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Travelling wave behaviour for a Porous-Fisher equation

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We study the propagation properties of the reaction-diffusion equation of Fisher type

$$u_t = (u^{m-1} u_x)_x + u^p(1-u) \quad \text{for} \quad x \in \mathbb{R}, t > 0,$$

with $p < 1 < m$. Taking into account that solutions of the Cauchy problem are nonunique if $m+p \geqslant 2$, we prove that the minimal solutions in this case tend to propagate with certain minimal speed $c_*(m, p)$. More precisely, if we translate any solution with a velocity $|c| < c_*$, we get the limit in time value one, and if the initial value $u(\cdot, 0)$ vanishes, say, for $x \geqslant 0$, the minimal solution translated with velocity $c > c_*$ tends to zero. Also, an interface appears for the minimal solutions, whose asymptotic velocity is c_*. This behaviour depends upon the existence of special solutions of travelling wave form. Travelling waves have been widely studied for diffusion equations related with the above. We characterize here the minimal velocity c_* for which travelling waves exist, as an analytic function of the parameters m and p, for every $m+p \geqslant 2$, by viewing it as an anomalous exponent. Some local properties of the minimal solutions and their interfaces in the case $m+p \geqslant 2$ are also proved.

1 Introduction

There is a vast number of phenomena in biology where a key element of a developmental process is the appearance of a travelling wave of chemical concentration. When reaction kinetics and diffusion are coupled, travelling waves of chemical concentration can effect a biochemical change very much faster than straight diffusional processes. This gives rise to reaction–diffusion equations like

$$u_t = Du_{xx} + f(u) \tag{1.1}$$

for a chemical concentration u, diffusion coefficient D, and where $f(u)$ represents the kinetics. The classical and simplest case is the so-called Fisher equation (1.1) with $f(u) = u(1-u)$. It was suggested as a deterministic version of a stochastic model for the spatial spread of a favoured gene in a population [1]. This equation has been widely studied since then, the seminal and now classical paper being [2]. In particular, in that paper, Kolmogorov *et al.* proved that any initial concentration which is one for large negative spatial variable x and vanishes for large x, evolves to a travelling wavefront with minimal velocity $c_* = 2\sqrt{D}$. Different initial values could propagate as different travelling waves, depending on the behaviour at $x \to \pm\infty$. All the waves are positive, their velocity being always $c \geqslant 2\sqrt{D}$.

In certain insect dispersal models, the diffusion coefficient D depends upon the

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population u (see Murray [3]). A natural extension to incorporate density-dependent diffusion is the porous medium type equation

$$u_t = (D(u)\, u_x)_x + f(u), \tag{1.2}$$

where typically $D(u) = u^\gamma$, with $\gamma > 0$.

This model implies that the population disperses to regions of lower density more rapidly as the population gets more crowded. The case $f(u) = u(1-u)$ was studied by Aronson *et al.* in several works [4, 5], where they proved that there is also a minimal speed of propagation $c_*(\gamma)$ that represents the asymptotic speed of propagation of any concentration that initially vanishes for large x. More precisely

$$\lim_{t\to\infty} u(x+ct, t) = \begin{cases} 0 & \text{if } c > c_* \\ 1 & \text{if } 0 \leqslant c < c_*. \end{cases} \tag{1.3}$$

The effect of the nonlinear diffusion is reflected in the phenomenon of finite speed of propagation, which holds for the travelling waves with exact velocity $c_*(\gamma)$. Actually, these waves vanish for large x and have a linear interface $s(t) = x_0 + c_*\, t$. The waves with $c > c_*$ are positive; no waves exist if $c < c_*$ (see also Atkinson *et al.* [6] and Sanchez-Garduño & Maini [7]).

We are interested in studying the large time behaviour of the solutions of the following porous medium version of the Fisher equation (1.2):

$$u_t = (u^{m-1} u_x)_x + u^p(1-u), \quad p < 1 < m. \tag{1.4}$$

The non-Lipschitz character of the reaction for $u = 0$ (it is even discontinuous if $p \leqslant 0$, in which case we consider $f(0) = 0$), makes equation (1.4) interesting in the coexistence of finite and infinite propagation, nonuniqueness, and the absence of a general comparison principle. A first study was performed by de Pablo & Vazquez [8] in connection with the strong reaction–slow diffusion equation

$$u_t = (u^{m-1} u_x)_x + u^p, \quad p < 1 < m. \tag{1.5}$$

It is proved in that paper that the relation between the parameters m and p differentiates two types of behaviour concerning the above questions. More precisely, for equation (1.5), if $m+p \geqslant 2$, the minimal solutions have finite speed of propagation, while the maximal solutions are positive, and if $m+p < 2$, there is a unique solution for every initial value $u(\cdot, 0) \not\equiv 0$, which is always positive (see also de Pablo & Vazquez [9]). We follow here the techniques of that paper to characterize the uniqueness for equation (1.4). Also, we complete the study of the local behaviour of the minimal solutions and their interfaces in the case $m+p \geqslant 2$.

As for the asymptotic behaviour for the solutions to equation (1.4), a first step is to characterize the existence of travelling wave solutions for this equation. This can be done using the techniques of Engler [10] and the results in Atkinson *et al.* [6]. A more detailed study is performed in de Pablo & Vazquez [8]. The result obtained in those papers is that there exist travelling waves for equation (1.4) only if $m+p \geqslant 2$, and only for velocities $c \geqslant c_*(m,p)$. In the present work, we give a new proof of existence of travelling waves, and describe the curve of minimal velocity c_* in terms of m and p. We then use these travelling waves to describe the asymptotic behaviour of the solutions to equation (1.4).

The paper is organized as follows. §2 considers the N-dimensional analogue of equation (1.4), and studies the uniqueness of the solutions in terms of the initial support and the value of the parameters m and p. In §3, a local analysis of the interface in the case $m+p \geqslant 2$ is performed. Some concavity properties are also proved.

The main sections 4 and 5 are devoted to the study of the properties of the travelling waves and their applications. In §4 we characterize which travelling waves are minimal solutions, and we also obtain the minimal velocity c_* as an analytic function of the parameter $\sigma = m+p-2$. We consider here equation (1.4) for every $m, p \in \mathbb{R}$, $m+p \geqslant 2$.

In §5 we prove the above-mentioned asymptotic behaviour of the minimal solutions as well as their interfaces. We remark on the influence of the Fisher-type coefficient $(1-u)$, which differentiates equations (1.5) and (1.4). For the former, the interfaces are asymptotically linear if $m+p = 2$ and superlinear if $m+p > 2$, while for the latter, they are always asymptotically linear for every $m+p \geqslant 2$.

2 Uniqueness

Consider the problem

$$\begin{cases} u_t = \Delta u^m + u^p(1-u) & \text{for} \quad (x,t) \in Q = \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for} \quad x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

with $p < 1 < m$, $N \geqslant 1$ and $0 \leqslant u_0 \leqslant 1$.

For the existence of (continuous, weak) minimal and maximal solutions to this problem, as well as comparison properties, we refer to de Pablo & Vazquez [8]. In particular, comparison holds when comparing minimal or maximal solution, not every solution. In general, this problem does not have a unique solution. This follows by considering the problem with null initial datum $u_0 \equiv 0$, for which the minimal solution is the null solution $u \equiv 0$, while the maximal solution is $\bar{u}(x,t) = \psi(t)$, where ψ is defined by the relation

$$t_+ = \int_0^{\psi(t)} \frac{ds}{s^p(1-s)}. \tag{2.2}$$

Moreover, a continuum of solutions appears in the form $u(x,t;\tau) = \psi(t-\tau), \tau \geqslant 0$.

The fact that this kind of bifurcation from the null solution is the only nonuniqueness possibility (called *almost uniqueness* in de Pablo & Vazquez [9] for the pure power-law reaction case; see also Aguirre & Escobedo [11]), depends upon the parameter $\sigma = m+p-2$. So we have

Theorem 2.1 (i) *If* $\sigma < 0$, *problem* (2.1) *has almost uniqueness.*
(ii) *If* $\sigma \geqslant 0$, *problem* (2.1) *has a unique solution if and only if the support of* u_0 *is the whole of* $\mathbb{R}^N$.

The uniqueness part of this theorem relies on proving that positive solutions are unique. Moreover, if $\sigma < 0$, every solution is positive for $t > 0$, while for $\sigma \geqslant 0$, we have a finite speed of propagation: if u_0 vanishes in some ball, then the minimal solution $\underline{u}$ also vanishes in some smaller ball for small times, thus being different from the positive maximal solution

$\bar{u}$. Observe also that uniqueness holds for *strictly positive* solutions, i.e. solutions with $u \geqslant \delta > 0$, since in that case the reaction term becomes regular.

We first prove a lower estimate, in the spirit of de Pablo & Vazquez [9].

Lemma 2.2 *Let u be a solution to problem* (2.1) *with $u_0(x_0) > 0$. Then there exist constants $c_1, c_2 > 0$ depending only on m, p, N, such that*

$$\text{(i)}\ u(x_0, t) \geqslant c_1 t^{\alpha} \quad \text{for} \quad 0 < t < 1, \tag{2.3}$$

$$\text{(ii)}\ u(x, t) > 0 \quad \text{for} \quad 0 < t < 1, \quad |x - x_0| < c_2 t^{\beta}, \tag{2.4}$$

where

$$\alpha = \frac{1}{1-p}, \quad \beta = \frac{m-p}{2(1-p)}.$$

Proof We will obtain estimates (2.3, 2.4) by comparing with a subsolution to an approximate (from below) regular problem, as in de Pablo & Vazquez [9], substituting the reaction term by

$$f_{\epsilon}(s) = \min\{s^p(1-s), \epsilon^{p-1}(1-\epsilon)\, s\}. \tag{2.5}$$

If we write $w(x, t) = t^{\alpha} \varphi(|x| t^{-\beta})$, we obtain the particular subsolution

$$\varphi(\xi) = (A - B\xi^2)_+^{\frac{1}{m-1}},$$

where $B = (m-1)\beta/2m$ and

$$A^{\frac{p-1}{m-1}}(1 - A^{\frac{1}{m-1}}) = \alpha + \beta N, \quad 1 \geqslant t \geqslant t_{\epsilon} \equiv (\alpha + \beta N)\frac{\epsilon^{1-p}}{1-\epsilon}.$$

Now, from the continuity of the solutions to problem (2.1), for any $\tau > 0$ there exists $\epsilon > 0$ small enough such that

$$w(x - x_0, t_{\epsilon}) \leqslant u(x, \tau) \quad \text{for every} \quad x \in \mathbb{R}^N,$$

and then

$$w(x - x_0, t + t_{\epsilon}) \leqslant u(x, t + \tau) \quad \text{for} \quad x \in \mathbb{R}^N, \quad 0 < t + t_{\epsilon} < 1.$$

Letting $\tau \to 0$, we obtain the required estimates with $c_1 = A^{\frac{1}{m-1}}$, $c_2 = \sqrt{A/B}$. □

A refinement of this proof allows us to get

Lemma 2.3 *Let u be a solution to problem* (2.1) *with $u_0(x_0) > 0$. Then there exists $T > 0$ such that*

$$u(x_0, t) \geqslant \left(\frac{t}{t+1}\right)^{\alpha} \quad \text{for} \quad t \geqslant T. \tag{2.6}$$

Proof Choosing $T = (1 + \beta N/\alpha)^{\alpha - 1}$ and $C = (T+1)^{-\alpha}$, we get

$$u(x_0, t) \geqslant Ct^{\alpha} \quad \text{for} \quad 0 \leqslant t \leqslant T. \quad \square$$

Observe that the best estimate of the support obtained by this method is

$$u(x, t) > 0 \quad \text{for} \quad |x - x_0| < D\sqrt{t}, \quad t \geqslant T.$$

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Though this is not sharp, as we will see in the following sections, we obtain in particular that the support of u tends to cover the whole $\mathbb{R}^N$ as $t \to \infty$.

Corollary 2.4 *Let u be a solution to problem* (2.1). *Then,*
(i) *if $u \not\equiv 0$ then for every $x \in \mathbb{R}^N$ there exists $T > 0$ such that $u(x, T) > 0$;*
(ii) *if $u \not\equiv 0$ then for every $x \in \mathbb{R}^N$ we have $\lim_{t\to\infty} u(x,t) = 1$.*

We also get the precise order of growth of positive solutions, since we can apply Lemma 2.2 at every point $(x_0, t_0) \in Q$.

Lemma 2.5 *If u is a solution to problem* (2.1) *with $u > 0$ in Q, then*

$$u(x,t) \geqslant \psi(t) \quad \text{in} \quad Q. \tag{2.7}$$

The final step in the uniqueness proof is the following:

Lemma 2.6 *If u is a solution to problem* (2.1) *with $u > 0$ in Q, then u coincides with the maximal solution to that problem.*

Proof Let $\bar{u}$ be the maximal solution to problem (2.1). We fix $T > 0$ and put $Q_T = \mathbb{R}^N \times (0, T)$. For every $\phi \in C^\infty(Q_T)$ with compact support in x and every $0 \leqslant t \leqslant T$, we have, after subtracting the equation (in its weak form) for both solutions,

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^N} (\bar{u}-u)(x,t)\,\phi(x,t)\,dx \\ &= \int_0^t \int_{\mathbb{R}^N} \{(\bar{u}-u)\,\phi_t + (\bar{u}^m - u^m)\,\Delta\phi + [\bar{u}^p(1-\bar{u}) - u^p(1-u)]\,\phi\}(x,s)\,dx\,ds. \end{aligned} \tag{2.8}$$

Proceeding as in de Pablo & Vazquez [9], i.e. choosing an appropriate test function, we get

$$0 \leqslant \int_{\mathbb{R}^N} (\bar{u}-u)(x,t)\,\phi(x,t)\,dx \leqslant \int_0^t \left\{ \int_{\mathbb{R}^N} [\bar{u}^p(1-\bar{u}) - u^p(1-u)]\,\phi(x,s)\,dx \right\} ds. \tag{2.9}$$

If $p \leqslant 0$ we have nothing more to do. Assume then $p > 0$. Using estimate (2.7), we obtain

$$\bar{u}^p(1-\bar{u}) - u^p(1-u) \leqslant p(\bar{u}-u)\,\psi^{p-1}, \tag{2.10}$$

and the above inequality becomes

$$0 \leqslant \int_{\mathbb{R}^N} (\bar{u}-u)(x,t)\,\phi(x,t)\,dx \leqslant p \int_0^t \left\{ \int_{\mathbb{R}^N} (\bar{u}-u)\,\phi(x,s)\,dx \right\} \psi^{p-1}(s)\,ds. \tag{2.11}$$

Putting

$$h(t) = \int_0^t \left\{ \int_{\mathbb{R}^N} (\bar{u}-u)\,\phi(x,s)\,dx \right\} \psi^{p-1}(s)\,ds,$$

this inequality reads

$$\psi^{1-p}(s)\,h'(s) \leqslant p h(s), \quad \text{for} \quad 0 \leqslant s \leqslant T.$$

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Integrating this equation for $\epsilon \leqslant s \leqslant t \leqslant T$, and using the differential equation satisfied by ψ, we have

$$h(t) \leqslant h(\epsilon)\left[\frac{\psi(t)}{1-\psi(t)}\,\frac{1-\psi(\epsilon)}{\psi(\epsilon)}\right]^p \leqslant c(T)\,b(\epsilon)\,\psi^{-p}(\epsilon), \tag{2.12}$$

for $\epsilon \leqslant t \leqslant T$. On the other hand, by equation (2.9) and the Hölder inequality, we get

$$\int_{\mathbb{R}^N} (\bar{u}-u)(x,s)\,\phi(x,s)\,dx \leqslant c(T)\int_0^s \left[\int_{\mathbb{R}^N} (\bar{u}-u)(x,\tau)\,\phi(x,\tau)\,dx\right]^p d\tau, \tag{2.13}$$

for every $0 \leqslant s \leqslant T$, which, when integrated means

$$\int_{\mathbb{R}^N} (\bar{u}-u)(x,s)\,\phi(x,s)\,dx \leqslant c(T)\,s^{\frac{1}{1-p}}. \tag{2.14}$$

Applying this estimate to bound $h(\epsilon)$, we get

$$h(\epsilon) \leqslant c(T)\int_0^\epsilon s^{\frac{1}{1-p}-1}\,ds = c(T)\,\epsilon^{\frac{1}{1-p}}.$$

Finally, from the definition of ψ, for $s > 0$ small we have $\psi(s) \geqslant \mu\, s^{\frac{1}{1-p}}$. Putting together all these estimates, equation (2.12) implies, for every $\epsilon > 0$,

$$0 \leqslant \int_{\mathbb{R}^N} (\bar{u}-u)(x,t)\,\phi(x,t)\,dx \leqslant h(t) \leqslant c(T)\,\epsilon.$$

Letting $\epsilon \to 0$ we get $u \equiv \bar{u}$ in Q. □

Proof of Theorem 2.1 The above lemma implies the uniqueness of positive solutions. Moreover, if $\sigma < 0$ and $u_0 \not\equiv 0$, every solution to problem (2.1) is positive in Q, and this is done exactly as in de Pablo & Vazquez [9, Lemma 2.4], using the main estimates of Lemma 2.2. Also, if $\sigma < 0$ and $u_0 \equiv 0$, it is easy to prove that the only nontrivial solutions are $u(x,t;\tau) = \psi(t-\tau), \tau \geqslant 0$.

The nonuniqueness result for $\sigma \geqslant 0$ follows from the finite speed of propagation of §3. □

3 The interface

In this section we study the propagation properties of the solutions to problem (2.1) in dimension one. We are interested here in the phenomenon of finite speed of propagation, which holds for the minimal solutions in the case $\sigma \geqslant 0$. Also, by a simple rescaling, we consider the equation

$$u_t = (u^{m-1}u_x)_x + u^p(1-u) \quad x \in \mathbb{R}, t > 0. \tag{3.1}$$

The formulae for the propagation will then be simpler.

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We assume that the initial value u_0 is positive in $(-\infty, 0)$ and vanishes in $[0, \infty)$. Then, for the minimal solution u to problem (3.1), an interface

$$s(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}$$

appears. The interface movement is best described in terms of the *pressure* variable $v = (u^{m-1})/(m-1)$. The equation satisfied by v is

$$v_t = (m-1)\, v v_{xx} + v_x^2 + \mu\, v^{\frac{\sigma}{m-1}} g(v), \tag{3.2}$$

$\mu = (m-1)^{\sigma/(m-1)}$, $g(v) = 1-((m-1)\,v)^{1/(m-1)}$ (interpreted in the obvious way if $\sigma = 0$).

The local analysis of the interface made for the power-law case (see de Pablo & Vazquez [8]) is carried over to our situation with only minor changes, thus producing

Proposition 3.1 (i) *$s(t)$ is nondecreasing and Hölder continuous.*
(ii) *The forward time derivative satisfies, for $t > 0$,*

$$D^+ s(t) \begin{cases} \geqslant 2 & \text{if} \quad \sigma = 0 \\ > 0 & \text{if} \quad \sigma > 0. \end{cases} \tag{3.3}$$

(iii) *If $v_x(s(t), t) = \lim_{x \nearrow s(t_0)} v_x(x, t_0) = -z > -\infty$, then the following equation of the interface holds*:

$$\sigma = 0 \Rightarrow D^+ s(t_0) = \begin{cases} z + 1/z & \text{if} \quad z \geqslant 1 \\ 2 & \text{if} \quad 0 \leqslant z \leqslant 1. \end{cases} \tag{3.4}$$

$$\sigma > 0 \Rightarrow D^+ s(t_0) = z. \tag{3.5}$$

The proofs are based on the fact that near the interface we have $(1-\epsilon) \leqslant g(v) \leqslant 1$. Moreover, if $\sigma = 0$ we can improve the equation of the interface to get

Theorem 3.2 *If $\sigma = 0$ we always have*

$$-v_x(s(t), t) \geqslant 1 \quad \text{for} \quad t > 0. \tag{3.6}$$

Proof The proof relies on comparison with some special solution to the pure power equation constructed in de Pablo & Vazquez [9], called the Absolute Minimal Solution.

Assume by contradiction that for some $t_0 > 0$ we have $-v_x(s(t_0), t_0) < 1$. Then, from the previous proposition, if $\tau > 0$ is small enough,

$$s(t) = s(t_0) + 2(t - t_0) \quad \text{for} \quad t_0 \leqslant t_0 + \tau.$$

On the other hand, for some point $y < s(t_0)$ we have

$$v(x,t) \leqslant \epsilon \quad \text{for} \quad x \geqslant y, \quad t_0 \leqslant t \leqslant t_0 + \tau;$$

thus v is a supersolution to the equation

$$v_t = (m-1)\, v v_{xx} + v_x^2 + (1-\epsilon)\, \mu\, v^{\frac{\sigma}{m-1}}.$$

Let W be the absolute minimal solution to that equation (see de Pablo & Vazquez [9, Theorem 4]), centred at the point $(s(t_0), t_0)$. It satisfies, for $t \geqslant t_0$,

$$\operatorname{supp} W(\cdot, t) = \{|x - s(t_0)| \leqslant 2\sqrt{(1-\epsilon)}\,(t-t_0)\}$$
$$-W_x(s(t_0) + 2\sqrt{(1-\epsilon)}\,(t-t_0), t) = \sqrt{(1-\epsilon)},$$

and also

$$W(x,t) \leqslant v(x,t) \quad \text{for} \quad x \geqslant y, \quad t \geqslant t_0.$$

This implies

$$-v_x(s(t_0), t_0) \geqslant \sqrt{(1-\epsilon)}, \quad \text{for} \quad t_0 < t \leqslant t_0 + \tau.$$

Letting $\epsilon \to 0$ we get equation (3.6). □

Another property of the minimal solution and its interface is the following concavity property:

Theorem 3.3 *If the initial pressure $v(x,0)$ is concave in its positivity set, then the same holds for $v(\cdot, t)$ for each $t > 0$. In the critical case $\sigma = 0$ the interface $s(t)$ is also concave.*

Proof A direct proof of this result via the equation satisfied by v_{xx} is hindered by the problem of bad behaviour at the interface. Instead, we use a technique based on the Lie–Trotter formula of composition of semigroups used in Bénilan & Vazquez [12] to prove the concavity of the solutions to the porous medium equation.

For that purpose we first consider the approximate problem

$$\begin{cases} u_t = (u^{m-1} u_x)_x + f(u) \\ u(x,0) = u_0(x) \end{cases} \tag{3.7}$$

with $f(s) = f_\epsilon(s) = \min\{s^p(1-s), \epsilon^{p-1}(1-\epsilon)\,s\}$, and the problems

$$(P_d) \quad \begin{cases} u_t = (u^{m-1} u_x)_x \\ u(x,0) = u_0(x) \end{cases} \qquad (Pr) \quad \begin{cases} u_t = f(u) \\ u(x,0) = u_0(x). \end{cases}$$

Let us consider the semigroups $S_d(t)$ and $S_r(t)$ associated, respectively, with problems (P_d) and (P_r), i.e. for each initial value u_0 they produce the solutions of those problems,

$$S_d(t)(u_0)(x) = u_d(x,t), \quad S_r(t)(u_0)(x) = u_r(x,t). \tag{3.8}$$

Fix $t > 0$ and consider, for each $n, j \in \mathbb{N}$, $n \geqslant j$, the function

$$u_{(n,j)} = [S_d(t/n)\, S_r(t/n)]^j (u_0), \tag{3.9}$$

which is the Lie–Trotter formula of composition of both semigroups. Observing that the operator

$$A(u) = -(u^{m-1} u_x)_x - f(u) + ku$$

is an m-accretive operator in $L^1(\mathbb{R})$ if k is larger than the Lipschitz constant of f (cf. Crandall [13]), then the limit $\hat{u} = \lim_{n\to\infty} u_{(n,n)}$ is well defined and produces the unique solution to the composed problem (3.7) (see Bénilan & Ismail [14, Theorem 1]).

Consider now the pressure $\hat{v}$ associated with $\hat{u}$, and define also $v_{(n,j)} = 1/(m-1)\, u_{(n,j)}^{m-1}$. We will study the concavity of $\hat{v}$ in terms of the concavity of each iteration $v_{(n,j)}$.

First, for the solution to the porous medium problem (P_d), we have that the pressure is concave if the initial datum is (cf. [12, 15]). On the other hand, the pressure v_r associated with problem (P_r) is implicitly given for each x in the initial support $P(0)$ by

$$t_+ = \int_{v_0(x)}^{v_r(x,t)} \frac{ds}{h(s)}, \tag{3.10}$$

where

$$h(s) = ((m-1)\,s)^{\frac{m-2}{m-1}} f(((m-1)\,s)^{\frac{1}{m-1}}) = \min\{\mu\, v^{\frac{\sigma}{m-1}} g(v), (m-1)\,\epsilon^{p-1}(1-\epsilon)\,s\}.$$

Moreover, the support is stationary, $P(t) \equiv P(0)$. From this we obtain

$$(v_r)_{xx} = [h'(v_r) - h'(v_0)]\frac{h(v_r)(v_0')^2}{h^2(v_0)} + \frac{h(v_r)}{h(v_0)} v_0''.$$

Observing that $h(s)$ is positive and concave, we have $(v_r)_{xx} \leqslant 0$.

Putting together these two concavity results, we obtain that $v_{(n,j)}$ is concave for each $n \geqslant j \geqslant 1$. Passing to the limit $j = n$, $n \to \infty$ we obtain $\hat{b}$ concave. Finally, passing to the limit $\epsilon \to 0$ we get v concave.

As for the interface in the case $\sigma = 0$, let $t_0 > 0$ be fixed, and let $-v_x(s(t_0), t_0) = z$. If $z \leqslant 1$, since $v_{xx} \leqslant 0$ we have $s(t) = s(t_0) + 2(t - t_0)$ for $t \geqslant t_0$, and we have nothing to prove. Assume then $z > 1$. From Proposition 3.1 we have $s'(t_0) = c = z + 1/z$. We consider now the linear front

$$w(x,t) = z[c(t-t_0) - (x - s(t_0))]_+,$$

which is a pressure supersolution to our equation. Its interface is $\eta(t) = s(t_0) + c(t - t_0)$.

Since v is concave we have $v(x, t_0) \leqslant w(x, t_0)$, and by comparison we also have $v(x,t) \leqslant w(x,t)$ for every $t \geqslant t_0$. We end with the comparison of the interfaces:

$$s(t_0) = \eta(t_0), \quad s'(t_0) = \eta'(t_0) \quad \text{and} \quad s(t) \leqslant \eta(t) \quad \text{for} \quad t \geqslant t_0.$$

This implies $s''(t) \leqslant 0$. □

4 The travelling waves (TWs)

One of the most interesting properties of equation (3.1) is the appearance of solutions in travelling wave form (see mainly de Pablo & Vazquez [8]). For the reader's convenience we summarize the analysis made in that paper in order to obtain the TWs; then we pass on to study their properties.

Introducing $u(x,t) = \varphi(\xi)$, $\xi = c - ct$, $x > 0$ into equation (3.1), and defining the variables

$$X(\tau) = \varphi(\xi), \quad Y(\tau) = -\varphi^{m-2}(\xi)\varphi'(\xi), \quad d\tau = X^{1-m}\, d\xi, \tag{4.1}$$

we obtain the system

$$\begin{cases} \dfrac{dX}{d\tau} = -XY \\[2ex] \dfrac{dY}{d\tau} = X^{\sigma}(1-X) - Y(c-Y). \end{cases} \tag{4.2}$$

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If $\sigma > 0$ it has critical points at

$$(X, Y) = (0,0), (1,0), (0,c). \tag{4.3}$$

These points are two saddles, $(1,0)$, $(0,c)$, and a degenerate node $(0,0)$. We look for admissible trajectories, i.e. solutions to

$$\frac{dY}{dX} = \frac{c-Y}{X} - \frac{X^{\sigma-1}(1-X)}{Y} \equiv H(X, Y, c, \sigma) \tag{4.4}$$

for $0 \leqslant X \leqslant 1$, joining the points $(1,0)$ and $(0,c)$ or $(1,0)$ and $(0,0)$. A saddle–saddle connection corresponds, via equation (4.1), to a finite TW, while a saddle-node connection will correspond to a finite or positive TW, depending on the value of p.

If $\sigma = 0$, and $c < 2$, the only critical point is $(X, Y) = (1,0)$, while for $c \geqslant 2$, the critical points are

$$(X, Y) = (1,0), (0,z_1), (0,z_2), \tag{4.5}$$

where $0 < z_1 \leqslant z_2$ are the roots of the equation $z^2 - cz + 1 = 0$. The finite TWs are given by the connection $(1,0)$ to $(0,z_1)$. In that way, in de Pablo & Vazquez [8] it is proved (see also [6, 10]):

Theorem 4.1 (de Pablo & Vazquez [8]) *Equation* (3.1) *with* $m > 1$ *and* $p \in \mathbb{R}$ *admits TW solutions if and only if* $\sigma \geqslant 0$, *and only for velocities* $c \geqslant c_*(\sigma)$, *the TW with* $c = c_*$ *being finite, all the other TWs being finite if and only if* $p < 1$.

We also note that the phase–plane (4.2) makes sense for every value of $m, p \in \mathbb{R}$. In that sense, we can also consider equation (3.1) with $m \leqslant 1$. The TWs in that case are all positive.

For future use, we write here the behaviour as $\xi \to -\infty$ of the TWs, which is obtained from equation (4.1), and the behaviour of the trajectory $Y(X)$ as $X \to 1$:

$$\varphi_c(\xi) \approx 1 - e^{\gamma\xi}, \quad \gamma - 1/\gamma = c. \tag{4.6}$$

It has been observed [8] that the TWs with velocity $c > c_*$ cannot be minimal solutions, since they do not satisfy the equation of the interface (see equations (3.4), (3.5)). On the other hand, the pressure associated with each TW satisfies $v_{xx}(x,t) = \phi''(\xi) = -dY/d\xi$. As is easily seen in equations (4.1), (4.2), Y is increasing in ξ when $c = c_*$, while it is not when $c > c_*$, which implies that v is concave, as the minimal solutions do, only if $c = c_*$. We complete here the classification of the minimal TWs by proving that, in fact, the TW with $c = c_*$ is a minimal solution.

Theorem 4.2 *The TWs are minimal solutions if and only if* $c = c_*(\sigma)$.

Proof The proof is done by comparison, and we use as subsolutions local TWs with velocities $0 < c < c_*(\sigma), c \approx c_*(\sigma)$.

Case $\sigma = 0$

First, we consider the approximate reaction function (2.5), i.e.

$$f_\epsilon(s) = \begin{cases} \epsilon^{p-1}(1-\epsilon)\,s & \text{if} \quad 0 \leqslant s \leqslant \epsilon \\ s^p(1-s) & \text{if} \quad \epsilon \leqslant s \leqslant 1, \end{cases} \tag{4.7}$$

and the differential equation

$$\frac{dY}{dX} = \frac{(c-Y)\,Y - X^{m-2} f_\epsilon(X)}{XY} \equiv H_\epsilon(X, Y). \tag{4.8}$$

We look for trajectories satisfying equation (4.8), passing through points $(X_0, 0)$, $0 < X_0 < 1$, and going to $\pm\infty$ as X goes to 0. We claim that if $\epsilon > 0$ is small enough, there exists a decreasing trajectory $Y_\epsilon(X)$ solution to equation (4.8), joining the points $(0, c)$ and $(X_1, 0)$ for some $0 < X_1 < \sqrt{\epsilon}$.

Assuming that this is true, for any $X_1 < X_0 < 1$, the trajectory in the first quadrant starting from the point $(X_0, 0)$ cannot cross Y_ϵ, and thus tends to ∞ as X tends to 0. The piece of the trajectory in the fourth quadrant is also easily seen to go to $-\infty$ as X goes to 0. Using the differential relation (4.1), we see that it takes the trajectory a finite ξ-interval. Thus, we get that, for each $0 < X_0 < 1$, there exists $\epsilon > 0$, and a local TW $\varphi_c(\xi)$, solution to

$$(\varphi^{m-1}\varphi')' + c\varphi' + f_\epsilon(\varphi) = 0, \quad \xi_1 \leqslant \xi \leqslant \xi_2, \tag{4.9}$$

and satisfying

$$\begin{gathered} \varphi(\xi_1) = \varphi(\xi_2) = 0, \quad \max_{\xi_1 \leqslant \xi \leqslant \xi_2} \varphi(\xi) = X_0, \\ (\varphi^{m-1})'(\xi_1) = -(\varphi^{m-1})'(\xi_2) = \infty. \end{gathered} \tag{4.10}$$

Now we prove the claim. Observe that $dY/dX = 0$ in equation (4.8) along the graph

$$\begin{cases} (c-Y)\,Y - \epsilon^{1-m}(1-\epsilon)\,X^{m-1} = 0 & \text{if} \quad 0 \leqslant X \leqslant \epsilon \\ (c-Y)\,Y - (1-X) = 0 & \text{if} \quad \epsilon \leqslant X \leqslant 1. \end{cases} \tag{4.11}$$

If $0 < \epsilon < 1 - c^2/4$, this graph consists of a curve joining $(0, 0)$ with $(0, c)$ and a curve joining $(1, 0)$ with $(1, c)$. The point of maximum in X of the first curve is

$$X = \epsilon\left(\frac{c^2}{4(1-\epsilon)}\right)^{1/(m-1)} < \epsilon.$$

To get the desired trajectory, we consider the curve $Y = T(X)$ given by

$$\tfrac{1}{2} T^2 + \left(1 - \frac{c^2}{4}\right) \log X + X + k = 0.$$

This curve satisfies, for $\epsilon \leqslant X \leqslant 1$,

$$\frac{dT}{dX} = \frac{c^2/4 - (1-X)}{XT} \geqslant \frac{(c-T)\,T - (1-X)}{XT} = H_\epsilon(X, T).$$

Choosing $k = (c^2/4 - 1) \log \sqrt{\epsilon} - \sqrt{\epsilon}$ and

$$\frac{c^2}{2} + \frac{1}{2}\left(1 - \frac{c^2}{4}\right) \log \epsilon + \epsilon - \sqrt{\epsilon} < 0, \tag{4.12}$$

we get $T(\sqrt{\epsilon}) = 0$ and $T(\epsilon) > c$.

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As a consequence, the decreasing trajectory starting from the point $(0, c)$ cannot cross T, and it goes down to $Y = 0$ at some point $0 < X_1 < \sqrt{\epsilon}$. We then need to select $\epsilon > 0$ small enough to satisfy equation (4.12), and also $\epsilon < 1 - c^2/4$, and the claim is proved.

Consider now the TW $u_*(x,t) = \varphi_*(x - c_* t)$ with velocity $c = c_*$, the associated pressure $\phi_* = 1/(m-1)\varphi_*^{m-1}$, and the corresponding trajectory $Y = Y_*(X)$. We want to compare the initial value $\phi_*(x)$ with the local TW $\phi_c(x) = 1/(m-1)\varphi_c^{m-1}(x)$ with velocity $c < c_*$ just constructed (4.10). Take any point $z < 0$. The previous phase–plane analysis implies that there exists $\epsilon > 0$ and a trajectory $Y_c(X)$ such that it cuts the trajectory Y_* at the point $X = \varphi(z)$. Observing that the phase–plane variables represent $X = \varphi$, $Y = -\phi'$, we get, by translation,

$$\varphi_c(z) = \varphi_*(z), \quad \phi_c(z) = \phi_*(z), \quad \phi'_c(z) = \phi'_*(z).$$

Also, comparing the slopes of Y_c and Y_*, we get $\phi''_c(z) < \phi''_*(z)$ and

$$\phi'_c(x) < \phi'_*(y) \quad \text{for} \quad z < x, y < \xi_2 \quad \text{with} \quad \phi_c(x) = \phi_*(y)$$
$$\phi'_c(x) > \phi'_*(y) \quad \text{for} \quad \xi_1 < x, y < z \quad \text{with} \quad \phi_c(x) = \phi_*(y).$$

This is easily seen, using equation (4.10), to imply

$$\phi_c(x) \leqslant \phi_*(x) \quad \text{for every} \quad \xi_1 \leqslant x \leqslant \xi_2.$$

Now, applying Lemma 5.3 of de Pablo & Vazquez [8] to any solution u to problem (3.1) with initial value φ_*, we obtain

$$u(z + ct, t) \geqslant \varphi_c(z) = \varphi_*(z) = u_*(z + c_* t, t).$$

We conclude by letting c tend to c_*.

Case $\sigma > 0$

In this case we cannot construct the desired subsolutions by shooting from any point $(X_0, 0)$, $0 < X_0 < 1$, since the curve of the zero slope in equation (4.8) does not tend to zero. Instead, we shoot from any point $(X_0, Y_*(X_0))$ in the trajectory. If $0 < \epsilon < X_0$, we get a trajectory Y_c going to ∞ for X tending to 0, and going down to $Y = 0$ for some $X_1 > X_0$. Observe that this method in the case $\sigma = 0$ will require us to prove that the trajectory Y_c does not go to the origin, which is done by constructing the trajectory Y_ϵ. The key point is that Y_* starts from $(0, c_*)$ if $\sigma > 0$ while for $\sigma = 0$ it starts from $(0, c_*/2)$.

The rest of the proof is the same. □

4.1 The curve $c = c_*(\sigma)$

We end this section by studying the function which gives the minimal velocity c_* in terms of the parameter σ.

If we put $x = \log y$, $t = \log \tau$, then looking for TWs involves looking for solutions $u(x,t) = g(y\tau^{-c})$. Thus, the velocity c appears as a similarity exponent. Since the value of c is not determined from physical or dimensional considerations, it is an example of what is known as an anomalous or *second-kind* exponent.

Recently, much attention has been paid to the existence of anomalous exponents in

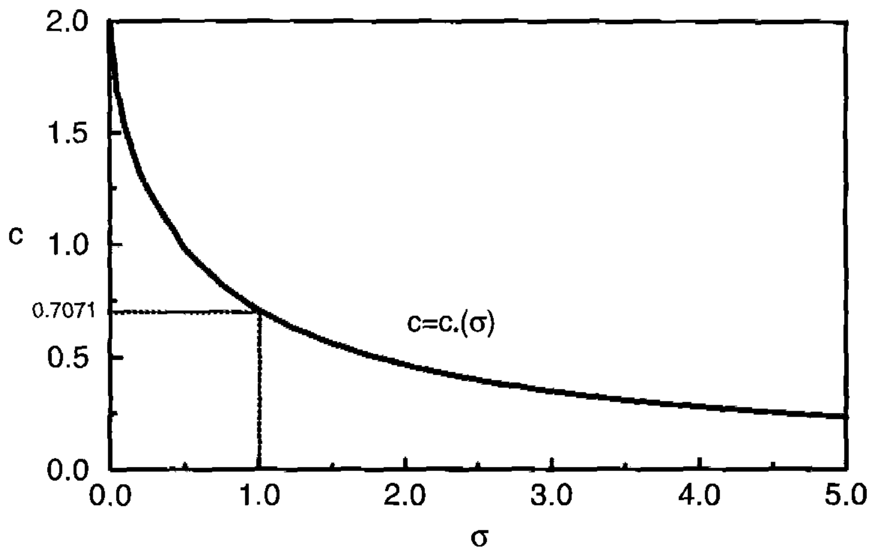

FIGURE 1. The curve $c = c_*(\sigma)$.

different contexts, and very different techniques (even formal approaches) have been used [16–19].

We follow here the method developed by Aronson & Vazquez [16] by applying the Implicit Function Theorem (IFT) to some function relating the velocity c_* with the parameter σ. The starting point will be the explicit solution for $\sigma = 1$:

$$\frac{1}{\sqrt{2}}(-\xi)_+ = \int_0^{\varphi(\xi)} \frac{s^{m-2}}{1-s}\,ds, \tag{4.13}$$

which is obtained by integrating the explicit trajectory $Y(X) = c(1-X)$ with $c = 1/\sqrt{2}$ in equation (4.4). For $m = 2$ and $p = 1$, this TW was obtained by Aronson [4]. Observe also that we obtain by this method a new proof of existence of TW.

Theorem 4.3 *The minimal velocity c_* is an analytic decreasing function of σ from* $[0, \infty)$ *onto* $(0, 2]$.

The curve $c = c_*(\sigma)$ obtained numerically is shown in Fig. 1.

Proof We shoot from the saddle points to obtain the connection for $\sigma \neq 1$. Since $\partial H/\partial c > 0$, it is obvious that, if a connection exists for $c = c_*$, the trajectories starting at $(1,0)$ and entering the region $D = (0,1) \times (0,\infty)$ will go to infinity as X tends to 0 if $c > c_*$, and to 0 if $0 < c < c_*$. On the other hand, the trajectories entering $(0,c)$ from D cross the line $X = 1$ at $Y > 0$ if $c > c_*$, and the axis $Y = 0$ at $0 < X < 1$ if $0 < c < c_*$.

Consider, then, for $c > 0$ and $\sigma \geqslant 0$ the trajectory Y_1 entering $(0,c)$ and the trajectory Y_2 leaving $(1,0)$, i.e. for some $0 < X_* < 1$ we have

$$\left.\begin{array}{ll} Y_1(\cdot, c, \sigma): [0, X_*] \to \mathbb{R}^+, & Y_1(0, c, \sigma) = c, \\ Y_2(\cdot, c, \sigma): [X_*, 1] \to \mathbb{R}^+, & Y_2(1, c, \sigma) = 0, \\ \partial Y_i(X, c, \sigma)/\partial X = H(X, Y_i, c, \sigma). & \end{array}\right\} \tag{4.14}$$

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It is clear that these two functions are analytic. We assume $Y_i(X_*, c, \sigma) > 0$. In fact for every $\sigma \geqslant 0$, there exists c_0 and X_* such that such assumption is true for $c \geqslant c_0$ (see equation (4.24)), though we do not need any information about X_* nor c_0.

Define now the (analytic) function l: $(0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$l(c, \sigma) = Y_1(X_*, c, \sigma) - Y_2(X_*, c, \sigma). \tag{4.15}$$

The existence of a connection for some c and σ is equivalent to the *matching condition*

$$l(c, \sigma) = 0. \tag{4.16}$$

The IFT allows us to define $c = c_*(\sigma)$ in a neighbourhood of any solution (c, σ) to equation (4.16) whenever the *transversality condition*

$$\frac{\partial l}{\partial c}(c_*(\sigma), \sigma) \neq 0 \tag{4.17}$$

holds. Moreover, we can obtain from equation (4.16) the derivative

$$c'_*(\sigma) = \frac{-\partial l/\partial \sigma}{\partial l/\partial c}(c_*(\sigma), \sigma). \tag{4.18}$$

We first prove the transversality condition. Defining

$$z_i(X) = \partial Y_i/\partial c(X, c, \sigma), \quad i = 1, 2, \tag{4.19}$$

we will show that $z_1(X_*) - z_2(X_*) \neq 0$. Differentiating equation (4.4) with respect to the parameter c, we get

$$\frac{dz_i}{dX} = \frac{\partial H}{\partial c} + \frac{\partial H}{\partial Y}\frac{\partial Y_i}{\partial c} = A_i z_i + B,$$

with

$$A_i(X) = \frac{-1}{X} + \frac{X^{\sigma-1}(1-X)}{Y_i^2}, \quad B(X) = \frac{1}{X}. \tag{4.20}$$

These are two linear ODEs explicitly solvable from the initial values

$$z_1(0) = 1, \quad z_2(1) = 0 \tag{4.21}$$

(see equation (4.14)). We have

$$z_1(X) = G_1(X)\int_0^X \frac{B(s)}{G_1(s)}ds, \quad z_2(X) = -G_2(X)\int_X^1 \frac{B(s)}{G_2(s)}ds \tag{4.22}$$

with $G_i(X) = \exp[\int_{1/2}^X A_i(s)\, ds]$. We immediately obtain $z_2 < 0 < z_1$, i.e.

$$\frac{\partial l}{\partial c}(c_*(\sigma), \sigma) = z_1(X_*) - z_2(X_*) > 0. \tag{4.23}$$

Since $(1/\sqrt{2}, 1)$ is a solution to equation (4.16), we get the existence of the analytic function $c_*(\sigma)$ defined in a neighbourhood of $\sigma = 1$. Moreover, the transversality condition is uniform for $\sigma > 0$, so we can extend the range of existence to a maximal open (in $[0, \infty)$) interval. Now take a sequence $(n \to \infty)$ of solutions Y_n to equation (4.4) with parameters $(c_n, \sigma_n) \to (c_\infty, \sigma_\infty)$, and satisfying $Y_n(0) = c_n$, $Y_n(1) = 0$. As long as $0 \leqslant \sigma_\infty < \infty$ and $0 < c_\infty \leqslant 2$, there exists a function $Y_\infty = \lim_{n\to\infty} Y_n$ which is again a solution to our problem. Then the maximal interval of existence is closed in $[0, \infty)$, and thus it is the whole $[0, \infty)$. Global existence then depends on the following estimate:

Lemma 4.4 *The minimal velocity satisfies, for* $\sigma > 0$,

$$c_*^2(\sigma) < \frac{4\sigma^\sigma}{(\sigma+1)^{\sigma+1}} \tag{4.24}$$

$$\frac{2}{(\sigma+1)(\sigma+2)} \leqslant c_*^2(\sigma) \leqslant \frac{2}{\sigma(\sigma+1)}. \tag{4.25}$$

Proof The first estimate is obtained by de Pablo & Vazquez [8] by studying the behaviour of the graph

$$(c-Y)\,Y = X^\sigma(1-X)$$

where $dY/dX = 0$. On the other hand, from equation (4.4) we have

$$\left.\begin{aligned} Y\frac{dY}{dX} &\geqslant -X^{\sigma-1}(1-X) \\ \frac{d(XY^2)}{dX} &\leqslant c^2 - 2X^\sigma(1-X). \end{aligned}\right\} \tag{4.26}$$

Now integrate these inequalities along the trajectory $Y = Y(X)$ for $c = c_*$, from $X = 0$, $Y = c_*$ to $X = 1$, $Y = 0$, to get equation (4.25). □

Observe that these estimates imply $0 < c_*(\sigma) < 2$ for $0 < \sigma < \infty$ and $\lim_{\sigma\to\infty} c_*(\sigma) = 0$.

The fact that $c_*(\sigma)$ is a decreasing function follows now from the study of the sign of $\partial l/\partial\sigma$. Analogously to equation (4.19) we define $\theta_i(X) = \partial Y_i/\partial\sigma(X, c, \sigma)$, $i = 1, 2$. We have

$$\frac{d\theta_i}{dX} = \alpha_i\,\theta_i + \beta_i,$$

with

$$\alpha_i(X) = A_i(X), \quad \beta_i(X) = \frac{-X^{\sigma-1}(1-X)\log X}{Y_i(X)} \tag{4.27}$$

and

$$\theta_1(0) = \theta_2(1) = 0. \tag{4.28}$$

As before, we get

$$\theta_1(X) = G_1(X)\int_0^X \frac{\beta_1(s)}{G_1(s)}ds, \quad \theta_2(X) = -G_2(X)\int_X^1 \frac{\beta_2(s)}{G_2,(s)}ds. \tag{4.29}$$

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Again, it is immediate to see that $\theta_2 < 0 < \theta_1$, i.e.

$$\frac{\partial l}{\partial \sigma}(c_*(\sigma), \sigma) = \theta_1(X_*) - \theta_2(X_*) > 0,$$

and the theorem is proved. □

The method also allows us to calculate the derivative at $\sigma = 1$.

Lemma 4.5

$$\frac{dc_*}{d\sigma}(1) = \frac{-13\sqrt{2}}{48}. \tag{4.30}$$

Proof The computation of the derivative (4.18) at $\sigma = 1$ follows by substituting the explicit functions in equations (4.22) and (4.29) given by the explicit solution (4.13). In fact we have, for $(c, \sigma) = (1/\sqrt{2}, 1)$:

$$Y_i(X) = \frac{1}{\sqrt{2}}(1-X), \quad A_i(X) = \frac{-1}{X} + \frac{2}{1-X},$$

$$G_i(X) = \frac{1}{X(1-X)^2}, \quad \beta_i(X) = -\sqrt{2}\log X,$$

and thus

$$(z_1 - z_2)(X) = \frac{1}{X(1-X)^2}\int_0^1 (1-s)^2\,ds = \frac{1}{3X(1-X)^2},$$

$$(\theta_1 - \theta_2)(X) = \frac{-\sqrt{2}}{X(1-X)^2}\int_0^1 s(1-s)^2 \log s\,ds = \frac{13\sqrt{2}}{144X(1-X)^2}.$$

This gives the value of equation (4.30), since $c_*'(\sigma) = (\theta_2 - \theta_1)/(z_1 - z_2)$. □

5 Asymptotic behaviour

In this section we investigate how the minimal solutions to problem (3.1) behave for large times. The main result is that they propagate with a certain limit velocity, precisely that of the minimal TW. We recall that in the pure power case [8, 9], this occurs only for the critical exponent $\sigma = 0$, while for $\sigma > 0$, the propagation is superlinear. This is explained by the fact that in the equation treated in those works, there exist TWs only when $\sigma = 0$.

We consider problem (3.1) with initial values u_0 vanishing for $x \geqslant 0$. Let u be the minimal solution to that problem, and $x = s(t)$ be its interface. A possible left-interface, when u_0 vanishes for large negative values, is treated in the same way.

We begin with the simpler result,

Theorem 5.1 *The interface of the minimal solution satisfies*

$$\lim_{t\to\infty} \frac{s(t)}{t} = c_*. \tag{5.1}$$

Proof

Case $\sigma = 0$

This case comes directly from the power-law equation.

For the lower estimate we recall equation (3.3). To get the upper estimate, we use a linear (in pressure) front with velocity 2, which is a supersolution to our equation.

Case $\sigma > 0$

If the initial value is not too close to one for $x \to -\infty$, i.e. if the TW with velocity c_* can be put above u_0 (see equation (4.6)), direct comparison implies, for some $a > 0$,

$$u(x,t) \leqslant \varphi_*(x-a-c_* t),$$

i.e. $s(t) \leqslant a + c_* t$.

If this is not the case, we consider, for each $0 < \delta < 1$, the rescaled function

$$w(x,t) = \delta^{\frac{-2}{m-p}} \varphi_*(\delta(x-a) - c_* \delta^{\frac{2(1-p)}{m-p}} t), \tag{5.2}$$

which satisfies the equation

$$w_t = (w^m)_{xx} + w^p(1-\delta^{\frac{2}{m-p}} w),$$

and thus it is a supersolution to our equation. Therefore, if we choose $a > 0$ such that $w(0,0) \geqslant 1$, we get by comparison

$$s(t) \leqslant a + \delta^{\frac{-\sigma}{m-p}} c_* t,$$

and

$$\lim_{t\to\infty} \frac{s(t)}{t} \leqslant \delta^{\frac{-\sigma}{m-p}} c_* \quad \text{for every} \quad \delta < 1.$$

To get a lower estimate, instead of insisting that a TW be put below u_0, we use a local TW with velocity $c < c_*$, as constructed in the proof of Theorem 4.2, and perform local comparison. In that way, we can also consider compactly supported initial values.

We first recall that in the proof of Theorem 4.2 we construct, for each $c < c_*$ and $\epsilon > 0$, a local TW $w(x,t) = \varphi_\epsilon(x-ct)$ subsolution to our equation in a region $\{\xi_1 + ct < x < \xi_2 + ct\}$, and satisfying

$$\varphi_\epsilon(\xi) \leqslant X_1 \quad \text{for} \quad \xi_1 < \xi < \xi_2.$$

Moreover, X_1 is not small for $\epsilon \to 0$.

We then use Lemmas 2.2 and 2.3 to get that, for some $T > 0$, the solution satisfies

$$u(x,T) \geqslant X_1 \quad \text{for} \quad \xi_1 < x < \xi_2.$$

Comparison now implies

$$u(x+ct, t+T) \geqslant \varphi_\epsilon(x),$$

i.e.

$$s(t) \geqslant \xi_2 + c(t-T) \quad \text{for every} \quad c < c_*. \qquad \square$$

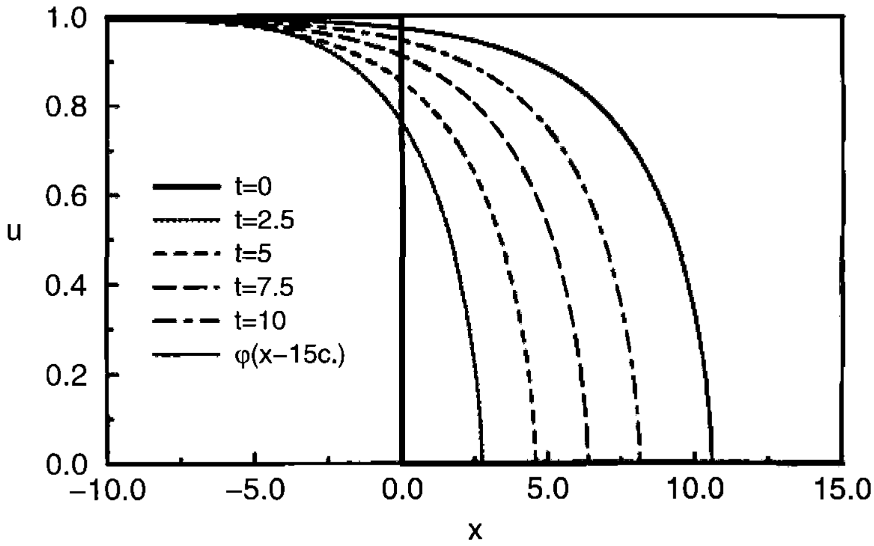

FIGURE 2. The minimal solution for $u_0(x) = 1$ if $x < 0$, $u_0(x) = 0$ if $x \geqslant 0$, $\sigma = 1$, $m = 2.5$, compared with the explicit TW.

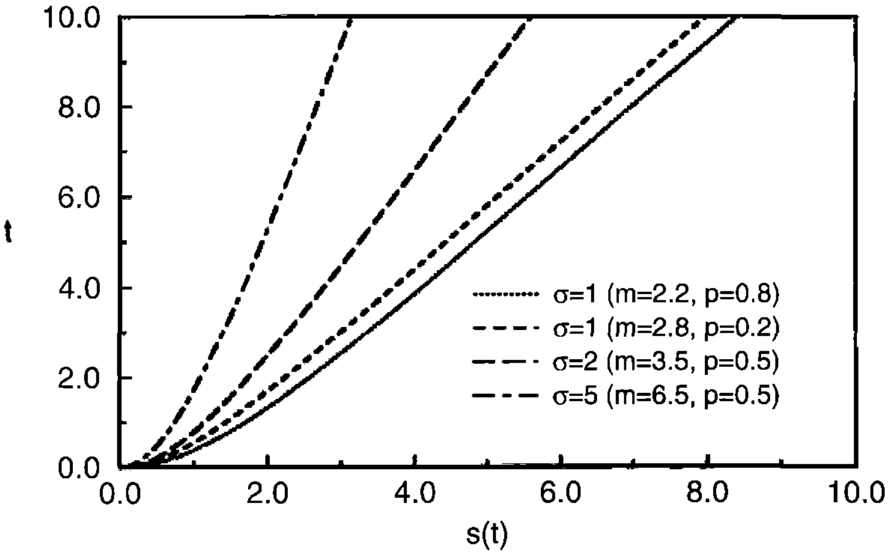

FIGURE 3. The interface of the minimal solution for the same data as in Fig. 2, and $\sigma = 1, 2, 5$.

In Fig. 2 we show the numerical minimal solution of problem (3.1) with $\sigma = 1$, for a Heaviside function initial condition, plotted at regular time intervals. It is compared with the explicit TW. Figure 3 shows the numerically calculated interface for different values of σ. We also give the calculated final speeds (Table 1).

More interesting is the following result, in the spirit of Aronson & Weinberger [5].

Theorem 5.2 *For each* $x \in \mathbb{R}$, *we have*

$$\lim_{t\to\infty} u(x+ct, t) = \begin{cases} 0 & \text{if} \quad c > c_* \\ 1 & \text{if} \quad c < c_*. \end{cases} \tag{5.3}$$

Proof The behaviour for $c > c_*$ is implied by the previous theorem. The proof that the translated solution with a velocity smaller than the minimal velocity is not only positive,

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Table 1. *The final velocity for the interfaces of Fig. 3, compared with the expected velocity, as calculated for Fig. 1*

σ	m	Calculated velocity	Expected velocity
1	2.2	0.711	0.707
1	2.8	0.708	0.707
2	3.5	0.467	0.463
5	6.5	0.236	0.232

but in fact tends to one, is again easier if the minimal TW can be put from below. Assuming that this is the case, we have, recalling Theorem 4.2,

$$u(x,t) \geqslant \varphi_*(x-c_*\,t),$$

i.e.

$$\lim_{t\to\infty} u(x+ct,t) \geqslant \lim_{t\to\infty} \varphi_*(x-(c_*-c)\,t) = \lim_{\xi\to-\infty} \varphi_*(\xi) = 1.$$

For the general case, let $x_0 \in \mathbb{R}$ be fixed, $a > 0$ any number, and $\delta > 0$ small. We know that there exists a time $T > 0$ such that

$$u(x,T) \geqslant 1-\delta \quad \text{for} \quad x_0 - a < x < x_0 + a.$$

Consider, then, a local TW φ with velocity $\bar{c}$, $(c < \bar{c} < c_*)$, satisfying

$$0 \leqslant \varphi(x) \leqslant \varphi(x_0) = 1-\delta, \quad \text{for} \quad x_0 - a < x < x_0 + a.$$

By comparison, we have

$$u(x_0 + \bar{c}(t-T), t) \geqslant 1-\delta \quad \text{for} \quad t \geqslant T.$$

We finally take $T_1 = \bar{c}/(\bar{c}-c)\,T$ to get

$$u(x_0+ct,t) \geqslant 1-\delta \quad \text{for} \quad t \geqslant T_1.$$

Observe that T depends upon x_0 if $x_0 \notin \operatorname{supp}(u_0)$. Therefore, if $u_0(x) > 0$ for $x < 0$, the convergence is uniform in sets of the form $(-\infty, a]$. □

6 Conclusion

We have characterized the uniqueness of solutions for the Cauchy problem associated to a nonlinear degenerate diffusion equation with a non-Lipschitz reaction term of Fisher type. We have also studied the properties of travelling wave solutions for the same problem, and characterized the minimal velocity for which these travelling waves exist as an analytic function of the parameters of the problem, in the context of anomalous exponents. The method gives a new proof of existence of such travelling waves. We have used these travelling waves to describe the asymptotic behaviour of the minimal solutions, as well as their interfaces, proving that they propagate with the minimal velocity.

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