TRACES ARISING FROM REGULAR INCLUSIONS

DANNY CRYTSER⊠ and GABRIEL NAGY

Abstract

We study the problem of extending a state on an abelian C^*-subalgebra to a tracial state on the ambient C^*-algebra. We propose an approach that is well suited to the case of regular inclusions, in which there is a large supply of normalizers of the subalgebra. Conditional expectations onto the subalgebra give natural extensions of a state to the ambient C^*-algebra; we prove that these extensions are tracial states if and only if certain invariance properties of both the state and conditional expectations are satisfied. In the example of a groupoid C^*-algebra, these invariance properties correspond to invariance of associated measures on the unit space under the action of bisections. Using our framework, we are able to completely describe the tracial state space of a Cuntz–Krieger graph algebra. Along the way we introduce certain operations called graph tightenings, which both streamline our description and provide connections to related finiteness questions in graph C^*-algebras. Our investigation has close connections with the so-called unique state extension property and its variants.

1. Introduction

A *trace* on a complex algebra A is a linear functional $\phi : A \to \mathbb{C}$ satisfying $\phi(xy) = \phi(yx)$ for all $x, y \in A$. If A is a C^*-algebra, and the trace ϕ is also a state, it is simply called a *tracial state*. Such objects play a fundamental role in Elliott's classification program, as well as in the study of K-theory for C^*-algebras [4, 5, 25, 31]; thus, considerable effort has been devoted to constructing them on various classes of C^*-algebras [11, 27, 28]. In this paper we study tracial states on C^*-algebras A by reconstructing them from their restrictions to *regular abelian* subalgebras $B \subset A$. Recall that an inclusion $B \subset A$ is regular if the normalizer of B in A generates A as a C^*-algebra. Regularity is an essential feature of Cartan inclusions [23] and their generalizations [20]; the concept first proved to be fruitful in the work of Feldman and Moore [6] in the von Neumann algebra setting.

The material is organized as follows. In Section 2 we focus on identifying those states $\phi \in S(B)$ which extend to tracial states on larger subalgebras of A. Throughout

the paper we assume the existence of a conditional expectation $\mathbb{E} : A \to B$, and the extensions we consider are those of the form $\phi \circ \mathbb{E}$. The natural conditions employed in our analysis are *invariance* (for ϕ) and *normalization* (for $\mathbb{E}$). The main result is Theorem 2.7, which provides the natural framework for extending a state ϕ to a tracial state on a subalgebra of the form $C^*(B \cup N_0)$, where N_0 is a set of normalizers for B.

Section 3 specializes our investigation to the case of étale groupoid C^*-algebras, where the natural abelian C^*-algebra to consider is $C_0(G^{(0)})$—the C^*-algebra of continuous functions that vanish at ∞ on the unit space $G^{(0)}$. In this framework, the invariance conditions treated in Section 2 become measure theoretical in nature, described as a *balancing* feature. Theorem 3.5 and Corollary 3.9 show how balanced measures naturally induce traces on groupoid C^*-algebras.

In Section 4 we explore the link between the invariance and normalization conditions from Section 2 and certain state extension properties, which simplify our earlier analysis by placing the emphasis solely on the state $\phi \in S(B)$, as illustrated in Corollary 4.3. Furthermore, when the so-called *extension property* holds, the tracial state space of A can be completely described by its restrictions to B, as seen in Corollary 4.6.

The paper concludes with Section 5, where the case of *graph C^*-algebras* is fully investigated, using results proved from Sections 2 and 4. Given some directed graph E, our main goal is the complete parametrization of the tracial state space of the associated C^*-algebra $C^*(E)$, solely in graph-theoretical language. Earlier work in this direction [8, 18, 27, 28] identified the notion of *graph traces* as a major ingredient. Following a lengthy review (which ends with Lemma 5.12), we apply the results from Section 2 to construct traces on $C^*(E)$ out of states defined on one of the two natural abelian subalgebras: $\mathcal{D}(E)$ the *diagonal* (see Corollary 5.16) or $\mathcal{M}(E)$ the *abelian core* (see Corollary 5.14). By combining this with Theorem 5.26, we provide a constructive alternative approach to that of [28]. As graph traces alone are insufficient for parametrizing the entire tracial state space of $C^*(E)$, it is necessary to augment them using what we term *cyclical tags*, and the main result concerning these objects is stated in Theorem 5.33. In order to tie up these results with those from Section 4, we need to decide when the inclusion $\mathcal{M}(E) \subset C^*(E)$ satisfies the hypotheses of Corollary 4.6. In Theorem 5.35 we are able to do this in purely graph-theoretical terms (on the graph E) by singling out the *tightness* condition. Our main goal of completely parametrizing all tracial states on $C^*(E)$ is achieved in Theorem 5.41, using a graph operation which we call *tightening*. Motivated by the observation that graph traces parametrize the *gauge-invariant traces*, the section concludes with Theorem 5.42, which provides a graph-theoretic characterization of the automatic gauge invariance for tracial states on $C^*(E)$.

2. Invariant states on abelian C^*-subalgebras

Following [13] and [24], given a C^*-algebra inclusion $B \subset A$, an element $n \in A$ is said to *normalize* B if $nBn^* \cup n^*Bn \subset B$. The collection of such normalizers is denoted

by $N_A(B)$, or simply $N(B)$ when there is no danger of confusion. Clearly, $N(B)$ is closed under products and adjoints, and contains B. A C^*-inclusion $B \subset A$ is said to be *regular* if $N(B)$ generates A as a C^*-algebra. (Equivalently, if the span of $N(B)$ is dense in A.)

Most of the C^*-algebra inclusions $B \subset A$ we are going to deal with in this paper are *nondegenerate*, in the sense that B contains an approximate unit for A. (Of course, if A is unital, then nondegeneracy of B is equivalent to the fact that B contains the unit of A.) Note that, if $B \subset A$ is a nondegenerate C^*-subalgebra, then $n^*n, nn^* \in B$ for any $n \in N(B)$.

DEFINITION 2.1. Assume that $B \subset A$ is nondegenerate and let ϕ be a state on $B \subset A$.

(1)   Given $n \in N(B)$, we say that ϕ is *n-invariant* if

$$\forall b \in B : \phi(nbn^*) = \phi(n^*nb). \tag{2.1}$$

(2)   Given $N_0 \subset N(B)$, we say that ϕ is *N_0-invariant* if ϕ is n-invariant for all $n \in N_0$.

(3)   Lastly, if ϕ is $N(B)$-invariant, then we simply say that ϕ is *fully invariant*.

The collection of fully invariant states on $B \subset A$ is denoted by $S^{\mathrm{inv}}(B)$.

REMARKS. The restriction $\tau|_B$ of any tracial state $\tau \in T(A)$ is clearly a fully invariant state on B, so we have an affine w^*-continuous map

$$T(A) \ni \tau \longmapsto \tau|_B \in S^{\mathrm{inv}}(B). \tag{2.2}$$

This paper aims at understanding when the map (2.2) is either surjective, or injective, or both.

The most important features of normalizers and invariant states are collected in Proposition 2.3 below. Both in its proof and elsewhere in the paper, we are going to employ the following facts, which can easily be obtained using continuous functional calculus.

FACTS 2.2. Assume that x is an element in some C^*-algebra A.

(i)   For any function $f \in C([0, \infty))$, the elements $f(xx^*), f(x^*x) \in \tilde{A}$, given by continuous functional calculus, satisfy the equality

$$xf(x^*x) = f(xx^*)x. \tag{2.3}$$

(ii)   When specializing to the kth root functions $f(t) = t^{1/k}$, we also have the equalities

$$\lim_{k\to\infty}(xx^*)^{1/k}x = \lim_{k\to\infty}x(x^*x)^{1/k} = x.$$

(iii)   If we fix a double sequence $(f_k^\ell)_{k,\ell=1}^\infty$ of polynomials in one variable, such that

$$\forall k \in \mathbb{N} : \lim_{\ell\to\infty} t f_k^\ell(t) = t^{1/k}, \text{uniformly on compact } K \subset [0, \infty)$$

(this is possible by the Stone–Weierstrass theorem), then

$$\lim_{k\to\infty}\lim_{\ell\to\infty} x f_k^\ell(x^*x)x^*x = \lim_{k\to\infty}\lim_{\ell\to\infty} xx^*x f_k^\ell(x^*x) = x,$$
$$\lim_{k\to\infty}\lim_{\ell\to\infty} f_k^\ell(xx^*)xx^*x = \lim_{k\to\infty}\lim_{\ell\to\infty} xx^* f_l^\ell(xx^*)x = x. \tag{2.4}$$

PROPOSITION 2.3. *Let $B \subset A$ be a nondegenerate abelian C^*-subalgebra of a C^*-algebra A. Then:*

(i)   *$\overline{nB} = \overline{Bn}$ for all $n \in N(B)$;*

(ii) *all states $\phi \in S(B)$ are B-invariant;*

(iii) *if $\phi \in S(B)$ is n-invariant for some $n \in N(B)$, then ϕ is also n^*-invariant;*

(iv) *if $\phi \in S(B)$ is both n_1-invariant and n_2-invariant, for some $n_1, n_2 \in N(B)$, then ϕ is also $n_1 n_2$-invariant;*

(v)   *if $N_0 \subset N(B)$ is a sub-$*$-semigroup, generated as a $*$-semigroup by some subset $W \subset N(B)$, and $\phi \in S(B)$ is W-invariant, then ϕ is N_0-invariant;*

(vi) *a state $\phi \in S(B)$ is fully invariant if and only if*

$$\forall n \in N(B) : \phi(nn^*) = \phi(n^*n). \tag{2.5}$$

PROOF. (i) It suffices to show that for any $n \in N(B)$ and any $b \in B$, we have $nb \in \overline{Bn}$ and $bn \in \overline{nB}$. If we fix n and b, then using the f_k^ℓ from Fact 2.2, combined with the commutativity of B,

$$nb = \lim_{k \to \infty} \lim_{\ell \to \infty} f_k^\ell(nn^*)nn^*nb = \lim_{k \to \infty} \lim_{\ell \to \infty} f_k^\ell(nn^*)nbn^*n. \tag{2.6}$$

Since n normalizes B, we know that $nbn^* \in B$, so the elements $b_k^\ell = f_k^\ell(nn^*)nbn^*$ all belong to B, and then (2.6), which now simply states that $nb = \lim_{k \to \infty} \lim_{\ell \to \infty} b_k^\ell n$, clearly proves that $nb \in \overline{Bn}$. The fact that $bn \in \overline{nB}$ is proved in exactly the same way.

(ii) This is obvious, since B is abelian.

(iii) Take a sequence $\{b_k\} \subset B$ such that $bn = \lim_k nb_k$. Then

$$\phi(n^*bn) = \lim_k \phi(n^*nb_k) = \lim_k \phi(nb_kn^*) = \phi(bnn^*) = \phi(nn^*b).$$

(iv) Suppose that $b \in B$. Take a sequence $\{c_k\} \subset B$ such that $(n_1^*n_1)n_2 = \lim_k n_2 c_k$. Then

$$\phi(n_1 n_2 b n_2^* n_1^*) = \phi(n_1^* n_1 n_2 b n_2^*) = \lim_k \phi(n_2 c_k b n_2^*) = \lim_k \phi(n_2^* n_2 c_k b)$$
$$= \phi(n_2^* n_1^* n_1 n_2 b),$$

so that ϕ is $n_1 n_2$-invariant.

Part (v) follows immediately from (iii) and (iv).

(vi) The 'if' implication (for which it suffices to prove (2.1) only for positive b) follows from the observation, that for any $n \in N(B)$ and any $b \in B^+$, the element $x = nb^{1/2}$ is again in $N(B)$, so applying condition (2.5) to x will clearly imply that

$$\phi(nbn^*) = \phi(b^{1/2}n^*nb^{1/2}) = \phi(n^*nb).$$

Conversely, if ϕ is fully invariant, then

$$\forall n \in N(B) : \phi(nn^*) = \lim_\lambda \phi(nu_\lambda n^*) = \lim_\lambda \phi(n^*nu_\lambda) = \phi(n^*n),$$

where $(u_\lambda) \subset B$ is an approximate identity for A.   □

Besides the notion of invariance for states on a C^*-subalgebra, we will also use the following two additional variants.

DEFINITION 2.4. Given a state $\psi \in S(A)$, we say that an element $x \in A$ *centralizes* ψ if $\psi(xa) = \psi(ax)$ for all $a \in A$. It is easy to see that the set

$$Z_\psi = \{x \in A : x \text{ centralizes } \psi\}$$

is a C^*-subalgebra of A. (Obviously, ψ is always tracial when restricted to Z_ψ. In particular, ψ is tracial on A if and only if its centralizer Z_ψ contains a set that generates A as a C^*-algebra.)

DEFINITION 2.5. If $B \subset A$ is a C^*-subalgebra and $n \in N(B)$, we will say that a map $\Phi : A \to B$ is *normalized by n* if $\Phi(nan^*) = n\Phi(a)n^*$ for all $a \in A$.

LEMMA 2.6. *Let $B \subset A$ be a nondegenerate abelian C*-subalgebra with a conditional expectation $\mathbb{E} : A \to B$, which is normalized by some $n \in N(B)$. For a state $\phi \in S(B)$, the following are equivalent:*

(i)   *ϕ is an n-invariant state on B;*

(ii)   *$\phi \circ \mathbb{E} \in S(A)$ is a state on A, which is centralized by n.*

PROOF. The implication (ii) $\Rightarrow$ (i) is pretty obvious, and holds even without the assumption that $\mathbb{E}$ is normalized by n. Indeed, if $b \in B$, then $nbn^* = \mathbb{E}(nbn^*)$ and $bn^*n = \mathbb{E}(bn^*n)$, so if $\phi \circ \mathbb{E}$ is centralized by n, then

$$\phi(nbn^*) = (\phi \circ \mathbb{E})(n(bn^*)) = (\phi \circ \mathbb{E})((bn^*)n) = \phi(bn^*n) = \phi(n^*nb).$$

For the proof of (i) $\Rightarrow$ (ii), we fix $a \in A$ and we show that $\phi(\mathbb{E}(an)) = \phi(\mathbb{E}(na))$. Fix polynomials (f_k^ℓ) as in Fact 2.2(iii). Since $\mathbb{E}$ is a conditional expectation, it follows that

$$\mathbb{E}(an) = \lim_{k \to \infty} \lim_{\ell \to \infty} \mathbb{E}(an f_k^\ell(n^*n)n^*n) = \lim_{k \to \infty} \lim_{\ell \to \infty} \mathbb{E}(an f_k^\ell(n^*n))n^*n.$$

By the *n*-invariance of ϕ,

$$\phi(\mathbb{E}(an)) = \lim_{k \to \infty} \lim_{\ell \to \infty} \phi(\mathbb{E}(an f_k^\ell(n^*n))n^*n)$$
$$= \lim_{k \to \infty} \lim_{\ell \to \infty} \phi(n\mathbb{E}(an f_k^\ell(n^*n))n^*).$$

Because $\mathbb{E}$ is normalized by *n*, with the help of (2.3) our computation continues as

$$\phi(\mathbb{E}(an)) = \lim_{k \to \infty} \lim_{\ell \to \infty} \phi(\mathbb{E}(nan f_k^\ell(n^*n)n^*))$$
$$= \lim_{k \to \infty} \lim_{\ell \to \infty} \phi(\mathbb{E}(na f_k^\ell(nn^*)nn^*)). \tag{2.7}$$

Since $\mathbb{E}$ is a conditional expectation onto an abelian C^*-subalgebra,

$$\mathbb{E}(na f_k^\ell(nn^*)nn^*) = \mathbb{E}(na) f_k^\ell(nn^*)nn^*$$
$$= f_k^\ell(nn^*)nn^*\mathbb{E}(na) = \mathbb{E}(f_k^\ell(nn^*)nn^*na),$$

so when we return to (2.7) and we also use (2.4), we finally get

$$\phi(\mathbb{E}(an)) = \lim_{k \to \infty} \lim_{\ell \to \infty} \phi(\mathbb{E}(f_k^\ell(nn^*)nn^*na)) = \phi(\mathbb{E}(na)). \qquad \square$$

THEOREM 2.7. *Let $B \subset A$ be a nondegenerate abelian C*-subalgebra with a conditional expectation $\mathbb{E} : A \to B$, which is normalized by some set $N_0 \subset N(B)$. For a state $\phi \in S(B)$, the following are equivalent:*

(i)   *ϕ is N_0-invariant;*

(ii)   *$\phi \circ \mathbb{E}$ is centralized by all elements of the C^*-subalgebra $C^*(B \cup N_0) \subset A$;*

(iii)   *the restriction $(\phi \circ \mathbb{E})|_{C^*(B \cup N_0)}$ is a tracial state on $C^*(B \cup N_0)$.*

PROOF. (i) $\Rightarrow$ (ii). Assume that ϕ is N_0-invariant. By Lemma 2.6, we clearly have the inclusion $N_0 \subset Z_{\phi \circ \mathbb{E}}$, so (using the fact that $Z_{\phi \circ \mathbb{E}}$ is a C^*-subalgebra of A) in order to prove statement (ii), it suffices to show that $\phi \circ \mathbb{E}$ is also centralized by B, which is pretty clear, since B is abelian.

The implication (ii) $\Rightarrow$ (iii) is trivial, since any state becomes tracial when restricted to its centralizer.

(iii) $\Rightarrow$ (i). Assume that $(\phi \circ \mathbb{E})|_{C^*(B \cup N_0)}$ is a tracial state. In particular, N_0 centralizes this restriction, so by Lemma 2.6 (applied to $C^*(B \cup N_0)$ in place of A), it again follows that ϕ is N_0-invariant.   □

3. Invariant states in the étale groupoid framework

The invariance conditions from Section 2 can be neatly described in the context of *étale groupoid C^*-algebras*, which we briefly recall here. A *groupoid* is a set G along with a subset $G^{(2)} \subset G \times G$ of *composable pairs* and two functions: a composition $G^{(2)} \ni (\alpha, \beta) \longmapsto \alpha\beta \in G$ and an involution $G \ni \gamma \longmapsto \gamma^{-1} \in G$ (the inversion), such that the following hold:

(i) $\gamma(\eta\zeta) = (\gamma\eta)\zeta$ whenever $(\gamma, \eta), (\eta, \zeta) \in G^{(2)}$;

(ii) $(\gamma, \gamma^{-1}) \in G^{(2)}$ for all $\gamma \in G$, and $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$ for $(\gamma, \eta) \in G^{(2)}$.

Elements satisfying $u = u^2 \in G$ are called *units* of G and the set of all such units is denoted by $G^{(0)} \subset G$ and called the *unit space* of G. There are maps $r, s : G \to G^{(0)}$ defined by

$$r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) = \gamma^{-1}\gamma$$

that are called, respectively, the *range* and *source* maps. If $A, B \subset G$, then

$$AB = \{\gamma \in G : \exists \alpha \in A, \beta \in B \text{ such that } \alpha\beta = \gamma\}.$$

It is not difficult to show that $(\alpha, \beta) \in G^{(2)}$ if and only if $s(\alpha) = r(\beta)$. For a given unit $u \in G^{(0)}$, there is an associated group $G(u) = \{\gamma \in G : r(\gamma) = s(\gamma) = u\}$; this is called the *isotropy* or *stabilizer group* of u. The union of all isotropy groups in G forms a subgroupoid of G called $\text{Iso}(G)$, the *isotropy bundle* of G. A groupoid is called *principal* (or an *equivalence relation*) if $\text{Iso}(G) = G^{(0)}$; that is, if no unit has nontrivial stabilizer group.

Throughout this present paper a groupoid G will be called *étale* if it is endowed with a locally compact and second countable topology so that:

(a)   the composition and inversion operations are continuous (the domain of ∘ is equipped with the relative product topology); and, furthermore,

(b)   the range and source maps are local homeomorphisms.

By condition (b), for each $\gamma \in G$, there exists an open set $\gamma \in X \subset G$ such that the maps $s(X) \xleftarrow{s|x} X \xrightarrow{r|x} r(X)$ are homeomorphisms onto open sets in G; such an X is called a *bisection*. Note that in the étale case, the unit space $G^{(0)}$ is in fact *clopen* in G, and all range and source fibers $r^{-1}(u)$, $s^{-1}(u)$, $u \in G^{(0)}$ are discrete in the relative topology; hence, compact subsets of G intersect any given range (or source) fiber at most finitely many times.

In order to define a C^*-algebra from an étale groupoid G, it is necessary to specify a $*$-algebra structure on $C_c(G)$. This is given by

$$(f \times g)(\gamma) = \sum_{(\alpha,\beta)\in G^{(2)}:\alpha\beta=\gamma} f(\alpha)g(\beta),$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

(Compactness of supports ensures that the sum involved in the definition of the product gives a well-defined element of $C_c(G)$.) As $G^{(0)}$ is open in G, we have an inclusion $C_c(G^{(0)}) \subset C_c(G)$, which turns $C_c(G^{(0)})$ into a $*$-subalgebra. However, the $*$-algebra operations on $C_c(G^{(0)})$ inherited from $C_c(G)$ coincide with the usual (pointwise) operations: $h^* = \bar{h}$ and $h \times k = hk$ for all $h, k \in C_c(G^{(0)})$. In fact, something similar can be said concerning the left and right $C_c(G^{(0)})$-module structures of $C_c(G)$: for all $f \in C_c(G)$, $h \in C_c(G^{(0)})$,

$$(f \times h)(\gamma) = f(\gamma)h(s(\gamma)),$$
$$(h \times f)(\gamma) = h(r(\gamma))f(\gamma).$$

Following Renault [23], for an étale groupoid G, the full C^*-norm on $C_c(G)$ is given as

$$\|f\| = \sup\left\{\|\pi(f)\| : \begin{matrix} \pi \text{ nondegenerate } \| \,.\, \|_I\text{-bounded} \\ *\text{-representation of } C_c(G) \end{matrix}\right\},$$

and the *full groupoid C^*-algebra* $C^*(G)$ is defined to be the completion of $C_c(G)$ in the full C^*-norm. When restricted to $C_c(G^{(0)})$, the full C^*-norm agrees with the usual sup-norm $\| \cdot \|_\infty$, so, by completion, the embedding $C_c(G^{(0)}) \subset C_c(G)$ gives rise to a nondegenerate inclusion $C_0(G^{(0)}) \subset C^*(G)$. At the same time, one can also consider the restriction map, which ends up being a contractive map $(C_c(G), \| \cdot \|) \ni f \longmapsto f|_{G^{(0)}} \in (C_c(G^{(0)}), \| \cdot \|_\infty)$, so, by completion, one obtains a contractive linear map $\mathbb{E} : C^*(G) \to C_0(G^{(0)})$, which is in fact a *conditional expectation*. We refer to $\mathbb{E}$ as the *natural expectation*. Using the KSGNS construction associated with $\mathbb{E}$ [15], we obtain a $*$-representation $\pi_{\mathbb{E}} : C^*(G) \to \mathcal{L}(L^2(C^*(G), \mathbb{E}))$, where $L^2(C^*(G), \mathbb{E})$ is the Hilbert $C_0(G^{(0)})$-module obtained by completing $C^*(G)$ in the norm given by the inner product $\langle a|b\rangle_{C_0(G^{(0)})} = \mathbb{E}(a^*b)$. With this representation in mind, the quotient $C^*(G)/\ker \pi_{\mathbb{E}}$ is the so-called *reduced* groupoid C^*-algebra, denoted by $C^*_{\text{red}}(G)$. An alternative description of the ideal $\ker \pi_{\mathbb{E}}$ is to employ the usual GNS representations $\pi_{ev_u \circ \mathbb{E}}$ associated

with the states $ev_u \circ \mathbb{E} \in S(C^*(G))$ that are obtained by composing $\mathbb{E}$ with evaluation maps $ev_u : C_0(G^{(0)}) \ni h \longmapsto h(u) \in \mathbb{C}$, $u \in G^{(0)}$. With these (honest) representations in mind, we have $\ker \pi_{\mathbb{E}} = \bigcap_{u \in G^{(0)}} \ker \pi_{ev_u \circ \mathbb{E}}$. As was the case with the full groupoid C^*-algebra, after composing with the quotient map $\pi_{\text{red}} : C^*(G) \to C^*_{\text{red}}(G)$, we still have an embedding $C_c(G) \subset C^*_{\text{red}}(G)$, so we can also view $C^*_{\text{red}}(G)$ as the completion of the convolution $*$-algebra $C_c(G)$ with respect to a (smaller) C^*-norm, denoted by $\| \cdot \|_{\text{red}}$. As before, when restricted to $C_c(G^{(0)})$, the norm $\| \cdot \|_{\text{red}}$ agrees with $\| \cdot \|_\infty$, so $C_0(G^{(0)})$ still embeds in $C^*_{\text{red}}(G)$ and, furthermore, since the natural expectation $\mathbb{E}$ vanishes on $\ker \pi_{\mathbb{E}}$, we will have a reduced version of natural expectation, denoted by $\mathbb{E}_{\text{red}} : C^*_{\text{red}}(G) \to C_0(G^{(0)})$, which satisfies $\mathbb{E}_{\text{red}} \circ \pi_{\text{red}} = \mathbb{E}$.

As pointed out for instance in [24], a large supply of normalizers for $C_0(G^{(0)})$ are those elements of the groupoid C^*-algebra represented by functions $f \in C_c(G)$ supported in bisections. We shall refer to such elements as *elementary normalizers* of $C_0(G^{(0)})$. Note that the collection $N_{\text{elem}}(C_0(G^{(0)}))$ of elementary normalizers, along with 0, is a $*$-subsemigroup of $N(C_0(G^{(0)}))$ and, furthermore, $N_{\text{elem}}(C_0(G^{(0)}))$ generate the ambient algebra—$C^*(G)$ or $C^*_{\text{red}}(G)$—as a C^*-algebra. Using the embedding of $C_c(G)$ in the groupoid (full or reduced) C^*-algebra, we interpret $N_{\text{elem}}(C_0(G^{(0)}))$ as a subset in $C_c(G)$, namely

$$N_{\text{elem}}(C_0(G^{(0)})) = \bigcup_{X \text{ bisection}} C_c(X) \subset C_c(G). \tag{3.1}$$

REMARKS. In order to avoid any unnecessary notational complications or duplications, the results and definitions in the remainder of this section are stated only using the reduced C^*-algebra $C^*_{\text{red}}(G)$ as the ambient C^*-algebra. However, with only a few explicitly noted exceptions, by composing with the quotient $*$-homomorphism $\pi_{\text{red}} : C^*(G) \to C^*_{\text{red}}(G)$, the same results will hold if we use the full C^*-algebra $C^*(G)$ instead; we leave it to the reader to write down the missing statements corresponding to the full case (by simply erasing the subscript 'red' from the statements).

The étale groupoid framework is particularly convenient because one of the hypotheses in Lemma 2.6 above is automatically satisfied.

PROPOSITION 3.1. *The natural conditional expectation* $\mathbb{E}_{\text{red}} : C^*_{\text{red}}(G) \to C_0(G^{(0)})$ *is normalized by all elementary normalizers. In particular, for a state ϕ on $C_0(G^{(0)})$, the following are equivalent:*

(i) ϕ *is an* $N_{\text{elem}}(C_0(G^{(0)}))$*-invariant state on* $C_0(G^{(0)})$;

(ii) $\phi \circ \mathbb{E}_{\text{red}}$ *is a tracial state on* $C^*_{\text{red}}(G)$.

PROOF. Assume that $n \in C_c(X)$ for some bisection $X \subset G$. In order to prove the first assertion, we must show that $\mathbb{E}_{\text{red}}(n \times f \times n^*) = n \times \mathbb{E}_{\text{red}}(f) \times n^*$ for all $f \in C_c(G)$. Fix f as well as $x \in G^{(0)}$. Then

$$\mathbb{E}_{\text{red}}(n \times f \times n^*)(u) = \begin{cases} |n(\gamma)|^2 f(s(\gamma)) & \text{if } \exists \gamma \in X \cap r^{-1}(u) \cap s^{-1}(\text{supp } f), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that this is the same as $(n \times \mathbb{E}_{\text{red}}(f) \times n^*)(u)$.

The second statement is a direct consequence of Theorem 2.7, combined with the fact that $N_{\text{elem}}(C_0(G^{(0)}))$ generates $C^*_{\text{red}}(G)$ as a C^*-algebra.   □

We want to characterize the $N_{\text{elem}}(C_0(G^{(0)}))$-invariant states on $C_0(G^{(0)})$—hereafter referred to as *elementary invariant* states—completely in measure-theoretical terms on G. We introduce the following terminology in parallel with Definition 2.1.

DEFINITION 3.2. Let G be an étale topological groupoid with unit space $G^{(0)}$, and let μ be a positive Radon measure on $G^{(0)}$.

(1)   Given an open bisection $X \subset G$, we say that μ is *X-balanced* if $\mu(XBX^{-1}) = \mu(s(X) \cap B)$ for any Borel set $B \subset G^{(0)}$.

(2)   If $\mathcal{X}$ is a family of open bisections, then we say that μ is *$\mathcal{X}$-balanced* if μ is X-balanced for all $X \in \mathcal{X}$.

(3)   If μ is X-balanced for every open bisection X, then we say that μ is *totally balanced*.

NOTATIONS. Given a proper continuous function between locally compact spaces $h : X \to Y$, and a Radon measure μ on X, we denote its h-pushforward by $h_*\mu$. This is a Radon measure on Y, given by $(h_*\mu)(A) = \mu(h^{-1}(A))$, for any Borel set $A \subset Y$. Note that the pushforward construction is covariant: $(g \circ f)_*\mu = g_*(f_*\mu)$.

By Riesz's theorem, we have a bijective correspondence

$$\text{Prob}(X) \ni \mu \longmapsto \phi_\mu \in S(C_0(X)) \tag{3.2}$$

between the space of *Radon probability measures on X* and the *state space of $C_0(X)$*, defined as follows. For each $\mu \in \text{Prob}(X)$, the associated state $\phi_\mu \in S(C_0(X))$ is

$$\phi_\mu(f) = \int_X f(x)\, d\mu(x), \quad f \in C_0(X).$$

On the level of positive linear functionals, the pushforward construction corresponds to *composition*:

$$(h_*\phi)(f) = \phi(f \circ h), \quad f \in C_0(Y), h : X \to Y.$$

LEMMA 3.3. *With G as above, let $X \subset G$ be an open bisection. For a finite Radon measure μ on $G^{(0)}$, the following are equivalent:*

(i) $\mu|_{s(X)} = (s \circ (r|_X)^{-1})_*(\mu|_{r(X)})$;

(ii) $\mu(s(B)) = \mu(r(B))$ *for all Borel subsets $B \subset X$;*

(iii) $\mu(s(K)) = \mu(r(K))$ *for all compact subsets $K \subset X$;*

(iv) μ *is X-balanced.*

(In condition (i), we use the restriction notation for measures: if μ is a finite Radon measure on $G^{(0)}$—thought of as a function $\mu : \text{Bor}(G^{(0)}) \to [0, \infty)$—and $D \subset G^{(0)}$ is some open subset, then $\mu|_D$ is the Radon measure on D obtained by restricting μ to $\text{Bor}(D)$.)

PROOF. The equivalence (i) ⇔ (ii) is trivial, because the maps $s(X) \xleftarrow{s|_X} X \xrightarrow{r|_X} r(X)$ are homeomorphisms onto open sets.

The equivalence (ii) ⇔ (iv) follows from the observation that, for any Borel set $B \subset G^{(0)}$, the set $B' = X \cap s^{-1}(B) \subset X$ is Borel and, furthermore, the sets that appear in the definition of X-invariance are precisely $XBX^{-1} = r(B')$ and $s(X) \cap B = s(B')$.

Lastly, the equivalence (ii) ⇔ (iii) follows from regularity and finiteness of μ. □

We are interested in balanced measures, because they are tied up with elementary invariance.

LEMMA 3.4. *Let G be an étale groupoid with unit space $G^{(0)}$, let μ be a Radon probability measure on $G^{(0)}$, and let ϕ_μ be the state on the C^*-subalgebra $C_0(G^{(0)}) \subset C^*_{\mathrm{red}}(G)$ given by (3.2). For an open bisection $X \subset G$, the following conditions are equivalent:*

(i)   *μ is X-balanced;*

(ii)   *ϕ_μ is $C_c(X)$-invariant. (As in (3.1), $C_c(X) \subset N_{C^*_{\mathrm{red}}(G)}(C_0(G^{(0)}))$.)*

PROOF. The entire argument will be based on the following claim.

CLAIM. For any $n \in C_c(X)$ and any $b \in C_c(G^{(0)})$, one has the equalities

$$\phi_\mu(n^* \times n \times b) = \int_{s(X)} |(n \circ (s|_X)^{-1})(u)|^2 b(u)\, d(\mu|_{s(X)})(u), \tag{3.3}$$

$$\phi_\mu(n \times b \times n^*) = \int_{r(X)} |(n \circ (r|_X)^{-1})(u)|^2 (b \circ s \circ (r|_X)^{-1})(u)\, d(\mu|_{r(X)})(u), \tag{3.4}$$

$$\phi_\mu(n \times b \times n^*) = \int_{s(X)} |(n \circ (s|_X)^{-1})(u)|^2 b(u)\, d(s \circ (r|_X)^{-1})_*(\mu|_{r(X)})(u). \tag{3.5}$$

The equality (3.3) follows from the definition of the convolution multiplication and $*$-involution, which yield

$$(n^* \times n)(u) = \begin{cases} |n((s|_X)^{-1}(u))|^2, & u \in s(X), \\ 0, & u \notin s(X), \end{cases}$$

so we can multiply the functions n^*n and b to obtain

$$(n^* \times n \times b)(u) = \begin{cases} |n((s|_X)^{-1}(u))|^2 b(u), & u \in s(X) \\ 0, & u \notin s(X). \end{cases}$$

Likewise, the equality in (3.4) follows from

$$(n \times b \times n^*)(u) = \begin{cases} |n((r|_X)^{-1}(u))|^2 \cdot b(s((r|_X)^{-1}(u))), & u \in r(X), \\ 0, & u \notin r(X), \end{cases}$$

which implies that the support of $n \times b \times n^*$ is contained in $X(\operatorname{supp} b)X^{-1} \subset r(X)$. Lastly, the equality between the right-hand sides of (3.4) and (3.5) follows immediately

by applying the definition of the pushforward,

$$\int_{s(X)} f \, d(s \circ (r|_X)^{-1})_*(\mu|_{r(X)}) = \int_{r(X)} (f \circ s \circ (r|_X)^{-1}) \, d(\mu|_{r(X)}),$$

to functions $f \in C_c(s(X))$ of the form $f(u) = |n \circ ((s|_X)^{-1})(u)|^2 b(u)$.

Having proved the claim, the implication (i) $\Rightarrow$ (ii) follows from Lemma 3.3, which yields

$$\forall n \in C_c(X), \quad b \in C_c(G^{(0)}) : \phi_\mu(n^* \times n \times b) = \phi_\mu(n \times b \times n^*). \qquad (3.6)$$

By density, (3.6) holds for all $n \in C_c(X)$, $b \in C_0(G^{(0)})$; thus, ϕ_μ is n-invariant for all $n \in C_c(X)$.

As for the implication (ii) $\Rightarrow$ (i), all we have to observe is that, if ϕ_μ is $C_c(X)$-invariant, then (3.6) is valid, which, by the identities (3.3) and (3.5), simply states that the equality

$$\int_{s(X)} f \, d(s \circ (r|_X)^{-1})_*(\mu|_{r(X)}) = \int_{s(X)} f \, d(\mu|_{s(X)}) \qquad (3.7)$$

holds for all functions of the form

$$f(u) = |(n \circ (s|_X)^{-1})(u)|^2 b(u), \quad n \in C_c(X), \ b \in C_c(G^{(0)}).$$

Since (using a partition of unity argument) the functions of the above form linearly span all functions in $C_c(s(X))$, the equality (3.7) simply states that

$$(s \circ (r|_X)^{-1})_*(\mu|_{r(X)}) = \mu|_{s(X)},$$

so, by Lemma 3.3(i), it follows that μ is indeed X-balanced. $\qquad \square$

Combining Proposition 3.1 with Lemma 3.4, we now reach the following conclusion.

THEOREM 3.5. *Let G be an étale groupoid with unit space $G^{(0)}$, let μ be a probability Radon measure on $G^{(0)}$, and let ϕ_μ be the state on the C^*-subalgebra $C_0(G^{(0)}) \subset C^*_{\mathrm{red}}(G)$ given by (3.2). The following conditions are equivalent:*

(i)   *μ is totally balanced;*
(ii)   *ϕ_μ is elementary invariant;*
(iii)   *ϕ_μ is fully invariant;*
(iv)   *$\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is a tracial state on $C^*_{\mathrm{red}}(G)$.*

In concrete situations, one would like to check condition (i) from the above theorem in an 'economical' way. To be more precise, assuming that a given measure $\mu \in \mathrm{Prob}(G^{(0)})$ is X-balanced, for some collection of bisections X, we seek a natural subalgebra on which $\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is tracial (as in Theorem 2.7), and furthermore find criteria on X which ensure that our subalgebra is in fact all of $C^*_{\mathrm{red}}(G)$. Parts of the lemma below mimic corresponding statements from Proposition 2.3. (Each one of the statements (i)–(iii) has an implicit statement built in: the new sets, such as X', X^{-1}, and $X_1 X_2$, are always bisections.)

PROPOSITION 3.6. *Let G be an étale groupoid with unit space $G^{(0)}$ and let μ be a Radon probability measure on $G^{(0)}$.*

(i)   *If μ is X-balanced for some bisection X, then μ is X'-balanced for any open subset $X' \subset X$.*

(ii) *If μ is X-balanced for some bisection X, then μ is X^{-1}-balanced.*

(iii) *If μ is both X_1- and X_2-balanced, for two bisections X_1, X_2, then μ is $X_1 X_2$-balanced.*

(iv) *Assume that X is an open set, written as a union $X = \bigcup_{j \in J} X_j$ of bisections, such that $s|_X, r|_X : X \to G^{(0)}$ are injective. Then X is a bisection and, if μ is X_j-balanced for all $j \in J$, then μ is X-balanced.*

PROOF. Statements (i) and (ii) are trivial from Lemma 3.3.

Before we prove (iii), we need some clarifications. First of all, the set $X_1 X_2$ is obtained as the image of the open set

$$X_1 \circ X_2 = \{(\alpha, \beta) \in X_1 \times X_2 : s(\alpha) = r(\beta)\} = X_1 \times X_2 \cap G^{(2)} \subset G^{(2)}$$

under the composition map $m : G^{(2)} \to G$. Secondly, by the bisection property, the restrictions of the coordinate maps $X_1 \xleftarrow{p_1} X_1 \times X_2 \xrightarrow{p_2} X_2$ give rise to two homeomorphisms $p_1(X_1 \circ X_2) \xleftarrow{p_1} X_1 \circ X_2 \xrightarrow{p_2} p_2(X_1 \circ X_2)$ onto open subsets of X_1 and X_2, respectively, and furthermore the compositions $s \circ p_1$ and $r \circ p_2$ agree on $X_1 \circ X_2$, and the resulting map, denoted here by $t : X_1 \circ X_2 \to \subset G^{(0)}$, is a homeomorphism onto an open subset $D \subset G^{(0)}$. (This open set is simply $D = t(X_1 \circ X_2) = s(X_1) \cap r(X_2)$. By construction, $X_1 X_2 = \varnothing \Leftrightarrow s(X_1) \cap r(X_2) = \varnothing$.) Furthermore, again by the bisection property, $m|_{X_1 \circ X_2} : X_1 \circ X_2 \to X_1 X_2$ is also a homeomorphism onto an open set, so composing its inverse with the coordinate maps, we obtain two homeomorphisms $q_k = p_k \circ (m|_{X_1 \circ X_2})^{-1} : X_1 X_2 \to X_k$, $k = 1, 2$, which satisfy $s|_{X_1 X_2} = s \circ q_1$ and $r|_{X_1 X_2} = r \circ q_2$. Using all these three homeomorphisms, the fact that $X_1 X_2$ is a bisection is obvious. Not only are the maps $s(X_1 X_2) \xleftarrow{s|_{X_1 X_2}} X_1 X_2 \xrightarrow{r|_{X_1 X_2}} r(X_1 X_2)$ homeomorphisms, but so is the map $r \circ q_2 = s \circ q_1 = t \circ (m|_{X_1 \circ X_2})^{-1} : X_1 X_2 \to D$.

After all these preparations, statement (iii) follows from the observation that the X_1- and X_2-balancing features imply that, for any Borel set $B \subset X_1 X_2$,

$$\mu(s(B)) = \mu(s(q_2(B))) = \mu(r(q_2(B)))$$
$$= \mu(s(q_1(B))) = \mu(r(q_1(B))) = \mu(r(B)),$$

so the desired conclusion follows from Lemma 3.3.

(iv) Since we have the equalities $s(X) = \bigcup_{j \in J} s(X_j)$ and $r(X) = \bigcup_{j \in J} s(X_j)$, it follows that $s(X)$ and $r(X)$ are open. The fact that both $s(X) \xleftarrow{s|_X} X \xrightarrow{r|_X} r(X)$ are homeomorphisms follows by their assumed injectivity and local compactness.

Finally, to prove that μ is X-balanced, we apply criterion (iii) from Lemma 3.3. Start with some compact set $K \subset X$ and, using compactness, write it as a finite disjoint union $K = \bigcup_{k=1}^{n} B_{j_k}$, where $B_{j_k} \subset X_{j_k}$, $k = 1, \ldots, n$, are Borel sets. Using the fact that

μ is X_j-balanced for all j, we know that $\mu(s(B_{j_k})) = \mu(r(B_{j_k}))$ for all k, so using that s and r are homeomorphisms, we also have $s(K) = \bigcup_{k=1}^n s(B_{j_k})$ and $r(K) = \bigcup_{k=1}^n r(B_{j_k})$ (disjoint unions of Borel sets in $s(X)$ and $r(X)$, respectively), so

$$\mu(s(K)) = \mu\Big(\bigcup_{k=1}^n s(B_{j_k})\Big) = \sum_{k=1}^n \mu(s(B_{j_k}))$$

$$= \sum_{k=1}^n \mu(r(B_{j_k})) = \mu\Big(\bigcup_{k=1}^n r(B_{j_k})\Big) = \mu(r(K)). \qquad \square$$

Using the above result, combined with Lemma 3.4, we immediately obtain the following measure-theoretic groupoid analogue of Theorem 2.7.

THEOREM 3.7. *Assume that $\mathcal{W}$ is a collection of bisections in the étale groupoid G, and let X be the inverse semigroup generated by $\mathcal{W}$. For a measure $\mu \in \mathrm{Prob}(G^{(0)})$, the following are equivalent:*

(i)   *μ is $\mathcal{W}$-balanced;*
(ii)   *μ is X-balanced;*
(iii)   *the state $\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is tracial when restricted to the subalgebra*

$$C^*\Big(C_0(G^{(0)}) \cup \bigcup_{W \in \mathcal{W}} C_c(W)\Big) = \overline{\mathrm{span}}\Big(C_0(G^{(0)}) \cup \bigcup_{X \in X} C_c(X)\Big).$$

REMARK 3.8. A sufficient condition for a collection X of bisections of G to satisfy the equality

$$\overline{\mathrm{span}}\Big(C_0(G^{(0)}) \cup \bigcup_{X \in X} C_c(X)\Big) = C^*_{\mathrm{red}}(G)$$

is that X *covers* $G \setminus G^{(0)}$. This follows using a standard partition of unity argument, which implies the equality $C_c(G) = \mathrm{span}(C_0(G^{(0)}) \cup \bigcup_{X \in X} C_c(X))$. As a consequence, the desired 'economical' criterion for traciality of $\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is as follows.

COROLLARY 3.9. *Assume that G, $\mathcal{W}$, and X are as in Theorem 3.7. If $\mu \in \mathrm{Prob}(G^{(0)})$ is $\mathcal{W}$-balanced, and X covers $G \setminus G^{(0)}$, then $\phi_\mu \circ \mathbb{E}_{\mathrm{red}}$ is tracial on $C^*_{\mathrm{red}}(G)$.*

4. Tracial states via extension properties

So far, assuming that a nondegenerate abelian C^*-subalgebra $B \subset A$ is the range of a conditional expectation $\mathbb{E} : A \to B$, we have examined certain conditions for both a state $\phi \in S(B)$ and for $\mathbb{E}$ that ensure that $\phi \circ \mathbb{E}$ is a trace. In the groupoid framework, the natural conditional expectation $\mathbb{E}$ exhibited nice behavior (elementary invariance), so the focus was solely placed on ϕ. In this section we provide another framework, in which again the conditional expectation in question will also be normalized by all $n \in N(B)$. (As a side issue, one should also be concerned with the *uniqueness* of conditional expectation.)

A natural class of subalgebras to which this analysis can be carried on nicely are Renault's Cartan subalgebras (see [24]; see also the comment following Corollary 4.3 below). As it turns out, very little from the Cartan subalgebra machinery is needed for our purposes: the *almost extension property* [17], which requires that the set

$$P_1(B \uparrow A) = \{\omega \in \hat{B} : \omega \text{ has a unique extension to a state on } A\}$$

is weak-$*$ dense in $\hat{B}$—the Gelfand spectrum of B. (A slight strengthening of the above condition will be introduced in the comment following Lemma 4.2 below.)

The utility of the almost extension property is exhibited by Lemma 4.2 below, in preparation of which we need the following simple fact.

FACT 4.1. *Let ω be a state on $B \subset A$ with extension $\theta \in S(A)$, so that $\theta|_B = \omega$. If $x, y \in A$ and satisfy either*

(1) $y^*y \in B$ and $\omega(y^*y) = 0$; or
(2) $xx^* \in B$ and $\omega(xx^*) = 0$,

then $\theta(xy) = 0$.

In particular, if $b \in B$ satisfies $0 \le b \le 1$ (in the unitized C^*-algebra $B^{\tilde{}}$) and $\omega(b) = 1$, then

$$\forall a \in A : \theta(a) = \theta(ab) = \theta(ba) = \theta(bab).$$

PROOF. Apply the Cauchy–Schwarz inequality for the sesquilinear form:

$$\langle a | a' \rangle = \theta(a^*a').$$

The second statement follows from the first one applied with $y = 1 - b$.   □

LEMMA 4.2 (compare to [13, Lemma 6]). *Let $B \subset A$ be a nondegenerate abelian C^*-subalgebra with the almost extension property, and let $\mathbb{E} : A \to B$ be a conditional expectation. Then $\mathbb{E}$ is normalized by all $n \in N(B)$.*

REMARKS. As noted in [17], the almost extension property implies that at most one conditional expectation $\mathbb{E} : A \to B$ can exist. In the case such an expectation does exist and the almost extension property holds, we say that the inclusion $B \subset A$ has the *conditional* almost extension property.

PROOF OF LEMMA 4.2. Fix some normalizer $n \in N(B)$; let us prove that

$$\mathbb{E}(nan^*) = n\mathbb{E}(a)n^* \tag{4.1}$$

for all $a \in A$. Fix polynomials (f_k^ℓ) as in Fact 2.2(iii), so

$$\mathbb{E}(nan^*) = \lim_{k \to \infty} \lim_{\ell \to \infty} \mathbb{E}(nn^*nf_k^\ell(n^*n)af_k^\ell(n^*n)n^*nn^*). \tag{4.2}$$

Likewise, and using also the fact that $\mathbb{E}$ is a conditional expectation, we also have

$$n\mathbb{E}(a)n^* = \lim_{k \to \infty} \lim_{\ell \to \infty} nn^*nf_k^\ell(n^*n)\mathbb{E}(a)f_k^\ell(n^*n)n^*nn^*$$

$$= \lim_{k \to \infty} \lim_{\ell \to \infty} n\mathbb{E}(n^*nf_k^\ell(n^*n)af_k^\ell(n^*n)n^*n)n^*. \tag{4.3}$$

Inspecting (4.2) and (4.3), we now see that it suffices to prove (4.1) for elements of the form $a = n^*a_1n$; in other words, instead of (4.1), it suffices to prove that

$$\forall a \in A : \mathbb{E}(nn^*ann^*) = n\mathbb{E}(n^*an)n^*.$$

As both sides of this equation belong to B, we only need to show that

$$\omega(\mathbb{E}(nn^*ann^*)) = \omega(n\mathbb{E}(n^*an)n^*) \qquad (*)$$

for all $\omega \in P_1(B \uparrow A)$.

Suppose that $\omega(nn^*) = 0$. In this case, we have by Fact 4.1 that both sides of (*) are zero. Suppose that $\omega(nn^*) > 0$ and define two states ψ_ω and θ_ω on A by

$$\psi_\omega(a) = \frac{(\omega \circ \mathbb{E})(nn^*ann^*)}{\omega(nn^*)^2} \quad \text{and} \quad \theta_\omega(a) = \frac{\omega(n\mathbb{E}(n^*an)n^*)}{\omega(nn^*)^2},$$

so (*) is equivalent to the equality $\psi_\omega = \theta_\omega$ (of states on A). Note that, if $b \in B$, then $\psi_\omega(b) = \theta_\omega(b) = \omega(b)$, so that both states ψ_ω and θ_ω are extensions of $\omega \in P_1(B \uparrow A)$, so, by uniqueness, we have $\psi_\omega = \theta_\omega$, and (*) is established. □

In the context of the conditional almost extension property, Theorem 2.7 has the following consequences.

COROLLARY 4.3. *Let $B \subset A$ be a nondegenerate abelian C*-subalgebra with the conditional almost extension property, let $\mathbb{E} : A \to B$ be its (unique) conditional expectation, and let ϕ be a state on B.*

(a) *For a subset $N_0 \subset N(B)$, the following are equivalent:*

   (i) *ϕ is N_0-invariant;*
   (ii) *$\phi \circ \mathbb{E}$ is centralized by all elements of $C^*(B \cup N_0) \subset A$;*
   (iii) *the restriction $(\phi \circ \mathbb{E})|_{C^*(B\cup N_0)}$ is a tracial state on $C^*(B \cup N_0)$.*

(b) *In particular, if B is regular, then $\phi \circ \mathbb{E}$ is a trace on A if and only ϕ is fully invariant.*

(Of course, statement (b) can be slightly relaxed, by requiring that ϕ is only N_0-invariant for a subset $N_0 \subset N(B)$ which together with B generates A as a C*-algebra.)

REMARKS. A natural class exhibiting the conditional almost extension property are Cartan subalgebras, as defined by Renault in [24]. They are regular nondegenerate inclusions $B \subset A$, in which:

- B is *maximal abelian* (masa) in A; and
- there exists a *faithful* conditional expectation $\mathbb{E} : A \to B$ (which is necessarily unique).

As pointed out for instance in [3], Cartan subalgebras do have the conditional almost extension property, but there are many examples of regular nondegenerate abelian C*-subalgebra inclusions $B \subset A$ with the conditional almost extension property

which are non-Cartan. In fact, for étale groupoids, the equivalent condition to the almost extension property is *topological principalness*: the set of units $u \in G^{(0)}$ with trivial isotropy $G(u)$ is dense in $G^{(0)}$. For topologically principal groupoids, both inclusions $C_0(G^{(0)}) \subset C^*_{red}(G)$ and $C_0(G^{(0)}) \subset C^*(G)$ have the conditional almost extension property. However, since the (full) conditional expectation $\mathbb{E} : C^*(G) \to C_0(G^{(0)})$ is not faithful in general, $C_0(G^{(0)})$ is generally not Cartan in $C^*(G)$. On the other hand, since the (reduced) expectation $\mathbb{E}_{red} : C^*_{red}(G) \to C_0(G^{(0)})$ is faithful, $C_0(G^{(0)})$ is Cartan in $C^*_{red}(G)$.

Up to this point, we have seen that for regular nondegenerate abelian C^*-subalgebras $B \subset A$ with the conditional almost extension property, Corollary 4.3(b) provides us with an injective w^*-continuous affine map

$$S^{inv}(B) \ni \phi \longmapsto \phi \circ \mathbb{E} \in T(A), \tag{4.4}$$

which is a right inverse of the restriction map (2.2); in particular, it follows that for such inclusions, the map (2.2) is surjective.

QUESTION. If $B \subset A$ is a regular nondegenerate abelian C^*-subalgebra with the conditional almost extension property, under what additional circumstances is the map (4.4) also surjective? (If this is the case, this would imply that the restriction map (2.2) is in fact an affine w^*-homeomorphism.)

As the example below suggests, even in the case of Cartan inclusions, the map (4.4) may fail to be surjective.

EXAMPLE 4.4. Let $B = C(\overline{\mathbb{D}}) \subset A = C(\overline{\mathbb{D}}) \rtimes_\alpha \mathbb{Z} = C^*(C(\overline{\mathbb{D}}), u)$, where α is a rotation of $\mathbb{D}$ by an irrational multiple of π and u is the unitary that implements the automorphism in the crossed product. Then B is a Cartan subalgebra, as can be directly verified. The conditional expectation is given on the dense set of Laurent polynomials in u by

$$\mathbb{E}\left(\sum f_n u^n\right) = f_0.$$

(It is obvious that $\mathbb{E}(u^n) = 0$ for all $n \neq 0$.) As 0 is a fixed point under the rotation α, we have that $(ev_0(\cdot)1, id)$ is a covariant representation of $(C(\overline{\mathbb{D}}), \alpha)$ in $C^*(\mathbb{Z}) \cong C(\mathbb{T})$; thus, it induces a $*$-homomorphism $\rho : A \to C(\mathbb{T})$. Any state ψ on $C(\mathbb{T})$ defines a state $\psi \circ \rho$ on A, which is clearly tracial since $C(\mathbb{T})$ is abelian and ρ is a $*$-homomorphism. A tracial state of this form factors through $\mathbb{E}$ if and only if it maps $\{u^n\}_{n \neq 0}$ to 0, so taking for instance $\psi = ev_z$ to be a point evaluation at $z \in \mathbb{T}$, then clearly $(ev_z \circ \rho)(u) = z \neq 0$, so the trace $\tau = ev_z \circ \rho \in T(A)$ does not belong to the range of the map (4.4).

REMARK 4.5. In connection with the above example, the reason that the map $\phi \to \phi \circ \mathbb{E}$ fails to be surjective is the fact that the state ev_0 on $C(\overline{\mathbb{D}})$ does not have a unique extension to a state on $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$. Such an obstruction can be avoided if we consider inclusions with the *(honest) extension property*, which are those nondegenerate abelian C^*-subalgebra inclusions $B \subset A$ for which *every* pure state on B has a unique extension to a state on A. As shown in [12] and [1], the extension property implies the following:

- B is maximal abelian;
- there exists a unique conditional expectation $\mathbb{E} : A \to B$;
- $\ker \mathbb{E} = [A, B]$ (the closed linear span of the set of elements of the form $ab - ba$, $a \in A, b \in B$).

From the last two properties it follows immediately that any tracial state $\tau \in T(A)$ vanishes on $\ker \mathbb{E}$. Thus, any tracial state factors through $\mathbb{E}$, and is completely determined by its restriction to B. Since restrictions of the form $\tau|_B$, $\tau \in T(A)$ are always fully invariant, Corollary 4.3 has the following immediate consequence.

COROLLARY 4.6. *If $B \subset A$ is a regular abelian C^*-subalgebra inclusion with the extension property, and $\mathbb{E} : A \to B$ is its associated conditional expectation, then the map*

$$S^{\mathrm{inv}}(B) \ni \phi \longmapsto \phi \circ \mathbb{E} \in T(A)$$

is an affine w^*-homeomorphism with inverse $\tau \to \tau|_B$.

EXAMPLE 4.7. For an étale groupoid G, the inclusions of $C_0(G^{(0)})$ into either the full or reduced C^*-algebra of G have the extension property if and only if G is *principal*: all units in G have trivial isotropy group. In the case when G is a principal groupoid, the above combined with Theorem 3.5 (in both its reduced and full versions) establishes a bijection between the set of totally balanced measures on $G^{(0)}$ and the tracial state spaces of both $C^*(G)$ and $C^*_{\mathrm{red}}(G)$. In particular, if Γ is a discrete group acting freely on X, then the tracial state spaces of both crossed-product C^*-algebras $C_0(X) \rtimes \Gamma$ and $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ are naturally identified with the Γ-invariant Radon probability measures on X.

The condition that the groupoid be principal (or, for crossed products, that the action be free) cannot be relaxed, especially in the nonamenable case, as the following example shows. Let $\mathbf{F}_2$—the free group on two generators—act by translation on its Alexandrov compactification $\mathbf{F}_2 \cup \{\infty\}$ (by keeping ∞ fixed), so that the associated action of $\mathbf{F}_2$ on $c_0(\mathbf{F}_2)^\sim$—the unitization of $c_0(\mathbf{F}_2)$—is given by $\alpha_g(f + c\mathbf{1}) = \lambda_g(f) + c\mathbf{1}$, where λ is the left-shift action on $c_0(\mathbf{F}_2)$.

CLAIM. $c_0(\mathbf{F}_2)^\sim \rtimes_{\mathrm{red}} \mathbf{F}_2$ has a unique tracial state.

Note that any tracial state on this crossed product is determined by its value on elements of the form fu_g, where $f \in c_0(\mathbf{F}_2)^\sim$ and $\{u_g\}_{g \in \mathbf{F}_2}$ are unitary generators. The key step in proving the claim is then showing that for any tracial state τ on $c_0(\mathbf{F}_2)^\sim \rtimes_{\mathrm{red}} \mathbf{F}_2$,

$$\forall f \in c_0(\mathbf{F}_2)^\sim, \quad g \neq e : \tau(fu_g) = 0. \tag{4.5}$$

Applying [19, Proposition 7.7.9] to the inclusion $\mathbb{C}\mathbf{1} \subset c_0(\mathbf{F}_2)^\sim$, it follows that the C^*-subalgebra $C^*(\{u_g\}_{g \in \mathbf{F}_2}) \subset c_0(\mathbf{F}_2)^\sim \rtimes_{\mathrm{red}} \mathbf{F}_2$ is isomorphic to $C^*_{\mathrm{red}}(\mathbf{F}_2)$, so using Powers' theorem [21] it follows that $C^*(\{u_g\}_{g \in \mathbf{F}_2})$ has a unique tracial state. In particular, any tracial state on $c_0(\mathbf{F}_2)^\sim \rtimes_{\mathrm{red}} \mathbf{F}_2$ must vanish on any u_g with $g \neq e$. Since any $f \in c_0(\mathbf{F}_2)^\sim$ can be written as $f = f^0 + c\mathbf{1}$, where $f^0 \in c_0(\mathbf{F}_2)$ and $c \in \mathbb{C}$, it suffices to prove (4.5) for

functions $f \in c_0(\mathbf{F}_2)$; in fact, by density, it suffices to prove (4.5) for $f = \delta_h$ for some $h \in \mathbf{F}_2$. But, for functions of this form,

$$\tau(fu_g) = \tau(f^2 u_g) = \tau(fu_g f) = \tau(f\lambda_g(f)u_g) = 0,$$

where the last equation follows from $\alpha_g(\delta_h) = \delta_{gh}$. Thus, a tracial state τ on $c_0(\mathbf{F}_2)^{\widetilde{}} \rtimes_{\mathrm{red}} \mathbf{F}_2$ is determined by its value on elements of the form $f = fu_e$, $f \in c_0(\mathbf{F}_2)^{\widetilde{}}$. If $f \in c_0(\mathbf{F}_2)$ and $g \in \mathbf{F}_2$, then $\tau(f) = \tau(u_g f u_g^*) = \tau(\alpha_g(f))$. Thus, τ, viewed as coming from a measure on the discrete space $\mathbf{F}_2$, must be invariant under left shift. As $\mathbf{F}_2$ is infinite, it must be the case that τ vanishes on any $f \in c_0(\mathbf{F}_2)$. So, a tracial state on $c_0(\mathbf{F}_2)^{\widetilde{}} \rtimes_{\mathrm{red}} \mathbf{F}_2$ is determined by its value on $\mathbb{C}u_e$, and must therefore be given by $\tau(\sum_g f_g u_g) = f_e(\infty)$, so the claim follows.

By contrast, the full crossed product $c_0(\mathbf{F}_2)^{\widetilde{}} \rtimes \mathbf{F}_2$ has many tracial states, since it has the full C^*-algebra $C^*(\mathbf{F}_2)$ as a quotient, and in turn it will also have $C(\mathbb{T}^2)$ as a quotient.

5. Graph C*-algebras

In this section we provide a method for parametrizing tracial state spaces on graph C^*-algebras. Our approach complements the treatment in [29] by giving an explicit parametrization of the tracial state space of a graph C^*-algebra.

We caution the reader that this section is quite long, mostly because of the extensive review of graph terminology. (Prior to Definition 5.2, we borrowed our material from [22]; the remainder of the review, ending with Lemma 5.12, is borrowed from [16] and [2].) Once two important abelian C^* subalgebras are identified (the *diagonal* and the *abelian core*), our analysis will be based on applying the results from Sections 2 and 4.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 as well as range and source maps $r, s : E^1 \to E^0$. A vertex is *regular* if $r^{-1}(v)$ is finite and nonempty. A vertex which is not regular is called *singular*; a singular vertex is either a source ($r^{-1}(v) = \varnothing$) or an infinite receiver ($r^{-1}(v)$ infinite).

A *finite path* in E is a sequence $\lambda = e_1 \ldots e_n$ of edges satisfying $s(e_k) = r(e_{k+1})$ for $k = 1, \ldots, n-1$. (Note that we are using the right-to-left convention.) The length $\lambda = e_1 \ldots e_n$ is defined to be $|\lambda| = n$, and the set of paths of length n in E is denoted by E^n; the collection $\bigcup_{n=0}^{\infty} E^n$ of all finite paths in E is denoted by E^*. (The vertices E^0 are included in E^* as the paths of length zero.) An *infinite path* in E is an infinite sequence $e_1 e_2 \ldots$ of edges in E satisfying $s(e_k) = r(e_{k+1})$ for all k; the set of these paths is denoted by E^∞. If $\lambda = e_1 \ldots e_n$ is a finite path, then we define its range $r(\lambda)$ to be $r(e_1)$ and its source $s(\lambda)$ to be $s(e_n)$. The range of an infinite path is defined in the same way. In order to avoid any confusion, for any vertex $v \in E^0$ and any $n \in \mathbb{N} \cup \{\infty\}$, the set $\{\lambda \in E^n : r(\lambda) = v, |\lambda| = n\}$ will be denoted by $r^{-n}(v)$.

If λ is a finite path and ν is a finite (or infinite) path with $s(\lambda) = r(\nu)$, then we can concatenate the paths to form $\lambda\nu$. Whenever a (finite or infinite) path σ can be decomposed as $\sigma = \lambda\nu$, we write $\lambda \prec \sigma$ (or $\sigma \succ \lambda$) and we denote ν by $\sigma \ominus \lambda$. A *cycle* is a finite path λ of positive length with $r(\lambda) = s(\lambda)$.

Given a cycle $\lambda = e_1 \dots e_n \in E^*$, an *entry* to λ is an edge $f \in E^1$, with $r(f) = r(e_k)$ and $f \neq e_k$, for some k. If no entry to λ exists, we say that λ is *entry-less*. It fairly easy to see that every entry-less cycle λ can be written uniquely as a repeated concatenation $\lambda = \nu^m$ of a *simple* entry-less cycle ν, that is, the number of vertices in ν equals $|\nu|$.

An infinite path x is called *periodic* if there exist $\alpha, \lambda \in E^*$, with $s(\alpha) = r(\lambda) = s(\lambda)$, such that $x = \alpha\lambda^\infty$ (that is, x is obtained by following α and then repeating the cycle λ forever). If $x = \alpha\lambda^\infty$, and λ has minimal length among any cycle in such a decomposition, then the period of x is defined to be $|\lambda|$ and is denoted by $\mathrm{per}(x)$.

DEFINITION 5.1. If B is a C^*-algebra, then a *Cuntz–Krieger E-family* in B is a set $\{S_e, P_v\}_{e \in E^1, v \in E^0}$, where the S_e are partial isometries with mutually orthogonal range projections and the P_v are mutually orthogonal projections which also satisfy:

(i) $S_e^* S_e = P_{s(e)}$;
(ii) $S_e S_e^* \leq P_{r(e)}$;
(iii) if v is regular, then $P_v = \sum_{r(e)=v} S_e S_e^*$.

The C^*-subalgebra of B generated by $\{S_e, P_v\}_{e \in E^1, v \in E^0}$ is denoted by $C^*(S, P)$. The graph algebra $C^*(E)$ is the universal C^*-algebra generated by a Cuntz–Krieger E-family, $C^*(E) = C^*(s, p)$, where $\{s_e, p_v\}$ are the *universal generators*. For any Cuntz–Krieger E-family $\{S_e, P_v\}_{e \in E^1, v \in E^0}$, there is a unique $*$-homomorphism $\pi_{S,P} : C^*(E) \to C^*(S, P)$ satisfying $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$.

For an E-family $\{S, P\}$ and a finite path $\lambda = e_1 \dots e_n$ in E^*, there is an associated partial isometry $S_\lambda = S_{e_1} S_{e_2} \dots S_{e_n}$ in $C^*(S, P)$. (If $|\lambda| = 0$, so λ reduces to a vertex $v \in E^0$, then $S_\lambda = P_v$.) When specializing to $C^*(E)$, we have partial isometries denoted by s_λ, $\lambda \in E^*$.

By construction, all $s_\lambda \in C^*(E)$, $\lambda \in E^*$ are partial isometries: the source projection of s_λ is $s_\lambda^* s_\lambda = p_{s(\lambda)}$; the range projection $s_\lambda s_\lambda^*$ will be denoted from now on by p_λ.

As it turns out, one has the equality

$$C^*(E) = \overline{\mathrm{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*, s(\alpha) = s(\beta)\}. \tag{5.1}$$

The products $s_\alpha s_\beta^*$ listed on the right-hand side of (5.1) are referred to as the *spanning monomials*, and the set of all these elements is denoted by $G(E)$. The equality (5.1) is due to the fact that $G(E) \cup \{0\}$ is a $*$-semigroup, which is a consequence of the following product rule:

$$(s_\alpha s_\beta^*)(s_\lambda s_\nu^*) = \begin{cases} s_\alpha s_{\nu(\beta\ominus\lambda)}^* & \text{if } \lambda \prec \beta, \\ s_{\alpha(\lambda\ominus\beta)} s_\nu^* & \text{if } \beta \prec \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Since all projections p_v, $v \in E^0$ are mutually orthogonal, for any finite set $V \subset E^0$, the sum $q_V = \sum_{v \in V} p_v$ will be again a projection and, furthermore, the net $(q_V)_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)}$ forms an approximate unit for $C^*(E)$, hereafter referred to as the *canonical approximate unit*. The $*$-subalgebra $\bigcup_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)} q_V C^*(E) q_V$ will be denoted by $C^*(E)_{\mathrm{fin}}$.

Passing from a graph to a subgraph does not always produce a meaningful link between the associated C^*-algebras. The best-suited objects that allow such links are those identified as follows: given some graph E, a subset $H \subset E^0$ is called:

- *hereditary* if $r(e) \in H$ implies that $s(e) \in H$;
- *saturated* if whenever $v \in E^0$ is regular and $\{s(e) : e \in r^{-1}(v)\} \subset H$, it follows that $v \in H$.

Any subset $H \subset E^0$ is contained in a minimal saturated set $\overline{H}$ called its *saturation*, which is the union $\overline{H} = \bigcup_{k=0}^{\infty} H_k$, where $H_0 = H$ and, for $k > 1$,

$$H_k = H_{k-1} \cup \{v \in E^0 : v \text{ regular and } s(r^{-1}(v)) \subset H_{k-1}\}.$$

Clearly, the saturation of an hereditary set is again hereditary. The main point about considering such sets is the fact (see [22]) that, whenever $H \subset E^0$ is saturated and hereditary, and we form the subgraph

$$E \setminus H = (E^0 \setminus H, s^{-1}(E^0 \setminus H), r, s),$$

then we have a natural surjective $*$-homomorphism $\rho_H : C^*(E) \to C^*(E \setminus H)$, defined on the generators as

$$\rho_H(p_v) = \begin{cases} p_v & \text{if } v \in E^0 \setminus H, \\ 0 & \text{otherwise,} \end{cases} \qquad \rho_H(s_e) = \begin{cases} s_e & \text{if } s(e) \in E^0 \setminus H, \\ 0 & \text{otherwise.} \end{cases}$$

(A subgraph of this form will be called *canonical*.) The ideal $\ker \rho_H$ is simply the closed two-sided ideal generated by $\{p_v\}_{v \in H}$; alternatively, it is also described as

$$\ker \rho_H = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*, s(\alpha) = s(\beta) \in H\}.$$

The *gauge action* on $C^*(E)$ is the point-norm continuous group homomorphism $\gamma : \mathbb{T} \ni z \longmapsto \gamma_z \in \text{Aut}(C^*(E))$, given on the generators by $\gamma_z(p_v) = p_v$, $v \in E^0$ and $\gamma_z(s_e) = z s_e$, $e \in E^1$. On the spanning monomials listed above, the automorphisms γ_z, $z \in \mathbb{T}$, act as $\gamma_z(s_\alpha s_\beta^*) = z^{|\alpha|-|\beta|} s_\alpha s_\beta^*$. The *gauge-invariant uniqueness theorem* of an Huef and Raeburn (see [9]) states that, given some C^*-algebra $\mathcal{A}$ equipped with a group homomorphism $\theta : \mathbb{T} \ni z \longmapsto \theta_z \in \text{Aut}(\mathcal{A})$, and a gauge invariant $*$-homomorphism $\pi : C^*(E) \to \mathcal{A}$ (that is, such that $\theta_z(\pi(x)) = \pi(\gamma_z(x))$ for all $x \in C^*(E)$, $z \in \mathbb{T}$), the condition that π is injective is equivalent to the condition that $\pi(p_v) \neq 0$ for all $v \in E^0$.

There are two distinguished abelian C^*-subalgebras of $C^*(E)$ which we use to define states on $C^*(E)$, the first of which is defined as follows.

DEFINITION 5.2. Let E be a directed graph. Then the *diagonal* $\mathcal{D} \subset C^*(E)$ is the C^*-subalgebra of $C^*(E)$ generated by the set $G_\mathcal{D}(E) = \{p_\alpha\}_{\alpha \in E^*}$. (We sometimes use the notation $\mathcal{D}(E)$ when specifying the graph is necessary.)

REMARK 5.3. As it turns out, $G_\mathcal{D}(E) \cup \{0\}$ is an abelian semigroup of projections; more specifically, by (5.2), the product rule for $G_\mathcal{D}(E)$ is

$$p_\alpha p_\beta = p_\beta p_\alpha = \begin{cases} p_\alpha & \text{if } \beta \prec \alpha, \\ p_\beta & \text{if } \alpha \prec \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Using the semigroup property, it follows that we can in fact present $\mathcal{D}(E) = \overline{\operatorname{span}} \, G_{\mathcal{D}}(E)$. We can also write $\mathcal{D}(E) = [\sum_{v \in E^0} \mathcal{D}(E) p_v]^-$, with each summand presented as

$$\mathcal{D}(E) p_v = \overline{\operatorname{span}}\{p_\alpha : \alpha \in E^*, p_\alpha \leq p_v\} = \overline{\operatorname{span}}\{p_\alpha : \alpha \in E^*, r(\alpha) = v\}.$$

As it turns out, each corner $\mathcal{D}(E) p_v$ is in fact a unital abelian AF-subalgebra, with unit p_v, so $\mathcal{D}$ itself is an abelian AF-algebra, which contains the canonical approximate unit $(q_V)_{V \in \mathcal{P}_{\text{fin}}(E^0)}$.

As explained for instance in [16], the Gelfand spectrum $\widehat{\mathcal{D}(E)}$ of the diagonal C^*-subalgebra $\mathcal{D}(E)$ can be identified with the set

$$E^{\leq \infty} = E^\infty \cup \{x \in E^* : s(x) \text{ is singular}\}$$

with evaluation maps defined by $ev_x^{\mathcal{D}}(p_\alpha) = 1$ if $\alpha \prec x$ and 0 otherwise. In other words, for each $\alpha \in E^*$, when we view $p_\alpha \in \mathcal{D}(E)$ as a continuous function on $\widehat{\mathcal{D}(E)} \simeq E^{\leq \infty}$, this function will be the indicator function of the compact–open set $Z(\alpha) = \{x \in E^{\leq \infty} : \alpha \prec x\}$. Furthermore, the sets $Z(\alpha)$, $\alpha \in E^*$ form a basis for the topology, so clearly $\widehat{\mathcal{D}(E)}$ is totally disconnected. When identifying $\mathcal{D}(E) \simeq C_0(\widehat{\mathcal{D}(E)})$, the algebraic sum (without closure) $\mathcal{D}(E)_{\text{fin}} = \sum_{v \in E^0} \mathcal{D}(E) p_v$ gets naturally identified with $C_c(\widehat{\mathcal{D}(E)})$, the algebra of continuous functions with compact support.

REMARK 5.4. Cylinder sets can be used to analyze path (in)comparability. To be more precise, given two paths $\alpha, \beta \in E^*$, the following statements hold.

(I) (Comparability rule) The inequality $\alpha \prec \beta$ is equivalent to the reverse inclusion $Z(\alpha) \supset Z(\beta)$.

(II) (Orthogonality rule) Conditions (i)–(iv) below are equivalent:

    (i) $s_\alpha^* s_\beta = 0$;

    (ii)   the projections p_α and p_β are orthogonal, that is, $p_\alpha p_\beta = 0$;

    (iii) α and β are *incomparable*, that is, $\alpha \nprec \beta$ and $\beta \nprec \alpha$;

    (iv) $Z(\alpha) \cap Z(\beta) = \varnothing$.

REMARK 5.5. Among all paths $x \in E^{\leq \infty}$, the ones of interest to us will be those that represent isolated points in the spectrum $\widehat{\mathcal{D}(E)}$. On the one hand, if E has *sources* (that is, vertices $v \in E^0$ with $r^{-1}(v) = \varnothing$), then all finite paths that start at sources are determined by isolated points in $\widehat{\mathcal{D}(E)}$. On the other hand, the infinite paths $x = e_1 e_2 \cdots \in E^\infty$ that produce isolated points in $\widehat{\mathcal{D}(E)}$ are precisely those with the property that there exists k such that $r^{-1}(r(e_n)) = \{e_n\}$ for all $n \geq k$. If this is the case, if we form $\alpha = e_1 e_2 \ldots e_{k-1}$, then $\{x\} = Z(\alpha)$. Among those paths, the periodic ones will play an important role in our discussion.

DEFINITION 5.6. A finite path $\alpha = e_1 e_2 \ldots e_n \in E^*$ (possibly of length zero) is called a *ray* if there is a simple entry-less cycle ν such that $s(\alpha) = s(\nu)$ and, furthermore, no edge e_k from α appears in ν. (Note: in [16], rays were called *distinguished* paths.) In this case, the cycle ν (which is uniquely determined by α) is referred to as the *seed*

of α. We caution the reader that zero-length rays are permitted: they are what we will call *cyclic vertices*. For reasons explained in the second paragraph below, the (possibly empty) set of all rays in E will be denoted by E_{IP}^*.

By definition, any two distinct rays $\alpha_1 \neq \alpha_2$ are incomparable, so by the orthogonality rule (Remark 5.4), they satisfy $s_{\alpha_1}^* s_{\alpha_2} = s_{\alpha_2}^* s_{\alpha_1} = 0$.

Clearly, rays parametrize the set E_{IP}^∞ of infinite periodic paths that yield isolated points in $\widehat{\mathcal{D}(E)}$: any such path can be uniquely presented as $x = \alpha v^\infty$, with α the ray and v the seed of α, and its period (as a function from $\mathbb{N}$ to E^1) is $\mathrm{per}(x) = |v|$. When it would be necessary to emphasize the sole dependence on α, we also denote the infinite path αv^∞ simply by ξ_α. When we collect the corresponding points in $\widehat{\mathcal{D}(E)}$, we obtain a countable open set $\Sigma_{\mathrm{IP}} = \{ev_x^{\mathcal{D}} : x \in E_{\mathrm{IP}}^\infty\} \subset \widehat{\mathcal{D}(E)}$.

REMARK 5.7. Associated with the space $E^{\leq\infty}$ we have the *path representation* $\pi_{\mathrm{path}} :$ $C^*(E) \to B(\ell^2(E^{\leq\infty}))$ given on generators by (see [22] for details)

$$\pi_{\mathrm{path}}(s_e)\delta_x = \begin{cases} \delta_{ex}, & r(x) = s(e), \\ 0 & \text{otherwise,} \end{cases} \qquad \pi_{\mathrm{path}}(p_v)\delta_x = \begin{cases} \delta_x, & r(x) = v, \\ 0 & \text{otherwise.} \end{cases}$$

In general, π_{path} is not faithful; however, it is always faithful on the diagonal subalgebra $\mathcal{D}(E)$. This embedding gives us a explicit form of the identification $\widehat{\mathcal{D}(E)} = E^{\leq\infty}$ as follows: for $x \in E^{\leq\infty}$, the associated character on $\mathcal{D}(E)$ is simply $ev_x^{\mathcal{D}}(a) = \langle \delta_x | \pi_{\mathrm{path}}(a)\delta_x \rangle$.

For future use, we denote the subalgebras $\pi_{\mathrm{path}}(\mathcal{D}(E))$ and $\pi_{\mathrm{path}}(C^*(E))$ of $B(\ell^2(E^{\leq\infty}))$ by $D_{\mathrm{path}}(E)$ and $A_{\mathrm{path}}(E)$, respectively.

NOTATION. As shown in [16, Proposition 3.1], a spanning monomial $b = s_\alpha s_\beta^* \in C^*(E)$ is normal (that is, it satisfies the equality $bb^* = b^*b$) if and only if one of the following holds:

(a) $\alpha = \beta$, so $b = s_\alpha s_\alpha^* \in G_{\mathcal{D}}(E)$;
(b) $\alpha \prec \beta$ and $\beta \ominus \alpha$ is an entry-less cycle;
(c) $\beta \prec \alpha$ and $\alpha \ominus \beta$ is an entry-less cycle.

The set of all such monomials is denoted by $G_{\mathcal{M}}(E)$.

DEFINITION 5.8. The *abelian core* $\mathcal{M}(E)$ is the C^*-subalgebra of $C^*(E)$ generated by $G_{\mathcal{M}}(E)$.

NOTATIONS. If $b \in G_{\mathcal{M}}(E) \setminus G_{\mathcal{D}}(E)$ (that is, b is of either type (b) or (c) above), then b is a normal partial isometry, so its adjoint b^* also acts as its pseudo-inverse. For this reason, we will denote b^* simply by b^{-1}. More generally, we will allow arbitrary negative integer exponents, by letting b^{-m} be an alternative notation for b^{*m}. We will also allow zero exponents, by agreeing that $b^0 = bb^* = b^*b$, a monomial which in fact belongs to $G_{\mathcal{D}}(E)$. (Equivalently, for any $b \in G_{\mathcal{M}}(E) \setminus G_{\mathcal{D}}(E)$, the C^*-subalgebra $C^*(b) \subset C^*(E)$ generated by b is a unital abelian C^*-algebra, and b is a unitary element in $C^*(b)$.)

REMARK 5.9. In general, for a monomial $b \in G_{\mathcal{M}}(E) \smallsetminus G_{\mathcal{D}}(E)$, there might be multiple ways to present it as $s_\alpha s_\beta^*$, with α and β as in (b) or (c) above, but, after careful inspection, one can show that b can be uniquely presented as $b = s_\alpha s_\nu^m s_\alpha^* = (s_\alpha s_\nu s_\alpha^*)^m$, where $\alpha \in E^*$ is a ray with seed ν and m is some nonzero integer, so if we let

$$b_\alpha = s_\alpha s_\nu s_\alpha^*$$

(recall that ν is uniquely determined by α), then we can present

$$G_{\mathcal{M}}(E) \smallsetminus G_{\mathcal{D}}(E) = \{b_\alpha^m : \alpha \text{ ray, } m \text{ nonzero integer}\}.$$

Clearly, using our exponent conventions, $G_{\mathcal{M}}(E) \smallsetminus G_{\mathcal{D}}(E)$ is closed under taking adjoints, because $(b_\alpha^m)^* = b_\alpha^{-m}$. As it turns out, $G_{\mathcal{M}}(E) \cup \{0\}$ is an abelian $*$-semigroup; besides the product rules (5.3) for $G_{\mathcal{D}}(E)$, the remaining rules which involve the monomials in $G_{\mathcal{M}}(E) \smallsetminus G_{\mathcal{D}}(E)$ are

$$b_\alpha^0 = p_\alpha \quad \text{for all rays } \alpha,$$

$$b_\alpha^m p_\beta = p_\beta b_\alpha^m = \begin{cases} b_\alpha^m & \text{if } \beta \prec \xi_\alpha, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

$$b_{\alpha_1}^{m_1} b_{\alpha_2}^{m_2} = b_{\alpha_2}^{m_2} b_{\alpha_1}^{m_1} = \begin{cases} b_{\alpha_1}^{m_1+m_2} & \text{if } \alpha_1 = \alpha_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

By the above $*$-semigroup property, $\mathcal{M}(E) \subset C^*(E)$ is an abelian C^*-subalgebra which contains $\mathcal{D}(E)$, and it can also be described as $\mathcal{M}(E) = \overline{\text{span}}\, G_{\mathcal{M}}(E)$. Furthermore, the images of $\mathcal{D}(E)$ and $\mathcal{M}(E)$ under the path representation agree; that is, $\pi_{\text{path}}(\mathcal{M}(E)) = D_{\text{path}}(E)$. In general, $\mathcal{M}(E)$ is much larger than $\mathcal{D}(E)$; in fact, $\mathcal{M}(E) = \mathcal{D}(E)'$, the commutant of $\mathcal{D}(E)$ in $C^*(E)$.

As was the case with the diagonal, we have $\mathcal{M}(E) = [\sum_{v \in E^0} \mathcal{M}(E) p_v]^-$, with the summand $\mathcal{M}(E) p_v$ now presented as

$$\overline{\text{span}}(\{b_\alpha^m : m \in \mathbb{Z}, \alpha \in E_{\text{rp}}^*, r(\alpha) = v\} \cup \{p_\alpha : \alpha \in E^*, r(\alpha) = v\}),$$

so, upon identifying $\mathcal{M}(E) \simeq C_0(\widehat{\mathcal{M}(E)})$, the (non-norm-closed) algebraic sum $\mathcal{M}(E)_{\text{fin}} = \sum_{v \in E^0} \mathcal{M}(E) p_v$ is naturally identified with $C_c(\widehat{\mathcal{M}(E)})$, the algebra of continuous functions with compact support.

DEFINITION 5.10 (twisted path representation). With the notation as above, define the *twisted representation* $\Theta : C^*(E) \to C(\mathbb{T}, A_{\text{path}}(E))$ by

$$\Theta(a)(z) = \pi_{\text{path}}(\gamma_z(a)), \quad z \in \mathbb{T}, a \in C^*(E).$$

For any pair $(z, x) \in \mathbb{T} \times E^{\leq \infty}$, we define the state $\omega_{z,x}$ on $C^*(E)$ by

$$\omega_{z,x}(a) = \langle \delta_x | \Theta(a)(z) \delta_x \rangle.$$

REMARK 5.11. As π_{path} is injective on $\mathcal{D}(E)$, the gauge-invariant uniqueness theorem implies that Θ is injective. (The gauge action on the codomain is by translation: $(\lambda_z(f))(w) = f(z^{-1}w)$.) In particular, Θ yields an injection of $\mathcal{M}(E)$ into $C(\mathbb{T}, D_{\text{path}}(E))$. Therefore, the spectrum of $\mathcal{M}(E)$ can be recovered as a quotient of the spectrum of $C(\mathbb{T}, D_{\text{path}}(E))$ (that is, $\mathbb{T} \times E^{\leq \infty}$), by the natural equivalence relation implemented by Θ. Specifically, if $(z, x) \in \mathbb{T} \times E^{\leq \infty}$, then the restriction $\omega_{z,x}|_{\mathcal{M}(E)}$ is a pure state on $\mathcal{M}(E)$. The equivalence relation $\sim$ on $\mathbb{T} \times E^{\leq \infty}$ is simply given by

$$(z_1, x_1) \sim (z_2, x_2) \Leftrightarrow \omega_{z_1, x_1}|_{\mathcal{M}(E)} = \omega_{z_2, x_2}|_{\mathcal{M}(E)}.$$

Since the restrictions of these states on the diagonal act as $\omega_{z,x}|_{\mathcal{D}(E)} = ev_x^{\mathcal{D}}$, it is fairly obvious that $(z_1, x_1) \sim (z_2, x_2)$ implies that $x_1 = x_2$. The precise description of the equivalence classes $(z, x)_\sim = \{(z_1, x_1) \in \mathbb{T} \times E^{\leq \infty} : (z_1, x_1) \sim (z, x)\}$ goes as follows.

$$(z, x)_\sim = \begin{cases} z\mathbb{U}_{\text{per}(x)} \times \{x\} & \text{if } x \in E_{\text{IP}}^\infty, \\ \mathbb{T} \times \{x\} & \text{if } x \in E^{\leq \infty} \smallsetminus E_{\text{IP}}^\infty. \end{cases}$$

(For any integer $n \geq 1$, the symbol $\mathbb{U}_n$ denotes the group of nth roots of unity.)

LEMMA 5.12. *Let E be a directed graph.*

(i) *When we equip the quotient space $\mathbb{T} \times E^{\leq \infty}/\sim$ with the quotient topology, the map $(z, x)_\sim \longmapsto \omega_{z,x}|_{\mathcal{M}(E)}$ is a homeomorphism onto the spectrum of $\mathcal{M}(E)$.*

(ii) *For every ray α, if we regard p_α as a continuous function on $\widehat{\mathcal{M}(E)}$, then p_α is the characteristic function of a compact–open subset $\mathbf{T}_\alpha$, which is homeomorphic to $\mathbb{T}$. Specifically, if ν is the seed of α, and $x = \alpha\nu^\infty \in E_{\text{IP}}^\infty$ is the associated periodic path, then $\mathbf{T}_\alpha = \{(z, x)_\sim\}_{z \in \mathbb{T}}$ and the map $\mathbb{T}/\mathbb{U}_{|\nu|} \ni z\mathbb{U}_{|\nu|} \longmapsto (z, x)_\sim \in \mathbf{T}_\alpha$ is a homeomorphism. Alternatively, $\mathbf{T}_\alpha$ is naturally identified with the spectrum—computed in the unital C^*-algebra $C^*(b_\alpha)$—of the normal partial isometry $b_\alpha = s_\alpha s_\nu s_\alpha^*$.*

(iii) *The compact–open sets $(\mathbf{T}_\alpha)_{\alpha \in E_{\text{IP}}^*}$ are mutually disjoint. When we consider $\Omega_{\text{IP}} = \bigcup_{\alpha \in E_{\text{IP}}^*} \mathbf{T}_\alpha$, and fix a positive Radon measure μ on $\widehat{\mathcal{M}(E)}$ with corresponding positive linear functional ϕ_μ on $\mathcal{M}(E)_{\text{fin}} = C_c(\widehat{\mathcal{M}(E)})$, then*

$$\int_{\Omega_{\text{IP}}} f \, d\mu = \sum_{\alpha \in E_{\text{IP}}^*} \phi(f p_\alpha)$$

for all $f \in \mathcal{M}(E)_{\text{fin}} = C_c(\widehat{\mathcal{M}(E)})$.

PROOF. Parts (i) and (ii) are established in [16] and [2]. For part (iii), we only need to justify the first statement, because the rest follows from the Lebesgue dominated convergence theorem. This follows immediately from the observation that any two distinct rays α_1, α_2 are incomparable, so by (5.2) the projections p_{α_1} and p_{α_2} are orthogonal; thus, the sets $\{\mathbf{T}_\alpha\}_{\alpha \text{ ray}}$ form a countable disjoint compact–open cover of Ω_{IP}.   □

REMARK 5.13. Both $\mathcal{D}(E)$ and $\mathcal{M}(E)$ are abelian regular C^*-subalgebras in $C^*(E)$, since all generators p_v, $v \in E^0$ and s_e, $e \in E^1$ normalize both of them. It is shown in [16] that $\mathcal{M}(E)$ is in fact a Cartan subalgebra of $C^*(E)$, with its (unique) conditional expectation acting on generators as

$$\mathbb{E}_{\mathcal{M}}(s_\alpha s_\beta^*) = \begin{cases} s_\alpha s_\beta^* & \text{if } s_\alpha s_\beta^* \in G_{\mathcal{M}}(E), \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Within this framework, Theorem 2.7 has the following consequence.

COROLLARY 5.14. *For a state ϕ on $\mathcal{M}(E)$, the following conditions are equivalent.*

(i)   *ϕ is s_e-invariant for all $e \in E^1$.*
(ii) *ϕ is fully invariant.*
(iii) *The composition $\phi \circ \mathbb{E}_{\mathcal{M}}$ is a tracial state on $C^*(E)$.*

REMARK 5.15. In general, $\mathcal{D}(E)$ is not Cartan, and there may exist more than one conditional expectation onto it. One expectation—hereafter referred to as the *Haar expectation*—always exists, defined as

$$\mathbb{E}_{\mathcal{D}}(a) = \int_{\mathbb{T}} \gamma_z(\mathbb{E}_{\mathcal{M}}(a)) \, dm(z) = \int_{\mathbb{T}} \mathbb{E}_{\mathcal{M}}(\gamma_z(a)) \, dm(z).$$

(Here m denotes the normalized Lebesgue measure on $\mathbb{T}$; the second equality follows from (5.6), which clearly implies that $\mathbb{E}_{\mathcal{M}}$ is gauge invariant.) The Haar expectation acts on the spanning monomials as

$$\mathbb{E}_{\mathcal{D}}(s_\alpha s_\beta^*) = \begin{cases} p_\alpha & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Since the integration map $\int_{\mathbb{T}} \gamma_z(a) \, dm(z)$ is always a faithful positive map, it follows that $\mathbb{E}_{\mathcal{D}}$ is faithful.

Using formulas (5.7), it is easy to see that $\mathbb{E}_{\mathcal{D}}$ is also normalized by all p_v, $v \in E^0$, and s_e, s_e^*, $e \in E^1$, so we also have the following analogue of Corollary 5.14.

COROLLARY 5.16. *For a state ψ on $\mathcal{D}(E)$, the following conditions are equivalent.*

(i)   *ϕ is s_e-invariant for all $e \in E^1$.*
(ii) *ϕ is fully invariant.*
(iii) *The composition $\psi \circ \mathbb{E}_{\mathcal{D}}$ is a tracial state on $C^*(E)$.*

REMARK 5.17. Either using Corollary 5.16 or directly from the definition, it follows that any fully invariant state ψ on $\mathcal{D}(E)$ satisfies

$$\forall \alpha \in E^* : \psi(p_\alpha) = \psi(p_{s(\alpha)}).$$

In particular, a fully invariant state on $\mathcal{D}(E)$ is completely determined by its values on the projections p_v, $v \in E^0$.

DEFINITION 5.18. Let E be a directed graph. A *graph trace* on E is a function $g : E^0 \to [0, \infty)$ such that:

(A)   for any $v \in E^0$, $g(v) \geq \sum_{e:r(e)=v} g(s(e))$;

(B)   for any regular v, we have equality in (A).

Note that, for any graph trace g, its null space $N_g = \{v \in E^0 : g(v) = 0\}$ is a saturated hereditary set.

Depending on the quantity $\|g\|_1 = \sum_{v \in E^0} g(v)$, a graph trace g is declared *finite*, if $\|g\|_1 < \infty$, or *infinite*, otherwise.

We denote the set of all graph traces on E by $T(E)$, and the set of finite graph traces on E by $T_{\text{fin}}(E)$. Lastly, we define the set $T_1(E) = \{g \in T(E) : \|g\|_1 = 1\}$, the elements of which are termed *normalized* graph traces.

THEOREM 5.19. *A map* $g : E^0 \to [0, \infty)$ *is a graph trace on* E *if and only if every finite tuple* $\Xi = (\xi_i, \lambda_i)_{i \in I}^n \subset \mathbb{R} \times E^*$ *satisfies*

$$\sum_{i \in I} \xi_i p_{\lambda_i} \geq 0 \Rightarrow \sum_{i \in I} \xi_i g(s(\lambda_i)) \geq 0. \tag{5.8}$$

PROOF. To prove the 'if' implication, assume that g satisfies condition (5.8); let us verify conditions (A) and (B) from Definition 5.18. To check condition (A), start off by fixing some $v \in E^0$ and notice that, since for every finite set $F \subset r^{-1}(v)$, we have $p_v \geq \sum_{e \in F} p_e$ (by the Cuntz–Krieger relations), then, by (5.8), it follows that $g(v) \geq \sum_{e \in F} g(s(e))$; this clearly implies the inequality $g(v) \geq \sum_{e \in r^{-1}(v)} g(s(e))$. In order to check (B), simply notice that, if v is regular (so $r^{-1}(v)$ is both finite and nonempty), then, by the Cuntz–Krieger relations, we have an equality $p_v = \sum_{e \in r^{-1}(v)} p_e$, so applying (5.8) both ways (writing the equality as two inequalities), we clearly get $g(v) = \sum_{e \in r^{-1}(v)} g(s(e))$.

To prove the 'only if' implication, we fix a graph trace g and we prove the implication (5.8). As a matter of terminology, if a tuple Ξ satisfies the inequality

$$\sum_{i \in I} \xi_i p_{\lambda_i} \geq 0, \tag{5.9}$$

we will call Ξ *admissible*. Our proof will use induction on the number $\langle \Xi \rangle = |I| + \sum_{i \in I} |\lambda_i|$.

If $\langle \Xi \rangle = 1$, then $|I| = 1$; thus, I is a singleton $\{i_0\}$ and λ_{i_0} is a path of length 0, that is, a vertex $v \in E^0$; in this case, (5.8) is the same as the implication '$\xi p_v \geq 0 \Rightarrow \xi g(v) \geq 0$', which is trivial, since g takes nonnegative values.

Assume that (5.8) holds whenever $\langle \Xi \rangle < N$ for some $N > 1$; we show that (5.8) holds when $\langle \Xi \rangle = N$. Fix an admissible tuple ξ with $\langle \Xi \rangle = N$ (so (5.9) is satisfied); let us prove the inequality

$$\sum_{i \in I} \xi_j g(s(\lambda_i)) \geq 0. \tag{5.10}$$

If we consider the set $W = \{r(\lambda_i) : i \in I\}$, then we can split (disjointly) $I = \bigcup_{v \in W} I_v$, where $I_v = \{i : r(\lambda_i) = v\}$, and we will have

$$\sum_{i \in I} \xi_j g(s(\lambda_i)) = \sum_{v \in W} \sum_{i \in I_v} \xi_i g(s(\lambda_i)),$$

with each tuple $\Xi_v = (\xi_i, \lambda_i)_{i \in I_v}$ admissible. (This is obtained by multiplying the inequality (5.9) by p_v; since all calculations take place in the abelian C^*-subalgebra $\mathcal{D}(E)$, the product $p_v(\sum_{i \in I} \xi_i p_{\lambda_i}) = \sum_{i \in I_v} \xi_i p_{\lambda_i}$ is indeed positive.) In the case when W has at least two vertices, we have $\langle \Xi_v \rangle < \langle \Xi \rangle$ for all $v \in W$, so the inductive hypothesis can be used, and the desired conclusion follows.

Based on the above argument, for the remainder of the proof we can assume that W is a singleton, so we have a vertex $v \in E^0$, such that $r(\lambda_i) = v$ for all $i \in I$. Split $I = I^0 \cup I^+$, where $I^0 = \{i \in I : |\lambda_i| = 0\}$ and $I^+ = \{i \in I : |\lambda_i| > 0\}$. Since W is a singleton, the set I^0 consists of all I for which $\lambda_i = v$. The case when $I^+ = \varnothing$ is trivial, because that would mean that all λ_i will be equal to v, so for the remainder of the proof we are going to assume that $I^+ \neq \varnothing$. With this set-up, the hypothesis (5.9) reads

$$\left(\sum_{i \in I^0} \xi_i\right) p_v + \sum_{i \in I^+} \xi_i p_{\lambda_i} \geq 0, \tag{5.11}$$

and the desired conclusion (5.10) reads

$$\left(\sum_{i \in I^0} \xi_i\right) g(v) + \sum_{i \in I^+} \xi_i g(s(\lambda_i)) \geq 0. \tag{5.12}$$

(In the case when $I^0 = \varnothing$, we let $\sum_{i \in I^0} \xi_i = 0$.)

Since I^+ is nonempty (and finite), we can find a finite nonempty set $F \subset E^1$ which allows us to split I^+ as a disjoint union of nonempty sets $I^+ = \bigcup_{e \in F} I_e$, where $I_e = \{i \in I : \lambda_i > e\}$. Using the Cuntz–Krieger relations, it follows that the element $q = \sum_{e \in F} s_e s_e^* \in \mathcal{D}$ is a projection satisfying $q \leq p_v$, so the difference $q' = p_v - q$ is also a (possibly zero) projection. In either case, it follows that $q' s_{\lambda_i} s_{\lambda_i}^* = 0$ for all $i \in I^+$, so when we multiply (5.11) by q',

$$\left(\sum_{i \in I^0} \xi_i\right) q' \geq 0. \tag{5.13}$$

Likewise, multiplying (5.11) by each $s_e s_e^*$,

$$\left(\sum_{i \in I^0} \xi_i\right) s_e s_e^* + \sum_{i \in I_e} \xi_j s_{\lambda_i} s_{\lambda_i}^* \geq 0,$$

so if we multiply on the left by s_e^* and on the right by s_e,

$$\left(\sum_{i \in I^0} \xi_i\right) p_{s(e)} + \sum_{i \in I_e} \xi_i s_{\lambda_i \ominus e} s_{\lambda_i \ominus e}^* \geq 0. \tag{5.14}$$

For each $e \in F$, we can form the tuple $\tilde{\Xi}_e = (\xi_i, \tilde{\lambda}_i)_{i \in I^0 \cup I_e}$ by letting

$$\tilde{\lambda}_i = \begin{cases} s(e) & \text{if } i \in I^0, \\ \lambda_i \ominus e & \text{if } j \in I_e \end{cases}$$

and then (5.14) shows that all $\tilde{\Xi}_e$ are admissible. Since we obviously have $\langle \tilde{\Xi}_e \rangle < \langle \Xi \rangle$, by the inductive hypothesis we obtain $(\sum_{i \in I^0} \xi_j) g(s(e)) + \sum_{i \in I_e} \xi_i g(s(\lambda_i \ominus e)) \geq 0$, which, combined with the obvious equality $s(\lambda_i \ominus e) = s(\lambda_i)$, yields

$$\left(\sum_{i \in I^0} \xi_j\right) g(s(e)) + \sum_{i \in I_e} \xi_j g(s(\lambda_i)) \geq 0. \tag{5.15}$$

When we sum all these inequalities (over $e \in E$),

$$\left(\sum_{i \in I^0} \xi_i\right) \left(\sum_{e \in F} g(s(e))\right) + \sum_{i \in I^+} \xi_i g(s(\lambda_i)) \geq 0. \tag{5.16}$$

Comparing this inequality with the desired conclusion (5.12), we see that it suffices to show that

$$\left(\sum_{i \in I^0} \xi_i\right) g(v) \geq \left(\sum_{i \in I^0} \xi_i\right) \left(\sum_{e \in F} g(s(e))\right). \tag{5.17}$$

The case when $I^0 = \varnothing$ is trivial, since both sides will equal zero, so for the remainder, we can assume that $I^0 \neq \varnothing$. In the case when $q' = 0$, that is, when $p_v = \sum_{e \in F} s_e s_e^*$, it follows that v is regular and $F = r^{-1}(v)$, so by condition (ii) in the graph trace definition, it follows that $g(v) = \sum_{e \in F} g(s(e))$ and again (5.17) becomes an equality. Lastly, in the case when $q' \neq 0$, we use condition (i) in the graph trace definition, which yields $g(v) \geq \sum_{e \in F} g(s(e))$; this means that the desired inequality would follow once we prove that $\sum_{i \in I^0} \xi_i \geq 0$, an inequality which is now (under the assumption that q' is a nonzero projection) a consequence of (5.13). $\square$

In preparation for Proposition 5.22 below, which contains two easy applications of Theorem 5.19, we introduce the following terminology.

DEFINITION 5.20. A vertex $v \in E^0$ is said to be *essentially left infinite* if there exists an infinite set $X \subset E^*$ of mutually incomparable paths such that $s(\alpha) = v$ for all $\alpha \in X$.

REMARK 5.21. One particular class of essentially left infinite vertices are those that *emit entries into cycles*, that is, vertices v that have some path $\alpha = e_1 e_2 \ldots e_m$ of positive length, with $s(\alpha) = v$, such that e_1 is an entry to a cycle. Indeed, if e_1 enters a cycle ν, then all paths $\nu^n \alpha$, $n \in \mathbb{N}$, are mutually incomparable.

Another class of essentially left infinite vertices are those that emit paths to infinitely many vertices. (In [30], such vertices are called *left infinite*.)

The following result generalizes [18, Lemma 3.3(i)] and part of the proof of [30, Theorem 3.2].

Proposition 5.22. *Let E be a directed graph, g be a graph trace on E, and $v \in E^0$ be some vertex. Assume that either one of the hypotheses below is satisfied:*

(a)   *v emits an entry to a cycle; or*

(b)   *g is finite and v is essentially left infinite.*

Then $g(v) = 0$.

Proof. The main ingredient in the proof is the observation that, for any finite set F of mutually incomparable paths starting at v, one has the inequality

$$\sum_{w \in r(F)} g(w) \geq |F| \cdot g(v). \qquad (5.18)$$

Indeed, if we list F as $\{\alpha_1, \ldots, \alpha_n\}$ (with all the α distinct, that is, $n = |F|$), then, by mutual incomparability, we have the inequality $\sum_{w \in r(F)} p_w \geq \sum_{j=1}^n p_{\alpha_j}$, and then (5.18) follows immediately from Theorem 5.19.

   By assumption, in either case, we can find an infinite set $Y \subset E^*$ of mutually incomparable paths starting at v such that the sum $M = \sum_{w \in r(Y)} g(w)$ is finite. (In case (a), as seen in the preceding remark, we can ensure that $r(Y)$ is a singleton; case (b) is trivial, by finiteness of g.) The desired conclusion now follows immediately from (5.18), which implies that $M \geq n \cdot g(v)$ for arbitrarily large n.   □

Remarks. As we will see shortly, graph traces on E correspond to certain maps on the 'compactly supported' diagonal subalgebra $\mathcal{D}(E)_{\mathrm{fin}} = \bigcup_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)} \mathcal{D}(E)q_V$, which will eventually yield tracial positive functionals on the dense $*$-subalgebra $C^*(E)_{\mathrm{fin}} \subset C^*(E)$.   Although neither $\mathcal{D}(E)_{\mathrm{fin}}$, nor $\mathcal{M}(E)_{\mathrm{fin}}$, nor $C^*(E)_{\mathrm{fin}}$ are C^*-algebras, they are nevertheless unions of increasing nets of unital C^*-algebras: $\mathcal{D}(E)_{\mathrm{fin}} = \bigcup_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)} \mathcal{D}(E)q_V$, $\mathcal{M}(E)_{\mathrm{fin}} = \bigcup_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)} \mathcal{M}(E)q_V$, and $C^*(E)_{\mathrm{fin}} = \bigcup_{V \in \mathcal{P}_{\mathrm{fin}}(E^0)} q_V C^*(E)q_V$. (Recall that, for any finite subset $V \subset E^0$, the projection q_V is defined to be $\sum_{v \in V} p_v$.) It is clear that the conditional expectations $\mathbb{E}_{\mathcal{M}}$ and $\mathbb{E}_{\mathcal{D}}$ map $C^*(E)_{\mathrm{fin}}$ onto $\mathcal{M}(E)_{\mathrm{fin}}$ and $\mathcal{D}(E)_{\mathrm{fin}}$, respectively, so Corollaries 5.14 and 5.16 have suitable statements applicable to $C^*(E)_{\mathrm{fin}}$, with the word 'state' replaced by 'positive linear functional'. By definition, positivity for linear functionals defined on each one of these $*$-algebras is equivalent to the positivity of their restrictions to each of the cut-off algebras corresponding to $V \in \mathcal{P}_{\mathrm{fin}}(E^0)$. Upon identifying $\mathcal{D}(E)_{\mathrm{fin}} = C_c(\widehat{\mathcal{D}(E)})$ and $\mathcal{M}(E)_{\mathrm{fin}} = C_c(\widehat{\mathcal{M}(E)})$, the positive cones $\mathcal{D}(E)_{\mathrm{fin}}^+$ and $\mathcal{M}(E)_{\mathrm{fin}}^+$ correspond precisely to the nonnegative continuous compactly supported functions.

   With this set-up in mind, Theorem 5.19 has the following consequence.

Theorem 5.23. *For any graph trace g on E, there exists a unique positive linear functional $\eta = \eta_g : \mathcal{D}(E)_{\mathrm{fin}} \to \mathbb{C}$ such that*

$$\eta_g(p_\lambda) = g(s(\lambda)) \quad \forall \lambda \in E^*.$$

When restricted to the unital C^*-algebras $\mathcal{D}(E)q_V$, $V \in \mathcal{P}_{\text{fin}}(E^0)$, the positive linear functionals η_g, $g \in T(E)$, have norms

$$\|\eta_g|_{\mathcal{D}(E)q_V}\| = \sum_{v \in V} g(v).$$

In particular, for $g \in T(E)$, the functional η_g is norm-continuous if and only if g is finite and, in this case, one has $\|\eta_g\| = \|g\|_1$.

PROOF. Let $\mathcal{A}$ be the complex span of $\{p_\lambda\}_{\lambda \in E^*}$, and let $\mathcal{A}_h$ be its Hermitean part, which is the same as the real span of $\{p_\lambda\}_{\lambda \in E^*}$. An application of Theorem 5.19 shows that there is a unique $\mathbb{R}$-linear functional $\theta : \mathcal{A}_h \to \mathbb{R}$ with $\theta(p_\lambda) = g(s(\lambda))$ for all $\lambda \in E^*$. If we fix $V \in \mathcal{P}_{\text{fin}}(E^0)$ and $x \in \mathcal{A}_h q_V$, another application of Theorem 5.19 to the inequality $-\|x\|q_V \le x \le \|x\|q_V$ shows that $|\theta(x)| \le \theta(q_V)\|x\|$. Thus, for each $V \in \mathcal{P}_{\text{fin}}(E^0)$, there is a unique $\mathbb{C}$-linear Hermitean functional $\eta_V : \mathcal{D}(E)q_V \to \mathbb{C}$ with $\|\eta_V\| = \eta_V(q_V)$, so that η_V is in fact positive with norm equal to $\sum_{v \in V} g(v)$. Clearly, if $V \subset W$ are both finite subsets of E^0, then $\eta_W|_{\mathcal{D}(E)q_V} = \eta_V$; thus, by density, there exists a unique positive linear functional η_g defined on all of $\mathcal{D}(E)$ such that $\eta_g|_{\mathcal{D}(E)q_V} = \eta_V$ if $V \in \mathcal{P}_{\text{fin}}(E^0)$.   □

REMARKS. As a $*$-subalgebra in $C^*(E)_{\text{fin}}$, both $\mathcal{D}(E)_{\text{fin}}$ and $\mathcal{M}(E)_{\text{fin}}$ are nondegenerate (since they both contain $\{q_V\}_{V \in \mathcal{P}_{\text{fin}}(E)}$, as well as regular, because they are normalized by all s_e, $e \in E^1$ and all p_v, $v \in E^0$. Given a positive linear functional η on either one of these algebras, it then makes sense to define what it means for it to be s_e-invariant.

REMARK 5.24. The map $g \longmapsto \eta_g$ establishes an affine bijective correspondence between $T(E)$ and the space of positive linear functionals on $\mathcal{D}(E)_{\text{fin}}$ that are s_e-invariant for all $e \in E^1$. The inverse of this correspondence is obtained as follows. Given a linear positive functional θ on $\mathcal{D}(E)_{\text{fin}}$ which is s_e-invariant for all $e \in E^1$, the associated graph trace is simply the map

$$g^\theta : E^0 \ni v \longmapsto \theta(p_v) \in [0, \infty). \tag{5.19}$$

REMARK 5.25. In [14], a groupoid model is exhibited for graph C^*-algebras, so that to any directed graph E one associates an étale groupoid G_E such that $C^*(G_E) \cong C^*(E)$. Furthermore, this construction identifies the diagonal $\mathcal{D}$ with $C_0(G_E^{(0)})$, where the unit space $G_E^{(0)}$ is naturally identified with $E^{\le \infty}$. Thus, the correspondence $g \mapsto \eta_g$ of the previous remark gives rise, with the help of Riesz's theorem, to the correspondence $g \mapsto \mu_g$, associating to each graph trace $g \in T(E)$ a totally balanced positive Radon measure μ_g on $G_E^{(0)} \cong E^{\le \infty}$.

When we specialize to the case of interest to us, Theorem 5.23 yields the following statement.

THEOREM 5.26. *For any normalized graph trace g, there exists a unique state $\psi_g \in S(\mathcal{D}(E))$ satisfying*

$$\psi_g(p_\lambda) = g(s(\lambda)) \quad \forall \lambda \in E^*. \tag{5.20}$$

All states ψ_g, $g \in T_1(E)$, are fully invariant and, furthermore, the correspondence

$$T_1(E) \ni g \longmapsto \psi_g \in S^{\text{inv}}(\mathcal{D}(E))$$

is an affine bijection, which has as its inverse the correspondence

$$S^{\text{inv}}(\mathcal{D}(E)) \ni \theta \longmapsto g^\theta \in T_1(E) \tag{5.21}$$

defined as in (5.19).

REMARKS. The reader who is familiar with [28] can see that our approach is lengthier than the one in that paper. However, we do not require any K-theory computations (or the related results on states on K_0, lower semicontinuous dimension functions, quasi-traces, etc); our approach is purely constructive.

REMARKS. Using Corollary 5.16, it follows that for any $g \in T_1(E)$, the composition $\chi_g = \psi_g \circ \mathbb{E}_{\mathcal{D}}$ defines a tracial state on $C^*(E)$; in this way we obtain an injective correspondence

$$T_1(E) \ni g \longmapsto \chi_g \in T(C^*(E)). \tag{5.22}$$

Of course, any tracial state $\tau \in T(C^*(E))$ becomes invariant when restricted to $\mathcal{D}(E)$, so using (5.21) we obtain a correspondence

$$T(C^*(E)) \ni \tau \longmapsto g^\tau \in T_1(E). \tag{5.23}$$

Theorem 5.26 shows that this map is surjective, because the correspondence (5.22) is clearly an affine right inverse for (5.23). The surjectivity of (5.23) is also proved in [28], by completely different means. In general, the map (5.22) may fail to be surjective (equivalently, the map (5.23) may fail to be injective). This will be clarified in Theorem 5.42 below.

REMARK 5.27. Using formulas (5.7), given a normalized graph trace $g \in T_1(E)$, the associated tracial state $\chi_g = \psi_g \circ \mathbb{E}_{\mathcal{D}}$—hereafter referred to as the *Haar trace induced by g*—acts on the spanning monomials as

$$\chi_g(s_\alpha s_\beta^*) = \begin{cases} g(s(\alpha)) & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Among other things, the above formulas prove that χ_g is in fact *gauge invariant*, that is, $\chi_g \circ \gamma_z = \chi_g$ for all $z \in \mathbb{T}$.

Conversely, every gauge-invariant tracial state $\tau \in T(C^*(E))$ arises in this way. Indeed, if τ is such a trace, then by gauge invariance it follows that, whenever $\alpha, \beta \in E^*$ are such that $|\alpha| \neq |\beta|$, we must have $\tau(s_\alpha s_\beta^*) = 0$; furthermore, if $|\alpha| = |\beta|$, then

$$\tau(s_\alpha s_\beta^*) = \tau(s_\beta^* s_\alpha) = \begin{cases} \tau(0) = 0 & \text{if } \alpha \neq \beta, \\ \tau(s_\alpha^* s_\alpha) = \tau(p_{s(\alpha)}) & \text{otherwise,} \end{cases}$$

so in all cases we get $\tau(s_\alpha s_\beta^*) = \chi_{g^\tau}(s_\alpha s_\beta^*)$.

To summarize:

- the range of the injective correspondence (5.22) is the set $T(C^*(E))^{\mathbb{T}}$ of gauge-invariant tracial states;
- when restricting the correspondence (5.23) to $T(C^*(E))^{\mathbb{T}}$, one obtains an affine *isomorphism*

$$T(C^*(E))^{\mathbb{T}} \ni \tau \longmapsto g^\tau \in T_1(E).$$

When searching for an analogue of Theorem 5.26, with $\mathcal{D}(E)$ replaced by $\mathcal{M}(E)$, it is obvious that the space $T(E)$ is not sufficient, so additional structure needs to be added to it.

DEFINITION 5.28. The *cyclic support* of a function $g : E^0 \to \mathbb{C}$ is defined to be the set

$$\mathrm{supp}^c g = \{v \in E^0 : v \in E^0 \text{ cyclic}, g(v) \neq 0\}.$$

(Recall that a cyclic vertex v is one visited by a simple entry-less cycle. Equivalently, v is a ray of length zero.) A *cyclically tagged graph trace* consists of a pair (g, μ), where g is a graph trace and $\mu : \mathrm{supp}^c g \ni v \longmapsto \mu_v \in \mathrm{Prob}(\mathbb{T})$ is a map—hereafter referred to as the *tag*. Note that our definition includes the possibility of an *empty* tag in the case when $\mathrm{supp}^c g = \varnothing$. (More on this is in Theorem 5.42 below.) The space of all such pairs will be denoted by $T^{\mathrm{ct}}(E)$. The adjective 'finite', 'infinite', or 'normalized' is attached to (g, μ) precisely when it applies to g.

Using this terminology, one has the following extension of Theorem 5.23. (Recall that, if v is the seed of the ray α, then b_α is the normal partial isometry $s_\alpha s_v s_\alpha^*$.) The structure of the proof is as follows: we fix a cyclically tagged graph trace and, roughly speaking, build part of the corresponding functional for each ray α. We then sum over these to obtain a functional θ on $\mathcal{M}_{\mathrm{fin}}(E)$ which satisfies some, but not all, of our desired invariance properties. We augment this functional with the 'graph-trace' part by use of the conditional expectation onto the diagonal, represented as a measure ν on $\widehat{\mathcal{M}(E)}$. Suitably combining θ with the functional represented by ν gives us the desired functional on $\mathcal{M}(E)$.

THEOREM 5.29. *For any cyclically tagged graph trace (g, μ) on E, there exists a unique positive linear functional $\tilde{\eta} = \tilde{\eta}_{(g,\mu)} : \mathcal{M}(E)_{\mathrm{fin}} \to \mathbb{C}$ such that:*

(i) $\tilde{\eta}_{(g,\mu)}(p_\lambda) = g(s(\lambda))$ *for every finite path $\lambda \in E^*$;*

(ii) *for any ray α and any integer $m \neq 0$,*

$$\tilde{\eta}_{(g,\mu)}(b_\alpha^m) = \begin{cases} g(s(\alpha)) \displaystyle\int_{\mathbb{T}} z^m \, d\mu_{s(\alpha)}(z) & \text{if } g(s(\alpha)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

When restricted to the unital C^*-algebras $\mathcal{M}(E)q_V$, $V \in \mathcal{P}_{\mathrm{fin}}(E^0)$, the positive linear functionals $\tilde{\eta}_{(g,\mu)}$, $(g, \mu) \in T^{\mathrm{ct}}(E)$, have norms

$$\|\tilde{\eta}_{(g,\mu)}|_{\mathcal{M}(E)q_V}\| = \sum_{v \in V} g(v).$$

In particular, for any $(g, \mu) \in T^{\text{cr}}(E)$, the functional $\tilde{\eta}_{(g,\mu)}$ is norm-bounded if and only if g is finite and, in this case, one has $\|\eta_{(g,\mu)}\| = \|g\|_1$.

PROOF. Assume that $(g, \mu) \in T^{\text{cr}}(E)$ is fixed throughout the entire proof. Fix for the moment a ray α with $g(s(\alpha)) \neq 0$, and consider the C^*-subalgebra $C^*(b_\alpha) \subset \mathcal{M}(E)$. As pointed out in Lemma 5.12, using the fact that the projection $b_\alpha^0 = p_\alpha$ is the characteristic function of the compact–open set $\mathbf{T}_\alpha \subset \widehat{\mathcal{M}(E)}$, we have of course the equality $\mathcal{M}(E)p_\alpha = C^*(b_\alpha)$, so using the surjective $*$-homomorphism

$$\pi_\alpha : \mathcal{M}(E) \ni a \longmapsto a p_\alpha \in C^*(b_\alpha) \xrightarrow{\sim} C(\mathbb{T})$$

we can define a state ω_α on $\mathcal{M}(E)$ by

$$\omega_\alpha(a) = \int_{\mathbb{T}} \pi_\alpha(a) \, d\mu_{s(\alpha)}.$$

Specifically, if we write the compression $a p_\alpha$ as $f(b_\alpha)$, for some $f \in C(\mathbb{T})$, then $\omega_\alpha(a) = \int_{\mathbb{T}} f(z) \, d\mu_{s(\alpha)}(z)$. Using the product rules (5.3), (5.4), and (5.5), it follows that on the generator set $G_{\mathcal{M}}(E)$, the state ω_α acts as

$$\omega_\alpha(p_\lambda) = \begin{cases} 1 & \text{if } \lambda \prec \xi_\alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \omega_\alpha(b_{\alpha_1}^m) = \begin{cases} \int_{\mathbb{T}} z^m \, d\mu_{s(\alpha)}(z) & \text{if } \alpha_1 = \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{5.24}$$

Define now the functional $\theta : \mathcal{M}(E)_{\text{fin}} \to \mathbb{C}$ by

$$\theta(a) = \sum_{\substack{\alpha \in E_{\text{IP}}^* \\ g(s(\alpha)) \neq 0}} g(s(\alpha)) \omega_\alpha(a), \quad a \in \mathcal{M}(E)_{\text{fin}}. \tag{5.25}$$

Concerning the pointwise convergence of the sum in (5.25), as well as its positivity, they are a consequence of the following fact.

CLAIM. For any vertex $v \in E^0$, one has the inequality

$$\sum_{\substack{\alpha \in E_{\text{IP}}^* \\ r(\alpha)=v}} g(s(\alpha)) \leq g(v). \tag{5.26}$$

In particular, the sum

$$\theta_v = \sum_{\substack{\alpha \in E_{\text{IP}}^* \\ r(\alpha)=v}} g(s(\alpha)) \omega_\alpha|_{\mathcal{M}(E)p_v}$$

is a norm-convergent sum; thus, θ_v is a positive linear functional on $\mathcal{M}(E)p_v$ with norm

$$\|\theta_v\| = \sum_{\substack{\alpha \in E_{\text{IP}}^* \\ r(\alpha)=v}} g(s(\alpha)). \tag{5.27}$$

The inequality (5.26) follows from the observation that, for any finite set F of rays with range v, the projections $\{p_\alpha\}_{\alpha \in F}$ satisfy the inequality $\sum_{\alpha \in F} p_\alpha \leq p_v$, which by Theorem 5.19 implies that $\sum_{\alpha \in F} g(s(\alpha)) \leq g(v)$. The equality (5.27) is now clear from the positivity of θ_v, which combined with (5.24) yields

$$\|\theta_v\| = \theta_v(p_v) = \sum_{\substack{\alpha \in E_{\mathrm{IP}}^* \\ r(\alpha)=v}} g(s(\alpha))\omega_\alpha(p_v) = \sum_{\substack{\alpha \in E_{\mathrm{IP}}^* \\ r(\alpha)=v}} g(s(\alpha)).$$

Using the claim, we see that θ given in (5.25) is indeed correctly defined, positive, and it can alternatively be presented as $\theta(a) = \sum_{v \in E^0} \theta_v(a)$ (a sum which has only finitely many nonzero terms for each $a \in \mathcal{M}(E)_{\mathrm{fin}}$). By construction, θ acts on the generator set $G_{\mathcal{M}}(E)$ as

$$\theta(p_\lambda) = \sum_{\substack{\alpha \in E_{\mathrm{IP}}^* \\ \lambda < \xi_\alpha}} g(s(\alpha)), \quad \lambda \in E^*, \tag{5.28}$$

$$\theta(b_\alpha^m) = \begin{cases} g(s(\alpha)) \displaystyle\int_{\mathbb{T}} z^m \, d\mu_{s(\alpha)}(z) & \text{if } \alpha \in E_{\mathrm{IP}}^* \text{ and } g(s(\alpha)) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.29}$$

Next we consider the positive linear functional $\eta_g : \mathcal{D}(E)_{\mathrm{fin}} \to \mathbb{C}$ associated to g, as constructed in Theorem 5.23, and the linear positive functional $\eta_g \circ \mathbb{E}_{\mathcal{D}} : \mathcal{M}(E)_{\mathrm{fin}} \to \mathbb{C}$. (Here we use the fact that $\mathbb{E}_{\mathcal{D}}$ maps $C^*(E)_{\mathrm{fin}}$ onto $\mathcal{D}(E)_{\mathrm{fin}}$.) Using Riesz' theorem, there is a positive Radon measure υ on $\widehat{\mathcal{M}(E)}$ such that $\eta_g(\mathbb{E}_{\mathcal{D}}(f)) = \int_{\widehat{\mathcal{M}(E)}} f \, d\upsilon$ for all $f \in C_c(\widehat{\mathcal{M}(E)}) = \mathcal{M}(E)_{\mathrm{fin}}$. Using this measure, we now define the desired positive linear functional $\tilde{\eta}$ on $C_c(\widehat{\mathcal{M}(E)}) = \mathcal{M}(E)_{\mathrm{fin}}$ by

$$\tilde{\eta}(f) = \theta(f) + \int_{\widehat{\mathcal{M}(E)} \setminus \Omega_{\mathrm{IP}}} f \, d\upsilon$$

$$= \theta(f) + \eta_g(\mathbb{E}_{\mathcal{D}}(f)) - \sum_{\alpha \in E_{\mathrm{IP}}^*} \eta_g(\mathbb{E}_{\mathcal{D}}(f p_\alpha)) \tag{5.30}$$

$$= \theta(f) + \eta_g(\mathbb{E}_{\mathcal{D}}(f)) - \sum_{\alpha \in E_{\mathrm{IP}}^*} \eta_g(\mathbb{E}_{\mathcal{D}}(f) p_\alpha). \tag{5.31}$$

(The equality (5.30) follows from Lemma 5.12.)

To check condition (i), start with some $\lambda \in E^*$ and observe that, for all rays α, we have the equalities

$$p_\lambda p_\alpha = \begin{cases} p_\alpha & \text{if } \lambda < \xi_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

which, by (5.28), imply that

$$\sum_{\alpha \in E_{\mathrm{IP}}^*} \eta_g(\mathbb{E}_{\mathcal{D}}(p_\lambda p_\alpha)) = \sum_{\substack{\alpha \in E_{\mathrm{IP}}^* \\ \lambda < \xi_\alpha}} \eta_g(p_\alpha) = \sum_{\substack{\alpha \in E_{\mathrm{IP}}^* \\ \lambda < \xi_\alpha}} g(s(\alpha)) = \theta(p_\lambda),$$

so by (5.30) we obtain the desired property

$$\tilde{\eta}(p_\lambda) = \eta_g(p_\lambda) = g(s(\lambda)).$$

In order to check condition (ii), we simply verify that, for any ray α and any integer m, we have the equality

$$\tilde{\eta}(b_\alpha^m) = \theta(b_\alpha^m). \tag{5.32}$$

In the case when $m = 0$, we have $b_\alpha^0 = p_\alpha$, so by condition (i) and (5.29), we have $\tilde{\eta}(b_\alpha^0) = \tilde{\eta}(p_\alpha) = g(s(\alpha)) = \theta(b_\alpha^0)$. In the case when $m \neq 0$, we notice that since $\mathbb{E}_{\mathcal{D}}$ vanishes on $G(E) \smallsetminus G_{\mathcal{D}}(E)$—by (5.7)—we have $\mathbb{E}_{\mathcal{D}}(b_\alpha^m) = 0$, and then (5.32) is trivial using (5.31).

The remaining statements in the theorem (including the uniqueness of $\tilde{\eta}$) are pretty clear, since any positive linear functional $\tilde{\eta}$ satisfying conditions (i) and (ii) must satisfy $\tilde{\eta}|_{\mathcal{D}(E)_{\text{fin}}} = \eta_g$, from which the continuity of the restrictions $\tilde{\eta}|_{\mathcal{M}(E)q_V}$ follows immediately.   □

One aspect not addressed so far is invariance of the states $\tilde{\eta}$. For this purpose, the following definition is well suited.

DEFINITION 5.30. Two cyclic vertices are said to be *equivalent* if they are visited by the same entry-less cycle. A cyclically tagged graph trace (g, μ) is said to be *consistent* if $\mu_v = \mu_w$ whenever v and w are equivalent. (Note that if two cyclic vertices v, w are equivalent, then $g(v) = g(w)$.) The space of all consistent cyclically tagged traces on E is denoted by $T^{\text{cct}}(E)$. As agreed earlier, the adjective 'finite', 'infinite', or 'normalized' is attached to an element $(g, \mu) \in T^{\text{cct}}(E)$ precisely when it applies to g. In particular, the space of normalized consistent cyclically tagged graph traces on E is denoted by $T_1^{\text{cct}}(E)$.

PROPOSITION 5.31. *A cyclically tagged graph trace (g, μ) is consistent if and only if the associated positive functional $\tilde{\eta}_{(g,\mu)} : \mathcal{M}(E)_{\text{fin}} \to \mathbb{C}$ constructed in Theorem 5.29 is s_e-invariant for all $e \in E^1$.*

PROOF. Assume that (g, μ) is consistent; let us show the invariance of $\tilde{\eta}_{(g,\mu)}$, which amounts to checking that for each $e \in E^1$:

(i) $\tilde{\eta}_{(g,\mu)}(s_e p_\lambda s_e^*) = \tilde{\eta}_{(g,\mu)}(p_e p_\lambda)$ for all $\lambda \in E^*$;

(ii) $\tilde{\eta}_{(g,\mu)}(s_e b_\alpha^m s_e^*) = \tilde{\eta}_{(g,\mu)}(p_e b_\alpha^m)$ for all $\alpha \in E_{\text{IP}}^*$, $m \in \mathbb{Z}$.

Property (i) is obvious, since $\tilde{\eta}_{(g,\mu)}$ agrees with the s_e-invariant functional η_g on $\mathcal{D}(E)_{\text{fin}}$. As for condition (ii), we only need to verify it if $s(e) = r(\alpha)$ (otherwise both sides are zero). Also notice that if $|\alpha| > 0$, then $e\alpha$ is also a ray with $s(e\alpha) = s(\alpha)$, which satisfies $s_e b_\alpha^m s_e^* = b_{e\alpha}^m$, so by condition (ii) in Theorem 5.29, we have $\tilde{\eta}_{(g,\mu)}(s_e b_\alpha^m s_e^*) = \tilde{\eta}_{(g,\mu)}(b_{e\alpha}^m) = g(s(e\alpha)) \int_{\mathbb{T}} z^m \, d\mu_{s(e\alpha)}(z) = g(s(\alpha)) \int_{\mathbb{T}} z^m \, d\mu_{s(\alpha)}(z) = \tilde{\eta}_{(g,\mu)}(b_\alpha^m)$. In the remaining case, $|\alpha| = 0$, so α reduces to a vertex $v = r(v)$ for some simple entry-less cycle v. If e is not an edge in v, then it is a ray; thus, the preceding argument still applies (we will have $s_e b_v^m s_e^* = b_e^m$). If e is an edge on v,

then $s_e b_v^m s_e^* = b_{r(e)}^m$, with $r(e)$ obviously equivalent to v, and the desired equality—which now reads $\tilde{\eta}_{(g,\mu)}(b_{r(e)}^m) = \tilde{\eta}_{(g,\mu)}(b_v^m)$—follows from the equalities $g(v) = g(r(e))$ and $\mu_v = \mu_{r(e)}$.

Conversely, notice first that, if $\tilde{\eta}_{(g,\mu)}$ is s_e-invariant for all $e \in E^1$, then it will also satisfy the identity

$$\tilde{\eta}_{(g,\mu)}(s_\lambda a s_\lambda^*) = \tilde{\eta}_{(g,\mu)}(p_\lambda a) \quad \forall \lambda \in E^*, a \in \mathcal{M}(E)_{\text{fin}}.$$

Secondly, observe that, if v, v' are equivalent cyclic vertices, presented as $v = s(v)$ and $v' = s(v')$ for two simple entry-less cycles, then we can write $v = \alpha\beta$ and $v' = \beta\alpha$ for two suitably chosen paths $\alpha, \beta \in E^*$. This clearly implies that $b_{v'} = s_\beta b_v s_\beta^*$, which also yields $b_{v'}^m = s_\beta b_v^m s_\beta^*$ for all $m \in \mathbb{Z}$.

Combining these two observations with condition (ii) from Theorem 5.29, it follows that, if $\tilde{\eta}_{(g,\mu)}$ is invariant, then for any two equivalent cyclic vertices v and v', we have (with α, β as above)

$$\int_{\mathbb{T}} z^m \, d\mu_{v'}(z) = \tilde{\eta}_{(g,\mu)}(b_{v'}^m) = \tilde{\eta}_{(g,\mu)}(s_\beta b_v^m s_\beta^*) = \tilde{\eta}_{(g,\mu)}(p_\beta b_v^m)$$

$$= \tilde{\eta}_{(g,\mu)}(b_v^m) = \int_{\mathbb{T}} z^m \, d\mu_v(z) \quad \forall m \in \mathbb{Z},$$

which clearly implies that $\mu_{v'} = \mu_v$. $\square$

REMARK 5.32. The map $(g, \mu) \longmapsto \tilde{\eta}_{(g,\mu)}$ establishes an affine bijective correspondence between $T^{\text{cct}}(E)$ and the space of positive linear functionals on $\mathcal{M}(E)_{\text{fin}}$ that are s_e-invariant for all $e \in E^1$. The inverse of this correspondence is the map $\theta \longmapsto (g^\theta, \mu^\theta)$ defined as follows. Given a linear positive functional θ on $\mathcal{M}(E)_{\text{fin}}$ which is s_e-invariant for all $e \in E^1$, the graph trace g^θ is given by (5.19), and the tag $\mu^\theta = (\mu_v^\theta)_{v\in\text{supp}^c g^\theta}$ is given (implicitly) by

$$\int_{\mathbb{T}} f(z) \, d\mu_v^\theta(z) = \frac{\theta(f(b_v))}{g^\theta(v)} \quad \forall v \in \text{supp}^c g^\theta, \ f \in C(\mathbb{T}). \tag{5.33}$$

When we specialize to states, we now have the following extension of Theorem 5.26.

THEOREM 5.33. *For any normalized consistent cyclically tagged graph trace* $(g, \mu) \in T_1^{\text{cct}}(E)$, *there exists a unique state* $\phi_{(g,\mu)} \in S(\mathcal{M}(E))$ *satisfying:*

(i) $\phi_{(g,\mu)}(p_\lambda) = g(s(\lambda))$ *for every finite path* $\lambda \in E^*$;

(ii)   *for any ray* α *and any integer* m,

$$\phi_{(g,\mu)}(b_\alpha^m) = \begin{cases} g(s(\alpha)) \displaystyle\int_{\mathbb{T}} z^m \, d\mu_{s(\alpha)}(z) & \text{if } g(s(\alpha)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

All states $\phi_{(g,\mu)}$, $(g,\mu) \in T_1^{\mathrm{cct}}(E)$, *are fully invariant and, furthermore, the correspondence*

$$T_1^{\mathrm{cct}}(E) \ni (g,\mu) \longmapsto \phi_{(g,\mu)} \in S^{\mathrm{inv}}(\mathcal{M}(E)) \tag{5.34}$$

is an affine bijection, which has as its inverse the correspondence

$$S^{\mathrm{inv}}(\mathcal{M}(E)) \ni \theta \longmapsto (g^{\theta}, \mu^{\theta}) \in T_1^{\mathrm{cct}}(E) \tag{5.35}$$

defined as in (5.19) and (5.33).

REMARKS. Using Corollary 5.14, it follows that for any $(g,\mu) \in T_1^{\mathrm{cct}}(E)$, the composition $\tau_{(g,\mu)} = \phi_{(g,\mu)} \circ \mathbb{E}_{\mathcal{M}}$ defines a tracial state on $C^*(E)$; in this way we obtain an injective correspondence

$$T_1^{\mathrm{cct}}(E) \ni (g,\mu) \longmapsto \tau_{(g,\mu)} \in T(C^*(E)). \tag{5.36}$$

Of course, any tracial state $\tau \in T(C^*(E))$ becomes invariant when restricted to $\mathcal{M}(E)$, so using (5.35) we obtain a correspondence

$$T(C^*(E)) \ni \tau \longmapsto (g^{\tau}, \mu^{\tau}) \in T_1^{\mathrm{cct}}(E). \tag{5.37}$$

Theorem 5.33 shows that this map is surjective, because the correspondence (5.36) is clearly an affine right inverse for (5.37).

REMARK 5.34. The range of (5.36) clearly contains the range of (5.22), which equals $T(C^*(E))^{\mathbb{T}}$. After all, any trace $g \in T_1(E)$ can be tagged using the constant map $\mu : \mathrm{supp}^c g \to \mathrm{Prob}(\mathbb{T})$ that takes μ_v to be the Haar measure for every v, and it is straightforward to verify that for this particular tagging one has $\tau_{(g,\mu)} = \chi_g$.

Concerning the range of (5.36), one legitimate question is whether it equals the whole tracial state space $T(C^*(E))$. Using the bijection (5.34), this question is equivalent to the surjectivity of the map

$$S^{\mathrm{inv}}(\mathcal{M}(E)) \ni \phi \longmapsto \phi \circ \mathbb{E}_{\mathcal{M}} \in T(C^*(E)). \tag{5.38}$$

As we have seen in Corollary 4.6, a sufficient condition for the surjectivity of (5.38) is the condition that the inclusion $\mathcal{M}(E) \subset C^*(E)$ has the (honest) extension property. As it turns out, this issue can be neatly described using the graph.

THEOREM 5.35. *The inclusion $\mathcal{M}(E) \subset C^*(E)$ has the extension property if and only if no cycle in E has an entry.*

PROOF. To prove the 'if' implication, assume that no cycle in E has an entry, fix a pure state ω on $\mathcal{M}(E)$, and let ϕ be an extension of ω to $C^*(E)$. In order to prove uniqueness of ϕ, it suffices to show that the value of ϕ on a standard generator $s_\alpha s_\beta^*$ is independent of the choice of ϕ. By assumption, there are an $x \in E^{\leq \infty}$ and a $z \in \mathbb{T}$ such that $\omega = \omega_{z,x}$ as in Lemma 5.12. On the one hand, by Fact 4.1 and the observation that $\omega(p_\gamma) = 1$ for all $\gamma \prec x$, it follows that

$$\forall \gamma \prec x : \phi(s_\alpha s_\beta^*) = \phi(p_\gamma s_\alpha s_\beta^* p_\gamma). \tag{5.39}$$

On the other hand, using the results from [16, Section 3], it follows that there is $\gamma \prec x$ such that $p_\gamma s_\alpha s_\beta^* p_\gamma$ belongs to $\mathcal{M}(E)$. (In the language of [16], x must be essentially aperiodic by our assumption on E.) Using (5.39), it follows that $\phi(s_\alpha s_\beta^*) = \omega(p_\gamma s_\alpha s_\beta^* p_\gamma)$, and the desired conclusion follows.

For the 'only if' direction, we show that if there is a cycle $\nu \in E^*$ that has an entry, then we can construct a pure state on $\mathcal{M}(E)$ which has multiple extensions to states on $C^*(E)$. Consider the path $x = \nu^\infty \in E^\infty$ formed by following ν infinitely many times. For each $z \in \mathbb{T}$, consider the state $\omega_{z,x} \in S(C^*(E))$ introduced in Definition 5.10, given by

$$\omega_{z,x}(a) = \langle \delta_x | \pi_{\text{path}}(\gamma_z(a)) \delta_x \rangle.$$

As explained in Remark 5.11, since $x \notin E_{\text{IP}}^\infty$, it follows that

$$(z, x) \sim (1, x) \quad \forall z \in \mathbb{T},$$

which, by Lemma 5.12, means that all restrictions $\omega_{z,x}|_{\mathcal{M}(E)}$, $z \in \mathbb{T}$, coincide, so they are all equal to the pure state $\vartheta \in \widehat{\mathcal{M}(E)}$ corresponding to the equivalence class $(1, x)_\sim = \mathbb{T} \times \{x\}$. However, as states on $C^*(E)$, the functionals $\omega_{z,x}$, $z \in \mathbb{T}$, cannot all be equal, since for example we have $\omega_{z,x}(\nu) = z^{|\nu|}$ for all $z \in \mathbb{T}$. $\square$

DEFINITION 5.36. A graph E is *tight* if every cycle is entry-less.

Combining Theorem 5.35 with Corollary 4.6 and Theorem 5.33, we now obtain the following statement.

THEOREM 5.37. *If E is tight, then the correspondence (5.36) is an affine isomorphism between the space $T_1^{\text{cct}}(E)$ and the tracial state space $T(C^*(E))$.*

REMARK 5.38. Tight graphs are interesting in other respects: they are the only graphs that yield finite, stably finite, quasi-diagonal, or AF-embeddable C^*-algebras [26], as well as the only graphs that yield graph algebras with stable rank one [10]. A graph which yields a C^*-algebra with Hausdorff spectrum must be tight, although this is not sufficient [7, Example 10].

In the remainder of this paper we aim to parametrize the entire tracial state space $T(C^*(E))$ for arbitrary graphs by employing Theorem 5.37 in conjunction with certain procedures that replace the graph E with a tight subgraph E', in such a way that the tracial state spaces $T(C^*(E))$ and $T(C^*(E'))$ coincide. Since the subgraphs that are best suited for analyzing how the trace spaces change are the *canonical* ones, the following terminology is all we need.

DEFINITION 5.39. If E is a directed graph, a *tightening* of E is a canonical subgraph, that is, one that can be presented as $E \setminus H$, for some saturated hereditary subset $H \subset E^0$, in such a way that:

(A) $E \setminus H$ is tight; and
(B)   the canonical $*$-homomorphism $\rho_H : C^*(E) \to C^*(E \setminus H)$ implements a bijective correspondence: $T(C^*(E \setminus H)) \ni \tau \longmapsto \tau \circ \rho_H \in T(C^*(E))$.

Since ρ_H is always surjective, the correspondence from (B) is always injective, so the only requirement in our definition is its *surjectivity*.

When it comes to parametrizing tracial states on graph C^*-algebras, the most useful and natural tightening is as follows.

EXAMPLE 5.40. Let E be a graph, and let $C = C_E$ be the set of vertices which emit entrances into cycles. The set C is obviously hereditary, but not saturated in general, so we need to take its saturation $\overline{C}$. As it turns out, $E \setminus \overline{C}$ constitutes a tightening of E. First of all, since passing from E to $E \setminus \overline{C}$ clearly removes all entries into the cycles in E, it is clear that $E \setminus \overline{C}$ is tight. Secondly, in order to justify the surjectivity of

$$T(C^*(E \setminus \overline{C})) \ni \tau \longmapsto \tau \circ \rho_{\overline{H}} \in T(C^*(E)),$$

all we must show is the fact that *all tracial states on $C^*(E)$ vanish on* $\ker \rho_{\overline{C}}$, for which it suffices to prove the inclusion $H \subset N_g$, which in itself is a consequence of Proposition 5.22.

The subgraph constructed in the above example is called the *minimal tightening*, and is denoted by E_{tight}. The canonical $*$-homomorphism will be denoted by ρ_{tight} : $C^*(E) \to C^*(E_{\text{tight}})$. Combining this construction with Theorem 5.37, we now obtain the following result.

THEOREM 5.41. *For any directed graph E, the map*

$$T_1^{\text{CCT}}(E_{\text{tight}}) \ni (g, \mu) \longmapsto \tau_{(g,\mu)} \circ \rho_{\text{tight}} \in T(C^*(E))$$

is an affine isomorphism.

The final result in this paper deals with a graph-theoretic characterization of automatic gauge invariance for tracial states, which as pointed out in Remark 5.27 is equivalent to the surjectivity of the map (5.22). In [30], it is shown that this feature is implied by condition (K), the condition that no vertex be the source of a unique simple cycle. However, as Theorem 5.42 below shows, this is not necessary.

THEOREM 5.42. *For a directed graph E, the following conditions are equivalent:*

(i)   *all tracial states on $C^*(E)$ are gauge invariant;*
(ii)   *the source of each cycle in E is essentially left infinite.*

PROOF. (i) $\Rightarrow$ (ii): Suppose that $\lambda = e_1 \ldots e_m$ is a cycle such that $v = s(\lambda) = r(e_1)$ is not essentially left infinite; we show how to construct a tracial state on $C^*(E)$ which is not gauge invariant. Note that as v is not essentially infinite, in particular, it does not emit an entrance to any cycle; therefore, none of the edges in λ will be removed when forming E_{tight}, and so we can assume that E is tight. (Since the canonical quotient $\pi : C^*(E) \to C^*(E_{\text{tight}})$ is equivariant for the respective gauge actions, a non-gauge-invariant tracial state on $C^*(E)_{\text{tight}}$) will give rise to a non-gauge-invariant trace on $C^*(E)$.)

We say that a path $\mu \in E^*$ is *acyclic* if it cannot be written as $\mu = \alpha \nu \beta$ for $\alpha, \beta \in E^*$ and ν a cycle. Let A denote the set of all acyclic paths with source v; note that any two paths in A are incomparable, and so A must be finite because v is not essentially left infinite. For $w \in E^0$, let $g(w) = |A \cap r^{-1}(w)|$; it is straightforward to verify that g is a finite graph trace with $g(v) = 1$, which we can normalize to obtain $g' \in T_1(E)$. Note that the cyclic support of g' is precisely $r(\{e_1, \ldots, e_m\})$ (as v is not essentially left infinite, it emits no entrances to cycles).

Now we can take any $z \in \mathbb{T} \setminus \mathbb{U}_{|\lambda|}$ and let $\mu_{s(e_i)} = \delta_z$ for all $i = 1, \ldots, m$. The affiliated tracial state $\tau_{(g,\mu)} \in T(C^*(E))$ will satisfy

$$\tau_{(g,\mu)}(b_\lambda) = g(s(\lambda)) z^{|\lambda|} \neq 0,$$

so that in particular $\tau_{(g,\mu)}$ is not gauge invariant.

(ii) $\Rightarrow$ (i): Suppose that the source of each cycle is essentially left infinite. Any finite graph trace must vanish on an essentially left infinite vertex as in Proposition 5.22; hence, if every source of every cycle is essentially left infinite, then there are no vertices in the cyclic support of any graph trace, and so there are no taggings to consider. Thus, every tracial state on $C^*(E_{\text{tight}})$ is gauge invariant, which shows that every tracial state on $C^*(E)$ is gauge invariant.   □

REMARKS. Besides the minimal tightening E_{tight} introduced in this paper, other tightenings could naturally be considered. The same arguments as those used in Example 5.40 can be used with C replaced by another hereditary subset $H \subset E^0$, as long as:

(A)   the canonical subgraph $E \setminus \overline{H}$ is tight; and
(B)   one has the inclusion $H \subset N_g$ for all $g \in T_1(E)$.

One way to ensure (A) is to take H to contain C_E. As far as condition (B) is concerned, we could use Proposition 5.22 as a guide. In particular, we can consider the set $L = L_E$ of *all* essentially left infinite vertices. Since L_E is potentially much larger than C_E, the resulting subgraph $E \setminus \overline{L_E}$ will potentially be considerably smaller than E_{tight} (and thus easier to analyze regarding graph traces).

References

[1]   R. J. Archbold, J. W. Bunce and K. D. Gregson, 'Extensions of states of C^*-algebras, II', *Proc. Edinb. Math. Soc.* **92A** (1982), 113–122.
[2]   J. Brown, G. Nagy and S. Reznikoff, 'A generalized Cuntz–Krieger uniqueness theorem for higher-rank graphs', *J. Funct. Anal.* **266**(4) (2013), 2590–2609.
[3]   J. Brown, G. Nagy, S. Reznikoff, A. Sims and D. Williams, 'Cartan subalgebras in C^*-algebras of Hausdorff etale groupoids', *Integral Equations Operator Theory* **85**(1) (2016), 109–126.
[4]   J. Cuntz and G. K. Pedersen, 'Equivalence and traces on C^*-algebras', *J. Funct. Anal.* **33**(2) (1979), 135–164.
[5]   G. Elliott and A. Toms, 'Regularity properties in the classification program for separable amenable C^*-algebras', *Bull. Amer. Math. Soc.* **45** (2008), 229–245.
[6]   J. Feldman and C. C. Moore, 'Ergodic equivalence relations, cohomology, and von Neumann algebras. II', *Trans. Amer. Math. Soc.* **234**(2) (1977), 325–359.

[7]   G. Goehle, 'Groupoid C^*-algebras with Hausdorff spectrum', *Bull. Aust. Math. Soc.* **88**(2) (2013), 232–242.

[8]   J. Hjelmborg, 'Purely infinite and stable C^*-algebras of graphs and dynamical systems', *Ergodic Theory Dynam. Systems* **21**(6) (2001), 1789–1808.

[9]   A. an Huef and I. Raeburn, 'The ideal structure of Cuntz–Krieger algebras', *Ergodic Theory Dynam. Systems* **17**(3) (1997), 611–624.

[10] J. Jeong, G. Park and D. Shin, 'Stable rank and real rank of graph C^*-algebras', *Pacific J. Math.* **200**(2) (2001), 331–343.

[11] M. Johnson, 'The graph traces of finite graphs and applications to tracial states of C^*-algebras', *New York J. Math.* **11** (2005), 649–658.

[12] R. Kadison and I. Singer, 'Extensions of pure states', *Amer. J. Math.* **81**(2) (1960), 383–400.

[13] A. Kumjian, 'On C^*-diagonals', *Canad. J. Math.* **38**(4) (1986), 969–1008.

[14] A. Kumjian, D. Pask, I. Raeburn and J. Renault, 'Graphs, groupoids, and Cuntz–Krieger algebras', *J. Funct. Anal.* **144**(2) (1997), 505–541.

[15] E. C. Lance, *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series, 210 (Cambridge University Press, 1994).

[16] G. Nagy and S. Reznikoff, 'Abelian core of graph algebras', *J. Lond. Math. Soc.* **85**(3) (2012), 889–908.

[17] G. Nagy and S. Reznikoff, 'Pseudo-diagonals and uniqueness theorems', *Proc. Amer. Math. Soc.* **142**(1) (2014), 263–275.

[18] D. Pask and A. Rennie, 'The noncommutative geometry of graph C^*-algebras I: the index theorem', *J. Funct. Anal.* **233**(1) (2006), 92–134.

[19] G.K. Pedersen, *C^*-Algebras and their Automorphism Groups* (Academic Press, 1979).

[20] D. Pitts, 'Structure for regular inclusions', Preprint, 2012, arXiv:1202.6413.

[21] R. Powers, 'Simplicity of the C^*-algebra associated with the free group on two generators', *Duke Math. J.* **42**(1) (1975), 151–156.

[22] I. Raeburn, *Graph Algebras*, CBMS Lecture Notes (American Mathematical Society, 2005).

[23] J. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Mathematics (Springer, 1980).

[24] J. Renault, 'Cartan subalgebras in C^*-algebras', *Irish Math. Soc. Bull.* **61** (2008), 29–63.

[25] M. Rieffel, 'C^*-algebras associated to irrational rotations', *Pacific J. Math.* **93**(2) (1981), 415–429.

[26] C. Schafhauser, 'AF-embeddings of graph algebras', Preprint, 2014, arXiv:1405.7757.

[27] C. Schafhauser, 'Traces on topological graph algebras', Preprint, 2016, arXiv:1605.03603.

[28] M. Tomforde, 'The ordered K_0-group of a graph C^*-algebra', *C. R. Math. Acad. Sci. Soc. R. Can.* **25** (2003), 19–25.

[29] M. Tomforde, 'A unified approach to Exel–Laca algebras and C^*-algebras associated to graphs', *J. Operator Theory* **50** (2003), 345–368.

[30] M. Tomforde, 'Stability of C^*-algebras affiliated to graphs', *Proc. Amer. Math. Soc.* **132**(6) (2004), 1787–1795.

[31] W. Winter and J. Zacharias, 'The nuclear dimension of C^*-algebras', *Adv. Math.* **224**(1) (2010), 461–498.

DANNY CRYTSER, Department of Mathematics,
Kansas State University, Manhattan, KS 66503, USA
e-mail: crytser@ksu.edu

GABRIEL NAGY, Department of Mathematics,
Kansas State University, Manhattan, KS 66503, USA
e-mail: nagy@ksu.edu