Ergod. Th. & Dynam. Sys. (2015), **35**, 2198–2215
doi:10.1017/etds.2014.16

Dynamics of piecewise contractions of the interval

ARNALDO NOGUEIRA† and BENITO PIRES‡

† *Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France*
(e-mail: arnaldo.nogueira@univ-amu.fr)
‡ *Departamento de Computação e Matemática, Faculdade de Filosofia, Ciências e Letras, Universidade de São Paulo, 14040-901, Ribeirão Preto - SP, Brazil*
(e-mail: benito@usp.br)

(*Received* 14 *January* 2013 *and accepted in revised form* 7 *February* 2014)

Abstract. We study the long-term behavior of injective piecewise contractions of the interval. We prove that every injective piecewise contraction with $n-1$ discontinuities has at most n periodic orbits and is topologically conjugate to a piecewise linear contraction.

1. *Introduction*

A map $f : [0, 1) \to [0, 1)$ is a *piecewise contraction (PC) of n intervals* if there exist $\kappa \in (0, 1)$ and a partition of the interval $[0, 1)$ into n intervals $I_1, \ldots, I_n$ such that for every $i \in \{1, \ldots, n\}$, the restriction $f|_{I_i}$ is κ-Lipschitz.

In this article we address two aspects of the dynamics of a PC. The first issue concerns an upper bound for the number of periodic orbits of a PC of n intervals. The first attempt to find such upper bound was done in [**2**]. However, the upper bound proposed therein, $2(n-1)$ for $n \geq 2$, besides not being sharp, holds only for typical elements of a special class of injective PCs. In this respect, we prove that n is the sharp upper bound for the number of periodic orbits of every injective PC of $n \geq 1$ intervals. The second issue is related to the linearization of an injective piecewise contraction.

Before stating our main results, we need the following definition. A point $p \in [0, 1]$ is an *ω-limit point* of x if there is a sequence of positive integers $n_1 < n_2 < \cdots$ such that $\lim_{\ell\to\infty} f^{n_\ell}(x) = p$. The collection of all such ω-limit points is the *ω-limit set of x*, denoted by $\omega(x)$. We say that f is *asymptotically periodic* if $\omega(x)$ is a periodic orbit of f for every $x \in [0, 1)$.

THEOREM 1.1. *Every injective PC of n intervals f has at most n periodic orbits. Moreover, if f has n periodic orbits, then f is asymptotically periodic.*

THEOREM 1.2. *Every injective piecewise contraction of n intervals is topologically conjugate to a piecewise linear contraction of n intervals whose slopes in absolute value equal* $\frac{1}{2}$.

The upper bound n given in Theorem 1.1 is sharp. In fact, the PC of n intervals f defined by $f(x) = \frac{1}{2}x + \frac{1}{2}((i-1)/n + 1/2n)$ if $x \in I_i = [(i-1)/n, i/n)$, $1 \le i \le n$, has n stable fixed points (and hence n periodic orbits). Although a typical injective PC of the interval is asymptotically periodic (see [**11**]), there are examples of order-preserving injective PCs of two intervals having irrational rotation number and no periodic orbit (see [**3**, Remark, p. 1391]). In these examples, instead of a periodic orbit, the attractor is a Cantor set. The second assertion in Theorem 1.1 says that if the PC has the maximum number of periodic orbits allowed then no space is left for a Cantor set; thus the PC has to be asymptotically periodic.

Theorem 1.2 is a generalization of [**5**, Lemma 3, p. 314] where it is proved that first return maps to a transverse interval of some Cherry flows are, up to topological conjugacy, piecewise linear contractions having no periodic orbit. The topological conjugacy can be made smooth in many cases (see [**6**]).

Injective PCs of the interval also arise as Poincaré maps of strange billiards governing switched server systems (see [**1**, **4**, **9**]), and in the study of a certain class of outer billiards (see [**8**]). Within the framework of interval exchange transformations with flip, it was proved in [**10**, p. 524] that periodic orbits are a typical phenomenon. Recently in [**12**, Theorem A, p. 3] it was shown that n is the sharp upper bound for the number of periodic components of every interval exchange transformation having n continuity intervals.

We call attention to two articles which are related to our work. In [**7**], all the possible itineraries of a piecewise contraction $f : X_1 \cup X_2 \to X_1 \cup X_2$ were listed, where X_i is a complete metric space and $f|_{X_i}$ is a contraction for $i = 1, 2$. In [**3**], the dynamics of a typical piecewise linear contraction of $\mathbb{C}$ was studied.

Henceforth, let $f : [0, 1) \to [0, 1)$ be an injective PC with partition intervals $I_1, \ldots, I_n$. Assume that I_i, $1 \le i \le n$, has endpoints at x_{i-1} and x_i, where $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. To simplify matters, suppose that every x_i, $1 \le i \le n-1$, is a jump discontinuity of f.

The proof of Theorem 1.1 is much easier in the special case where $I_i = [x_{i-1}, x_i)$ and $f|_{I_i}$ is increasing for every $i \in \{1, \ldots, n\}$. In this case, all the periodic orbits are attractive and thus easily detected: we count them by counting the attractors defined by them.

Here we consider the general case where I_i can be any of the intervals (x_{i-1}, x_i), $[x_{i-1}, x_i)$, $(x_{i-1}, x_i]$, $[x_{i-1}, x_i]$. We also allow the restriction $f|_{I_i}$ to be decreasing for some $i \in \{1, \ldots, n\}$. The general case turns out to be much more difficult to deal with because a new phenomenon appears: the presence of degenerate periodic orbits which attract no other points besides those in themselves; thus their basins of attraction have empty interiors. Since such orbits cannot be detected through their basins of attraction, our approach is to show that each such orbit rules out an attractive periodic orbit. That is achieved through a Combinatorial Lemma (Lemma 5.1). Counting attractive periodic orbits in the general case is not as easy as counting them in the piecewise increasing case: the result is only provided in §4, by means of Theorem 4.1.

Note that $f(x_i) \in \{f(x_i^-), f(x_i^+)\}$ for every $i \in \{1, \ldots, n-1\}$, where $f(x_i^-) = \lim_{\epsilon\to 0^+} f(x_i - \epsilon)$ and $f(x_i^+) = \lim_{\epsilon\to 0^+} f(x_i + \epsilon)$. Theorem 1.1 states that, no matter how we define f at its jump discontinuities, f has at most n periodic orbits.

This article is organized in the following way. Theorem 3.3 describes the geometric structure of stable manifolds of regular periodic orbits of f. Theorem 4.1 provides the optimal upper bound for the number of regular (and thus attractive) periodic orbits of f. Lemma 5.8, which is obtained using Lemma 5.1, is a stronger version of Theorem 4.1. Theorem 1.1 is a corollary of Lemma 5.8. The proof of Theorem 1.2 depends only on Lemma 3.6.

2. *Trapping intervals and trapping regions*

For a set $G \subset [0, 1)$, denote by $\text{int}(G)$ the interior of G and by $\overline{G}$ its closure, with respect to the topology of the real line $\mathbb{R}$. The boundary of G is the set $\partial G = \overline{G}\backslash \text{int}(G)$. In this way, if $I \subset [0, 1)$ is an interval with endpoints at $a < b$ then $\text{int}(I) = (a, b)$ and $\overline{I} = [a, b]$. We omit double parentheses by setting $f(a, b) = f((a, b)) = \{f(x) \mid x \in (a, b)\}$.

Let f^0 be the identity map on $[0, 1)$ and let $f^\ell = f \circ f \circ \cdots \circ f$ be the ℓth-iterate of f. The *orbit* of a point $p \in [0, 1)$ is the set $O_f(p) = \{f^\ell(p) \mid \ell \geq 0\}$. The point p is *periodic* if there exists a positive integer k such that $f^k(p) = p$. If $k = \min\{\ell \geq 1 \mid f^\ell(p) = p\}$, then p is called a *k-periodic point*. An orbit is *periodic* (respectively, k-periodic) if its points are periodic (respectively, k-periodic).

A periodic point p is *internal* if $p \in (0, 1)\backslash\{x_1, \ldots, x_{n-1}\}$, otherwise p is an *external periodic point*. Hence, an external periodic point is either 0 or a discontinuity of f. A periodic orbit $\gamma = O_f(p)$ is *internal* if $\gamma \subset (0, 1)\backslash\{x_1, \ldots, x_{n-1}\}$, otherwise γ is said to be an *external periodic orbit*. In this way, a periodic orbit is internal if it contains only internal periodic points.

Throughout this article, interval means an interval with non-empty interior.

Definition 2.1. (Regular/degenerate periodic point) A periodic point p of f is *regular* if there exists an interval J containing p whose iterates $f^\ell(J)$, $\ell \geq 1$, are intervals. A periodic point is *degenerate* if it is not regular.

LEMMA 2.2. *A periodic point p of f is regular if and only if every point in its orbit is regular.*

Proof. Let p be a regular k-periodic point. By Definition 2.1, there exists an interval J such that for every $i \in \{0, \ldots, k-1\}$, the k-periodic point $f^i(p)$ is contained in the interval $f^i(J)$. Moreover, $f^\ell(f^i(J))$ is an interval for every $\ell \geq 0$. Thus $f^i(p)$ is also regular. □

By Lemma 2.2, it makes sense to define regular periodic orbit.

Definition 2.3. (Regular/degenerate periodic orbit) An orbit $\gamma = O_f(p)$ is *regular* if p is a regular periodic point, otherwise γ is said to be *degenerate*.

PROPOSITION 2.4. *Every periodic orbit of f that contains no discontinuity is regular.*

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Proof. Let $\gamma = O_f(p)$ be a k-periodic orbit of f containing no discontinuity. Firstly suppose that γ is internal, thus γ is contained in the interior of the set $A = [0, 1) \setminus \bigcup_{\ell=0}^{k-1} f^{-\ell}(\{x_1, \ldots, x_{n-1}\})$. Let $\epsilon > 0$ be so small that $J := [p - \epsilon, p + \epsilon]$ is contained in A. Thus, for every $\ell \in \{0, \ldots, k-1\}$, there exists $i(\ell) \in \{1, \ldots, n\}$ such that $f^\ell(J)$ is contained in the continuity interval $I_{i(\ell)}$. Consequently, the first k iterates $f(J), \ldots, f^k(J)$ of J are intervals. Moreover, $f^k(J)$ is an interval centered at p of ratio less than $\kappa^k \epsilon$, where $\kappa \in (0, 1)$ is the Lipschitz constant of f. Thus, $f^k(J) \subset J$. Therefore,

$$f^\ell(J) \subset f^{\ell \bmod k}(J) \subset I_{i(\ell \bmod k)} \quad \text{for every } \ell \geq 0.$$

In this way, $f^\ell(J)$ is an interval for every $\ell \geq 0$. Now suppose that γ is external, thus $\gamma = O_f(0)$ and $\gamma \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$. Therefore, there exists $\epsilon > 0$ such that $J := [0, \epsilon]$ is contained in A. By proceeding as above, we obtain that $f^\ell(J)$ is an interval for every $\ell \geq 0$; thus γ is regular. □

Besides the internal periodic orbits, there exist external periodic orbits that are regular. We will prove later that regular periodic orbits are attractive (and thus have basins of attraction with non-empty interiors), whereas degenerate periodic orbits may have basins of attraction reduced to the periodic orbits themselves.

Definition 2.5. (Trapping interval) We say that an interval J containing a k-periodic point p is a *trapping interval* of p if its iterates $f(J), \ldots, f^k(J)$ are intervals and $f^k(J) \subset J$.

Next we prove the existence of a trapping interval which contains every trapping interval of p.

LEMMA 2.6. *Let $\{J_\lambda : \lambda \in \Lambda\}$ be the family of all trapping intervals of the k-periodic point p, then $\bigcup_{\lambda \in \Lambda} J_\lambda$ is a trapping interval of p.*

Proof. By Definition 2.5, $p \in \bigcup_{\lambda \in \Lambda} J_\lambda$ and $f^\ell(\bigcup_{\lambda \in \Lambda} J_\lambda) = \bigcup_{\lambda \in \Lambda} f^\ell(J_\lambda)$ is an interval containing $f^\ell(p)$ for all $0 \leq \ell \leq k$. Moreover, $f^k(\bigcup_{\lambda \in \Lambda} J_\lambda) = \bigcup_{\lambda \in \Lambda} f^k(J_\lambda) \subset \bigcup_{\lambda \in \Lambda} J_\lambda$. □

LEMMA 2.7. (Existence of trapping intervals) *If p is a regular periodic point of f then p admits a maximal trapping interval J_p.*

Proof. Let p be a regular k-periodic point of f. By Definition 2.1, there exists an interval K containing p such that the iterates $f^\ell(K)$, $\ell = 0, 1, 2, \ldots$ are intervals. Let $J = \bigcup_{\ell \geq 0} f^{\ell k}(K)$, thus $f^m(J) = \bigcup_{\ell \geq 0} f^{m + \ell k}(K)$ is an interval for all $m \geq 1$. Moreover, $f^k(J) = \bigcup_{\ell \geq 1} f^{\ell k}(K) \subset J$. This proves that J is a trapping interval of p. The existence of the maximal trapping interval follows now from Lemma 2.6. □

Definition 2.8. We denote by J_p the maximal trapping interval of a regular periodic point p.

Definition 2.9. (Maximal trapping region) Let γ be a regular periodic orbit. We call the set $\Omega(\gamma) = \bigcup_{p \in \gamma} J_p$ the *maximal trapping region* of γ.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

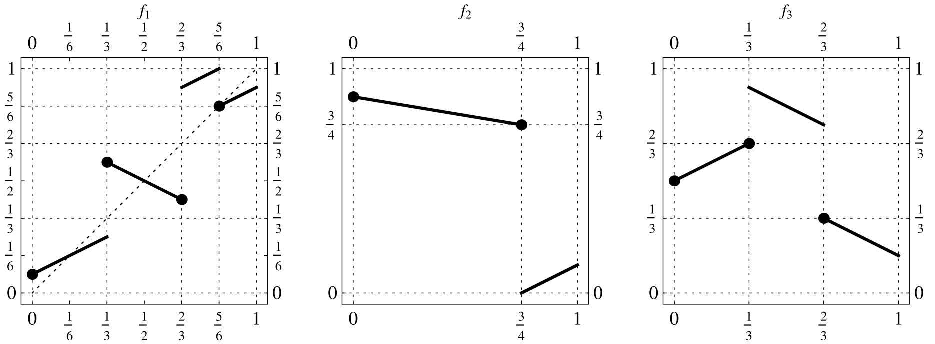

FIGURE 1. Distinct types of periodic points.

PROPOSITION 2.10. (Trapping region structure) *Let γ be a regular periodic orbit. Then its maximal trapping region $\Omega(\gamma)$ has the following properties:*

(TR1) $f(\Omega(\gamma)) \subset \Omega(\gamma)$;

(TR2) $\gamma = \bigcap_{\ell=0}^{\infty} f^{\ell}(\Omega(\gamma))$;

(TR3) $\Omega(\gamma)$ *is the union of k disjoint intervals, where k is the period of γ.*

Proof. We have that $f^{\ell}(f(J_p))$ is an interval for all $\ell \geq 0$. Moreover,

$$f^k(f(J_p)) = f(f^k(J_p)) \subset f(J_p),$$

and thus $f(J_p)$ is a trapping interval of $f(p)$, so $f(J_p) \subset J_{f(p)}$. Therefore,

$$f(\Omega(\gamma)) = f\bigg(\bigcup_{p\in\gamma} J_p\bigg) = \bigcup_{p\in\gamma} f(J_p) \subset \Omega(\gamma),$$

which proves (TR1). Let $p \in \gamma$, thus $p \in \bigcap_{\ell\geq 0} T^{\ell k}(J_p)$ and

$$|T^{\ell k}(J_p)| \leq \kappa^{\ell k}|J_p|,$$

where $|\cdot|$ stands for the length of the interval. Hence, $\bigcap_{\ell\geq 0} T^{\ell k}(J_p) = \{p\}$, which proves (TR2).

The item (TR3) follows straightforwardly from the Definition 2.9. □

Example 1. Figure 1 shows the graphs of three PCs f_1, f_2 and f_3. The points $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{5}{6}$ are regular periodic points of f_1. Their maximal trapping intervals are, respectively, $J_{p_1} = [0, 1/3)$, $J_{p_2} = [1/3, 2/3]$ and $J_{p_3} = [5/6, 1)$. The existence of such trapping intervals are ensured by Lemma 2.7.

The map f_2 shows that the claim of Lemma 2.7 is false for the degenerate periodic point $p_4 = \frac{3}{4}$. More precisely, the point p_4 is a degenerate external 1−periodic orbit of f_2 that attracts no other orbit (there is another periodic orbit that attracts all orbits different from $\{p_4\}$).

The point $p_5 = 1/3$ is an external 2-periodic point of f_3 that is also degenerate.

Remark. The next example shows that it may happen that $\overline{\Omega(\gamma)} \cap \{x_0, \ldots, x_n\}$ is a one-point-set for some regular periodic orbit γ.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Example 2. Let $g:[0,1)\to[0,1)$ be the 2-interval PC defined by $g(x)=-0.4x+0.6$ if $x\in[0,0.5)$, otherwise $g(x)=0.2\,(x-0.5)$. The point $p_1=\frac{3}{7}$ is a 1-periodic point of g whereas $p_2=\frac{16}{27}$ is a 2-periodic point of g. Moreover, $J_{p_1}=(\frac{1}{4},\frac{1}{2})$ is the maximal trapping interval of p_1 and $J_{p_2}=[\frac{1}{2},1)$ is the maximal trapping interval of p_2. For $\gamma=O_g(p_1)$ we have that $\overline{\Omega(\gamma)}=\overline{J_{p_1}}=[\frac{2}{5},\frac{1}{2}]$. Thus $\overline{\Omega(\gamma)}\cap\{x_0,x_1,x_2\}=\{x_1\}$, where $x_0=0$, $x_1=\frac{1}{2}$ and $x_2=1$.

LEMMA 2.11. *If γ_1 and γ_2 are two distinct regular periodic orbits of f then $\Omega(\gamma_1)\cap\Omega(\gamma_2)=\emptyset$.*

Proof. The proof follows easily from Proposition 2.10. □

3. *Stable manifolds of periodic orbits*

The *stable manifold* (also called the *basin of attraction*) of a periodic orbit γ of f is the set

$$W^s(\gamma)=\{x\in[0,1)\mid\omega(x)=\gamma\}\quad\text{where }\omega(x)=\bigcap_{m\in\mathbb{N}}\overline{\{f^\ell(x)\mid\ell\geq m\}}.$$

The following lemmas are immediate.

LEMMA 3.1. *Let γ be a periodic orbit; then $f(W^s(\gamma))\subset W^s(\gamma)$.*

Proof. Let $x\in W^s(\gamma)$. Then $\omega(x)=\gamma$, that is, $\bigcap_{m\in\mathbb{N}}\overline{\{f^\ell(x)\mid\ell\geq m\}}=\gamma$. Hence,

$$\omega(f(x))=\bigcap_{m\in\mathbb{N}}\overline{\{f^{\ell+1}(x)\mid\ell\geq m\}}=\bigcap_{m\in\mathbb{N}}\overline{\{f^\ell(x)\mid\ell\geq m\}}=\gamma,$$

and hence $f(x)\in W^s(\gamma)$. □

LEMMA 3.2. *If γ_1 and γ_2 are two distinct regular periodic orbits of f then $W^s(\gamma_1)\cap W^s(\gamma_2)=\emptyset$.*

Proof. It follows from the fact that regular periodic orbits are attractive (see Lemma 2.7). □

The stable manifold of a regular periodic orbit γ contains the trapping region of γ, that is, $\Omega(\gamma)\subset W^s(\gamma)$. The stable manifold of a periodic orbit may also include finite sets or intervals that are attracted by the trapping region.

Example 3. In Figure 2, the map $h_1:[0,1)\to[0,1)$ is a PC of 4 intervals. The point $p=\frac{3}{8}$ is a fixed point of h_1. In addition, the 1-periodic orbit $\gamma=O_{h_1}(p)$ is internal and hence regular. It is easy to show that $W^s(\gamma)=[\frac{1}{4},\frac{1}{2})\cup\{\frac{3}{4}\}$.

In figure 2, the map $h_2:[0,1)\to[0,1)$ is a PC of 5 intervals having positive constant slope. Notice that the 1-periodic points $p_1=\frac{1}{10}$ and $p_2=\frac{9}{10}$ are regular whereas the 3-periodic point $p_3=\frac{1}{5}$ is degenerate. Moreover, the stable manifolds of $\gamma_1=O_{h_2}(p_1)$ and $\gamma_2=O_{h_2}(p_2)$ satisfy $W^s(\gamma_1)\cup W^s(\gamma_2)=[0,1)\backslash\gamma_3$. In Figure 1, $\gamma=\{\frac{3}{4}\}$ is a degenerate 1-periodic orbit of f_2 and $W^s(\gamma)=\{\frac{3}{4}\}$.

In general, the geometric structure of the stable manifold of a regular periodic orbit is given by the next result, which turns out to be of paramount importance for the proof of Theorem 1.1.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

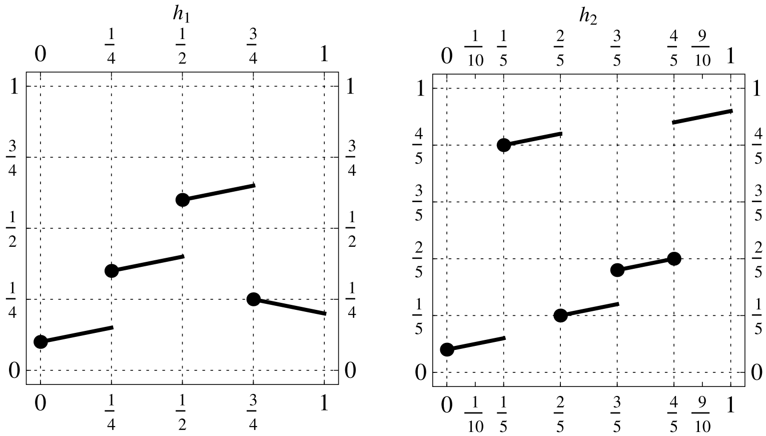

FIGURE 2. Stable manifolds of periodic orbits.

THEOREM 3.3. *If γ is a regular periodic orbit of f, then the interior of $W^s(\gamma)$ is the union of finitely many open intervals.*

We postpone the proof of Theorem 3.3 to the end of this section. Now we will describe the key points necessary for its proof.

First, we will define a family of finitely many pairwise disjoint open intervals $F_1, F_2, \ldots, F_r$ whose iterates $f^\ell(F_j)$ never meet the discontinuity set $\{x_1, \ldots, x_{n-1}\}$ of f. In this way, $f^\ell(F_j)$ is an interval for every $j \in \{1, \ldots, r\}$ and $\ell \geq 0$. The next step is to show that the union of the forward orbits $O_f(F_j) = \bigcup_{\ell=0}^{\infty} f^\ell(F_j)$ covers the interval $[0, 1)$ up to a null Lebesgue measure set. In this way, eventually some of these intervals will enter the trapping regions of the regular periodic orbits and stay there thereafter. As we show in Lemma 3.7, the orbit of an interval F_j can enter at most one trapping region. The time that the interval F_j takes to be captured by a trapping region $\Omega(\gamma)$ of a regular periodic orbit γ is called the *target time* and is denoted by $\tau(F_j, \gamma)$. We set $\tau(F_j, \gamma) = +\infty$ if $O_f(F_j) \cap \Omega(\gamma) = \emptyset$.

Theorem 3.3 will then follow once we prove that for each regular periodic orbit γ

$$\text{int}(W^s(\gamma)) = \text{int}(\Omega(\gamma)) \cup \bigcup_{j \in \Lambda(\gamma)} \bigcup_{\ell=0}^{\tau(F_j,\gamma)-1} f^\ell(F_j) \times (\text{up to a null Lebesgue measure set}), \tag{3.1}$$

where $\Lambda(\gamma) = \{j \in \{1, \ldots, r\} \mid \tau(F_j, \gamma) < +\infty\}$.

Hereafter, we will implement the key points described above in order to prove Theorem 3.3.

Let E be the open set defined by

$$E = \text{int}([0, 1) \backslash f([0, 1)))$$

which is the union of at most $n + 1$ open intervals $E_1, E_2, \ldots, E_s$. Then the following holds.

LEMMA 3.4. *For every positive integer ℓ, $E \cap f^\ell(E) = \emptyset$.*

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Proof. The assertion follows from the fact that $E \subset [0,1)\backslash f([0,1))$ and $f^{\ell}(E) = f(f^{\ell-1}(E)) \subset f([0,1))$. □

Now let B be the set consisting of those points of E which are taken by some iterate of f into a discontinuity of f, that is:

$$B = E \cap \bigcup_{\ell=0}^{+\infty} f^{-\ell}(\{x_1, \ldots, x_{n-1}\}). \tag{3.2}$$

LEMMA 3.5. *The set B has at most $n-1$ elements.*

Proof. We claim that the set $E \cap \bigcup_{\ell=0}^{+\infty} f^{-\ell}(\{x_j\})$ has at most one element for each $j \in \{1, \ldots, n-1\}$, otherwise the injectivity of f would imply that there exist $x, y \in E$, $x \neq y$, and $m > \ell$ such that $f^m(x) = x_j = f^{\ell}(y)$. Hence $y = f^{m-\ell}(x)$. In particular, $E \cap f^{m-\ell}(E) \neq \emptyset$, which contradicts Lemma 3.4. Therefore, the claim is true and B has at most $n-1$ elements. □

A *measurable partition of* $[0,1)$ *into intervals* is a denumerable family of pairwise disjoint open intervals $A_1, A_2, A_3, \ldots$ such that $[0,1)\backslash \bigcup_{j=1}^{\infty} A_j$ has Lebesgue measure zero.

LEMMA 3.6. *The set $F = E\backslash B$ is the union of $r \leq 2n$ pairwise disjoint open intervals $F_1, F_2, \ldots, F_r$. Moreover:*

(i) *$F \cap f^{\ell}(F) = \emptyset$ for every positive integer ℓ;*

(ii) *$f^{\ell}(F_j) \subset (0,1)\backslash\{x_1, \ldots, x_{n-1}\}$ for every $\ell \geq 0$ and $j \in \{1, \ldots, r\}$;*

(iii) *$\{f^{\ell}(F_j) \mid \ell \geq 0$ and $j \in \{1, \ldots, r\}\}$ is a measurable partition of $[0,1)$ into open intervals.*

Proof. It follows from Lemma 3.5 that F is the union of $r \leq 2n$ disjoint intervals $F_1, \ldots, F_r$. By Lemma 3.4, $F \cap f^{\ell}(F) \subset E \cap f^{\ell}(E) = \emptyset$ for every $\ell > 0$. Item (ii) follows immediately from the definition of F. It follows from (i), (ii) and the injectivity of f that the sets $f^{\ell}(F_j)$ form a family of pairwise disjoint intervals. Let $I = [0,1)$. It remains to prove that $I\backslash \bigcup_{\ell,j} f^{\ell}(F_j)$ has Lebesgue measure zero. Recall that the sets $\bigcup_{j=1}^{r} F_j$ and $I\backslash f(I)$ are equal up to a finite set. Let A be the set

$$A = I \Bigg\backslash \left(\bigcup_{\ell \geq 0} f^{\ell}(I\backslash f(I)) \right).$$

Notice that

$$\begin{aligned} y \in I\backslash A \Leftrightarrow\ &\text{there exist } x \in I\backslash f(I), \text{ there exist } \ell \geq 0 \text{ such that } y = f^{\ell}(x) \\ &\text{thus } y \in f^{\ell}(I)\backslash f^{\ell+1}(I). \end{aligned}$$

In this way,

$$I\backslash A = \bigcup_{\ell \geq 0} f^{\ell}(I)\backslash f^{\ell+1}(I) \quad \text{thus } A = \bigcap_{k \geq 1} f^k(I).$$

Since the Lebesgue measure of $f^k(I)$ is not greater than κ^k, where $\kappa < 1$ is the Lipschitz constant of f, we have that A has null Lebesgue measure. Moreover, A and

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

$I \setminus \bigcup_{\ell,j} f^\ell(F_j)$ are equal up to a countable set; thus $I \setminus \bigcup_{\ell,j} f^\ell(F_j)$ has Lebesgue measure zero. □

LEMMA 3.7. *Let $\Omega(\gamma)$ be the maximal trapping region of a regular k-periodic orbit γ. For each $j \in \{1, \ldots, r\}$ and for each $\ell \geq 0$, either $f^\ell(F_j) \cap \Omega(\gamma) = \emptyset$ or $f^\ell(F_j) \subset \Omega(\gamma)$.*

Proof. Suppose that $f^\ell(F_j) \cap \Omega(\gamma) \neq \emptyset$. We claim the following.

(i) There exists $\ell' \geq \ell$ such that $f^{\ell'}(F_j) \subset \Omega(\gamma)$. In fact, by (TR3) of Proposition 2.10, $\partial\Omega(\gamma)$ is a finite point-set. By item (iii) of Lemma 3.6, $f^\ell(F_j)$, $f^{\ell+1}(F_j), \ldots$ is a family of pairwise disjoint open intervals. Hence there exists $\ell' \geq \ell$ such that $f^{\ell'}(F_j) \cap \partial\Omega(\gamma) = \emptyset$. By (TR1) of Proposition 2.10 and by the injectivity of f,

$$f^{\ell'}(F_j) \cap \Omega(\gamma) = f^{\ell'-\ell}(f^\ell(F_j)) \cap \Omega(\gamma) \supset f^{\ell'-\ell}(f^\ell(F_j) \cap \Omega(\gamma)),$$

and thus $f^{\ell'}(F_j) \cap \Omega(\gamma) \neq \emptyset$. This together with $f^{\ell'}(F_j) \cap \partial\Omega(\gamma) = \emptyset$ yields $f^{\ell'}(F_j) \subset \Omega(\gamma)$, which proves Claim (i).

By Definition 2.9, $\Omega(\gamma) = \bigcup_{p\in\gamma} J_p$. The hypothesis that $f^\ell(F_j) \cap \Omega(\gamma) \neq \emptyset$ implies that there exists $q \in \gamma$ such that $J_q \cap f^\ell(F_j) \neq \emptyset$; hence $J_q \cup f^\ell(F_j)$ is an interval. We claim that

(ii) The interval $K := J_q \cup f^\ell(F_j)$ is a trapping interval of q. Let $m > 0$ be an even integer such that $mk + \ell \geq \ell'$. By Definition 2.5, $f^\rho(J_q)$ is an interval for every integer $\rho \geq 0$. Moreover, $f^\rho(f^\ell(F_j))$ is an interval for every integer $\rho \geq 0$. Hence $f^\rho(K)$ is an interval for every $\rho \geq 0$. By Claim (i) and by (TR1) of Proposition 2.10, $f^{mk}(f^\ell(F_j)) \subset \Omega(\gamma)$. By Definition 2.5, $f^{mk}(J_q) \subset J_q$. Hence, $f^{mk}(K) \subset J_q \cup \Omega(\gamma) \subset \Omega(\gamma)$. Since $f^{mk}(K)$ is an interval that contains q, it is contained in the connected component of $\Omega(\gamma)$ that contains q, that is, $f^{mk}(K) \subset J_q$. This proves Claim (ii).

Since J_q is a maximal trapping interval, by Claim (ii) we have that $K \subset J_q$, and thus $f^\ell(F_j) \subset J_q \subset \Omega(\gamma)$. This is the end of the proof of Lemma 3.7. □

COROLLARY 3.8. *Let $\Omega(\gamma)$ be the maximal trapping region of a regular periodic orbit γ. Then there exist $\ell \geq 0$ and $j \in \{1, \ldots, r\}$ such that $f^\ell(F_j) \subset \Omega(\gamma)$.*

Proof. The proof follows from item (iii) of Lemma 3.6 and from Lemma 3.7. □

Proof of Theorem 3.3. For each $j \in \{1, \ldots, r\}$, let $\tau(F_j, \gamma) = \inf\{\ell \in \mathbb{N} \mid f^\ell(F_j) \subset \Omega(\gamma)\}$, where $\inf \emptyset = +\infty$. Let $\Lambda(\gamma) = \{j \in \{1, \ldots, r\} \mid \tau(F_j, \gamma) < +\infty\}$. It follows from Corollary 3.8 that $\Lambda(\gamma) \neq \emptyset$. Now Proposition 2.10 and Lemmas 3.6 and 3.7 ensure that the following claims are true:

(I) $f^\ell(F_j) \cap \Omega(\gamma) = \emptyset$ if $j \in \Lambda(\gamma)$ and $0 \leq \ell < \tau(F_j, \gamma)$;
(II) $f^\ell(F_j) \subset \Omega(\gamma)$ if $j \in \Lambda(\gamma)$ and $\ell \geq \tau(F_j, \gamma)$;
(III) $O_f(F_j) \cap \Omega(\gamma) = \emptyset$ if $j \notin \Lambda(\gamma)$.

Let

$$S = \operatorname{int}(\Omega(\gamma)) \cup \bigcup_{j\in\Lambda(\gamma)} \bigcup_{\ell=0}^{\tau(F_j,\gamma)-1} f^\ell(F_j).$$

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

By (TR3) of Proposition 2.10 and by item (iii) of Lemma 3.6, S is the union of finitely many open intervals. By Claim (II) and (TR2) of Proposition 2.10, we have that $S \subset W^s(\gamma)$. In particular, $S \subset \operatorname{int}(W^s(\gamma))$ because S is open.

It follows from item (iii) of Lemma 3.6 and from Claims (I)–(III) above that

$$\Omega(\gamma) = \bigcup_{j\in\Lambda(\gamma)} \bigcup_{\ell\geq\tau(F_j,\gamma)} f^\ell(F_j) \quad \text{(up to a null Lebesgue measure set).}$$

Hence, $S = \bigcup_{j\in\Lambda(\gamma)} \bigcup_{\ell=0}^{+\infty} f^\ell(F_j)$ (up to a null Lebesgue measure set). This, together with Claim (III), yields

$$\begin{aligned} W^s(\gamma)\backslash S &= W^s(\gamma) \Big\backslash \bigcup_{j\in\Lambda(\gamma)} \bigcup_{\ell=0}^{+\infty} f^\ell(F_j) \\ &= W^s(\gamma) \Big\backslash \bigcup_{j=1}^{r} \bigcup_{\ell=0}^{+\infty} f^\ell(F_j) \quad \text{(up to a null measure set).} \end{aligned}$$

By (iii) of Lemma 3.6, $\bigcup_{j=1}^{r} \bigcup_{\ell=0}^{+\infty} f^\ell(F_j)$ has Lebesgue measure one, and thus $W^s(\gamma)\backslash S$ has Lebesgue measure zero.

Suppose that $\operatorname{int}(W^s(\gamma))$ is not the union of finitely many open intervals. Then there exist denumerably many pairwise disjoint open intervals $U_1, U_2, \ldots$ such that $\operatorname{int}(W^s(\gamma)) = \bigcup_{j=1}^{\infty} U_j$. Moreover, because S is the union of finitely many pairwise disjoint open intervals and $S \subset \operatorname{int}(W^s(\gamma))$, there exists a positive integer d such that $S \subset \bigcup_{j=1}^{d} U_j$. Then $W^s(\gamma)\backslash S$ contains the open set U_{d+1}, which is a contradiction since $W^s(\gamma)\backslash S$ has Lebesgue measure zero. □

4. *A tight upper bound for the number of regular periodic orbits*

In this section we will present a proof of the following result.

THEOREM 4.1. *Every injective PC of n intervals has at most n regular periodic orbits.*

By Proposition 2.4, Theorem 4.1 asserts that an injective PC of n intervals has at most n internal periodic orbits. The proof of the main Theorem (Theorem 1.1) is a variation of the proof of Theorem 4.1. The steps necessary for obtaining it from the proof of Theorem 4.1 will be outlined in the next section.

Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be a collection of pairwise distinct regular periodic orbits of f. Set $W_j = \operatorname{int}(W^s(\gamma_j))$ for every $j \in \{1, \ldots, m\}$ and let $W_{m+1} = \operatorname{int}([0, 1)\backslash \bigcup_{j=1}^{m} \overline{W_j})$. By Theorem 3.3, W_j, $1 \leq j \leq m+1$, is the union of finitely many intervals. Moreover, $\bigcup_{j=1}^{m+1} \overline{W_j} = [0, 1]$.

The claim of Theorem 4.1 is that $m \leq n$. Its proof follows straightforwardly from the next lemmas.

LEMMA 4.2. *For every $j \in \{1, \ldots, m+1\}$ we have that $f(W_j) \subset \overline{W_j}$.*

Proof. First, let $j \in \{1, \ldots, m\}$. By Theorem 3.3, W_j is the union of finitely many open intervals. Therefore, as f is injective and $f|_{I_i}$ continuous for every $i \in \{1, \ldots, n\}$, $f(W_j)$ is the union of finitely many intervals. Assume by contradiction that $f(W_j) \cap (\mathbb{R}\backslash\overline{W_j}) \neq \emptyset$, thus there exist $0 \leq a < b \leq 1$ such that $(a, b) \subset f(W_j) \cap (\mathbb{R}\backslash\overline{W_j})$.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Lemma 3.1 together with the definition $W_j = \mathrm{int}(W^s(\gamma_j))$ yields

$$(a, b) \subset f(W_j) \subset f(W^s(\gamma_j)) \subset W^s(\gamma_j) \quad \text{and thus } (a, b) \subset W_j,$$

which contradicts the fact that $(a, b) \subset \mathbb{R}\backslash\overline{W_j}$. Hence, $f(W_j) \subset \overline{W_j}$ for every $1 \le j \le m$.

Now let us prove that $f(W_{m+1}) \subset \overline{W_{m+1}}$. Let $x \in [0, 1)$ and $1 \le j \le m$. We claim that:

(a) if $f(x) \in W_j$ then $x \in \overline{W_j}$.

As $x \in [0, 1)$, there exists $1 \le i \le n$ such that $x \in I_i$. Suppose that $f(x) \in W_j$. Since W_j is open and $f|_{I_i}$ is monotone continuous, there exists an interval $I_x \subset I_i$, with $x \in \overline{I_x}$, such that $f(I_x) \subset W_j$. Then $\omega(y) = \gamma_j$ for every $y \in I_x$. In this way, $I_x \subset \mathrm{int}(W^s(\gamma_j))$, that is, $I_x \subset W_j$. The fact that $x \in \overline{I_x}$ yields $x \in \overline{W_j}$, which proves Claim (a).

It follows from Claim (a) that if $x \in [0, 1)\backslash \bigcup_{j=1}^m \overline{W_j}$ then $f(x) \in [0, 1)\backslash \bigcup_{j=1}^m W_j$. By Theorem 3.3, $[0, 1)\backslash \bigcup_{j=1}^m W_j = \overline{W_{m+1}}$, which concludes the proof. □

LEMMA 4.3. *If $z \in \overline{W_i} \cap \overline{W_j}$ for some $i \neq j$ then there exists an integer $q \ge 0$ such that $f^q(z) \in \{x_1, \ldots, x_{n-1}\} \cap \partial W_i \cap \partial W_j$.*

Proof. We may assume that $z \notin \{x_1, \ldots, x_{n-1}\}$, otherwise the proof is finished by taking $q = 0$. Thus, f is continuous in a neighborhood of z. By continuity of f and Lemma 4.2, we have that $f(z) \in \overline{W_i} \cap \overline{W_j}$ and the reasoning can be repeated. Hence, we may assume that $f(z) \notin \{x_1, \ldots, x_{n-1}\}$, otherwise we set $q = 1$ and the proof is finished. By repeating this reasoning over and over again, we obtain that either $f^q(z) \in \{x_1, \ldots, x_{n-1}\}$ for some $q \ge 0$ (and the proof is finished) or $O_f(z) \cap \{x_1, \ldots, x_{n-1}\} = \emptyset$. This together with Lemma 4.2 yields $O_f(x) \subset \overline{W_i} \cap \overline{W_j}$. By Theorem 3.3, $\overline{W_i} \cap \overline{W_j}$ is a finite point-set; thus $O_f(z)$ is a periodic orbit. By Proposition 2.4, $O_f(z)$ is regular periodic orbit, which contradicts $O_f(z) \subset \overline{W_i} \cap \overline{W_j}$. Hence, there exists an integer $q \ge 0$ such that $f^q(z) \in \{x_1, \ldots, x_{n-1}\}$. By Lemma 4.2 and by Theorem 3.3, $\overline{W_i} \cap \overline{W_j} \subset \partial W_i \cap \partial W_j$. □

LEMMA 4.4. *The following statements are true:*

(i) *if $W_{m+1} \neq \emptyset$ then $m \le n - 1$;*

(ii) *if $W_{m+1} = \emptyset$ then $m \le n$.*

Proof. First, let us prove (i). Let $\mathcal{W} = \{W_1, \ldots, W_{m+1}\}$. We will define an injective map

$$\beta : \mathcal{W} \to \{x_0, \ldots, x_{n-1}\}. \tag{4.1}$$

Set $y_j = \inf W_j$ for all $j \in \{1, \ldots, m + 1\}$. By definition and by Lemma 3.2,

$$W_1, \ldots, W_{m+1} \text{ are pairwise disjoint and } [0, 1] = \bigcup_{j=1}^{m+1} \overline{W_j}. \tag{4.2}$$

Let $j_0 \in \{1, \ldots, m + 1\}$ be the index that satisfies $y_{j_0} = x_0 = 0$. Set $\beta(W_{j_0}) = x_0$. By equation (4.2) and by Theorem 3.3, $y_1, y_2, \ldots, y_{m+1}$ are pairwise disjoint. Let $i \in \{1, \ldots, m + 1\}$, $i \neq j_0$; thus there exists $W^{(i)} \in \mathcal{W}$, $W^{(i)} \neq W_i$, such that $y_i \in \partial W^{(i)} \cap \partial W_i$. Moreover, for ϵ small enough, we have

$$(y_i - \epsilon, y_i) \subset W^{(i)} \text{ and } (y_i, y_i + \epsilon) \subset W_i.$$

Using Lemma 4.3, let $q_i = \min\{q \geq 0 : f^q(y_i) \in \{x_1, \ldots, x_{n-1}\}\}$ and set $\beta(W_i) = f^{q_i}(y_i)$.

Now we show that the map β is injective. Let $1 \leq i, k \leq m+1$, with $q_i \leq q_k$, be such that $\beta(W_i) = \beta(W_k)$. It is easy to see that $i = j_0$ or $k = j_0$ imply $i = k = j_0$. Thus we may assume that $i \neq j_0$ and $k \neq j_0$. By the injectivity of f,

$$f^{q_k - q_i}(y_k) = y_i,$$

where $0 \leq q_k - q_i \leq q_k$. Notice that $q_k = 0$ implies $q_i = 0$. In this case, $y_i = y_k$, which contradicts $q_1, q_2, \ldots, q_{m+1}$ are pairwise disjoint. Hence, we may assume that $q_k \geq 1$. We have that f is continuous on a neighborhood of $f^j(y_k)$ for every $0 \leq j \leq q_k - 1$. Therefore, by Lemma 4.2, $\{W_i, W^{(i)}\} = \{W_k, W^{(k)}\}$. There are two possibilities: either (a) $W_i = W_k$ or (b) $W_i = W^{(k)}$ and $W_k = W^{(i)}$. Suppose that (b) happens. If $y_i < y_k$ then by equation (4.2), there exists $\epsilon > 0$ such that $(y_i - \epsilon, y_i) \subset W_k$ and thus $\inf W_k < y_i < y_k$, which is a contradiction. By analogy, assuming $y_k < y_i$ also yields a contradiction. Therefore, (b) cannot happen and hence $W_i = W_k$. This proves that β is a well defined injective map, and thus $m \leq n - 1$. To prove (ii), we neglect W_{m+1} and define $\mathcal{W} = \{W_1, \ldots, W_m\}$. By replacing in the above proof $m+1$ by m, we obtain that $m - 1 \leq n - 1$, and thus $m \leq n$. □

Notice that Theorem 4.1 is a corollary of Lemma 4.4.

5. *Proof of Theorem 1.1*

In this section we will prove Theorem 1.1. In this respect, the combinatorial lemma we present now is going to be of paramount importance. We will keep the notation of previous sections.

5.1. *The Combinatorial Lemma.*

An *s-chain* is a sequence of $s \geq 1$ ordered pairs of positive integers $A_0 = (a_0, b_0), A_1 = (a_1, b_1), \ldots, A_{s-1} = (a_{s-1}, b_{s-1})$, where $a_{\ell-1} \in \{a_\ell, b_\ell\}$, for every $1 \leq \ell \leq s-1$, and $a_{s-1} \in \{a_0, b_0\}$. The set $S = \{a_\ell : 0 \leq \ell \leq s-1\} \cup \{b_\ell : 0 \leq \ell \leq s-1\}$ is called the *set of coordinates* of the s-chain whose cardinality is denoted by $\#S$.

Example 4. Let $s \geq 1$; then the sequence $A_0 = (1, 2), A_1 = (1, 3), \ldots, A_{s-1} = (1, s+1)$ is an s-chain and its set of coordinates is $S = \{1, 2, \ldots, s+1\}$.

Example 5. The sequence $A_0 = (1, 2), A_1 = (1, 3), A_2 = (4, 1), A_3 = (2, 4)$ is a 4-chain and its set of coordinates is $S = \{1, 2, 3, 4\}$.

We would like to know how large the set S can be in the general case.

LEMMA 5.1. (Combinatorial Lemma) *If $A_0 = (a_0, b_0)$, $A_1 = (a_1, b_1), \ldots, A_{s-1} = (a_{s-1}, b_{s-1})$ is an s-chain then $\#S \leq s+1$. Moreover, $\#S = s+1$ if and only if $a_0 = a_1 = \cdots = a_{s-1}$ and the elements $a_0, b_0, b_1, \ldots, b_{s-1}$ are pairwise distinct.*

Proof. The assertion follows by induction on s. The claim holds for $s = 1$. Now assume that the claim holds for some $s \geq 1$. Let $A_0 = (a_0, b_0), \ldots, A_{s-1} = (a_{s-1}, b_{s-1})$,

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

$A_s = (a_s, b_s)$ be an $(s+1)$-chain and let S be its set of coordinates. We have to prove that $\#S \le s+2$.

If $a_{s-1} = a_0$ or $a_{s-1} = b_0$, then $A_0, A_1, \ldots, A_{s-1}$ is an s-chain, then by the induction hypothesis the set $\bigcup_{\ell=0}^{s-1}\{a_\ell\} \cup \{b_\ell\}$ has at most $s+1$ elements. Now if we add a_s and b_s, as at least one of them is also equal to a_0 or b_0, the set S must have at most $(s+1)+1$ elements.

Otherwise, $a_{s-1} \ne a_0$ and $a_{s-1} \ne b_0$, thus $b_s = a_{s-1}$ and $a_s = a_0$ or $a_s = b_0$ which means that $S = \bigcup_{\ell=0}^{s-1}\{a_\ell\} \cup \{b_\ell\}$. One of the coordinates of A_{s-1} equals a_{s-2}. Now we replace the couple (a_{s-1}, b_{s-1}) by the couple (a_0, a_{s-2}), so the sequence

$$(a_0, b_0), (a_1, b_1), \ldots, (a_{s-2}, b_{s-2}), (a_0, a_{s-2})$$

becomes an s-chain. By the induction hypothesis, the set $\bigcup_{\ell=0}^{s-2}\{a_\ell\} \cup \{b_\ell\} \cup \{a_0\} \cup \{a_{s-2}\} = \bigcup_{\ell=0}^{s-2}\{a_\ell\} \cup \{b_\ell\}$ has at most $s+1$ elements. The set S of the $(s+1)$-chain has in addition at most one more new element which implies that $\#S \le (s+1)+1$. This proves the claim. □

5.2. *An application of the Combinatorial Lemma.* In what follows, let γ be a degenerate k-periodic orbit of f and let $x = \min \gamma$. For the next results, we assume that

$$\gamma \subset [0, 1)\backslash W_{m+1}. \tag{5.1}$$

The hypothesis (5.1) will be removed in Lemma 5.7.

LEMMA 5.2. $\gamma \subset \bigcup_{j=1}^{m+1} \partial W_j$.

Proof. By equation (4.2), it is enough to prove that $\gamma \subset [0, 1)\backslash W_j$ for all $j \in \{1, \ldots, m+1\}$. Firstly we consider $j \in \{1, \ldots, m\}$. In this case, there exists a regular periodic orbit γ_j such that $\omega(y) = \gamma_j$ for all $y \in W_j$. In particular, if $\gamma \cap W_j \ne \emptyset$ and $y \in \gamma \cap W_j$ then $\gamma = \omega(x) = \omega(y) = \gamma_j$, which is a contradiction, because γ is a degenerate periodic orbit. Thus, $\gamma \subset [0, 1)\backslash W_j$ for every $j \in \{1, \ldots, m\}$. By (5.1), $\gamma \subset [0, 1)\backslash W_{m+1}$. Hence, $\gamma \subset [0, 1)\backslash \bigcup_{j=1}^{m+1} W_j$. By equation (4.2), $[0, 1)\backslash \bigcup_{j=1}^{m+1} W_j = \bigcup_{j=1}^{m+1} \partial W_j$. □

LEMMA 5.3. *There exist integers $s \ge 1$ and $0 \le \ell_0 < \ell_1 < \cdots < \ell_{s-1} \le k-1$ such that $\gamma \cap \{x_0, \ldots, x_{n-1}\} = \{f^{\ell_0}(x), f^{\ell_1}(x), \ldots, f^{\ell_{s-1}}(x)\}$.*

Proof. The proof follows immediately from Proposition 2.4. □

Because $[0, 1) = \bigcup_{j=1}^{n} I_j$, for each $\ell \in \{0, \ldots, k-1\}$, there exists a unique $j(\ell) \in \{1, \ldots, n\}$ such that $f^\ell(x) \in I_{j(\ell)}$.

LEMMA 5.4. *Let $\{\ell_0, \ell_1, \ldots, \ell_{s-1}\}$ be as in Lemma 5.3. For each $\ell \in \{\ell_0, \ell_1, \ldots, \ell_{s-1}\}$, there exists a uniquely defined ordered pair $(a_\ell, b_\ell) \in \{1, \ldots, m+1\} \times \{1, \ldots, m+1\}$ satisfying the following conditions:*

(i) $f^\ell(x) \in \operatorname{int}(\overline{W_{a_\ell}} \cup \overline{W_{b_\ell}})$ *if* $f^\ell(x) \ne 0$;

(ii) $f^\ell(x) \in \overline{W}_{a_\ell}$ *and* $a_\ell = b_\ell$ *if* $f^\ell(x) = 0$;

(iii) $I_{j(\ell)} \cap (f^\ell(x) - \epsilon, f^\ell(x) + \epsilon) \subset \overline{W}_{a_\ell}$ *for small enough* $\epsilon > 0$.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Proof. Let $\ell \in \{\ell_0, \ell_1, \ldots, \ell_{s-1}\}$; thus there exists a unique integer $j(\ell) \in \{1, \ldots, n\}$ such that $f^\ell(x) \in I_{j(\ell)} \cap \partial I_{j(\ell)}$. By Lemma 5.2, $\gamma \subset \bigcup_{j=1}^{m+1} \partial W_j$. By Theorem 3.3 and by equation (4.2), there exists a unique index $a_\ell \in \{1, \ldots, m+1\}$ such that

$$I_{j(\ell)} \cap (f^\ell(x) - \epsilon, f^\ell(x) + \epsilon) \subset \overline{W_{a_\ell}}$$

for small enough $\epsilon > 0$. If $f(x) = 0$ or if $\overline{W_{a_\ell}}$ contains the whole interval $(f(x) - \epsilon, f(x) + \epsilon)$, we set $b_\ell = a_\ell$.

Otherwise, there exists a unique index $b_\ell \in \{1, \ldots, m+1\}$, $b_\ell \neq a_\ell$, such that

$$(f^\ell(x) - \epsilon, f^\ell(x) + \epsilon) \cap \overline{W}_{b_\ell} \neq \emptyset$$

for all small enough $\epsilon > 0$. We have proved there exists a unique pair of indices $(a_\ell, b_\ell) \in \{1, \ldots, m+1\} \times \{1, \ldots, m+1\}$ which satisfies (i), (ii) and (iii). □

LEMMA 5.5. *Let (a_ℓ, b_ℓ), and $0 \leq \ell_0 < \ell_1 < \cdots < \ell_{s-1} \leq k-1$ be as in Lemmas 5.3 and 5.4. The following hold:*

(i) $A_0 = (a_{\ell_0}, b_{\ell_0})$, $A_1 = (a_{\ell_1}, b_{\ell_1})$, $\ldots$, $A_{s-1} = (a_{\ell_{s-1}}, b_{\ell_{s-1}})$ *is an s-chain;*
(ii) $\#S \leq s$;
(iii) *if $0 \in \gamma$ then $\#S \leq s-1$.*

Proof. Let $r \in \{0, \ldots, s-1\}$. For convenience we set $\ell_s = \ell_0 + k$, $a_{\ell_s} = a_{\ell_0}$ and $b_{\ell_s} = b_{\ell_0}$. Notice that, because x is k-periodic, $f^{\ell_s}(x) = f^{\ell_0}(x)$.

By Lemma 4.2 and by the continuity of f on $f^\ell(x)$ for all $\ell \in \{0, \ldots, k-1\} \backslash \{\ell_0, \ldots, \ell_{s-1}\}$, we have that $f^{\ell_{r+1}}(x) \in \overline{W_{a_{\ell_r}}}$ for all $r \in \{0, \ldots, s-1\}$. By the unicity in the definition of $(a_{\ell_{r+1}}, b_{\ell_{r+1}})$ (see Lemma 5.4), we have that $a_{\ell_{r+1}} = a_{\ell_r}$ or $b_{\ell_{r+1}} = a_{\ell_r}$. Thus, $A_0, A_1, \ldots, A_{s-1}$ is an s-chain. By Lemma 5.1, $\#S \leq s+1$, where S is the set of coordinates of the chain. Moreover, if $\#S = s+1$ then

$$a_{\ell_0} = a_{\ell_1} = \cdots = a_{\ell_{s-1}}. \tag{5.2}$$

By equation (5.2), there exists $\epsilon > 0$ and an interval U containing $f^{\ell_0}(x)$ such that $f^\ell(U)$ is an interval containing $f^{\ell+\ell_0}(x)$ for all $\ell \in \{0, \ldots, k\}$. Now there are two possibilities: either (a) $f^k(U) \subset U$ or (b) $f^k(U) \cap U = \{f^{\ell_0}(x)\}$. Case (a) implies that $f^{\ell_0}(x)$ is a regular periodic point, which contradicts the assumption that $\gamma = O_f(x)$ is a degenerate periodic orbit. In case (b) we have that $a_{\ell_0} = b_{\ell_0}$, which together with the second statement of Lemma 5.1 implies that $\#S \leq s$. Items (i) and (ii) of the assertion of the lemma are proved.

Now suppose that $0 \in \gamma$. By item (iii) of Lemma 5.4, $a_{i_0} = b_{i_0}$. Consequently,

$$A_1 = (a_{\ell_1}, b_{\ell_1}), \ldots, A_{s-1} = (a_{\ell_{s-1}}, b_{\ell_{s-1}})$$

is an $(s-1)$-chain. By the above, $\bigcup_{r=1}^{s-1}\{a_r\} \cup \{b_r\}$ has at most $s-1$ elements. Moreover, as $a_{i_0} \in \{a_{i_1}, b_{i_1}\}$, we have that $S = \bigcup_{r=0}^{s-1}\{a_{\ell_r}\} \cup \{b_{\ell_r}\} = \bigcup_{r=1}^{s-1}\{a_{\ell_r}\} \cup \{b_{\ell_r}\}$ and so $\#S \leq s-1$, which proves item (iii). □

LEMMA 5.6. *The cardinality of the set $\{j \in \{1, \ldots, m+1\} : \inf W_j \in \gamma\}$ is at most $s-1$.*

Proof. We claim that

$$\#\{j \in \{1, \ldots, m+1\} : \inf W_j \in \gamma\} = \#\{i \in S : \inf W_i \in \gamma\}, \tag{5.3}$$

where $S = \bigcup_{r=0}^{s-1}\{a_{\ell_r}\} \cup \{b_{\ell_r}\}$.

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

Suppose that $\inf W_j \in \gamma$; thus there exist $r \in \{0, 1, \ldots, s-1\}$ and $\ell_r < \ell \le \ell_{r+1}$ such that $f^{\ell}(x) = \inf W_j$, where for convenience we set $\ell_s = \ell_0 + k$, $a_{\ell_s} = a_{\ell_0}$ and $b_{\ell_s} = b_{\ell_0}$. Notice that, because the point $x = \min \gamma$ is k-periodic, $f^{\ell_s}(x) = f^{\ell_0}(x)$. By Lemma 4.2 and the continuity of f at $f^{\ell}(x)$ for every $\ell \in \{0, \ldots, k-1\} \backslash \{\ell_0, \ldots, \ell_{s-1}\}$, we have that $f^{\ell_{r+1}}(x) \in \overline{W_j}$ for every $r \in \{0, \ldots, s-1\}$. By the definition of $(a_{\ell_{r+1}}, b_{\ell_{r+1}})$ (see Lemma 5.4), we have that $a_{\ell_{r+1}} = j$ or $b_{\ell_{r+1}} = j$. Hence,

$$\inf W_j \in \{\inf W_{a_{\ell_{r+1}}}, \inf W_{b_{\ell_{r+1}}}\} \subset \{i \in S : \inf W_i \in \gamma\},$$

which proves (5.3).

By equation (5.3), it suffices to prove that $\#\{i \in S : \inf W_i \in \gamma\} \le s - 1$. It follows from item (iii) of Lemma 5.5, that if $0 \in \gamma$ then

$$\#\{i \in S : \inf W_i \in \gamma\} \le \#S \le s - 1.$$

Otherwise, $0 \notin \gamma$ and $f^{\ell_0}(x) > 0$. Moreover, there exists $i(x) \in S$ such that $x \in \overline{W_{i(x)}}$ and $\inf W_{i(x)} < x$. This together with the item (ii) of Lemma 5.5 yields

$$\#\{i \in S : \inf W_i \in \gamma\} \le \#S - 1 \le s - 1. \qquad \square$$

Let $\beta : \mathcal{W} \to \{x_0, x_1, \ldots, x_{n-1}\}$ be the map defined in equation (4.1), where $\mathcal{W} = \{W_1, \ldots, W_{m+1}\}$ if $W_{m+1} \neq \emptyset$, otherwise $\mathcal{W} = \{W_1, \ldots, W_m\}$.

Let $\gamma_1, \ldots, \gamma_m$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_d$ be, respectively, collections of regular and degenerate periodic orbits of f.

LEMMA 5.7. *The image of the map β contains at most $n - d$ elements.*

Proof. Let $\ell \in \{1, \ldots, d\}$. We claim that

$$\#(\tilde{\gamma}_\ell \cap \text{image}(\beta)) \le \#(\tilde{\gamma}_\ell \cap \{x_0, \ldots, x_{n-1}\}) - 1. \tag{5.4}$$

We split the proof of the claim into three cases.

Case 1. $W_{m+1} \neq \emptyset$ and $\tilde{\gamma}_\ell \subset [0, 1) \backslash W_{m+1}$. Let $x_i \in \tilde{\gamma}_\ell \cap \text{image}(\beta)$ and let $j \in \{1, \ldots, m+1\}$ be such that $x_i = \beta(W_j)$. By the definition of x_i and β, we have that $x_i \in \tilde{\gamma}_\ell \cap O_f(\inf W_j)$; thus $O_f(\inf W_j) = \tilde{\gamma}_\ell$. In particular, $\inf W_j \in \tilde{\gamma}_\ell$. This together with the fact that $\text{image}(\beta) \subset \{x_0, \ldots, x_{n-1}\}$, Lemma 5.6 and the injectivity of β yields

$$\begin{aligned}\#(\tilde{\gamma}_\ell \cap \text{image}(\beta)) &= \#(\{x_0, \ldots, x_{n-1}\} \cap \tilde{\gamma}_\ell \cap \text{image}(\beta)) \\ &\le \#(\{j \in \{1, \ldots, m+1\} : \inf W_j \in \tilde{\gamma}_\ell\}) \\ &\le \#(\tilde{\gamma}_\ell \cap \{x_0, \ldots, x_{n-1}\}) - 1,\end{aligned}$$

which proves the claim in Case 1.

Case 2. $W_{m+1} \neq \emptyset$ and $\tilde{\gamma}_\ell \cap W_{m+1} \neq \emptyset$. In this case, by Lemma 4.2, we have that $\tilde{\gamma}_\ell \subset \overline{W_{m+1}}$. Moreover, as $\tilde{\gamma}_\ell \cap W_{m+1} \neq \emptyset$, we cannot have $\tilde{\gamma}_\ell \subset \partial W_{m+1}$. Hence, there are two possibilities: either (a) $\tilde{\gamma}_\ell \subset W_{m+1}$ or (b) $\tilde{\gamma}_\ell \cap W_{m+1} \cap \partial W_{m+1} \neq \emptyset$. In case (a), because W_{m+1} is open and $\text{image}(\beta) \subset \bigcup_{j=1}^{m+1} \partial W_j$, we have that $\tilde{\gamma}_\ell \cap \text{image}(\beta) = \emptyset$, and thus equation (5.4) holds. In case (b), by Lemma 4.3, $\tilde{\gamma}_\ell \cap \{x_0, \ldots, x_{n-1}\} \cap \partial W_{m+1} \neq \emptyset$. Moreover, by the hypothesis of case (b), there exists $z \in W_{m+1}$ and $x_i \in \tilde{\gamma}_\ell \cap \{x_0, \ldots, x_{n-1}\} \cap \partial W_{m+1}$ such that $f(z) = x_i$. If $z \in \{x_0, \ldots, x_{n-1}\}$ then, by proceeding

https://doi.org/10.1017/etds.2014.16 Published online by Cambridge University Press

as above, we can see that $z \notin \text{image}(\beta)$ and so equation (5.4) holds. Otherwise, f is continuous on a neighborhood of z and so $x_i \in \text{int}(\overline{W_{m+1}})$. In this case, by the definition of β, $x_i \notin \text{image}(\beta)$, hence equation (5.4) holds. This proves the claim in Case 2.

Case 3. $W_{m+1} = \emptyset$. The proof of Case 1 holds word-by-word for Case 3, provided we replace $\{1, \ldots, m+1\}$ by $\{1, \ldots, m\}$ in that proof.

By the claim, for each $\ell \in \{1, \ldots, d\}$, there exists $\tilde{x_\ell} \in \tilde{\gamma_\ell} \cap \{x_0, \ldots, x_{n-1}\}$ such that $\tilde{x_\ell} \notin \text{image}(\beta)$. Therefore,

$$\text{image}(\beta) \subset \{x_0, \ldots, x_{n-1}\} \backslash \{\tilde{x_1}, \ldots, \tilde{x_d}\}.$$

In this way, $\#\text{image}(\beta) \leq n - d$. □

By Lemma 4.4, f has at most n regular periodic orbits; thus $m \leq n$. By Proposition 2.4, every degenerate periodic orbit of f contains a discontinuity, and so $d \leq n$. Therefore, a corollary of these two results is that the number of periodic orbits of f is bounded by $2n$, that is, $m + n \leq 2n$. By using Lemma 5.7, we provide now a stronger version of Lemma 4.4.

LEMMA 5.8. *The following statements are true:*
(i) *if* $W_{m+1} \neq \emptyset$ *then* $m + d \leq n - 1$;
(ii) *if* $W_{m+1} = \emptyset$ *then* $m + d \leq n$.

Proof. By Lemma 5.7, the image of the injective map $\beta : \mathcal{W} \to \{x_0, \ldots, x_{n-1}\}$ has at most $n - d$ elements. In case (i), $\mathcal{W} = \{W_1, \ldots, W_{m+1}\}$ and so $m + 1 \leq n - d$, that is to say, $m + d \leq n - 1$. In case (ii), $\mathcal{W} = \{W_1, \ldots, W_m\}$ and $m \leq n - d$, that is, $m + d \leq n$. □

Proof of Theorem 1.1. By items (i) and (ii) of Lemma 5.8, f has at most n periodic orbits. Moreover, by item (i) of Lemma 5.8, if f has n periodic orbits, then $W_{m+1} = \emptyset$. In this case, $\bigcup_{i=1}^{m} \overline{W_i} = [0, 1]$. For every $x \in W_i$, we have that $\omega(x)$ is the periodic orbit γ_i. Now if $x \in \partial W_i$, then either $O_f(x) \cap W_i \neq \emptyset$ (and so $\omega(x) = \gamma_i$) or $O_f(x)$ is contained in the finite set $\bigcup_{i=1}^{n} \partial W_i$ (see Theorem 3.3), and thus $O_f(x)$ is periodic. □

6. *Proof of Theorem 1.2*

By item (iii) of Lemma 3.6, $\{f^\ell(F_j) \mid \ell \geq 0,\ j \in \{1, \ldots, r\}\}$ is a denumerable family of pairwise disjoint open intervals whose union $G = \bigcup_{\ell \geq 0} \bigcup_{j=1}^{r} f^\ell(F_j)$ has Lebesgue measure one. Moreover, the subintervals of G generate the Borel σ-algebra in $[0, 1)$. Let $K \subset G$ be an interval; then there exist $\ell \geq 0$, $1 \leq j \leq r$, and a subinterval J of F_j such that $K = f^\ell(J)$. We set

$$\nu(K) = \nu(f^\ell(J)) = \frac{1}{2^{\ell+1} r} \frac{|J|}{|F_j|} \quad \text{thus } \nu(f(K)) = \frac{1}{2}\nu(K). \tag{6.1}$$

The set function $K \mapsto \nu(K)$ can be extended to a non-atomic Borel probability measure positive on open intervals, as

$$\nu(G) = \sum_{\ell \geq 0} \sum_{j=1}^{r} \frac{1}{2^{(\ell+1)} r} = \sum_{j=1}^{r} \frac{1}{r} = 1.$$

In this way, the map $h : [0, 1) \to [0, \infty)$ defined by

$$h(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ \nu((0, x)) & \text{if} \quad 0 < x < 1 \end{cases}$$

is continuous and strictly increasing. Moreover, $h(1) = \nu((0, 1)) = \nu(G) = 1$. Therefore, $h : [0, 1) \to [0, 1)$ is a homeomorphism. Let $\hat{f} : [0, 1) \to [0, 1)$ be the map defined by $\hat{f} = h \circ f \circ h^{-1}$. We have that $\hat{f}$ is continuous on $[0, 1)\backslash\{h(x_1), \ldots, h(x_{n-1})\}$ and its continuity intervals are $\hat{I}_i = h(I_i)$, $1 \le i \le n$.

Let $B \subset [0, 1)$ be an interval. Being Lispchitz, f takes ν-null measure set onto ν-null measure set, thus $\nu(f(B)) = \nu(f(B \cap G))$. Now it follows from equation (6.1) that

$$\nu(f(B)) = \tfrac{1}{2}\nu(B) \quad \text{for every interval } B \subset [0, 1). \tag{6.2}$$

Let $(u, v) \subset h(I_i)$ be an interval. If $f|_{I_i}$ is increasing then

$$(f(h^{-1}(u)), f(h^{-1}(v))) = f(h^{-1}(u), h^{-1}(v)). \tag{6.3}$$

By equations (6.2) and (6.3),

$$\begin{aligned} \hat{f}(v) - \hat{f}(u) &= h(f(h^{-1}(v))) - h(f(h^{-1}(u))) \\ &= \nu((0, f(h^{-1}(v)))) - \nu((0, f(h^{-1}(u)))) \\ &= \nu((f(h^{-1}(u)), f(h^{-1}(v)))) = \nu(f(h^{-1}(u), h^{-1}(v))) \\ &= \tfrac{1}{2}\nu(h^{-1}(u), h^{-1}(v)) = \tfrac{1}{2}[\nu((0, h^{-1}(v))) - \nu((0, h^{-1}(u)))] \\ &= \tfrac{1}{2}[h(h^{-1}(v)) - h(h^{-1}(u))] = \tfrac{1}{2}(v - u). \end{aligned}$$

Otherwise, $f|_{I_i}$ is decreasing and

$$\hat{f}(v) - \hat{f}(u) = -\tfrac{1}{2}(v - u).$$

We have proved that $\hat{f}|_{\hat{I}_i}$ is linear for every $i \in \{1, \ldots, n\}$. □

Acknowledgements. The first author is partially supported by ANR Perturbations and EurDynBraz. The second author is partially supported by FAPESP-BRAZIL (2009/02380-0 and 2013/12359-3) and DynEurBraz.

References

[1] B. Blank and L. Bunimovich. Switched flow systems: pseudo-billiard dynamics. *Dyn. Syst.* **19**(4) (2004), 359–370.

[2] J. Brémont. Dynamics of injective quasi-contractions. *Ergod. Th. & Dynam. Sys.* **26** (2006), 19–44.

[3] H. Bruin and J. H. B. Deane. Piecewise contractions are asymptotically periodic. *Proc. Amer. Math. Soc.* **137**(4) (2009), 1389–1395.

[4] C. Chase, J. Serrano and P. J. Ramadge. Periodicity and chaos from switched flow systems: contrasting examples of discretely controlled continuous systems. *IEEE Trans. Automat. Control.* **38**(1) (1993), 70–83.

[5] C. Gutiérrez. *Smoothability of Cherry Flows (Lecture Notes in Mathematics, 1007)*. Springer, Berlin, 1983, pp. 308–331.

[6] C. Gutiérrez. On two-dimensional recurrence. *Bull. Braz. Math. Soc.* **10**(1) (1979), 1–16.

[7] J.-M. Gambaudo and C. Tresser. On the dynamics of quasi-contractions. *Bull. Braz. Math. Soc.* **19**(1) (1988), 61–114.

[**8**] I.-J. Jeong. Outer billiards with contraction. *Senior Thesis*, Brown University, 2012.
[**9**] I. M. MacPhee, M. V. Menshikov, S. Popov and S. Volkov. Periodicity in the transient regime of exhaustive polling systems. *Ann. Appl. Probab.* **16**(4) (2006), 1816–1850.
[**10**] A. Nogueira. Almost all interval exchange transformation with flips are nonergodic. *Ergod. Th. & Dynam. Sys.* **9** (1989), 515–525.
[**11**] A. Nogueira, B. Pires and R. Rosales. Asymptotically periodic piecewise contractions of the interval. *Preprint*, http://arxiv.org/pdf/1310.5784v1.pdf, pp. 1–8.
[**12**] A. Nogueira, B. Pires and S. Troubetzkoy. Orbit structure of interval exchange transformations with flip. *Nonlinearity* **26**(2) (2013), 525–537.