A GENERALISATION OF MINKOWSKI'S SECOND INEQUALITY IN THE GEOMETRY OF NUMBERS

A. C. WOODS

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Let K be a bounded open convex set in euclidean n-space R_n symmetric in the origin 0. Further let L be a discrete point set in R_n containing 0 and at least n linearly independent points of R_n. Put $m_i = \inf u_i$ extended over all positive real numbers u_i for which $u_i K$ contains i linearly independent points of L, $i = 1, 2, \cdots, n$.

Denote by P a parallelopiped in R_n centred at 0. For each positive integer N denote by $L(NP)$ the number of points of L contained in NP. Put

$$d(L, P) = \liminf \left(V(NP)/L(NP)\right), \qquad N \to \infty,$$

where $V(NP)$ denotes the Jordan content of NP and

$$d(L) = \inf d(L, P)$$

extended over all nondegenerate parallelopipeds P. It is assumed here that L is such that $d(L)$ is finite and positive. In particular if L is a lattice then $d(L)$ is the determinant of L and in this case Minkowski's second inequality in the geometry of numbers asserts that

$$m_1 m_2 \cdots m_n V(K) \leqq 2^n d(L). \tag{1}$$

The object here is to show that (1) remains true if the restriction of L to a lattice is replaced by the weaker condition

(A) if $X \in L$ and $Y \in L$ then either $X-Y$ or $Y-X$ is in L. Examples of such sets are obtained by taking a lattice Λ, a positive integer m and a fixed point X of R_n such that the sets

$$\Lambda, \Lambda+X, \Lambda+2X, \cdots, \Lambda+mX$$

are pairwise disjoint. With L as the union of these $m+1$ sets it is evident that L satisfies (A) and that $d(L) = d(\Lambda)/(m+1)$. In particular with $m = 1$, L becomes the familiar double lattice relating homogeneous and inhomogeneous problems in the geometry of numbers and (1) becomes a transference theorem. M. Bleicher has pointed out in correspondence that there are sets L other than those given here with property A.

Minkowski's original proof of (1) for lattices has been simplified by Weyl [3] and Cassels [4], and a quite different proof has been given by Davenport [1]. Further simplifications of both lines of proof are given in Bambah, Woods and Zassenhaus [5]. It is the version by Bambah of Davenports proof that is used here to obtain the generalisation stated above and it is interesting to note that the Minkowski method of proof appears to break down for this wider class of sets.

2. Proof of the generalisation

THEOREM 1. *Suppose that S is a bounded set in R_n. Let $\chi(X)$ be its characteristic function. Suppose further that*

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi(X)\, dx_1 \cdots dx_n$$

exists and $I > d(L)$. Then the sets $S+A$, $A \in L$ overlap so that there exists points X, Y in S such that $X-Y \in L$.

PROOF. By way of contradiction assume that the sets $S+A$, $A \in L$ do not overlap. Let P denote the cube given by

$$\max\,(|x_1|, \cdots, |x_n|) \leqq 1.$$

Since S is bounded there exists a positive integer k such that S is contained in kP. For a fixed positive integer N consider the set Z of points A of L that lie in the cube NP. Define

$$F(X) = \sum_{A \in Z} \chi(X-A).$$

By hypothesis the sets $S+A$ do not overlap, hence $F(X) \leqq 1$ for all X. Further $F(X) = 0$ if X is not in the cube $(N+k)P$. Therefore

$$\begin{aligned}
(2N+2k)^m &\geqq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(X)\, dx_1 \cdots dx_n \\
&= \sum_{A \in Z} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi(X-A)\, dx_1 \cdots dx_n \\
&= \sum_{A \in Z} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi(X)\, dx_1 \cdots dx_n \\
&= L(NP) I.
\end{aligned}$$

Thus

$$I \leqq (V(NP)/L(NP))\,((2N+2k)/2N)^n).$$

Let N tend to infinity; this implies that

$$I \leqq d(L, P) \text{ and therefore also } I \leqq d(L),$$

the contradiction that proves the theorem.

LEMMA 1. *If T is a nonsingular transformation of R_n into itself then $d(T(L)) = ||T||d(L)$.*

PROOF. Let P be a parallelopiped centred at 0. For a fixed positive integer N the number of points of $T(L)$ in NP is the same as the number of points of L in $NT^{-1}(P)$. Hence

$$\begin{aligned} d(T(L), P) &= \liminf \,[V(NP)/L(NT^{-1}(P))] \\ &= ||T|| \liminf \,[V(NT^{-1}(P))/L(NT^{-1}(P))] \\ &= ||T||d(L, T^{-1}(P)). \end{aligned}$$

Thus $d(T(L)) = \inf_P ||T||d(L, T^{-1}(P)) = ||T||d(L)$, which proves the lemma.

LEMMA 2. *Let $n_1, n_2, \cdots, n_n$ be n positive real numbers such that*

$$n_1 \leqq n_2 \leqq \cdots \leqq n_n.$$

There exist sets $K_1, K_2, \cdots, K_n$ such that

(i) $K_1 = \frac{1}{2}n_1 K$,

(ii) $K_i \subset \frac{1}{2}n_i K$ *for* $i = 1, 2, \cdots, n$,

(iii) *If $X, Y \in K_i$ for $i > 1$ and $x_i = y_i, \cdots, x_n = y_n$ then there exist points X', Y' in K_{i-1} such that $X-Y = X'-Y'$,*

(iv) *If $\chi_i(X)$ is the characteristic function of K_i then*

$$V_i = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \chi_i(X)\, dx_1 \cdots dx_n$$

exists and
$$V_i = \begin{cases} (\frac{1}{2}n_1)^n V(K) & \text{if } i = 1 \\ (n_i/n_{i-1})^{n-i+1} V_{i-1} & \text{if } i > 1. \end{cases}$$

A proof of this lemma is given in [5].

THEOREM 2. *If L has property (A) then* (1) *holds.*

PROOF. Since L is discrete and contains n linearly independent points of R_n, it follows that there exist n linearly independent points $F_1, F_2, \cdots, F_n$ of L such that F_i lies on the boundary of $m_i K$ for each $i = 1, 2, \cdots, n$. From lemma 1 it follows that (1) is an invariant inequality under nonsingular linear transformations of R_n, so without loss of generality it may be assumed that the coordinate system is such that F_i has coordinates of the form $(f_1, f_2, \cdots, f_i, 0, \cdots, 0)$ for each $i = 1, 2, \cdots, n$. Now if $\frac{1}{2}m_i K$ contains two points $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ such that $X-Y$ is in L then, by the convexity and symmetry of K, $X-Y$ is in $m_i K$ and so cannot be linearly independent of $F_1, \cdots, F_{i-1}$; hence $x_i = y_i, \cdots, x_n = y_n$. Apply lemma 2 with $n_i = m_i$ for $i = 1, 2, \cdots, n$ to obtain the sets $K_1, \cdots, K_n$. Assume by way of contradiction that (1) is false so that

$$m_1 \cdots m_n V(K) > 2^n d(L).$$

Then

$$V_n = m_1 \cdots m_n (\tfrac{1}{2})^n V(K) > d(L)$$

and by theorem 1 there exist points X, Y in K_n such that $0 \neq X-Y \in L$. Since $K_n \subset \frac{1}{2} m_n K$ this implies that $x_n = y_n$, and by property (1) of the sets K_i, K_{n-1} contains points $X^{(1)}$, $Y^{(1)}$ such that

$$X-Y = X^{(1)} - Y^{(1)}.$$

Since $K_{n-1} \subset \frac{1}{2} m_{n-1} K$ this implies that $x^{(1)}_{n-1} = y^{(1)}_{n-1}$, $x^{(1)}_n = y^{(1)}_n$ and there exist points $X^{(2)}$, $Y^{(2)}$ in K_{n-2} with

$$X-Y = X^{(2)} - Y^{(2)}.$$

Repeating this argument a number of times we obtain points X^*, Y^* in $\frac{1}{2} m_1 K$ such that $X-Y = X^* - Y^*$. But $X^* - Y^* \in L$ implies that $X^* = Y^*$ and $X - Y = 0$, which is a contradiction. This proves the theorem.

3. A comment on Minkowski's method

Minkowski's method relies upon the fact that the measure of any measurable subset of R_n in the quotient space of R_n modulo a lattice of dimension $\leqq n$ is monotone, that is to say, if $C < C'$ then the measure of C does not exceed the measure of C' in the quotient space. Such a measure can be generalised to sets L other than lattices as follows. Let C be a measurable subset of R_n. If C is bounded there exist at most a finite number of points $Z_1, \cdots, Z_k$ other than 0 such that $C + Z_i$ intersects C. Denote by M_{i+1} the measure of the set of points X of C such that C lies in exactly i of the sets $C+Z_1, \cdots, C+Z_k$ and define

$$M(C) = M_1 + \tfrac{1}{2} M_2 + \tfrac{1}{3} M_3 + \cdots \tfrac{1}{k} M_k.$$

Now if $M(C)$ is monotone so that $C \subset C'$ implies $M(C) \leqq M(C')$ for bounded measurable sets C, C' then it is possible to show that (1) holds provided $d(L)$ is replaced by the upper bound of $M(C)$ taken over all measurable bounded sets C. However the author has been unable to find any interesting sets L other than lattices that have this property.

References

[1] H. Davenport; "Minkowski's inequality for the minimum associated with a convex body", *Quart. J. Math.* (*Oxford*) 10 (1939), 199—121.
[2] H. Minkowski, *Geometrie der Zahlen* (Teubner 1896), Chapter 5.
[3] H. Weyl, "On geometry of numbers", *Proc. Lond. Math. Soc.* (2) 47 (1942), 268—289.
[4] J. W. S. Cassels, *An introduction to the geometry of numbers* (Springer, 1959).
[5] R. P. Bambah, A. C. Woods and H. Zassenhaus, "Three proofs of Minkowski's second inequality in the geometry of numbers", J. Aust. Math. Soc. 5 (1965), 453—462.

The Ohio State University
Columbus, Ohio