes–Allen–Cahn equations, respectively. In [22], Luo, Yin and Zhu proved that the solutions to the Cauchy problem of 1D compressible Navier–Stokes/Allen–Cahn system tend time-asymptotically to the rarefaction wave, where the strength of the rarefaction wave is not required to be small. In addition, for the inflow problem of 1D compressible Navier–Stokes/Allen–Cahn system, Yin and Zhu [30] analysed the large-time behaviour of the solution, proved the existence of the stationary solution and the asymptotic stability of the nonlinear wave. However, to our knowledge, there's no result on the Cauchy problem for the 3D compressible Navier–Stokes–Allen–Cahn system.

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In this paper, we consider the Cauchy problem of 3D compressible Navier–Stokes–Allen–Cahn equations [2, 17, 28]

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla p - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u \\ \qquad\qquad = -\ell\nabla\cdot\left(\nabla\chi\otimes\nabla\chi = -\dfrac{|\nabla\chi|^2}{2}\mathbb{I}_3\right), \\ \partial_t(\rho\chi) + \mathrm{div}(\rho\chi u) = -\omega, \\ \rho\omega = -\ell\Delta\chi + \rho\dfrac{\partial\Phi(\rho,\chi)}{\partial\chi}, \\ (\rho,u,\chi)|_{t=0} = (\rho_0,u_0,\chi_0), \end{cases} \tag{1.1}$$

where ρ denotes the total fluid density, u implies the mean velocity of the fluid mixture, χ is the concentration of one selected constituent and the pressure $p = p(\rho)$ is a smooth function in a neighbourhood of 1 with $p'(1) = 1$, respectively. Moreover, ω is the chemical potential and $\mathbb{I}_3$ denotes a 3×3 identity matrix. μ and λ are two viscosity coefficients, which satisfy

$$\mu > 0, \quad 2\mu + 3\lambda \geqslant 0.$$

The specific free energy $f(\rho,\chi)$ can be defined as

$$\Phi(\rho,\chi) = \frac{\rho^{\gamma-1}}{\gamma-1} + \frac{1}{\ell}\left(\frac{\chi^4}{4} - \frac{\chi^2}{2}\right), \tag{1.2}$$

where $\gamma > 1$ is the adiabatic constant and the constant $\sqrt{\ell}$ denotes the thickness of the interfacial region. In this paper, we take

$$\rho\frac{\partial\Phi(\rho,\chi)}{\partial\chi} = \frac{\rho}{\ell}(\chi^3 - \chi).$$

For simplicity, we let $\ell \equiv 1$ throughout the rest of this paper.

Notation 1.1. In the following, we use $H^k(\mathbb{R}^3$ $(k\in\mathbb{R})$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1\leqslant p\leqslant\infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. We also introduce the homogeneous negative index Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$:

$$\dot{H}^{-s}(\mathbb{R}^3) := \{f\in L^2(\mathbb{R}^3) : \||\xi|^{-s}\hat{f}(\xi)\|_{L^2} < \infty\}$$

endowed with the norm $\|f\|_{\dot{H}^{-s}} := \||\xi|^{-s}\hat{f}(\xi)\|_{L^2}$. The symbol ∇^l with an integer $l\geqslant 0$ stands for the usual spatial derivatives of order l. For instance, define

$$\nabla^l z = \{\partial_x^\alpha z_i | |\alpha| = l, i = 1,2,3\}, \quad z = (z_1,z_2,z_3).$$

If $l < 0$ or l is not a positive integer, ∇^l stands for Λ^l defined by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s\hat{f}(\xi)\,\mathrm{e}^{2\pi i x\cdot\xi}\,\mathrm{d}\xi, \tag{1.3}$$

where $\hat{f}$ is the Fourier transform of f (see [13]). Moreover, we use the notation $A\lesssim B$ to mean that $A\leqslant cB$ for a universal constant $c > 0$ that only depends on the parameters coming from the problem and the indexes N and s coming from the regularity on the data.

For this system, we first show the local well-posedness in the following lemma.

LEMMA 1.1. *Let the initial data*$(\varrho_0, u_0, \chi_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$. *Then there exit positive constants* $\nu_0 > 0$ *and* $T > 0$ *depending only on* $\|(\varrho_0, u_0, \chi_0, \nabla\chi_0)\|_{H^3}$ *such that the system* (1.4) *has a unique solution*

$$(\varrho, u, \chi, \nabla\chi) \in L^\infty(0, T; H^3), \quad (u, \chi, \nabla\chi) \in L^2(0, T; H^4),$$

satisfying

$$\begin{aligned}&\|(\varrho, u, \chi, \nabla\chi)(t)\|_{H^3}, \left(\nu_0 \int_0^t \|\nabla(u, \chi, \nabla\chi)(\tau)\|_{H^3}^2 \, \mathrm{d}\tau\right)^{1/2} \\ &\quad \leqslant C(\|(\varrho_0, u_0, \chi_0, \nabla\chi_0)\|_{H^3}), \quad \forall t \in [0, T].\end{aligned}$$

Lemma 1.1 can be proved by using the Banach fixed point theorem. One will sketch the proof in the second part of § 2.

Next, denote $\varrho = \rho - 1$, rewrite (1.1) in the perturbation form as

$$\begin{cases} \varrho_t + \operatorname{div} u = -\varrho \operatorname{div} u - u \cdot \nabla\varrho, \\ u_t - \mu\Delta u - (\mu + \lambda)\nabla\operatorname{div} u + \nabla\varrho \\ \quad = -u \cdot \nabla u - h(\varrho)(\mu\Delta u + (\mu + \lambda)\nabla\operatorname{div} u) \\ \qquad -g(\varrho)\nabla\varrho - \phi(\varrho)\operatorname{div}\left(\nabla\chi \otimes \nabla\chi - \dfrac{|\nabla\chi|^2}{2}\mathbb{I}_3\right), \\ \chi_t - \Delta\chi = -u \cdot \nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3 - \chi), \\ (\varrho, u, \chi)|_{t=0} = (\varrho_0, u_0, \chi_0) = (\rho_0 - 1, u_0, \chi_0), \end{cases} \tag{1.4}$$

where

$$h(\varrho) = \frac{\varrho}{\varrho + 1}, \quad g(\varrho) = \frac{p'(\varrho + 1)}{\varrho + 1} - 1, \quad \phi(\varrho) = \frac{1}{\varrho + 1}, \quad \varphi(\varrho) = \frac{\varrho(\varrho + 2)}{(\varrho + 1)^2},$$

The main purpose of this paper is to study the small initial data global well-posedness and optimal decay estimates of strong solutions for system (1.1) in the whole space $\mathbb{R}^3$. We use a general energy method, Kato–Ponce inequality and the Gagliardo–Nirenberg interpolation techniques to obtain the global well-posedness and the optimal time-decay rates of the solution to system (1.1) when the initial data are sufficiently small.

Next, one state the global well-posedness and decay estimates of solutions for system (1.4):

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THEOREM 1.1. *Let $N \geqslant 3$, assume that $(\varrho_0, u_0, \chi_0) \in H^N(\mathbb{R}^3) \times H^N(\mathbb{R}^3) \times H^{N+1}(\mathbb{R}^3)$, and there exists a constant $\delta_0 > 0$ such that if*

$$\|\varrho_0\|_{H^3} + \|u_0\|_{H^3} + \|\chi_0\|_{H^3} + \|\nabla\chi_0\|_{H^3} \leqslant \delta_0, \tag{1.5}$$

then there exists a unique global solution (ϱ, u, χ) satisfying that for all $t \geqslant 0$,

$$\begin{aligned}&\|\varrho(t)\|_{H^N}^2 + \|u(t)\|_{H^N}^2 + \|\chi\|_{H^N}^2 + \|\nabla\chi\|_{H^N}^2\\&\quad + \int_0^t (\|\nabla u(s)\|_{H^N}^2 + \|\chi\|_{H^N}^2 + \|\nabla\chi\|_{H^N}^2)\,\mathrm{d}s\\&\qquad \leqslant C(\|\varrho_0\|_{H^N}^2 + \|u_0\|_{H^N}^2 + \|\chi_0\|_{H^N}^2 + \|\nabla\chi_0\|_{H^N}^2).\end{aligned} \tag{1.6}$$

If further, $(\varrho, u_0, \chi_0, \nabla\chi_0) \in \dot{H}^{-s}(\mathbb{R}^3)$ for some $s \in [0, \frac{3}{2})$, then for all $t \geqslant 0$,

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\chi(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla\chi(t)\|_{L^2}^2 \leqslant C, \tag{1.7}$$

and

$$\begin{aligned}&\|\nabla^l\varrho(t)\|_{H^{N-l}} + \|\nabla^l u(t)\|_{H^{N-l}} + \|\nabla^l\chi(t)\|_{H^{N-l}}\\&\quad + \|\nabla^{l+1}\chi(t)\|_{H^{N-l}} \leqslant C(1+t)^{-(l+1)/2}, \quad \text{for } l = 0, 1, \ldots, N-1.\end{aligned} \tag{1.8}$$

Note that the Hardy–Littlewood-Sobolev theorem implies that for $p \in (1, 2]$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{3}{2})$. Then, on the basis of theorem 1.1, we easily obtain the following corollary of the optimal decay estimates.

COROLLARY 1.1. *Under the assumptions of theorem 1.1, if we replace the $\dot{H}^{-s}(\mathbb{R}^3)$ assumption by*

$$(\varrho, u_0, \chi_0, \nabla\chi_0) \in L^p(\mathbb{R}^3), \quad 1 < p \leqslant 2,$$

then the following decay estimate holds:

$$\begin{aligned}&\|\nabla^l\varrho(t)\|_{H^{N-l}} + \|\nabla^l u(t)\|_{H^{N-l}} + \|\nabla^l\chi(t)\|_{H^{N-l}} + \|\nabla^{l+1}\chi(t)\|_{H^{N-l}}\\&\quad \leqslant C(1+t)^{-[3/2(1/p-1/2)+l/2]}, \quad \text{for } l = 0, 1, \ldots, N-1.\end{aligned} \tag{1.9}$$

REMARK 1.1. The compressible Hall-magnetohydrodynamics system [7] has the following form:

$$\begin{cases}\rho_t + \operatorname{div}(\rho u) = 0,\\(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u = (\nabla \times b) \times b,\\b_t + \nabla \times (b \times u) + \nabla \times \left(\dfrac{(\nabla \times b) \times b}{\rho}\right) = \Delta b,\\\operatorname{div} b = 0,\\(\rho, u, b)|_{t=0} = (\rho_0, u_0, b_0),\end{cases} \tag{1.10}$$

where ρ is the density of the fluid, u is the fluid velocity field and b is the magnetic field. In [7], the authors considered the local well-posedness, small initial data global well-posedness and large time behaviour of strong solutions for system (1.10). It is worth pointing out that the structures of systems (1.1) and (1.10) are different.

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Especially, the principle part of (1.10)$_3$ is a linear term $-\Delta b$, however, it is a nonlinear function $-(\ell/\rho)\Delta\chi$ in (1.1)$_3$. One can't use the tools of the dissipative equation with linear principle part to study the properties of this equation. Here, we borrow a linear term from the right-hand side of (1.1)$_3$, rewrite this equation as a second-order PDE

$$\chi_t - \Delta\chi = -u\cdot\nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3 - \chi), \tag{1.11}$$

where $\varrho = \rho - 1$, $\phi(\varrho) = 1/(\varrho + 1)$ and $\varphi(\varrho) = (\varrho(\varrho+2))/((\varrho+1)^2)$. It is worth pointing out that there exists a linear principle part $-\Delta\chi$ in (1.11). Then, by using the pure energy method, one can obtain suitable energy estimates. Moreover, the right-hand term of system (1.1)$_2$ is different from system (1.10)$_2$. One can use the usual Sobolev embedding inequality, Kato–Ponce inequality together with the Gagliardo–Nirenberg interpolation techniques to deal with this term.

Remark 1.2. There are some classical results on the initial-boundary value problem of compressible Navier–Stokes–Allen–Cahn equations in bounded domains, for example [**3**, **5**, **6**, **8**, **17**, **24**] and references therein. However, only a few results related to the Cauchy problem. In terms of well-posedness of compressible Navier–Stokes–Allen–Cahn equations on bounded domains, the known constructions make use of some compactness properties in an essential manner, and more specifically of the compact Sobolev embeddings. However, if we consider the equations in the whole space, such properties are no longer valid, it is more difficult to obtain suitable *a priori* estimates to develop a general theory of well-posedness. In this paper, we adopt the Gagliardo–Nirenberg inequality in $\mathbb{R}^3$ (lemma 2.2), negative Sobolev norm estimates and pure energy method, overcome this difficulty, obtain the global well-posedness and the decay rate of higher-order derivatives of strong solutions. We remark that since the decay estimate is same as the heat equation, it is optimal.

The structure of this paper is organized as follows. In § 2, we introduce some preliminary results and give a brief proof on the local well-posedness. Section 3 is devoted to establish some refined energy estimates for the solution. In § 4, we derive the evolution of the negative Sobolev norms of the solution. Finally, the proof of theorem 1.1 is postponed in § 5.

2. Preliminaries

2.1. Useful inequalities

In this section, we introduce some helpful results in $\mathbb{R}^3$.

First of all, the following Kato–Ponce inequality is of great importance in our paper.

Lemma 2.1 [**16**]. *Let $1 < p < \infty$, $s > 0$. There exists a positive constant C such that*

$$\|\nabla^s(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\nabla^s g\|_{L^{p_2}} + \|\nabla^s f\|_{L^{q_1}}\|g\|_{L^{q_2}}, \tag{2.1}$$

where $p_2, q_2 \in (1,\infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

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The following Gagliardo–Nirenberg inequality in $\mathbb{R}^3$ was proved in [23].

LEMMA 2.2 [23]. *Let* $0 \leqslant m, \alpha \leqslant l$*, then we have*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^q}^{1-\theta} \|\nabla^l f\|_{L^r}^{\theta}, \tag{2.2}$$

where $\theta \in [0, 1]$ *and* α *satisfies*

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta. \tag{2.3}$$

Here, when $p = \infty$*, we require that* $0 < \theta < 1$*.*

We recall the following commutator estimate:

LEMMA 2.3 [26]. *Let* $m \geqslant 1$ *be an integer and define the commutator*

$$[\nabla^m, f]g = \nabla^m(fg) - f\nabla^m g. \tag{2.4}$$

Then, the following inequality holds:

$$\|[\nabla^m, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{2.5}$$

where $p,\ p_2,\ p_3 \in (1, \infty)$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Wang [26], Wei, Li and Yao [27] introduced the following result:

LEMMA 2.4. *Suppose that* $\|\varrho\|_{L^\infty} \leqslant 1$ *and* $p > 1$*. Let* $f(\varrho)$ *be a smooth function of* ϱ *with bounded derivatives of any order, then for any integer* $n \geqslant 1$*, the following inequality holds:*

$$\|\nabla^m(f(\varrho))\|_{L^p} \lesssim \|\nabla^m \varrho\|_{L^p}. \tag{2.6}$$

We also introduce the Hardy–Littlewood–Sobolev theorem, which implies the following L^p type inequality.

LEMMA 2.5 [13, 25]. *Let* $0 \leqslant s < \frac{3}{2}$, $1 < p \leqslant 2$ *and* $\frac{1}{2} + \frac{s}{3} = \frac{1}{p}$*, then*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}. \tag{2.7}$$

The special Sobolev interpolation lemma will be used in the proof of theorem 1.1.

LEMMA 2.6 [25, 26]. *Let* $s, k \geqslant 0$ *and* $l \geqslant 0$*, then*

$$\|\nabla^l f\|_{L^2} \leqslant \|\nabla^{l+k} f\|_{L^2}^{1-\theta} \|f\|_{\dot{H}^{-s}}^{\theta}, \quad \text{with } \theta = \frac{k}{l+k+s}. \tag{2.8}$$

2.2. Local well-posedness

In the following, we give a brief proof of lemma 1.1 on the local well-posedness of system (1.1), which is similar to the arguments in [7]. For completeness, we outline the proof here.

First, consider the following system

$$\begin{cases}\rho_t + \nabla\cdot(\rho\tilde{u}) = 0,\\ \partial_t(\rho u) + \nabla\cdot(\rho\tilde{u}\otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div}\, u\\ \quad -\ell\nabla\cdot\left(\nabla\chi\otimes\nabla\chi - \dfrac{|\nabla\chi|^2}{2}\mathbb{I}_3\right),\\ \partial_t(\rho\chi) + \mathrm{div}(\rho\chi\tilde{u}) = -\omega,\\ \rho\omega = -\ell\Delta\chi + \dfrac{\rho}{\ell}[(\tilde{\chi})^2 - 1]\chi,\\ (\rho, u, \chi)|_{t=0} = (\rho_0, u_0, \chi_0),\end{cases} \tag{2.9}$$

where $(\tilde{u}, \tilde{\varphi}) \in R_T^*$ are known functions with $(\tilde{u}, \tilde{\varphi})(x, 0) = (u_0, \varphi_0)$ and

$$R_T^* = \left\{ v \in H^3 | \sup_{0\leqslant t\leqslant T} \|v\|_{H^3}^2 + \int_0^T \|\nabla v\|_{H^3}^2 \,\mathrm{d}s \leqslant R \right\},$$

where $R > 1$ and $T > 0$ will be decided later.

Note that $(2.9)_1$ is linear with regular $\tilde{u}$. The existence and uniqueness are well-known and we also have

$$0 < \underline{\rho} \leqslant \rho, \quad \|\rho(x,t)\|_{H^3} + \|\rho_t\|_{H^2} \leqslant C\exp\left(C\int_0^t \|\nabla\tilde{u}\|_{H^2}\,\mathrm{d}s\right).$$

Then, if T is suitably small, we can obtain the estimates for ρ. Next, taking advantage of the estimate for ρ and the classical theory of linear parabolic system, one can get the existence and uniqueness of $(u, \varphi, \nabla\varphi)$ by $(2.9)_2$–$(2.9)_4$.

Define a fixed point map $F : (\tilde{u}, \tilde{\varphi}, \tilde{\nabla\varphi}) \in R_T^* \times R_T^* \times R_T^* \to (u, \varphi, \nabla\varphi)$. We will prove that the map F mapping $R_T^* \times R_T^* \times R_T^*$ into itself for suitable constant R and small time T and F is a contraction mapping on $R_T^* \times R_T^* \times R_T^*$. Thus, F has a unique fixed point in $R_T^* \times R_T^* \times R_T^*$ and prove the local well-posedness result.

First, in order to prove F mapping $R_T^* \times R_T^* \times R_T^*$ into itself, one need to establish some *a priori* estimates on $(u, \varphi, \nabla\varphi)$. In fact, simple calculations show that

$$\begin{aligned}&\|(u, \varphi, \nabla\varphi)\|_{L^\infty(0,T;H^3)} + \|(u_t, \varphi_t, \nabla\varphi_t)\|_{L^\infty(0,T;H^1)}\\ &\quad + \|(u, \varphi, \nabla\varphi)\|_{L^2(0,T;H^4)} + \|(u_t, \varphi_t, \nabla\varphi_t)\|_{L^2(0,T;H^2)} \leqslant C,\end{aligned}$$

for sufficiently small $T \in (0, 1]$, and thus F maps $R_T^* \times R_T^* \times R_T^*$ into $R_T^* \times R_T^* \times R_T^*$.

Second, in order to show F is contracted in the sense of weaker norm, one suppose that $(\rho_i, u_i, \varphi_i,)$ $(i = 1, 2)$ are the solutions to system (2.9) corresponding to $(\tilde{u}_i, \tilde{\varphi}_i)$.

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Denote

$$\rho := \rho_1 - \rho_2, \quad u := u_1 - u_2, \quad \varphi := \varphi_1 - \varphi_2,$$
$$\omega := \omega_1 - \omega_2, \quad \tilde{u} := \tilde{u}_1 - \tilde{u}_2, \quad \tilde{\varphi} := \tilde{\varphi}_1 - \tilde{\varphi}_2.$$

We obtain

$$\begin{cases} \rho_t + \nabla\cdot(\rho\tilde{u}_1) = -\nabla\cdot(\rho_2\tilde{u}), \\ \rho_1\partial_t u + \rho_1\tilde{u}_1\cdot\nabla u + \nabla(p(\rho_1) - p(\rho_2)) - \mu\Delta u - (\lambda+\mu)\nabla\nabla\cdot u \\ \quad = -\rho\partial_t u_2 - (\rho_1\tilde{u} - \rho_2\tilde{u}_2)\cdot\nabla u_2 - \ell\nabla\cdot\left(\nabla\chi_1\otimes\nabla\chi_1 - \dfrac{|\nabla\chi_1|^2}{2}\mathbb{I}_3\right) \\ \qquad p + \ell\nabla\cdot\left(\nabla\chi_2\otimes\nabla\chi_2 - \dfrac{|\nabla\chi_2|^2}{2}\mathbb{I}_3\right), \\ \rho_1\partial_t\chi + \rho\partial_t\chi_2 + \nabla\cdot(\rho\chi_1\tilde{u}_1 + \rho_2\chi\tilde{u}_1 + \rho_2\chi_2\tilde{u}) = -\omega, \\ \rho_1\omega + \rho\omega_2 = -\ell\Delta\chi + \dfrac{1}{\ell}\rho(\tilde{\chi}_1+1)(\tilde{\chi}_1-1)\chi_1 + \dfrac{1}{\ell}\rho_2(\tilde{\chi}+1)(\tilde{\chi}_1-1)\chi_1 \\ p + \dfrac{1}{\ell}\rho_2(\tilde{\chi}_2+1)(\tilde{\chi}-1)\chi_1 + \dfrac{1}{\ell}\rho_2(\tilde{\chi}_2+1)(\tilde{\chi}_2-1)\chi_2. \end{cases} \tag{2.10}$$

Testing $(2.10)_1$, $(2.10)_2$ by ρ, u, testing $(2.10)_3$ by φ and $\Delta\varphi$ respectively, after simple calculations, one obtains

$$\begin{aligned} &\|F(\tilde{u}_1, \tilde{\varphi}_1, \tilde{\nabla\varphi}_1) - F(\tilde{u}_2, \tilde{\varphi}_2, \tilde{\nabla\varphi}_2)\|_{L^2(0,T;H^1)} \\ &\quad \leqslant \theta\|(\tilde{u}_1 - \tilde{u}_2, \tilde{\varphi}_1 - \tilde{\varphi}_2, \nabla\tilde{\varphi}_1 - \nabla\tilde{\varphi}_2\|_{L^2(0,T;H^1)}, \end{aligned}$$

where $\theta \in (0,1)$ is a constant and $T \in (0,1]$ is a small time. The above inequality implies that F is contracted in the sense of weaker norm.

Next, by using Banach fixed point theorem, we complete the proof of lemma 1.1.

3. Energy estimates

In this section, we derive the *a priori* energy estimates for system (1.4). Suppose that there exists a small positive constant $\delta > 0$ such that

$$\sqrt{\mathcal{E}_0^3(t)} = \|\varrho(t)\|_{H^3} + \|u(t)\|_{H^3} + \|\chi(t)\|_{H^3} + \|\nabla\chi(t)\|_{H^3} \leqslant \delta, \tag{3.1}$$

which, together with Sobolev's inequality, yields directly

$$\frac{1}{2} \leqslant \varrho + 1 \leqslant 2.$$

Simple calculations show that

$$|h(\varrho)|, |g(\varrho)|| \leqslant C|\varrho|, \tag{3.2}$$

and

$$|\phi^{(l)}(\varrho)|, |\varphi^{(l)}(\varrho)|, |h^{(k)}(\varrho)|, |g^{(k)}(\varrho)| \leqslant C \text{ for any } l \geqslant 0,\ k \geqslant 1. \tag{3.3}$$

Next, we establish the following energy estimates including ϱ, u and χ themselves.

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LEMMA 3.1. *If $\sqrt{\mathcal{E}_0^3(t)} \leqslant \delta$, then for $k = 0, 1, \dots, N-1$, we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^k \varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k \chi\|_{L^2}^2 + \|\nabla^k \nabla\chi\|_{L^2}^2) \\
&\quad + C(\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2 + \|\nabla^{k+1}\nabla\chi\|_{L^2}^2) \\
&\qquad \lesssim (\delta + \delta^3)(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2 + \|\nabla^{k+1}\nabla\chi\|_{L^2}^2). \qquad (3.4)
\end{aligned}
$$

Proof. Taking k-th spatial derivatives to $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$, $k+1$th spatial derivatives to $(1.1)_3$, multiplying the resulting identities by $\nabla^k\varrho$, $\nabla^k u$, $\nabla^k\chi$ and $\nabla^{k+1}\chi$ respectively, summing up and then integrating over $\mathbb{R}^3$, we derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\nabla^k\varrho|^2 + |\nabla^k u|^2 + |\nabla^k\chi|^2 + |\nabla^{k+1}\chi|^2)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}(|\nabla^{k+1}u|^2 + |\nabla^{k+1}\chi|^2 + |\nabla^{k+2}\chi|^2)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^3}\Bigg[\nabla^k(-\varrho\mathrm{div}\, u - u\cdot\nabla\varrho)\cdot\nabla^k\varrho \\
&\qquad - \nabla^k\Bigg[u\cdot\nabla u + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}\, u) \\
&\qquad + g(\varrho)\nabla\varrho + \phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\Bigg]\cdot\nabla^k u \\
&\qquad + \nabla^k\left(-u\cdot\nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3-\chi)\right)\cdot\nabla^k\chi \\
&\qquad + \nabla^{k+1}\left(-u\cdot\nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3-\chi)\right)\cdot\nabla^{k+1}\chi\Bigg]\,\mathrm{d}x \\
&\quad = \sum_{i=1}^{12} I_i. \qquad (3.5)
\end{aligned}
$$

The right-hand side terms of (3.5) will be estimated one by one in the following. The main idea is that we will carefully interpolate the spatial derivatives between the higher-order derivatives and the lower-order derivatives to bound these nonlinear terms by the right-hand side of (3.4). First, for I_1, by using Kato–Ponce inequality (lemma 2.1), Gagliardo–Nirenberg inequality (lemma 2.2) and Sobolev embedding theorem, we can estimate as

$$
\begin{aligned}
I_1 &\lesssim \|\nabla^k\varrho\|_{L^6}\|\nabla^k(\varrho\nabla\cdot u)\|_{L^{6/5}} \lesssim \|\nabla^k\varrho\|_{L^6}(\|\nabla^k\varrho\|_{L^3}\|\nabla u\|_{L^2} + \|\varrho\|_{L^3}\|\nabla^{k+1}u\|_{L^2}) \\
&\lesssim \|\nabla^{k+1}\varrho\|_{L^2}\Big(\|\nabla^{k+1}\varrho\|_{L^2}^{k/(k+1/2)}\|\Lambda^{1/2}\varrho\|_{L^2}^{(1/2)/(k+1/2)}\| \\
&\quad \times\Lambda^{1/2}u\|_{L^2}^{k/(k+1/2)}\|\nabla^{k+1}u\|_{L^2}^{(1/2)/(k+1/2)} + \|\Lambda^{1/2}\varrho\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\Big) \\
&\lesssim (\|\Lambda^{1/2}\varrho\|_{L^2} + \|\Lambda^{1/2}u\|_{L^2})(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}) \\
&\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}). \qquad (3.6)
\end{aligned}
$$

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Similarly, by using Kato–Ponce inequality (lemma 2.1), Gagliardo–Nirenberg inequality (lemma 2.2) and Sobolev embedding theorem again, we estimate the terms I_2 and I_3 as

$$\begin{aligned} I_2 &\lesssim \|\nabla^k \varrho\|_{L^6}\|\nabla^k(u\cdot\nabla\varrho)\|_{L^{6/5}} \lesssim \|\nabla^k\varrho\|_{L^6}(\|\nabla^k u\|_{L^3}\|\nabla\varrho\|_{L^2} + \|u\|_{L^3}\|\nabla^{k+1}\varrho\|_{L^2}) \\ &\lesssim \|\nabla^{k+1}\varrho\|_{L^2}\Big(\|\nabla^{k+1}u\|_{L^2}^{k/(k+1/2)}\|\Lambda^{1/2}u\|_{L^2}^{(1/2)/(k+1/2)}\| \\ &\quad\times \Lambda^{1/2}\varrho\|_{L^2}^{k/(k+1/2)}\|\nabla^{k+1}\varrho\|_{L^2}^{(1/2)/(k+1/2)} + \|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}\Big) \\ &\lesssim (\|\Lambda^{1/2}\varrho\|_{L^2} + \|\Lambda^{1/2}u\|_{L^2})(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}) \\ &\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} I_3 &\lesssim \|\nabla^k u\|_{L^6}\|\nabla^k(u\cdot\nabla u)\|_{L^{6/5}} \lesssim \|\nabla^k u\|_{L^6}(\|\nabla^k u\|_{L^3}\|\nabla u\|_{L^2} + \|u\|_{L^3}\|\nabla^{k+1}u\|_{L^2}) \\ &\lesssim \|\nabla^{k+1}u\|_{L^2}\Big(\|\nabla^{k+1}u\|_{L^2}^{k/(k+1/2)}\|\Lambda^{1/2}u\|_{L^2}^{(1/2)/(k+1/2)}\| \\ &\quad\times \Lambda^{1/2}u\|_{L^2}^{k/(k+1/2)}\|\nabla^{k+1}u\|_{L^2}^{(1/2)/(k+1/2)} + \|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\Big) \\ &\lesssim \|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \lesssim \delta\|\nabla^{k+1}u\|_{L^2}. \end{aligned} \tag{3.8}$$

Next, for the term I_4, we do the approximation to simplify the presentations by

$$I_4 = -\int_{\mathbb{R}^3}\nabla^k[h(\varrho)(\nu\Delta u + \eta\nabla\operatorname{div} u)]\cdot\nabla^k u\,\mathrm{d}x \approx -\int_{\mathbb{R}^3}\nabla^k[h(\varrho)\nabla^2 u]\cdot\nabla^k u\,\mathrm{d}x. \tag{3.9}$$

If $k=0$, on the basis of the fact (3.2), Hölder's together with Sobolev embedding theorem, we deduce that

$$\begin{aligned} I_4 &\approx -\int_{\mathbb{R}^3} h(\varrho)\nabla^2 u\cdot u\,\mathrm{d}x \lesssim \|\nabla h(\varrho)\|_{L^2}\|\nabla u\|_{L^3}\|u\|_{L^6} + \|\varrho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u\|_{L^2} \\ &\lesssim (\|\nabla u\|_{L^3} + \|\varrho\|_{L^\infty})(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\ &\lesssim (\|\Lambda^{3/2}u\|_{L^2} + \|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2})(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\ &\lesssim \delta(\|\nabla\varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \end{aligned} \tag{3.10}$$

If $k=1$, by the fact (3.2), integrating by parts, one obtains

$$\begin{aligned} I_4 &\approx -\int_{\mathbb{R}^3}\nabla[h(\varrho)\nabla^2 u]\cdot\nabla u\,\mathrm{d}x \approx \int_{\mathbb{R}^3} h(\varrho)|\nabla^2 u|^2\,\mathrm{d}x \\ &\lesssim \|h(\varrho)\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2 \lesssim \|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^2 \lesssim \delta\|\nabla^2 u\|_{L^2}^2. \end{aligned} \tag{3.11}$$

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If $k \geqslant 2$, by using Hölder's inequality, lemmas 2.1 and 2.4 and Sobolev embedding theorem, we deduce that

$$\begin{aligned}
I_4 &\lesssim \|\nabla^{k+1}u\|_{L^2}\|\nabla^{k-1}[h(\varrho)\nabla^2 u]\|_{L^2}\\
&\lesssim \|\nabla^{k+1}u\|_{L^2}(\|\nabla^k\varrho\|_{L^2}\|\Lambda^{5/2}u\|_{L^2} + \|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^{k+1}u\|_{L^2})\\
&\lesssim \|\nabla^{k+1}u\|_{L^2}\left(\|\nabla^{k+1}\varrho\|_{L^2}^{(k-3/2)/(k-1/2)}\|\Lambda^{3/2}\varrho\|_{L^2}^{1/(k-1/2)}\|\nabla^{k+1}u\|_{L^2}^{1/(k-1/2)}\|\right.\\
&\quad\left.\times\ \Lambda^{3/2}u\|_{L^2}^{(k-3/2)/(k-1/2)} + \|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^{k+1}u\|_{L^2}\right)\\
&\lesssim (\|\Lambda^{3/2}\varrho\|_{L^2} + \|\Lambda^{3/2}u\|_{L^2} + \|\nabla\varrho\|_{L^2} + \|\Delta\varrho\|_{L^2})(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2)\\
&\lesssim \delta(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2).
\end{aligned} \tag{3.12}$$

Moreover, applying lemma 2.4, Hölder's inequality, the Kato–Ponce inequality (lemma 2.1) together with Sobolev embedding theorem, the term I_5 can be bounded as

$$\begin{aligned}
I_5 &= -\int_{\mathbb{R}^3}\nabla^k(g(\varrho)\nabla\varrho)\cdot\nabla^k u\,\mathrm{d}x \lesssim \|\nabla^k u\|_{L^3}\|\nabla^k(g(\varrho)\nabla\varrho)\|_{L^{3/2}}\\
&\lesssim \|\nabla^k u\|_{L^3}(\|\nabla^k g(\varrho)\|_{L^6}\|\nabla\varrho\|_{L^2} + \|g(\varrho)\|_{L^6}\|\nabla^{k+1}\varrho\|_{L^2})\\
&\lesssim \left(\|\nabla^{k+1}u\|_{L^2}^{k/(k+1/2)}\|\Lambda^{1/2}u\|_{L^2}^{(1/2)/(k+1/2)}\right)\\
&\quad\times\left(\|\nabla^{k+1}\varrho\|_{L^2}\|\Lambda^{1/2}\varrho\|_{L^2}^{k/(k+1/2)}\|\nabla^{k+1}\varrho\|_{L^2}^{(1/2)/(k+1/2)}\right)\\
&\lesssim (\|\Lambda^{1/2}\varrho\|_{L^2} + \|\Lambda^{1/2}u\|_{L^2})(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2)\\
&\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2).
\end{aligned} \tag{3.13}$$

For the term I_6, since $|\phi(\varrho)| \leqslant C$, by using lemmas 2.4 and 2.1 and Sobolev embedding theorem, we have

$$\begin{aligned}
I_6 &\lesssim \|\nabla^k u\|_{L^6}\left\|\nabla^k\left[\phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\right\|_{L^{6/5}}\\
&\lesssim \|\nabla^{k+1}u\|_{L^2}\|\phi(\varrho)\|_{L^\infty}\left\|\nabla^k\mathrm{div}\left(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^{6/5}}\\
&\quad+\|\nabla^{k+1}u\|_{L^2}\|\nabla^k\phi(\varrho)\|_{L^6}\left\|\mathrm{div}\left(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^{3/2}}\\
&\lesssim \|\nabla^{k+1}u\|_{L^2}\|\nabla\chi\|_{L^3}\|\nabla^{k+1}\nabla\chi\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}\|\nabla^k\varrho\|_{L^6}\|\nabla\chi\|_{L^3}\|\nabla^2\chi\|_{L^3}\\
&\lesssim \delta^2(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2).
\end{aligned} \tag{3.14}$$

Using Kato–Ponce inequality (lemma 2.1), Gagliardo–Nirenberg inequality (lemma 2.2) and Sobolev embedding theorem, we can estimate I_7 as

$$\begin{aligned}
I_7 &= -\int_{\mathbb{R}^3}\nabla^k(u\nabla\cdot\chi)\cdot\nabla^k\chi\,\mathrm{d}x\\
&\lesssim \|\nabla^k\chi\|_{L^6}\|\nabla^k(u\nabla\cdot\chi)\|_{L^{6/5}}
\end{aligned}$$

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$$
\begin{aligned}
&\lesssim \|\nabla^k\chi\|_{L^6}(\|\nabla^k u\|_{L^3}\|\nabla\chi\|_{L^2}+\|u\|_{L^3}\|\nabla^{k+1}\chi\|_{L^2})\\
&\lesssim \|\nabla^{k+1}\chi\|_{L^2}\Big(\|\nabla^{k+1}u\|_{L^2}^{k/(k+1/2)}\|\Lambda^{1/2}u\|_{L^2}^{(1/2)/(k+1/2)}\|\\
&\quad\times \Lambda^{1/2}\chi\|_{L^2}^{k/(k+1/2)}\|\nabla^{k+1}\chi\|_{L^2}^{(1/2)/(k+1/2)}+\|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+1}\chi\|_{L^2}\Big)\\
&\lesssim (\|\Lambda^{1/2}u\|_{L^2}+\|\Lambda^{1/2}\chi\|_{L^2})(\|\nabla^{k+1}\chi\|_{L^2}+\|\nabla^{k+1}u\|_{L^2})\\
&\lesssim \delta(\|\nabla^{k+1}\chi\|_{L^2}+\|\nabla^{k+1}u\|_{L^2}). \qquad (3.15)
\end{aligned}
$$

For I_8, if $k=0$, Hölder's inequality and Sobolev embedding theorem imply

$$
\begin{aligned}
I_8 &\approx -\int_{\mathbb{R}^3}\varphi(\varrho)\cdot\nabla^2\chi\cdot\chi\,\mathrm{d}x\\
&\lesssim \|\nabla\varphi(\varrho)\|_{L^2}\|\nabla\chi\|_{L^3}\|\chi\|_{L^6}+\|\varrho\|_{L^6}\|\nabla\chi\|_{L^\infty}\|\nabla\chi\|_{L^2}\\
&\lesssim \delta(\|\nabla\varrho\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2). \qquad (3.16)
\end{aligned}
$$

On the other hand, if $k\geqslant 1$, employing the Leibniz formula and Hölder's inequality, we arrive at

$$
\begin{aligned}
I_8 &= -\int_{\mathbb{R}^3}\nabla^{k-1}[\varphi(\varrho)\nabla^2\chi]\cdot\nabla^{k+1}\chi\,\mathrm{d}x\\
&= -\sum_{l=0}^{k-1}C_{k-1}^l\int_{\mathbb{R}^3}\nabla^{k-1-l}\varphi(\varrho)\cdot\nabla^l\nabla^2\chi\cdot\nabla^{k+1}\chi\,\mathrm{d}x\\
&\lesssim \sum_{l=0}^{[k/2]-1}C_{k-1}^l\|\nabla^{l+2}\chi\|_{L^3}\|\nabla^{k-l-1}\varphi(\varrho)\|_{L^6}\|\nabla^{k+1}\chi\|_{L^2}\\
&\quad+\sum_{l=[k/2]}^{k-2}C_{k-1}^l\|\nabla^{l+2}\chi\|_{L^6}\|\nabla^{k-1-l}\varphi(\varrho)\|_{L^3}\|\nabla^{k+1}\chi\|_{L^2}\\
&\quad+\underbrace{\int_{\mathbb{R}^3}|\varphi(\varrho)||\nabla^{k+1}\chi|^2\mathrm{d}x}_{l=k-1}. \qquad (3.17)
\end{aligned}
$$

Gagliardo–Nirenberg's inequality (lemma 2.2) implies that

$$
\begin{aligned}
&\sum_{l=0}^{[k/2]-1}C_{k-1}^l\|\nabla^{l+2}\chi\|_{L^3}\|\nabla^{k-l-1}\varphi(\varrho)\|_{L^6}\|\nabla^{k+1}\chi\|_{L^2}\\
&\lesssim \sum_{l=0}^{[k/2]-1}\|\nabla^\alpha\chi\|_{L^2}^{1-((l+1)/(k+1))}\|\nabla^{k+1}\chi\|_{L^2}^{(l+1)/(k+1)}\|\varrho\|_{L^2}^{(l+1)/(k+1)}\|\\
&\quad\times\nabla^{k+1}\varrho\|_{L^2}^{1-((l+1)/(k+1))}\|\nabla^{k+1}u\|_{L^2}\\
&\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}\chi\|_{L^2}^2), \qquad (3.18)
\end{aligned}
$$

where α satisfies

$$\frac{l+2}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right)\left(1 - \frac{l+1}{k+1}\right) + \left(\frac{k+1}{3} - \frac{1}{2}\right)\frac{l+1}{k+1},$$

that is

$$\alpha = \frac{3k+3}{2k-2l} \in \left(\frac{3}{2}, 3\right).$$

Moreover, also by Gagliardo–Nirenberg's inequality (lemma 2.2), we derive that

$$\begin{aligned}
&\sum_{l=[k/2]}^{k-2} C_{k-1}^{l} \|\nabla^{l+2}\chi\|_{L^6} \|\nabla^{k-1-l}\varphi(\varrho)\|_{L^3} \|\nabla^{k+1}\chi\|_{L^2} \\
&\lesssim \sum_{l=[k/2]}^{k-2} \|\chi\|_{L^2}^{1-((l+3)/(k+1))} \|\nabla^{k+1}\chi\|_{L^2}^{(l+3)/(k+1)} \|\nabla^{\alpha}\varrho\|_{L^2}^{(l+3)/(k+1)} \| \\
&\quad \times \nabla^{k+1}\varrho\|_{L^2}^{1-((l+3)/(k+1))} \|\nabla^{k+1}\chi\|_{L^2} \\
&\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2),
\end{aligned} \tag{3.19}$$

where α satisfies

$$\frac{k-1-l}{3} - \frac{1}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right)\frac{l+3}{k+1} + \left(\frac{k+1}{3} - \frac{1}{2}\right)\left(1 - \frac{l+3}{k+1}\right),$$

that is

$$\alpha = \frac{3k+3}{2l+6} \in \left[\frac{3}{2}, 3\right).$$

For the last term of the right-hand side of (3.17), we have

$$\int_{\mathbb{R}^3} |\varphi(\varrho)||\nabla^{k+1}\chi|^2 \, dx \leqslant \|\varphi(\varrho)\|_{L^\infty} \|\nabla^{k+1}\chi\|_{L^2}^2 \lesssim \delta \|\nabla^{k+1}\chi\|_{L^2}^2. \tag{3.20}$$

Combining (3.16)–(3.20) together, we easily obtain

$$I_8 \lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2). \tag{3.21}$$

Note that I_9 satisfies

$$\begin{aligned}
I_9 &= -\int_{\mathbb{R}^3} \nabla^k[\phi(\varrho)(\chi^3 - \chi)] \cdot \nabla^k \chi \, dx \\
&\lesssim \left(\|\nabla^k[\phi(\varrho)\chi^3]\|_{L^{6/5}} + \|\nabla^k[\phi(\varrho)\chi]\|_{L^{6/5}}\right) \|\nabla^k\chi\|_{L^6} \\
&\lesssim (\|\phi(\varrho)\|_{L^3} \|\nabla^k\chi^3\|_{L^2} + \|\chi^2\|_{L^6} \|\chi\|_{L^2} \|\nabla^k\phi(\varrho)\|_{L^6} \\
&\quad + \|\nabla^k[\phi(\varrho)\chi]\|_{L^{6/5}}) \|\nabla^{k+1}\chi\|_{L^2} \\
&\lesssim (I_{91} + I_{92} + I_{93}) \|\nabla^{k+1}\chi\|_{L^2},
\end{aligned} \tag{3.22}$$

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where we have used Hölder's inequality and Kato–Ponce inequality (lemma 2.1) in (3.22). Next, we first estimate I_{91}–I_{92} term by term:

$$\begin{aligned} I_{91} &\lesssim \|\varrho\|_{L^3}\|\chi\|_{L^6}^2\|\nabla^k\chi\|_{L^6} \\ &\lesssim \|\Lambda^{1/2}\varrho\|_{L^2}\|\nabla\chi\|_{L^2}^2\|\nabla^{k+1}\chi\|_{L^2} \lesssim \delta^3\|\nabla^{k+1}\chi\|_{L^2}, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} I_{92} &\lesssim \|\varrho\|_{L^\infty}\|\chi\|_{L^6}\|\chi\|_{L^2}\|\nabla^k\varrho\|_{L^6} \\ &\lesssim \|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla\chi\|_{L^2}\|\chi\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim \delta^3\|\nabla^{k+1}\varrho\|_{L^2}. \end{aligned} \tag{3.24}$$

Since $\phi(\varrho) = 1/(\varrho+1)$, we obtain $\zeta(\varrho) := \sqrt{\phi(\varrho)} = 1/\sqrt{(\varrho+1)}$. It is easy to see that $\zeta(\varrho)$ is a smooth function of ϱ with bounded derivatives of any order. Hence, lemma 2.4 holds for $\zeta(\varrho)$. By using Sobolev embedding theorem and Kato–Ponce inequality, we have

$$\begin{aligned} I_{93} &= \|\nabla^k[\phi(\varrho)\chi]\|_{L^{6/5}} = \|\nabla^k[\zeta(\varrho)\zeta(\varrho)\chi]\|_{L^{6/5}} \\ &\lesssim \|\zeta(\varrho)\|_{L^2}\|\zeta(\varrho)\|_{L^6}\|\nabla^k\chi\|_{L^6} + \|\chi\|_{L^2}\|\zeta(\varrho)\|_{L^6}\|\nabla^k\zeta(\varrho)\|_{L^6} \\ &\lesssim \|\varrho\|_{L^2}\|\nabla\varrho\|_{L^2}\|\nabla^{k+1}\chi\|_{L^2} + \|\chi\|_{L^2}\|\nabla\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\ &\lesssim \delta^2(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}\chi\|_{L^2}). \end{aligned} \tag{3.25}$$

Combining (3.23)–(3.25) together, we derive that

$$I_9 \lesssim \left(\delta^3+\delta\right)(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2 + \|\nabla^{k+2}\chi\|_{L^2}^2). \tag{3.26}$$

The term I_{10} satisfies

$$\begin{aligned} I_{10} &= -\int_{\mathbb{R}^3}\nabla^{k+1}[u\cdot\nabla\chi]\cdot\nabla^{k+1}\chi\,\mathrm{d}x \\ &\lesssim \|\nabla^{k+1}\chi\|_{L^6}\|\nabla^{k+1}[u\cdot\nabla\chi]\|_{L^{6/5}} \\ &\lesssim \|\nabla^{k+1}\chi\|_{L^6}(\|\nabla^{k+1}u\|_{L^2}\|\nabla\chi\|_{L^3} + \|u\|_{L^3}\|\nabla^{k+1}\nabla\chi\|_{L^2}) \\ &\lesssim (\|\nabla\chi\|_{L^3} + \|u\|_{L^3})(\|\nabla^{k+1}\nabla\chi\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2) \\ &\lesssim \delta(\|\nabla^{k+1}\nabla\chi\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2), \end{aligned} \tag{3.27}$$

where we have used Kato–Ponce inequality (lemma 2.1) and Sobolev embedding theorem in (3.27). Similar to (3.27), I_{11} and I_{12} can be bounded as

$$\begin{aligned} I_{11} &= -\int_{\mathbb{R}^3}\nabla^{k+1}[\varphi(\varrho)\Delta\chi]\cdot\nabla^{k+1}\chi\,\mathrm{d}x \\ &\lesssim \|\nabla^{k+2}\chi\|_{L^2}\|\nabla^k[\varphi(\varrho)\Delta\chi]\|_{L^2} \\ &\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\varphi(\varrho)\|_{L^\infty}\|\nabla^k\Delta\chi\|_{L^2} + \|\Delta\chi\|_{L^3}\|\nabla^k\varphi(\varrho)\|_{L^6}) \\ &\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^{k+2}\chi\|_{L^2} + \|\nabla^3\chi\|_{L^2}^{1/2}\|\nabla^2\chi\|_{L^2}^{1/2}\|\nabla^{k+1}\varrho\|_{L^2}) \\ &\lesssim \delta(\|\nabla^{k+2}\chi\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2), \end{aligned} \tag{3.28}$$

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and

$$
\begin{aligned}
I_{12} &= -\int_{\mathbb{R}^3} \nabla^{k+1}[\phi(\varrho)(\chi^3-\chi)]\cdot\nabla^{k+1}\chi\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\nabla^k[\phi(\varrho)\chi^3]\|_{L^2}+\|\nabla^k[\phi(\varrho)\chi]\|_{L^2}) \\
&\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\phi(\varrho)\|_{L^\infty}\|\nabla^k\chi^3\|_{L^2}+\|\chi^3\|_{L^3}\|\nabla^k\phi(\varrho)\|_{L^6} \\
&\quad + \|\zeta(\varrho)\|_{L^2}\|\zeta(\varrho)\|_{L^6}\|\nabla^k\chi\|_{L^6}+\|\chi\|_{L^2}\|\zeta(\varrho)\|_{L^6}\|\nabla^k\zeta(\varrho)\|_{L^6}). \qquad (3.29)
\end{aligned}
$$

It then follows from (3.23) that

$$
\|\nabla^k\chi^3\|_{L^2}\lesssim \delta^2\|\nabla^{k+1}\chi\|_{L^2}. \qquad (3.30)
$$

Adding (3.29) and (3.30) together gives

$$
\begin{aligned}
I_{12} &\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\phi(\varrho)\|_{L^\infty}\delta^2\|\nabla^{k+2}\chi\|_{L^2}+(\|\chi\|_{L^9}^3+\|\chi\|_{L^3})\|\nabla^{k+1}\varrho\|_{L^2}) \\
&\lesssim \left(\delta^3+\delta\right)(\|\nabla^{k+1}\chi\|_{L^2}^2+\|\nabla^{k+2}\chi\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2). \qquad (3.31)
\end{aligned}
$$

Summing up the estimates for I_1–I_{12}, we deduce (3.4), this yields the desired result. □

We also need to derive the second type of energy estimates excluding ϱ, u and χ themselves.

LEMMA 3.2. *If* $\sqrt{\mathcal{E}_0^3(t)}\leqslant\delta$. *Then, for* $k=0,1,\dots,N-1$, *the following inequality holds:*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}\chi\|_{L^2}^2+\|\nabla^{k+1}\nabla\chi\|_{L^2}^2) \\
&\quad + C(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+2}\chi\|_{L^2}^2+\|\nabla^{k+2}\nabla\chi\|_{L^2}^2) \\
&\qquad \lesssim (\delta^3+\delta)(\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+2}\chi\|_{L^2}^2+\|\nabla^{k+2}\nabla\chi\|_{L^2}^2). \quad (3.32)
\end{aligned}
$$

Proof. Taking $k+1$th spatial derivatives to $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$, $k+2$th spatial derivatives to $(1.1)_3$, multiplying the resulting identities by $\nabla^{k+1}\varrho$, $\nabla^{k+1}u$, $\nabla^{k+1}\chi$ and $\nabla^{k+2}\chi$ respectively, summing up and then integrating over $\mathbb{R}^3$ by parts, we derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\nabla^{k+1}\varrho|^2+|\nabla^{k+1}u|^2+|\nabla^{k+1}\chi|^2+|\nabla^{k+2}\chi|^2)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}(|\nabla^{k+2}u|^2+|\nabla^{k+2}\chi|^2+|\nabla^{k+3}\chi|^2)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \big[\nabla^{k+1}(-\varrho \operatorname{div} u - u\cdot\nabla\varrho)\cdot\nabla^{k+1}\varrho \\
&\quad -\nabla^{k+1}\left[u\cdot\nabla u + h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\operatorname{div} u) + g(\varrho)\nabla\varrho\right.\\
&\quad \left. +\,\phi(\varrho)\operatorname{div}\left(\nabla\chi\otimes\nabla\chi - \frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\cdot\nabla^{k+1}u \\
&\quad +\nabla^{k+1}\left(-u\cdot\nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3-\chi)\right)\cdot\nabla^{k+1}\chi \\
&\quad +\nabla^{k+2}\left(-u\cdot\nabla\chi - \varphi(\varrho)\Delta\chi - \phi(\varrho)(\chi^3-\chi)\right)\cdot\nabla^{k+2}\chi\big]\,\mathrm{d}x \\
&= \sum_{i=1}^{12} K_i. \qquad (3.33)
\end{aligned}
$$

We will estimate the term K_1–K_{11} on the right-hand side of (3.33) one by one. First, through Hölder's inequality and lemma 2.1, we arrive at

$$
\begin{aligned}
K_1 &= -\int_{\mathbb{R}^3} \nabla^{k+1}(\varrho\operatorname{div} u)\cdot\nabla^{k+1}\varrho\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+1}\varrho\|_{L^2}\|\nabla^{k+1}(\varrho\nabla\cdot u)\|_{L^2} \\
&\lesssim \|\nabla^{k+1}\varrho\|_{L^2}(\|\nabla^{k+1}\varrho\|_{L^2}\|\nabla u\|_{L^\infty} + \|\varrho\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}) \\
&\lesssim (\|\nabla u\|_{L^\infty} + \|\varrho\|_{L^\infty})(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2) \\
&\lesssim \delta(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2). \qquad (3.34)
\end{aligned}
$$

Next, for the term J_2, we utilize the commutator notation (2.4) to rewrite it, then integrate by part and use Sobolev's inequality, obtain the following inequality:

$$
\begin{aligned}
K_2 &= -\int_{\mathbb{R}^3} \nabla^{k+1}(u\cdot\nabla\varrho)\cdot\nabla^{k+1}\varrho\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} (u\cdot\nabla\nabla^{k+1}\varrho + [\nabla^{k+1},u]\cdot\nabla\varrho)\nabla^{k+1}\varrho\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} u\cdot\nabla\frac{|\nabla^{k+1}\varrho|^2}{2}\,\mathrm{d}x \\
&\quad + (\|\nabla u\|_{L^\infty}\|\nabla^k\nabla\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}\|\nabla\varrho\|_{L^\infty})\|\nabla^{k+1}\varrho\|_{L^2} \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \nabla\cdot u|\nabla^{k+1}\varrho|^2\,\mathrm{d}x \\
&\quad + (\|\nabla u\|_{L^\infty}\|\nabla^k\nabla\varrho\|_{L^2} + \|\nabla^{k+1}u\|_{L^2}\|\nabla\varrho\|_{L^\infty})\|\nabla^{k+1}\varrho\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2}^2 + (\|\nabla u\|_{L^\infty} + \|\nabla\varrho\|_{L^\infty})\|\nabla^{k+1}\varrho\|_{L^2}^2 \\
&\lesssim \delta\|\nabla^{k+1}\varrho\|_{L^2}^2. \qquad (3.35)
\end{aligned}
$$

Integrating by parts, applying Hölder's inequality, Kato–Ponce inequality (lemma 2.1), Gagliardo–Nirenberg inequality (lemma 2.2) and Sobolev embedding

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theorem, one estimates K_3–K_8 as

$$\begin{aligned}
K_3 &= -\int_{\mathbb{R}^3} \nabla^{k+1}(u\cdot\nabla u)\cdot\nabla^{k+1}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \nabla^{k}(u\cdot\nabla u)\cdot\nabla^{k+2}u\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}\|\nabla^k(u\cdot\nabla u)\|_{L^2} \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}(\|\nabla^k u\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^3}\|\nabla^{k+1}u\|_{L^6}) \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}\Big(\|\nabla^{k+2}u\|_{L^2}^{(k-1/2)/(k+1/2)}\|\Lambda^{1/2}u\|_{L^2}^{1/(k+1/2)}\| \\
&\quad\times\Lambda^{1/2}u\|_{L^2}^{(k-1/2)/(k+1/2)}\|\nabla^{k+2}u\|_{L^2}^{1/(k+1/2)}+\|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\Big) \\
&\lesssim \|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}^2 \lesssim \delta\|\nabla^{k+2}u\|_{L^2}^2, \qquad (3.36)\\
K_4 &\approx -\int_{\mathbb{R}^3} \nabla^{k+1}[h(\varrho)\nabla^2 u]\cdot\nabla^{k+1}u\,\mathrm{d}x \\
&\approx \int_{\mathbb{R}^3} \nabla^{k}[h(\varrho)\nabla^2 u]\cdot\nabla^{k+2}u\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}\|\nabla^k[h(\varrho)\nabla^2 u]\|_{L^2} \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}(\|\nabla^k h(\varrho)\|_{L^6}\|\nabla^2 u\|_{L^3}+\|h(\varrho)\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}) \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}(\|\nabla^{k+1}\varrho\|_{L^2}\|\nabla^{5/2}u\|_{L^2}+\|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^{k+2}u\|_{L^2}) \\
&\lesssim (\|\nabla^{5/2}u\|_{L^2}+\|\nabla\varrho\|_{L^2}+\|\Delta\varrho\|_{L^2})(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2) \\
&\lesssim \delta(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2), \qquad (3.37)\\
K_5 &= -\int_{\mathbb{R}^3} \nabla^{k+1}[g(\varrho)\nabla\varrho]\cdot\nabla^{k+1}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \nabla^{k}[g(\varrho)\nabla\varrho]\cdot\nabla^{k+2}u\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}(\|\nabla^k g(\varrho)\|_{L^6}\|\nabla\varrho\|_{L^3}+\|g(\varrho)\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2}) \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}(\|\nabla^{k+1}\varrho\|_{L^2}\|\Lambda^{3/2}\varrho\|_{L^2}+\|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^{k+1}\varrho\|_{L^2}) \\
&\lesssim (\|\Lambda^{3/2}\varrho\|_{L^2}+\|\nabla\varrho\|_{L^2}+\|\Delta\varrho\|_{L^2})(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2\|_{L^2}^2) \\
&\lesssim \delta(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2\|_{L^2}^2), \qquad (3.38)\\
K_6 &= \int_{\mathbb{R}^3} \nabla^{k+1}\left[\phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\cdot\nabla^{k+1}u\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}\left\|\nabla^k\left[\phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\right\|_{L^2} \\
&\lesssim \|\nabla^{k+2}u\|_{L^2}\|\phi(\varrho)\|_{L^\infty}\left\|\nabla^k\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^2}
\end{aligned}$$

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$$+\|\nabla^{k+2}u\|_{L^2}\|\nabla^k\phi(\varrho)\|_{L^6}\left\|\operatorname{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^3}$$
$$\lesssim \|\nabla^{k+2}u\|_{L^2}\|\varrho\|_{L^\infty}\|\nabla\chi\|_{L^3}\|\nabla^{k+2}\chi\|_{L^2}+\|\nabla^{k+2}u\|_{L^2}\|$$
$$\times\nabla^{k+1}\varrho\|_{L^2}\|\nabla\chi\|_{L^6}\|\nabla^2\chi\|_{L^6}$$
$$\lesssim \delta^2(\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}\chi\|_{L^2}^2+\|\nabla^{k+2}\varrho\|_{L^2}^2), \tag{3.39}$$
$$K_7=-\int_{\mathbb{R}^3}\nabla^{k+1}(u\cdot\nabla\chi)\cdot\nabla^{k+1}\chi\,\mathrm{d}x$$
$$\lesssim \|\nabla^{k+1}\chi\|_{L^6}\|\nabla^{k+1}(u\cdot\nabla\chi)\|_{L^{6/5}}$$
$$\lesssim \|\nabla^{k+2}\chi\|_{L^2}(\|\nabla^{k+1}u\|_{L^2}\|\nabla\chi\|_{L^3}+\|u\|_{L^3}\|\nabla^{k+1}\nabla\chi\|_{L^2})$$
$$\lesssim \|\nabla^{k+2}\chi\|_{L^2}\Big(\|\nabla^{k+2}u\|_{L^2}^{(k+1/2)/(k+3/2)}\|\Lambda^{1/2}u\|_{L^2}^{1/(k+3/2)}\|$$
$$\times\nabla^{k+2}\chi\|_{L^2}^{1/(k+3/2)}\|\Lambda^{1/2}\chi\|_{L^2}^{(k+1/2)/(k+3/2)}+\|\Lambda^{1/2}u\|_{L^2}\|\nabla^{k+2}\chi\|_{L^2}\Big)$$
$$\lesssim (\|\Lambda^{1/2}\chi\|_{L^2}+\|\Lambda^{1/2}u\|_{L^2})(\|\nabla^{k+1}\nabla\chi\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2)$$
$$\lesssim \delta(\|\nabla^{k+2}\chi\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2), \tag{3.40}$$

and

$$K_8=-\int_{\mathbb{R}^3}\nabla^{k+1}[\varphi(\varrho)\Delta\chi]\cdot\nabla^{k+1}\chi\,\mathrm{d}x$$
$$\lesssim \|\nabla^{k+1}[\varphi(\varrho)\Delta\chi]\|_{L^{6/5}}\|\nabla^{k+1}\chi\|_{L^6}$$
$$\lesssim (\|\varphi(\varrho)\|_{L^3}\|\nabla^{k+1}\Delta\chi\|_{L^2}+\|\Delta\chi\|_{L^3}\|\nabla^{k+1}\varphi(\varrho)\|_{L^2})\|\nabla^{k+2}\chi\|_{L^2}$$
$$\lesssim (\|\varrho\|_{L^3}\|\nabla^{k+3}\chi\|_{L^2}+\|\nabla^2\chi\|_{L^3}\|\nabla^{k+1}\varrho\|_{L^2})\|\nabla^{k+2}\chi\|_{L^2}$$
$$\lesssim \delta(\|\nabla^{k+2}\chi\|_{L^2}^2+\|\nabla^{k+3}\chi\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2). \tag{3.41}$$

Next, we consider the term K_9. Hölder's inequality implies that

$$K_9=-\int_{\mathbb{R}^3}\nabla^{k+1}[\phi(\varrho)(\chi^3-\chi)]\cdot\nabla^{k+1}\chi\,\mathrm{d}x$$
$$\lesssim (\|\nabla^{k+1}[\phi(\varrho)\chi^3]\|_{L^{6/5}}+\|\nabla^{k+1}[\zeta(\varrho)\zeta(\varrho)\chi]\|_{L^{6/5}})\|\nabla^{k+1}\chi\|_{L^6}$$
$$=:(K_{91}+K_{92})\|\nabla^{k+2}\chi\|_{L^2}. \tag{3.42}$$

By using Kato–Ponce inequality of lemma 2.1 and Sobolev embedding theorem, we arrive at

$$K_{91}=\|\nabla^{k+1}[\phi(\varrho)\chi^3]\|_{L^{6/5}}$$
$$\lesssim \|\phi(\varrho)\|_{L^3}\|\nabla^{k+1}\chi^3\|_{L^2}+\|\chi^3\|_{L^3}\|\nabla^{k+1}\phi(\varrho)\|_{L^2}$$
$$\lesssim \|\phi(\varrho)\|_{L^3}\|\chi\|_{L^6}^2\|\nabla^{k+1}\chi\|_{L^6}+\|\chi\|_{L^\infty}\|\chi\|_{L^6}^2\|\nabla^{k+1}\varrho\|_{L^2}$$
$$\lesssim \|\Lambda^{1/2}\varrho\|_{L^2}\|\nabla\chi\|_{L^2}^2\|\nabla^{k+2}\chi\|_{L^2}+\|\nabla\chi\|_{L^2}^{1/2}\|\Delta\chi\|_{L^2}^{1/2}\|\nabla\chi\|_{L^2}^2\|\nabla^{k+1}\varrho\|_{L^2}$$
$$\lesssim \delta^3(\|\nabla^{k+2}\chi\|_{L^2}+\|\nabla^{k+1}\varrho\|_{L^2}). \tag{3.43}$$

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Moreover, the term K_{92} can be bounded as

$$\begin{aligned}
K_{92} &= \|\nabla^{k+1}[\phi(\varrho)\chi]\|_{L^{6/5}} = \|\nabla^{k+1}[\zeta(\varrho)\zeta(\varrho)\chi]\|_{L^{6/5}} \\
&\lesssim \|\zeta(\varrho)\|_{L^2}\|\zeta(\varrho)\|_{L^6}\|\nabla^{k+1}\chi\|_{L^6} + \|\chi\|_{L^6}\|\zeta(\varrho)\|_{L^6}\|\nabla^{k+1}\zeta(\varrho)\|_{L^2} \\
&\lesssim \|\varrho\|_{L^2}\|\nabla\varrho\|_{L^2}\|\nabla^{k+2}\chi\|_{L^2} + \|\nabla\chi\|_{L^2}\|\nabla\varrho\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2} \\
&\lesssim \delta^2(\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+2}\chi\|_{L^2}). \qquad (3.44)
\end{aligned}$$

Combining (3.43)–(3.44) together, we derive that

$$K_9 \lesssim \left(\delta^3+\delta^2\right)(\|\nabla^{k+1}\varrho\|_{L^2}^2 + \|\nabla^{k+2}\chi\|_{L^2}^2). \qquad (3.45)$$

Next, employing Hölder's inequality, Kato–Ponce inequality of lemma 2.1 and Sobolev embedding theorem, we estimate the term K_{10} as

$$\begin{aligned}
K_{10} &= -\int_{\mathbb{R}^3} \nabla^{k+2}[u\cdot\nabla\chi]\cdot\nabla^{k+2}\chi\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+2}\chi\|_{L^6}\|\nabla^{k+2}(u\cdot\nabla\chi)\|_{L^{6/5}} \\
&\lesssim \|\nabla^{k+2}\nabla\chi\|_{L^2}(\|\nabla^{k+2}u\|_{L^2}\|\nabla\chi\|_{L^3} + \|u\|_{L^3}\|\nabla^{k+2}\nabla\chi\|_{L^2}) \\
&\lesssim (\|\Lambda^{3/2}\chi\|_{L^2} + \|\Lambda^{1/2}u\|_{L^2})(\|\nabla^{k+2}\nabla\chi\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2) \\
&\lesssim \delta(\|\nabla^{k+2}\nabla\chi\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2). \qquad (3.46)
\end{aligned}$$

Using Hölder's inequality, Kato–Ponce inequality, Gagliardo–Nirenberg inequality together with Sobolev embedding theorem again, we have

$$\begin{aligned}
K_{11} &= -\int_{\mathbb{R}^3} \nabla^{k+2}[\varphi(\varrho)\Delta\chi]\cdot\nabla^{k+2}\chi\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}\|\nabla^{k+1}[\varphi(\varrho)\Delta\chi]\|_{L^2} \\
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}(\|\varphi(\varrho)\|_{L^\infty}\|\nabla^{k+1}\Delta\chi\|_{L^2} + \|\Delta\chi\|_{L^\infty}\|\nabla^{k+1}\varphi(\varrho)\|_{L^2}) \\
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}(\|\nabla\varrho\|_{L^2}^{1/2}\|\nabla^2\varrho\|_{L^2}^{1/2}\|\nabla^{k+3}\chi\|_{L^2} \\
&\quad + \|\nabla^3\chi\|_{L^2}^{1/2}\|\nabla^3\nabla\chi\|_{L^2}^{1/2}\|\nabla^{k+1}\varrho\|_{L^2}) \\
&\lesssim \delta(\|\nabla^{k+3}\chi\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2), \qquad (3.47)
\end{aligned}$$

and

$$\begin{aligned}
K_{12} &= -\int_{\mathbb{R}^3} \nabla^{k+2}[\phi(\varrho)(\chi^3-\chi)]\cdot\nabla^{k+2}\chi\,\mathrm{d}x \\
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}(\|\nabla^{k+1}[\phi(\varrho)\chi^3]\|_{L^2} + \|\nabla^{k+1}[\zeta(\varrho)\zeta(\varrho)\chi]\|_{L^2})
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}(\|\phi(\varrho)\|_{L^\infty}\|\chi\|_{L^6}^2\|\nabla^{k+1}\chi\|_{L^6}+\|\chi\|_{L^\infty}\|\nabla^{k+1}\phi(\varrho)\|_{L^2}\\
&\quad+\|\zeta(\varrho)\|_{L^6}^2\|\nabla^{k+1}\chi\|_{L^6}+\|\zeta(\varrho)\|_{L^\infty}\|\chi\|_{L^\infty}\|\nabla^{k+1}\zeta(\varrho)\|_{L^2})\\
&\lesssim \|\nabla^{k+3}\chi\|_{L^2}(\|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla\chi\|_{L^2}^2\|\nabla^{k+1}\chi\|_{L^6}\\
&\quad+\|\nabla\chi\|_{L^2}^{1/2}\|\Delta\chi\|_{L^2}^{1/2}\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla\varrho\|_{L^2}^2\|\nabla^{k+2}\chi\|_{L^2}\\
&\quad+\|\nabla\chi\|_{L^2}^{1/2}\|\Delta\chi\|_{L^2}^{1/2}\|\nabla^{k+1}\varrho\|_{L^2})\\
&\lesssim \left(\delta^3+\delta\right)(\|\nabla^{k+3}\chi\|_{L^2}^2+\|\nabla^{k+1}\varrho\|_{L^2}^2).
\end{aligned}
\tag{3.48}
$$

Summing up the estimates for K_1–K_{12}, we deduce (3.32), this yields the desired result. □

The following lemma provides the dissipation estimate for ϱ.

Lemma 3.3. *If $\sqrt{\mathcal{E}_0^3(t)}<\delta$, then for $k=0,1,\dots,N-1$, we have*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla^{k+1}\varrho\,\mathrm{d}x+C\|\nabla^{k+1}\varrho\|_{L^2}^2\lesssim\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+3}\chi\|_{L^2}^2.
\tag{3.49}
$$

Proof. Applying ∇^k to $(1.4)_2$, multiplying $\nabla\nabla^k\varrho$, integrating over $\mathbb{R}^3$ by parts, it yields that

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla^{k+1}\varrho|^2\,\mathrm{d}x&\leqslant-\int_{\mathbb{R}^3}\nabla^k u_t\cdot\nabla\nabla^k\varrho\,\mathrm{d}x+C\|\nabla^{k+2}u\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}\\
&\quad+\Bigg\|\nabla^k\left[u\cdot\nabla u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)+g(\varrho)\nabla\varrho\right.\\
&\quad\left.+\psi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\Bigg\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}.
\end{aligned}
\tag{3.50}
$$

For the first term of the right-hand side of (3.50), using $(1.4)_1$, integrating by parts for both the t and x variables, one obtains

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\nabla^k u_t\cdot\nabla\nabla^k\varrho\,\mathrm{d}x\\
&\quad=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^k\mathrm{div}\,u\cdot\nabla^k\varrho_t\,\mathrm{d}x\\
&\quad=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k\varrho\,\mathrm{d}x+\|\nabla^k\mathrm{div}\,u\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^k\mathrm{div}\,u\cdot\nabla^k\mathrm{div}(\varrho u)\,\mathrm{d}x.
\end{aligned}
\tag{3.51}
$$

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Employing Hölder's inequality, Kato–Ponce inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla^k \mathrm{div}\, u \cdot \nabla^k \mathrm{div}(\varrho u)\, \mathrm{d}x \\
&\quad\lesssim \|\nabla^{k+1}u\|_{L^2}(\|\nabla^{k+1}\varrho\|_{L^2}\|u\|_{L^\infty} + \|\varrho\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}) \\
&\quad\lesssim \|\nabla^{k+1}u\|_{L^2}(\|\nabla^{k+1}\varrho\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + \|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\nabla^{k+1}u\|_{L^2}) \\
&\quad\lesssim (\|\nabla u\|_{L^2} + \|\Delta u\|_{L^2} + \|\nabla\varrho\|_{L^2} + \|\Delta\varrho\|_{L^2})(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2) \\
&\quad\lesssim \delta(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2).
\end{aligned} \tag{3.52}$$

It then follows from (3.51) and (3.52) that

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \nabla^k u_t \cdot \nabla\nabla^k \varrho\, \mathrm{d}x \\
&\quad\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \nabla^k u \cdot \nabla\nabla^k\varrho\, \mathrm{d}x + C(\|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2) + C\delta\|\nabla^{k+1}\varrho\|_{L^2}^2.
\end{aligned} \tag{3.53}$$

Next, one need to estimate the last term of the right-hand side of (3.50). Note that

$$\begin{aligned}
\|\nabla^k(u\cdot\nabla u)\|_{L^2} &\lesssim \|u\|_{L^3}\|\nabla^{k+1}u\|_{L^6} + \|\nabla u\|_{L^3}\|\nabla^k u\|_{L^6} \\
&\lesssim (\|\Lambda^{1/2}u\|_{L^2} + \|\Lambda^{3/2}u\|_{L^2})(\|\nabla^{k+1}u\|_{L^2} + \|\nabla^{k+2}u\|_{L^2}) \\
&\lesssim \delta(\|\nabla^{k+1}u\|_{L^2} + \|\nabla^{k+2}u\|_{L^2}).
\end{aligned} \tag{3.54}$$

We also have

$$\begin{aligned}
&\|\nabla^k[h(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}\, u)]\|_{L^2} \\
&\quad\approx \|\nabla^k(h(\varrho)\nabla^2 u)\|_{L^2} \\
&\quad\lesssim \|h(\varrho)\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla^k h(\varrho)\|_{L^6} \\
&\quad\lesssim \|\varrho\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2} + \|\nabla^2 u\|_{L^3}\|\nabla^{k+1}\varrho\|_{L^2} \\
&\quad\lesssim \delta(\|\nabla^{k+2}u\|_{L^2} + \|\nabla^{k+1}\varrho\|_{L^2}),
\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}
\|\nabla^k(g(\varrho)\nabla\varrho)\|_{L^2} &\lesssim \|g(\varrho)\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^k g(\varrho)\|_{L^6} \\
&\lesssim \|\varrho\|_{L^\infty}\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla\varrho\|_{L^3}\|\nabla^{k+1}\varrho\|_{L^2} \\
&\lesssim \delta\|\nabla^{k+1}\varrho\|_{L^2}.
\end{aligned} \tag{3.56}$$

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Moreover, Kato–Ponce inequality of lemma 2.1 and Sobolev inequality of lemma 2.2 imply that

$$\begin{aligned}
&\left\|\nabla^k\left[\psi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\right\|_{L^2}\\
&\quad\lesssim \|\psi(\varrho)\|_{L^3}\left\|\nabla^k\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^6}\\
&\qquad+\left\|\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right\|_{L^3}\|\nabla^k\psi(\varrho)\|_{L^6}\\
&\quad\approx \|\psi(\varrho)\|_{L^3}\|\nabla^{k+1}|\nabla\chi|^2\|_{L^6}+\||\nabla\chi||\nabla^2\chi|\|_{L^3}\|\nabla^k\psi(\varrho)\|_{L^6}\\
&\quad\lesssim \|\varrho\|_{L^3}\|\nabla\chi\|_{L^\infty}\|\nabla^{k+2}\nabla\chi\|_{L^2}+\|\nabla\chi\|_{L^6}\|\nabla^2\chi\|_{L^6}\|\nabla^{k+1}\varrho\|_{L^2}\\
&\quad\lesssim \delta^3(\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla^{k+2}\nabla\chi\|_{L^2}). \qquad (3.57)
\end{aligned}$$

Combining (3.54)–(3.57) together, we easily obtain

$$\begin{aligned}
&\left\|\nabla^k\left[u\cdot\nabla u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)+g(\varrho)\nabla\varrho\right.\right.\\
&\quad\left.\left.+\psi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\right\|_{L^2}\|\nabla^{k+1}\varrho\|_{L^2}\\
&\qquad\lesssim (\delta^3+\delta)(\|\nabla^{k+1}\varrho\|_{L^2}+\|\nabla^{k+1}u\|_{L^2}+\|\nabla^{k+2}u\|_{L^2}+\|\nabla^{k+2}\nabla\chi\|_{L^2}). \quad (3.58)
\end{aligned}$$

Plugging the estimates (3.53) and (3.58) into (3.50), by using Cauchy's inequality and the smallness of δ, we then complete the proof of lemma 3.3. □

4. Negative Sobolev estimates

In this section, we derive the evolution of the negative Sobolev norms of the solution.

LEMMA 4.1. *If* $\sqrt{\mathcal{E}_0^3(t)}\leqslant\delta$. *Then for* $s\in[0,\frac{1}{2}]$, *we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{-s}\varrho\|_{L^2}^2+\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s}\chi\|_{L^2}^2+\|\Lambda^{-s}\nabla\chi\|_{L^2}^2)\\
&\quad+C(\|\nabla\Lambda^{-s}u\|_{L^2}^2+\|\nabla\Lambda^{-s}\chi\|_{L^2}^2+\|\nabla\Lambda^{-s}\chi\|_{L^2}^2)\\
&\qquad\lesssim (\|\varrho\|_{H^2}^2+\|\nabla u\|_{H^1}^2+\|\nabla\chi\|_{H^3}^2)(\|\Lambda^{-s}\varrho\|_{L^2}+\|\Lambda^{-s}u\|_{L^2}\\
&\qquad\quad+\|\Lambda^{-s}\chi\|_{L^2}+\|\Lambda^{-s}\nabla\chi\|_{L^2}). \qquad (4.1)
\end{aligned}$$

Moreover, for $s\in(\frac{1}{2},\frac{3}{2})$, *we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{-s}\varrho\|_{L^2}^2+\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s}\chi\|_{L^2}^2+\|\Lambda^{-s}\nabla\chi\|_{L^2}^2)\\
&\quad+C(\|\nabla\Lambda^{-s}u\|_{L^2}^2+\|\nabla\Lambda^{-s}\chi\|_{L^2}^2+\|\nabla\Lambda^{-s}\chi\|_{L^2}^2)\\
&\quad\lesssim\Big[\|(\varrho,u,\chi,\nabla\chi)\|_{L^2}^{s-1/2}(\|\varrho\|_{H^2}+\|\nabla u\|_{H^1}+\|\nabla\chi\|_{H^1}+\|\nabla^2\chi\|_{H^1})^{5/2-s}\\
&\qquad+(\|\nabla\varrho\|_{H^1}^2+\|\Delta\chi\|_{L^2}^2)\Big]\\
&\qquad\times(\|\Lambda^{-s}\varrho\|_{L^2}+\|\Lambda^{-s}u\|_{L^2}+\|\Lambda^{-s}\chi\|_{L^2}+\|\Lambda^{-s}\nabla\chi\|_{L^2}). \qquad (4.2)
\end{aligned}$$

Proof. Applying Λ^{-s} to $(1.4)_1$, $(1.4_2$ and $(1.4)_3$, $\Lambda^{-s}\nabla$ to $(1.4)_3$, multiplying the resulting identities by $\Lambda^{-s}\varrho$, $\Lambda^{-s}u$, $\Lambda^{-s}\chi$ and $\Lambda^{-s}\nabla\chi$, respectively, summing up and then integrating by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|\Lambda^{-s}\varrho|^2+|\Lambda^{-s}u|^2+|\Lambda^{-s}\chi|^2+|\Lambda^{-s}\nabla\chi|^2)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(|\Lambda^{-s}\nabla u|^2+|\Lambda^{-s}\nabla\chi|^2+|\Lambda^{-s}\nabla^2\chi|^2)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\Bigg[\Lambda^{-s}(-\varrho\mathrm{div}\,u-u\cdot\nabla\varrho)\cdot\Lambda^{-s}\varrho\\
&\quad-\Lambda^{-s}\Bigg[u\cdot\nabla u+h(\varrho)(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)+g(\varrho)\nabla\varrho\\
&\quad+\phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\Bigg]\cdot\Lambda^{-s}u\\
&\quad+\Lambda^{-s}\left(-u\cdot\nabla\chi-\varphi(\varrho)\Delta\chi-\phi(\varrho)(\chi^3-\chi)\right)\cdot\Lambda^{-s}\chi\\
&\quad+\Lambda^{-s}\nabla\left(-u\cdot\nabla\chi-\varphi(\varrho)\Delta\chi-\phi(\varrho)(\chi^3-\chi)\right)\cdot\Lambda^{-s}\nabla\chi\Bigg]\,\mathrm{d}x\\
&=\sum_{i=1}^{12}J_i.
\end{aligned}\tag{4.3}$$

The main tool to estimate the nonlinear terms in the right-hand side of (4.3) is the estimate in lemma 2.5. This forces us to require that $s\in(0,\frac{3}{2})$. If $s\in(0,\frac{1}{2}]$, we easily obtain $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\geqslant 6$. Then, applying lemmas 2.5, 2.2 together with Hölder's and Young's inequalities, it yields that

$$\begin{aligned}
J_1&=-\int_{\mathbb{R}^3}\Lambda^{-s}(\varrho\nabla\cdot u)\Lambda^{-s}\varrho\,\mathrm{d}x\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\Lambda^{-s}(\varrho\nabla\cdot u)\|_{L^2}\\
&\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\varrho\nabla\cdot u\|_{L^{1/(1/2+s/3)}}\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\varrho\|_{L^{3/s}}\|\nabla u\|_{L^2}\\
&\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla u\|_{L^2}\lesssim(\|\nabla\varrho\|_{H^1}^2+\|\nabla u\|_{L^2}^2)\|\Lambda^{-s}\varrho\|_{L^2}.
\end{aligned}\tag{4.4}$$

Similarly, by using lemmas 2.5 and 2.2 together with Hölder's and Young's inequalities, the term J_2-J_{12} can be bound by

$$\begin{aligned}
J_2&=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla\varrho)\Lambda^{-s}\varrho\,\mathrm{d}x\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla\varrho)\|_{L^2}\\
&\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|u\cdot\nabla\varrho\|_{L^{1/(1/2+s/3)}}\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|u\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}\\
&\lesssim\|\Lambda^{-s}\varrho\|_{L^2}\|\nabla u\|_{L^2}^{1/2+s}\|\nabla^2u\|_{L^2}^{1/2-s}\|\nabla\varrho\|_{L^2}\\
&\lesssim(\|\nabla u\|_{H^1}^2+\|\nabla\varrho\|_{L^2}^2)\|\Lambda^{-s}\varrho\|_{L^2},
\end{aligned}\tag{4.5}$$

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$$
\begin{aligned}
J_3 &= -\int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla u)\Lambda^{-s}u\,\mathrm{d}x \lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|u\cdot\nabla u\|_{L^{1/(1/2+s/3)}} \lesssim \|\Lambda^{-s}u\|_{L^2}\|u\|_{L^{3/s}}\|\nabla u\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\nabla u\|_{L^2}^{1/2+s}\|\nabla^2 u\|_{L^2}^{1/2-s}\|\nabla u\|_{L^2} \lesssim \|\nabla u\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
J_4 &= -\int_{\mathbb{R}^3} \Lambda^{-s}(h(\varrho)(\nu\Delta u+(\nu+\eta)\nabla\mathrm{div}\,u))\Lambda^{-s}u\,\mathrm{d}x\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(h(\varrho)(\nu\Delta u+(\nu+\eta)\nabla\mathrm{div}\,u))\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|h(\varrho)(\nu\Delta u+(\nu+\eta)\nabla\mathrm{div}\,u)\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|h(\varrho)\|_{L^{3/s}}\|\Delta u\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla^2 u\|_{L^2}\\
&\lesssim (\|\nabla\varrho\|_{H^1}^2+\|\nabla^2 u\|_{L^2}^2)\|\Lambda^{-s}u\|_{L^2},
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
J_5 &= -\int_{\mathbb{R}^3} \Lambda^{-s}(g(\varrho)\nabla\varrho)\Lambda^{-s}u\,\mathrm{d}x \lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(g(\varrho)\nabla\varrho)\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|g(\varrho)\nabla\varrho\|_{L^{1/(1/2+s/3)}} \lesssim \|\Lambda^{-s}u\|_{L^2}\|g(\varrho)\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla\varrho\|_{L^2} \lesssim \|\nabla\varrho\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
J_6 &= \int_{\mathbb{R}^3} \Lambda^{-s}\left[\phi(\varrho)\mathrm{div}\left(\nabla\chi\otimes\nabla\chi-\frac{|\nabla\chi|^2}{2}\mathbb{I}_3\right)\right]\cdot\Lambda^{-s}u\,\mathrm{d}x\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}[\phi(\varrho)|\nabla\chi||\nabla^2\chi|]\|_{L^2} \lesssim \|\Lambda^{-s}u\|_{L^2}\|\phi(\varrho)|\nabla\chi||\nabla^2\chi|\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\||\nabla\chi||\nabla^2\chi|\|_{L^{1/(1/2+s/3)}} \lesssim \|\nabla\chi\|_{L^{3/s}}\|\nabla^2\chi\|_{L^2}\|\Lambda^{-s}u\|_{L^2}\\
&\lesssim \|\nabla^2\chi\|_{L^2}^{1/2-s}\|\nabla^3\chi\|_{L^2}^{1/2+s}\|\nabla^2\chi\|_{L^2}\|\Lambda^{-s}u\|_{L^2}\\
&\lesssim (\|\nabla^2\chi\|_{L^2}^2+\|\nabla^3\chi\|_{L^2}^2)\|\Lambda^{-s}u\|_{L^2},
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
J_7 &\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla\chi)\|_{L^2} \lesssim \|\Lambda^{-s}\chi\|_{L^2}\|u\cdot\nabla\chi\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla\chi\|_{L^2} \lesssim \|\nabla u\|_{L^2}^{1/2-s}\|\nabla^2 u\|_{L^2}^{1/2+s}\|\nabla\chi\|_{L^2}\|\Lambda^{-s}\chi\|_{L^2}\\
&\lesssim (\|\nabla u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2)\|\Lambda^{-s}\chi\|_{L^2},
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
J_8 &= \int_{\mathbb{R}^3} \Lambda^{-s}(\varphi(\varrho)\Delta\chi)\cdot\Lambda^{-s}\chi\,\mathrm{d}x\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\Lambda^{-s}(\varphi(\varrho)\Delta\chi)\|_{L^2} \lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\varphi(\varrho)\Delta\chi\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\varphi(\varrho)\|_{L^{3/s}}\|\Delta\chi\|_{L^2} \lesssim \|\nabla\varrho\|_{L^2}^{1/2-s}\|\nabla^2\varrho\|_{L^2}^{1/2+s}\|\Delta\chi\|_{L^2}\|\Lambda^{-s}\chi\|_{L^2}\\
&\lesssim (\|\nabla\varrho\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^2\chi\|_{L^2}^2)\|\Lambda^{-s}\chi\|_{L^2},
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
J_9 &\lesssim \|\Lambda^{-s}\chi\|_{L^2}(\|\Lambda^{-s}(\phi(\varrho)\chi^3\|_{L^2}+\|\Lambda^{-s}(\zeta(\varrho)\zeta(\varrho)\chi\|_{L^2})\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\left(\|\phi(\varrho)\chi^3\|_{L^{1/(1/2+s/3)}}+\|\zeta(\varrho)\zeta(\varrho)\chi\|_{L^{1/(1/2+s/3)}}\right)\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\left(\|\phi(\varrho)\|_{L^\infty}\|\chi\|_{L^\infty}\|\chi\|_{L^2}\|\chi\|_{L^{3/s}}+\|\zeta(\varrho)\|_{L^{3/s}}\|\zeta(\varrho)\|_{L^\infty}\|\chi\|_{L^2}\right)
\end{aligned}
$$

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$$\begin{aligned}
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\Big(\|\nabla\chi\|_{L^2}^{1/2}\|\Delta\chi\|_{L^2}^{1/2}\|\chi\|_{L^2}\|\nabla\chi\|_{L^2}^{1/2-s}\|\nabla^2\chi\|_{L^2}^{1/2+s}\\
&\quad+\|\nabla\varrho\|_{L^2}^{1/2-s}\|\nabla^2\varrho\|_{L^2}^{1/2+s}\|\nabla\varrho\|_{L^2}^{1/2}\|\Delta\varrho\|_{L^2}^{1/2}\|\chi\|_{L^2}\Big)\\
&\lesssim (\delta+1)(\|\nabla\varrho\|_{L^2}^2+\|\Delta\varrho\|_{L^2}^2+\|\nabla^2\chi\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2)\|\Lambda^{-s}\chi\|_{L^2}, \qquad (4.12)\\
J_{10} &\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\|\Lambda^{-s}(\nabla u\cdot\nabla\chi+u\cdot\nabla^2\chi)\|_{L^2}\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla u\cdot\nabla\chi\|_{L^{1/(1/2+s/3)}}+\|u\cdot\nabla^2\chi\|_{L^{1/(1/2+s/3)}})\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla\chi\|_{L^{3/s}}\|\nabla u\|_{L^2}+\|\nabla^2\chi\|_{L^2}\|u\|_{L^{3/s}})\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla^2\chi\|_{L^2}^{1/2+s}\|\nabla^3\chi\|_{L^2}^{1/2-s}\|\nabla u\|_{L^2}\\
&\quad+\|\nabla^2\chi\|_{L^2}\|\nabla u\|_{L^2}^{1/2+s}\|\nabla^2 u\|_{L^2}^{1/2-s})\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla^2\chi\|_{L^2}^2+\|\nabla^3\chi\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2), \qquad (4.13)\\
J_{11} &\lesssim \|\Lambda^{-s}(\varphi(\varrho)\nabla^3\chi+\varphi'(\varrho)\nabla\varrho\nabla^2\chi)\|_{L^2}\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\varphi(\varrho)\nabla^3\chi\|_{L^{1/(1/2+s/3)}}+\|\varphi'(\varrho)\nabla\varrho\nabla^2\chi\|_{L^{1/(1/2+s/3)}})\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\nabla^3\chi\|_{L^2}\|\varrho\|_{L^{3/s}}+\|\nabla\varrho\|_{L^2}\|\nabla^2\chi\|_{L^{3/s}})\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\nabla^3\chi\|_{L^2}\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\\
&\quad+\|\nabla\varrho\|_{L^2}\|\nabla^3\chi\|_{L^2}^{1/2+s}\|\nabla^4\chi\|_{L^2}^{1/2-s})\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\nabla^3\chi\|_{L^2}^2+\|\nabla\varrho\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2+\|\nabla^4\chi\|_{L^2}^2)\|\Lambda^{-s}\nabla\chi\|_{L^2}, \qquad (4.14)
\end{aligned}$$

and

$$\begin{aligned}
J_{12} &= \int_{\mathbb{R}^3}\Lambda^{-s}[\phi'(\varrho)\nabla\varrho(\chi^3-\chi)+\phi(\varrho)(3\chi^2-1)\nabla\chi]\cdot\Lambda^{-s}\nabla\chi\,\mathrm{d}x\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\big(\|\phi'(\varrho)\nabla\varrho\chi^3\|_{L^{1/(1/2+3/s)}}+\|\phi'(\varrho)\nabla\varrho\chi\|_{L^{1/(1/2+3/s)}}\\
&\quad+\|\phi(\varrho)\chi^2\nabla\chi\|_{L^{1/(1/2+s/3)}}+\|\zeta(\varrho)\zeta(\varrho)\nabla\chi\|_{L^{1/(1/2+s/3)}}\big)\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\Big(\|\chi\|_{L^\infty}^2\|\nabla\varrho\|_{L^2}\|\nabla\chi\|_{L^2}^{1/2+s}\|\nabla^2\chi\|_{L^2}^{1/2-s}\\
&\quad+\|\nabla\varrho\|_{L^2}\|\nabla\chi\|_{L^2}^{1/2+s}\|\nabla^2\chi\|_{L^2}^{1/2-s}\\
&\quad+\|\chi\|_{L^\infty}^2\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla\chi\|_{L^2}+\|\nabla\varrho\|_{L^2}^{1/2+s}\|\nabla^2\varrho\|_{L^2}^{1/2-s}\|\nabla\chi\|_{L^2}\Big)\\
&\lesssim (1+\delta^2)\|\Lambda^{-s}\nabla\chi\|_{L^2}(|\nabla\varrho\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2+\|\nabla^2\chi\|_{L^2}^2+\|\nabla^2\varrho\|_{L^2}^2). \qquad (4.15)
\end{aligned}$$

Plugging the estimates (4.4)–(4.15) into (4.3), we obtain (4.1).

Next, if $s\in(\frac{1}{2},\frac{3}{2})$, we can estimate J_1–J_{12} in a different way. Since $s\in(\frac{1}{2},\frac{3}{2})$, it is easy to see that $\frac{1}{2}+\frac{s}{3}<1$ and $2<\frac{3}{s}<6$. Then, lemmas 2.5 and 2.2 implies that

$$\begin{aligned}
J_1 &\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|\Lambda^{-s}(\varrho\nabla\cdot u)\|_{L^2}\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|\varrho\nabla\cdot u\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|\varrho\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}. \qquad (4.16)
\end{aligned}$$

Similarly, by using lemmas 2.5 and 2.2, the term J_2–J_{12} can be bound by

$$
\begin{aligned}
J_2 &\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla\varrho)\|_{L^2} \lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|u\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}\\
&\lesssim \|\Lambda^{-s}\varrho\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}, \qquad (4.17)\\
J_3 &\lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2} \lesssim \|\Lambda^{-s}u\|_{L^2}\|u\|_{L^{3/s}}\|\nabla u\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}, \qquad (4.18)\\
J_4 &\lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(h(\varrho)(\nu\Delta u+(\nu+\eta)\nabla\mathrm{div}\, u))\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|h(\varrho)\|_{L^{3/s}}\|\Delta u\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla^2 u\|_{L^2}, \qquad (4.19)\\
J_5 &\lesssim \|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(g(\varrho)\nabla\varrho)\|_{L^2} \lesssim \|\Lambda^{-s}u\|_{L^2}\|g(\varrho)\|_{L^{3/s}}\|\nabla\varrho\|_{L^2}\\
&\lesssim \|\Lambda^{-s}u\|_{L^2}\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\varrho\|_{L^2}, \qquad (4.20)\\
J_6 &\lesssim \|\Lambda^{-s}u\|_{L^2}\||\nabla\chi||\nabla^2\chi|\|_{L^{1/(1/2+s/3)}} \lesssim \|\nabla\chi\|_{L^{3/s}}\|\nabla^2\chi\|_{L^2}\|\Lambda^{-s}u\|_{L^2}\\
&\lesssim \|\nabla\chi\|_{L^2}^{s-1/2}\|\nabla^2\chi\|_{L^2}^{3/2-s}\|\nabla^2\chi\|_{L^2}\|\Lambda^{-s}u\|_{L^2}, \qquad (4.21)\\
J_7 &\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|u\cdot\nabla\chi\|_{L^{1/(1/2+s/3)}} \lesssim \|\Lambda^{-s}\chi\|_{L^2}\|u\|_{L^{3/s}}\|\nabla\chi\|_{L^2}\\
&\lesssim \|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla\chi\|_{L^2}\|\Lambda^{-s}\chi\|_{L^2}, \qquad (4.22)\\
J_8 &\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\varphi(\varrho)\Delta\chi\|_{L^{1/(1/2+s/3)}} \lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\varphi(\varrho)\|_{L^{3/s}}\|\Delta\chi\|_{L^2}\\
&\lesssim \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\Delta\chi\|_{L^2}\|\Lambda^{-s}\chi\|_{L^2}, \qquad (4.23)\\
J_9 &\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\phi(\varrho)(\chi^3-\chi)\|_{L^{1/(1/2+s/3)}}\\
&\lesssim \|\Lambda^{-s}\chi\|_{L^2}\|\phi(\varrho)\|_{L^2}(\|\chi\|_{L^\infty}^2+1)\|\chi\|_{L^{3/s}}\\
&\lesssim \|\varrho\|_{L^2}(\|\chi\|_{L^\infty}^2+1)\|\chi\|_{L^2}^{s-1/2}\|\nabla\chi\|_{L^2}^{3/2-s}\|\Lambda^{-s}\chi\|_{L^2}\\
&\lesssim (\delta^2+1)\|\varrho\|_{L^2}\|\chi\|_{L^2}^{s-1/2}\|\nabla\chi\|_{L^2}^{3/2-s}\|\Lambda^{-s}\chi\|_{L^2}, \qquad (4.24)\\
J_{10} &\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla u\cdot\nabla\chi\|_{L^{1/(1/2+s/3)}}+\|u\cdot\nabla^2\chi\|_{L^{1/(1/2+s/3)}})\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla\chi\|_{L^{3/s}}\|\nabla u\|_{L^2}+\|\nabla^2\chi\|_{L^2}\|u\|_{L^{3/s}})\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}(\|\nabla\chi\|_{L^2}^{s-1/2}\|\nabla^2\chi\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\\
&\quad +\|\nabla^2\chi\|_{L^2}\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}), \qquad (4.25)\\
J_{11} &\lesssim (\|\varphi(\varrho)\nabla^3\chi\|_{L^{1/(1/2+s/3)}}+\|\nabla\varrho\nabla^2\chi\|_{L^{1/(1/2+s/3)}})\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\nabla^3\chi\|_{L^2}\|\varrho\|_{L^{3/s}}+\|\nabla\varrho\|_{L^{3/s}}\|\nabla^2\chi\|_{L^2})\|\Lambda^{-s}\nabla\chi\|_{L^2}\\
&\lesssim (\|\nabla^3\chi\|_{L^2}\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\\
&\quad +\|\nabla^2\chi\|_{L^2}\|\nabla\varrho\|_{L^2}^{s-1/2}\|\nabla^2\varrho\|_{L^2}^{3/2-s})\|\Lambda^{-s}\nabla\chi\|_{L^2}, \qquad (4.26)
\end{aligned}
$$

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and

$$\begin{aligned}
J_{12} &\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\left(\|\phi'(\varrho)\nabla\varrho(\chi^3-\chi)\|_{L^{1/(1/2+3/s)}} + \|\phi(\varrho)(3\chi^2-1)\nabla\chi\|_{L^{1/(1/2+s/3)}}\right)\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\left(\|\phi'(\varrho)\|_{L^\infty}(1+\|\chi\|_{L^\infty}^2)\|\nabla\varrho\|_{L^2}\|\chi\|_{L^{3/s}}\right.\\
&\quad\left. + (1+\|\chi\|_{L^\infty}^2)\|\varrho\|_{L^{3/s}}\|\nabla\chi\|_{L^2}\right)\\
&\lesssim \|\Lambda^{-s}\nabla\chi\|_{L^2}\Big((1+\|\chi\|_{L^\infty}^2)\|\nabla\varrho\|_{L^2}\|\chi\|_{L^2}^{s-1/2}\|\nabla\chi\|_{L^2}^{3/2-s}\\
&\quad + (1+\|\chi\|_{L^\infty}^2)\|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\chi\|_{L^2}\Big)\\
&\lesssim (1+\delta^2)\|\Lambda^{-s}\nabla\chi\|_{L^2}\Big(\|\nabla\varrho\|_{L^2}\|\chi\|_{L^2}^{s-1/2}\|\nabla\chi\|_{L^2}^{3/2-s}\\
&\quad + \|\varrho\|_{L^2}^{s-1/2}\|\nabla\varrho\|_{L^2}^{3/2-s}\|\nabla\chi\|_{L^2}\Big).
\end{aligned} \tag{4.27}$$

Plugging the estimates (4.16)–(4.27) into (4.3), we obtain (4.2). Hence, the proof is complete. □

5. Proof of theorem 1.1

We first close the energy estimates at each l-th level in our weaker sense. Suppose that $N \geqslant 3$ and $0 \leqslant l \leqslant m-1$ with $1 \leqslant m \leqslant N$. Summing up the estimates (3.4) from $k = l$ to $m-1$, since $\sqrt{\mathcal{E}_0^3(t)} \leqslant \delta$ is sufficiently small, we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\leqslant k\leqslant m-1}(\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^k\nabla\chi\|_{L^2}^2)\\
&\quad + C\sum_{l+1\leqslant k\leqslant m}(\|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2)\\
&\qquad \lesssim (\delta^3+\delta)\sum_{l+1\leqslant k\leqslant m}\left(\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2\right).
\end{aligned} \tag{5.1}$$

Moreover, let $k = m-1$ in the estimates (3.32). Hence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^m\varrho\|_{L^2}^2 + \|\nabla^m u\|_{L^2}^2 + \|\nabla^m\chi\|_{L^2}^2 + \|\nabla^{m+1}\chi\|_{L^2}^2)\\
&\quad + C(\|\nabla^{m+1}u\|_{L^2}^2 + \|\nabla^{m+1}\chi\|_{L^2}^2 + \|\nabla^{m+2}\chi\|_{L^2}^2)\\
&\qquad \lesssim (\delta^3+\delta)(\|\nabla^m\varrho\|_{L^2}^2 + \|\nabla^{m+1}u\|_{L^2}^2 + \|\nabla^{m+1}\chi\|_{L^2}^2 + \|\nabla^{m+2}\chi\|_{L^2}^2).
\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2) together gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\leqslant k\leqslant m}(\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^k\nabla\chi\|_{L^2}^2)\\
&\quad + C_1\sum_{l+1\leqslant k\leqslant m+1}(\|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^{k+1}\chi\|_{L^2}^2)\\
&\qquad \lesssim C_2(\delta^3+\delta)\sum_{l+1\leqslant k\leqslant m}\|\nabla^k\varrho\|_{L^2}^2.
\end{aligned} \tag{5.3}$$

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Summing up the estimates (3.49) from $k = l$ to $m - 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\leqslant k\leqslant m-1}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla^{k+1}\varrho\,\mathrm{d}x + C_3\sum_{l+1\leqslant k\leqslant m}\|\nabla^k\varrho\|_{L^2}^2 \lesssim C_4\left(\sum_{l+1\leqslant k\leqslant m+1}\|\nabla^k u\|_{L^2}^2 + \sum_{l+2\leqslant k\leqslant m+1}\|\nabla^k\nabla\chi\|_{L^2}^2\right). \tag{5.4}$$

Multiplying (5.4) by $C_5 \equiv 2C_2\delta(\delta^2 + 1)/C_3$, adding the resulting inequality with (5.1), since $\delta > 0$ is sufficiently small, we deduce that there exists a positive constant C_6 such that for $0 \leqslant l \leqslant m - 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{l\leqslant k\leqslant m}(\|\nabla^k\varrho\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^k\nabla\chi\|_{L^2}^2) + C_5\sum_{l\leqslant k\leqslant m-1}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla^{k+1}\varrho\,\mathrm{d}x\right\} + C_6\left\{\sum_{l+1\leqslant k\leqslant m}\|\nabla^k\varrho\|_{L^2}^2 + \sum_{l+1\leqslant k\leqslant m+1}(\|\nabla^k u\|_{L^2}^2 + \|\nabla^k\chi\|_{L^2}^2 + \|\nabla^k\nabla\chi\|_{L^2}^2\right\} \leqslant 0. \tag{5.5}$$

Define $\mathcal{E}_l^m(t)$ to be $1/C_6$ times the expression under the time derivative in (5.5). It is easy to see that since $\delta > 0$ is so small, $\mathcal{E}_l^m(t)$ is equivalent to

$$\|\nabla^l\varrho\|_{H^{m-l}}^2 + \|\nabla^l u\|_{H^{m-l}}^2 + \|\nabla^l\chi\|_{H^{m-l}}^2 + \|\nabla^l\nabla\chi\|_{H^{m-l}}^2.$$

Then, (5.5) can be rewritten as that for $0 \leqslant l \leqslant m - 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l^m(t) + \|\nabla^l\varrho\|_{H^{m-l}}^2 + \|\nabla^{l+1}u\|_{H^{m-l}}^2 + \|\nabla^{l+1}\chi\|_{H^{m-l}}^2 + \|\nabla^{l+1}\nabla\chi\|_{H^{m-l}}^2 \leqslant 0. \tag{5.6}$$

Taking $l = 0$ and $m = 3$ in (5.6), integrating directly in time, it yields that

$$\begin{aligned}\|\varrho\|_{H^3}^2 + \|u\|_{H^3}^2 + \|\chi\|_{H^3}^3 + \|\nabla\chi\|_{H^3}^2 &\lesssim \mathcal{E}_0^3(0)\\ &\lesssim \|\varrho_0\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|\chi_0\|_{H^3}^3 + \|\nabla\chi_0\|_{H^3}^2.\end{aligned} \tag{5.7}$$

Then, by a standard continuity argument, this closes the *a priori* estimates (3.1) if at the initial time we assume that $\|\varrho_0\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|\chi_0\|_{H^3}^3 + \|\nabla\chi_0\|_{H^3}^2 \leqslant \delta_0$ is sufficiently small. This in turn allows us to take $l = 0$ and $m = N$ in (5.6), and then integrate it directly in time to obtain (1.6).

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Next, we prove the decay rate of solutions for $s \in [0, \frac{1}{2}]$. Define

$$\mathcal{E}_{-s}(t) = \|\Lambda^{-s}\varrho\|_{L^2}^2 + \|\Lambda^{-s}u\|_{L^2}^2 + \|\Lambda^{-s}\chi\|_{L^2}^2 + \|\Lambda^{-s}\nabla\chi\|_{L^2}^2.$$

Then, integrating in time (4.1), by the bound (1.6), we derive that

$$\begin{aligned} \mathcal{E}_{-s}(t) &\leqslant \mathcal{E}_{-s}(0) + C\int_0^t (\|\varrho\|_{H^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla\chi\|_{H^3}^2)\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \\ &\leqslant C_0\left(1 + \sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)}\right), \end{aligned} \tag{5.8}$$

which implies (1.7) for $s \in [0, \frac{1}{2}]$, that is

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\chi(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla\phi(t)\|_{L^2}^2 \leqslant C_0. \tag{5.9}$$

Moreover, if $l = 1, 2, \dots, N-1$, we may use lemma 2.6 to have

$$\|\nabla^{l+1}f\|_{L^2} \geqslant C\|\Lambda^{-s}f\|_{L^2}^{-(1/(l+s))}\|\nabla^l f\|_{L^2}^{1+(1/(l+s))}.$$

Then, by this facts and (5.9), we get

$$\|\nabla^{l+1}(u,\chi,\nabla\chi)\|_{L^2}^2 \geqslant C_0(\|\nabla^l(u,\chi,\nabla\chi)\|_{L^2}^2)^{1+(1/(k+s))}. \tag{5.10}$$

Thus, for $1 = 1, 2, \dots, N-1$,

$$\|\nabla^{l+1}(u,\chi,\nabla\chi)\|_{H^{N-l-1}}^2 \geqslant C_0(\|\nabla^l(u,\chi,\nabla\chi)\|_{H^{N-l}}^2)^{1+(1/(l+s))}.$$

Thus, we deduce from (5.6) with $m = N$ the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_l^N + C_0\left(\mathcal{E}_l^N\right)^{1+(1/(l+s))} \leqslant 0, \quad \text{for } l = 1, 2, \dots, N-1, \tag{5.11}$$

which implies

$$\mathcal{E}_l^N(t) \leqslant C_0(1+t)^{-l-s}, \quad \text{for } l = 1, 2, \dots, N-1. \tag{5.12}$$

Hence, (1.7) holds.

On the other hand, the arguments for $s \in [0, \frac{1}{2}]$ cannot be applied to $s \in (\frac{1}{2}, \frac{3}{2})$. However, observing that $\varrho_0, u_0, \chi_0, \nabla\chi_0 \in \dot{H}^{-(1/2)}$ hold since $\dot{H}^{-s}\bigcap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$, we can deduce from what we have proved for (1.7)–(1.8) with $s = \frac{1}{2}$

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that the following estimate holds for $l = 0, 1, \ldots, N-1$:

$$\|\nabla^l \varrho\|_{H^{N-l}}^2 + \|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l \chi\|_{H^{N-l}}^2 + \|\nabla^l \nabla\chi\|_{H^{N-l}}^2 \leqslant C_0(1+t)^{-(1/2)-l}. \tag{5.13}$$

Therefore, we deduce from (4.2) that for $s \in (\frac{1}{2}, \frac{3}{2})$,

$$\begin{aligned} \mathcal{E}_{-s}(t) &\leqslant \mathcal{E}_{-s}(0) + C\int_0^t (\|\nabla\varrho\|_{H^1}^2 + \|\Delta\chi\|_{L^2}^2)\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \\ &\quad + C\int_0^t \|(\varrho, u, \chi, \nabla\chi)\|_{L^2}^{s-1/2}(\|\varrho\|_{H^2} + \|\nabla u\|_{H^1} \\ &\quad + \|\nabla\chi\|_{H^1} + \|\nabla^2\chi\|_{H^1})^{5/2-s}\sqrt{\mathcal{E}_{-s}(\tau)}\,\mathrm{d}\tau \\ &\leqslant C + C\sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)} + C\int_0^t (1+\tau)^{-(7/4)-(s/2)}\,\mathrm{d}\tau \sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)} \\ &\lesssim 1 + \sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)}, \end{aligned} \tag{5.14}$$

which implies that (1.7) holds for $s \in (\frac{1}{2}, \frac{3}{2})$, i.e.

$$\|\Lambda^{-s}\varrho(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}\chi(t)\|_{L^2}^2 + \|\Lambda^{-s}\nabla\phi(t)\|_{L^2}^2 \leqslant C_0. \tag{5.15}$$

Since we have proved (5.15), we may repeat the arguments leading to (1.8) for $s \in [0, \frac{1}{2}]$ to prove that they also hold for $s \in (\frac{1}{2}, \frac{3}{2})$. Therefore, the proof of theorem 1.1 is complete.

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