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ON VALUE GROUPS AND RESIDUE FIELDS OF SOME VALUED FUNCTION FIELDS*

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Let $K=K_0(x,y)$ be a function field of transcendence degree one over a field K_0 with x,y satisfying $y^2=F(x)$, $F(x)$ being any polynomial over K_0. Let v_0 be a valuation of K_0 having a residue field k_0 and v be a prolongation of v_0 to K with residue field k. In the present paper, it is proved that if $G_0 \subseteq G$ are the value groups of v_0 and v, then either G/G_0 is a torsion group or there exists an (explicitly constructible) subgroup G_1 of G containing G_0 with $[G_1:G_0]<\infty$ together with an element γ of G such that G is the direct sum of G_1 and the cyclic group $\mathbb{Z}_\gamma$. As regards the residue fields, a method of explicitly determining k has been described in case k/k_0 is a non-algebraic extension and char $k_0 \neq 2$. The description leads to an inequality relating the genus of K/K_0 with that of k/k_0: this inequality is slightly stronger than the one implied by the well-known genus inequality (cf. [*Manuscripta Math.* **65** (1989), 357–376], [*Manuscripta Math.* **58** (1987), 179–214]).

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0. Introduction

Let v_0 be a valuation of a field K_0 and v be a prolongation of v_0 to a simple transcendental extension K of K_0. Let $G_0 \subseteq G$ and $k_0 \subseteq k$ be the value groups and residue fields of v_0 and v respectively. In 1983 Ohm [**12**] proved a conjecture made by Nagata that either k is an algebraic extension of k_0 or it is a simple transcendental extension of a finite extension of k_0. Analogously for value groups, Khanduja [**7**] has proved that either G/G_0 is a torsion group or there exists an explicitly constructible subgroup G_1 of G containing G_0 with $[G_1:G_0]<\infty$ such that G is the direct sum of G_1 and an infinite cyclic group. In this paper, we prove similar results for value groups and residue fields of $(K,v)/(K_0,v_0)$ when $K=K_0(x,y)$ is a function field of transcendence degree 1 over K_0 with x,y satisfying a relation $y^2=F(x)$, $F(x)$ being any polynomial over K_0. In the case that the extension k/k_0 is non-algebraic, we describe a method to determine explicitly the residue field k of (K,v) and thereby establish an inequality relating the genus of K/K_0 with that of k/k_0; in certain cases this relation happens to be slightly stronger than the one implied by the genus inequality of Matignon (cf. [**6**, Theorem 3.1], [**10**, p. 201, Theorem 4]) which was obtained by entirely different techniques.

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1. Notation and statements of results

We shall prove:

Theorem 1.1. *Let v_0 be a valuation of a field K_0 and v be an extension of v_0 to an overfield $K=K_0(x,y)$ of transcendence degree one over K_0, where $y^2=F(x)$ is in $K_0[x]$. If $G_0 \subseteq G$ are the value groups of v_0 and v, then either G/G_0 is a torsion group or there exists a subgroup G_1 of G containing G_0 with $[G_1;G_0]<\infty$ and an element γ of G (both G_1 and γ explicitly constructible) such that G is the direct sum of G_1 and the cyclic group $\mathbb{Z}\gamma$ generated by γ.*

Notation. For a finite extension $(L,w)/(L_0,w_0)$ of valued fields the henselian defect of the extension is defined to be $[L^h:L_0^h]/ef$ where "h" stands for henselisation and e, f denote the ramification index and the residual degree of w/w_0. We shall denote it by $\mathrm{def}^h((L,w)/(L_0,w_0))$ or by $\mathrm{def}^h(L/L_0)$ when the underlying valuations are clear.

Throughout the paper (K_0,v_0), (K,v) and $G_0 \subseteq G$ will be as in Theorem 1.1 and $k_0 \subseteq k$ will denote the residue fields of v_0 and v, respectively. For any ξ in the valuation ring of v, ξ^* will stand for its v-residue, i.e., the image of ξ in the residue field of v. In the remaining part of this section, it is assumed that the field k_0 is of characteristic (to be abbreviated as char)$\neq 2$ and that k is not algebraic over k_0. We shall denote by Δ the algebraic closure of k_0 in k and by I, R the numbers $[G:G_0]$ and $[\Delta:k_0]$ respectively.

Let ξ be an element of the valuation ring of v such that ξ^* is transcendental (to be written as *tr.*) over k_0. We shall denote by D (more precisely by $D(v/v_0)$ the henselian defect of the finite extension $(K,v)/(K_0(\xi),v_0^\xi)$ (v_0^ξ denotes the restriction of v to $K_0(\xi)$); in view of the Independence Theorem [13, p. 299], D is independent of the choice of the residually *tr.* element ξ.

With the above notation, we shall prove:

Theorem 1.2. *Let v_0 be a valuation of a field K_0 with residue field k_0 of char$\neq 2$ and let v be an extension of v_0 to an overfield $K=K_0(x,\sqrt{F(x)})$, $F(x)$ being a non-constant polynomial in an indeterminate x over K_0. Assume that the residue field k of v is not algebraic over k_0. Then one can determine (by an explicit algorithm) an element u transcendental over k_0 and a polynomial $A(u)$ over the algebraic closure Δ of k_0 in k with $\deg A(u) \leqq \delta + (\deg F(x))/IRD$ such that $k=\Delta(u,\sqrt{A(u)})$ where $\delta=0$ or 1; indeed δ can be chosen to be 0 when $I=1$.*

Throughout the paper, when we refer to the genus of a function field, we shall mean the genus over the exact constant field as in [1] or [4] and shall denote the genus of L by g_L.

The following theorem will be deduced from Theorem 1.2.

Theorem 1.3. *Let $K_0 \subseteq K$, v_0, v and $k_0 \subseteq k$ be as in Theorem 1.2. Assume that K_0 is algebraically closed in K. Then*

(i) $IRD(g_k-1) \leqq g_K - IRD + 1$

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(ii) $I=R=D=1$ *implies that* $g_k \leqq g_K$.

If K_0, K etc. are as above with K_0 an algebraically closed field, then G_0 is a divisible group and k_0 is an algebraically closed field, so that $I=R=1$; in this case $D=1$ by the Stability Theorem (see [**13**, Theorem 2.1]). Thus the following corollary is an immediate consequence of the 2nd assertion of Theorem 1.3.

Corollary 1.4. *Let* K_0, K, v_0, v *etc. be as in the above theorem. Assume further that* K_0 *is an algebraically closed field. Then* $g_k \leqq g_K$.

Remark 1.5. The relation between g_k and g_K given by the well-known genus inequality (cf. [**6**, Theorem 3.1.], [**10**, p. 201, Theorem 4]) is

$$IRD(g_k - 1) \leqq g_K - 1. \tag{1}$$

If $IRD \geqq 2$, then clearly Theorem 1.3(i) implies (1). In view of the fact that the henselian defect is always a non-negative integral power of the characteristic of the residue field (see [**2**, p. 180, Prop. 15]), we conclude that $IRD < 2$, if and only if, $I=R=D=1$, in which case (1) follows from assertion (ii) of Theorem 1.3.

Remark 1.6. We shall give examples in the last section to show that the bound on g_k given by Theorem 1.3 is indeed the best possible and stronger than the one given by (1). In fact $(K,v)/(K_0,v_0)$ will be constructed so that $g_k = [(g_K+1)/IRD] < [(g_K-1)/IRD]+1$; here $[r]$ stands for the largest integer not exceeding r.

2. Proof of Theorem 1.1

Assume that G/G_0 is not a torsion group. Let H denote the value group of the valuation v restricted to the subfield $K_0(x)$ of K. Then $[G:H] \leqq [K:K_0(x)] \leqq 2$, and H/G_0 is not a torsion group. It is known (cf. [**7**, Corollary 1.2.] or [**8**, Remark 3.2]) that there exists an (explicitly constructible) subgroup H_1 of H containing G_0 with $[H_1:G_0] < \infty$ and an element of θ of H such that H is the direct sum of H_1 and $\mathbb{Z}\theta$. So we need to prove the theorem when $[G:H]=2$.

Two cases are distinguished.

If $(\lambda+\theta)/2 = \theta_1$(say) belongs to G for some λ in H, then

$$H = H_1 \oplus \mathbb{Z}\theta \subsetneqq H_1 \oplus \mathbb{Z}\theta_1 \subseteq G$$

and hence $G = H_1 \oplus \mathbb{Z}\theta_1$ in this case.

Suppose that $(h_1+\theta)/2 \notin G$ for any h_1 in H_1. It will be shown that $G = (G \cap \tfrac{1}{2}H_1) \oplus \mathbb{Z}\theta$ in this case. Let g be any element of G. Since $2g \in H$, we can write

$$g = \frac{h_1}{2} + \frac{n\theta}{2}$$

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for some h_1 in H_1 and some integer n. The claim is that n must be even. If n were odd, then on writing g as

$$g=\frac{h_1+\theta}{2}+\frac{n-1}{2}\theta,$$

we derive that $(h_1+\theta)/2\in G$, contrary to the supposition. This proves the claim and the theorem follows.

3. Proof of Theorem 1.2

We first introduce some notation and state a couple of lemmas.

Let v_0 be a valuation of a field K_0 with value group G_0 and v' be a prolongation of v_0 to a simple *tr.* extension $K_0(x)$. (Later on we shall take v' to be the restriction of v to the subfield $K_0(x)$ of K). For any ξ in the valuation ring of v', we denote by ξ^* its v'-residue. It is assumed that the residue field k' of v' is not algebraic over the residue field k_0 of v_0. For such an extension v'/v_0, we define a number E' (more precisely written as $E'(v'/v_0)$) by

$$E'=\min\{[K_0(x):K_0(\xi)]\mid \xi\in K_0(x), v'(\xi)\geqq 0, \xi^* \textit{ tr. over } k_0\}.$$

Fix an element ξ of the valuation ring of v' with ξ^* *tr.* over k_0. We shall denote by D' the henselian defect of the extension $(K_0(x),v')/(K_0(\xi),v_0^\xi)$; in view of the Independence Theorem [**13**, p. 299], D' is independent of the choice of ξ, we shall denote by Δ' the algebraic closure of k_0 in k' and by G' the value group v'. It may be recalled that, by the Ruled Residue Theorem [**12**], k' is a simple *tr.* extension of Δ'. The following inequality which is due to Matignon and Ohm [**11**, p. 353, Corollary 2.2.3] is quoted for future reference:

$$E'\geqq[G':G_0][\Delta':k_0]D'. \tag{2}$$

For the proof of the following lemma see [**9**, Lemma 2.2].

Lemma 3.1. *Let v_0, v', G', k' and E' be as above. Then to any $\lambda\in G'$, there corresponds a polynomial $h(x)\in K_0[x]$ of degree $\leqq E'-1$ such that $\lambda=v'(h(x))$.*

After introducing some notation we recall a few results proved in [**8**]. Let v_0, v', G', k' and E' be as before. Fix an algebraic closure $\bar{K}_0$ of K_0 and an extension v'' of v' to $\bar{K}_0(x)$. We denote by $\bar{v}_0$ the restriction of v'' to $\bar{K}_0$. The extension k'/k_0 is given to be non-algebraic, therefore so is $k''/\bar{k}$, where $\bar{k}\subseteq k''$ are the residue fields of $\bar{v}_0$, v'' respectively. Arguing exactly as in [**14**, p. 205, §2.5], it can be easily proved that there exist α and a in $\bar{K}_0$ such that the v''-residue $((x-\alpha)/a)^*$ of $(x-\alpha)/a$ is *tr.* over k_0. If $v''(x-\alpha)=\bar{v}_0(a)$ is denoted by μ then μ is torsion mod G_0, i.e., $m\mu\in G_0$ for some positive integer m. As in

[3, §10.1, Proposition 2], it can be easily seen that for any polynomial $f(x)=\sum_i c_i(x-\alpha)^i$ over $\bar{K}_0$,

$$v''(f(x))=\min_i(\bar{v}_0(c_i)+i\mu),$$

since the assumption $v''(f(x))>\min_i(\bar{v}_0(c_i)+i\mu)$, would lead to $((x-\alpha)/a)^*$ being algebraic over $\bar{k}_0$. This also shows that $v''(f(x))$ is torsion mod G_0 for $f(x)\in\bar{K}_0[x]$.

Define a subset D_0 of $\bar{K}_0$ by

$$D_0=\{\gamma\in\bar{K}_0:\bar{v}_0(\gamma-\alpha)\geqq\mu\}.$$

Fix an element β of D_0 such that $[K_0(\beta):K_0]\leqq[K_0(\gamma):K_0]$ for all γ in D_0. We shall denote by $P(x)$ the minimal polynomial of β over K_0 of degree n (say), by θ the element $v'(P(x))$ of G' and by G_1 the value group of the valuation $\bar{v}_0$ restricted to $K_0(\beta)$. As shown above θ is torsion mod G_0; let s be the smallest positive integer such that $s\theta\in G_1$. It is clear from the proof of Theorem 1.3 of [**8**] that

$$E'=sn=s\deg P(x).$$

In view of the choice of β any polynomial over K_0 having degree less than n has no root in D_0. So by assertion (ii) of Lemma 2.1 of [**8**] for such a polynomial $g(x)$, one has

$$v'(g(x))=\bar{v}_0(g(\beta)).$$

We now prove:

Lemma 3.2. *Let v_0, v', E', k' and Δ' be as above and let $\eta=f(x)/g(x)$ be a unit of the valuation ring of v' with $f(x)$, $g(x)$ in $K_0[x]$ and $\deg g(x)\leqq 2E'-1$. Then one can determine (by an explicit algorithm) a generator t of the simple tr. extension k'/Δ' together with polynomials $B(t)$, $C(t)$ over Δ' satisfying $\deg B(t)\leqq(\deg f(x))/E'$, $\deg C(t)\leqq 1$ such that the v'-residue η^* of η is given by $\eta^*=B(t)/C(t)$.*

Proof. Let v'', $\bar{v}_0$, α, β, $P(x)$ n,θ and s be as above, so that $E'=sn$. Let $q(x)\in K_0[x]$ be a polynominal of degree less than n such that $\bar{v}_0(q(\beta))=s\theta$. By [**8**, Theorem 1.3(i)] the v'-residue of $P(x)^s/q(x)$ is a generator of the simple *tr.* extension k'/Δ' and Δ' equals the residue field of the valuation $\bar{v}_0$ restricted to $K_0(\beta)$; we shall denote this generator of k'/Δ' by t.

Observe that any polynomial $h(x)\in K_0[x]$ can be uniquely written as a finite sum

$$h(x)=\sum_{i=0}^{r}h_i(x)P(x)^i$$

where, for $0\leqq i\leqq r$, the polynomial $h_i(x)\in K_0[x]$ is either 0 or of degree less than that of

$P(x)$ and $h_r(x) \neq 0$. This will be referred to as the canonical representation of $h(x)$ with respect to $P(x)$.

By hypothesis $\deg g(x) \leqq 2E' - 1$, so the index i in the canonical representation of $g(x)$ with respect to $P(x)$ cannot vary beyond $2s-1$. Arguing similarly for $f(x)$, we can rewrite the canonical representations of $f(x)$ and $g(x)$ (after adding zero terms, if necessary) as

$$f(x) = \sum_{i=0}^{m} f_i(x) P(x)^i, g(x) = \sum_{i=0}^{2s-1} g_i(x) P(x)^i.$$

where the integer m does not exceed $1/n \deg f(x)$.

It is given that $v'(f(x)) = v'(g(x)) = \lambda$(say). In view of [**8**, Lemma 2.1(ii), (iii)], we have

$$\lambda = \min_i (\bar{v}_0(f_i(\beta)) + i\theta) = \min_i (\bar{v}_0(g_i(\beta)) + i\theta).$$

Let j be the smallest non-negative index such that at least one of the minimum of the above equation is attained at j, i.e., λ is either $\bar{v}_0(f_j(\beta)) + j\theta$ or $\bar{v}_0(g_j(\beta)) + j\theta$ and j is the smallest with this property. Observe that $0 \leqq j \leqq 2s-1$. Since s is the smallest positive integer for which $s\theta \in \bar{v}_0(K_0(\beta))$, it follows that

$$\lambda \leqq \bar{v}_0(f_j(\beta)) + j\theta \text{ for } 0 \leqq i \leqq m \text{ with strict inequality if } i \not\equiv j \bmod s \tag{3}$$

and

$$\lambda \leqq \bar{v}_0(g_j(\beta)) + j\theta \text{ for } 0 \leqq i \leqq 2s-1 \text{ with strict inequality if } i \not\equiv j \bmod s. \tag{4}$$

Define $h(x) = f_j(x) P(x)^j$, when $\lambda = \bar{v}_0(f_j(\beta)) + j\theta$ and $h(x) = g_j(x) P(x)^j$, if $\lambda > \bar{v}_0(f_j(\beta)) + j\theta$; in the latter case λ must equal $\bar{v}_0(g_j(\beta)) + j\theta$ by choice of j. We write $\eta = \eta_1/\eta_2$ where $\eta_1 = f(x)/h(x)$, $\eta_2 = g(x)/h(x)$. Observe that $v'(\eta_1) = v'(\eta_2) = 0$. The lemma is proved as soon as it is shown that η_2^* is a polynomial in t over Δ' of degree $\leqq 1$ and that η_1^* is a polynomial over Δ' of degree $\leqq m/s \leqq (\deg f)/sn = (\deg f)/E'$. For this distinguish two cases.

Consider first the case when $h(x) = f_j(x) P(x)^j$. Keeping in view (4) and using the fact (proved in [**8**, Lemma 2.1(ii)]) that for any non-zero polynomial $R(x) \in K_0[x]$ of degree less than n, the v''-residue of $R(x)/R(\beta)$ is 1, it can be easily checked that in the case $j < s, \eta_2^* = (g_j(\beta)/f_j(\beta))^* + t(q(\beta) g_{j+s}(\beta)/f_j(\beta))^*$ and $\eta_2^* = (g_j(\beta)/f_j(\beta))^*$, otherwise. Arguing similarly and using (3), it can be easily seen that η_1^* is a polynomial in t over Δ' of degree $\leqq m/s$. This completes the proof of the lemma in the first case.

The proof in the second case, i.e., when $h(x) = g_j(x) P(x)^j$ is similar and is omitted.

Remark 3.3. Let v', k', $\eta = f(x)/g(x)$ be as in Lemma 3.2. If we further assume that $g(x)$ is a constant polynomial (in fact if $\deg g(x) \leqq n-1$ then it is clear from the proof of the above lemma that η^* will be a polynomial in t over Δ' of degree $\leqq (\deg f(x))/E'$.

The following lemma (whose proof is omitted) is an immediate consequence of

Theorem 17.17 and Corollary 16.6 of [5]. A simple proof of this lemma which was suggested to the author by Professor A. Wadsworth is given in [9, Lemma 2.4].

Lemma 3.4. *Let $L=L'(\sqrt{\eta})$ be a quadratic extension of a field L of char $\neq 2$, $\eta \in L'$. Let w' be a valuation of L' having $w'(\eta)=0$ such that the residue field k' of w' has char $\neq 2$. Suppose that w' can be uniquely extended to a valuation w of L, then the w-residue of $\sqrt{\eta}$ is not in k'.*

Proof of Theorem 1.2. We write $K=K_0(x,y)$, where $y^2=F(x)\in K_0[x]\backslash K_0$. We denote by v' the valuation v restricted to $K_0(x)$ and by k', G' the residue field and the value group of v'. Then $[k:k']\leqq[K:K_0(x)]\leqq 2$, and k'/k_0 is a non-algebraic extension as k/k_0 is given to be so. By the Ruled Residue Theorem [12], k' is a simple *tr.* extension of a finite extension Δ' of k_0. Throughout the proof, t will stand for the particular generator of k'/Δ' described in the opening lines of the proof of Lemma 3.2. If $k=k'$, the theorem needs no proof. From now on, it is assumed that $[k:k']=2$ and that $\Delta'=\Delta$, for $\Delta'\subsetneqq\Delta$ yields $k=\Delta(t)$.

Since

$$[K:K_0(x)]=[k:k'] \tag{5}$$

it follows from the fundamental inequality [3, §8.3, Theorem 1(b)] that the value group of v is G'; in particular $v(y)\in G'$. By Lemma 3.1 we can choose a non-zero polynomial $h(x)\in K_0[x]$ of degree $<E'=E'(v'/v_0)$ such that $v(y)=v'(h(x))$; in the case $G=G_0$, we choose $h(x)$ of degree 0. Set

$$z=y/h(x), \eta=F(x)/h(x)^2.$$

Then $z^2=\eta$ and $v'(\eta)=0$. In view of (5) and the fundamental inequality [3, §8.3, Theorem 1(b)], v is the only extension to $K=K_0(x,z)$ of the valuation v' defined on $K_0(x)$. It follows from Lemma 3.4 applied to the extension $K/K_0(x)$ that $z^*=\sqrt{\eta^*}$ is not in k'. Keeping in view the assumptions $[k:k']=2$ and $\Delta=\Delta'$, it is now clear that

$$k=k'(\sqrt{\eta^*})=\Delta(t,\sqrt{\eta^*}).$$

Recall that $\eta=F(x)/h(x)^2$, where $\deg h(x)^2\leqq 2E'-2$; in fact $\deg h(x)^2=0$ if $G=G_0$. By Lemma 3.2, $\eta^*=B(t)/C(t)$ with $B(t)$, $C(t)$ in $\Delta[t]$ satisfying $\deg B(t)\leqq(\deg F)/E'$ and $\deg C(t)\leqq 1$. Further by Remark 3.3, the polynomial $C(t)$ may be chosen to be of degree 0 when $G=G_0$.

Let us assume the inequality $E'\geqq IRD$ to be proved below.

If $\deg C(t)=1$, on taking $u=C(t)$ and writing the polynomial $B(t)$ as $B_1(u)$, we see that

$$k=\Delta(u,\sqrt{B_1(u)/u})=\Delta(u,\sqrt{uB(u)})$$

as desired, for $\deg B_1(u)=\deg B(t)\leqq(\deg F)/E'\leqq(\deg F)/IRD$.

In case $\deg C(t)=0$, say $C(t)=C\in\Delta$, then the theorem is proved on taking $u=t$ and $A(u)=B(t)/C$.

It only remains to verify the inequality $E'\geqq IRD$ with the assumptions $\Delta=\Delta'$ and $[K:K_0(x)]=[k:k']$. The latter implies that $G=G'$ and that the henselian defect of the extension $(K,v)/(K_0(x),v')$ is 1. Fix any element ξ of $K_0(x)$ with $v'(\xi)=0$ and ξ^* *tr.* over k_0. Then as remarked in the first section, $D=\operatorname{def}^h(K/K_0(\xi))$. Since the henselian defect is multiplicative, it follows that

$$D=\operatorname{def}^h(K/K_0(x))\operatorname{def}^h(K_0(x)/K_0(\xi))=\operatorname{def}^h(K_0(x)/K_0(\xi)).$$

Thus D equals the number D' defined in the beginning of the third section and the inequality (2) quoted there can be rewritten as $E'\geqq IRD$, as $G'=G$ and $\Delta'=\Delta$.

4. Proof of Theorem 1.3

The following lemma is probably known; we merely give reference of the results leading to its proof.

Lemma 4.1. *Let $L=L_0(x,\sqrt{f(x)})$ be an extension of a field L_0 of char$\neq 2$, where x is transcendental over L_0 and $f(x)$ is a non-constant polynomial over L_0. Suppose that L_0 is algebraically closed in L. Then*

(i) *there exist x_0, y_0 in L such that $L=L_0(x_0,y_0)$ where $y_0^2=h(x_0)$ is a polynomial in x_0 over L_0* of degree $\leqq 2g_L+2$;

(ii) $g_L\leqq(\deg f(x))-1/2$.

Proof. If $g_L=0$, then keeping in view that char $L\neq 2$, (i) is immediate from [**1**, Chapter 16, §4, Theorem 6]. If $g_L=1$, then since L has a divisor of degree $\leqq 2$, assertion (i) follows from cases 1 and 2 of [**1**, Chapter 16, §5]. In case $g_L\geqq 2$, L being a quadratic extension of $L_0(x)$, has a desired set of generators over L_0 in view of case 1 of [1, Chapter 16, §7, Theorem 14].

If $\deg f(x)\leqq 2$, then $g_L=0$ by a well-known result referred to above and hence (ii) holds in this case. Suppose (ii) is false, so that $\deg f(x)\geqq 3$ and $g_L>((\deg f(x))-1)/2\geqq 1$. Since L contains the subfield $L_0(x)$ of co-dimension 2 and $g_L\geqq 2$, it is a hyperelliptic field (cf. [**1**, Chapter 16]. So by case 1 of [**1**, Chapter 16, §7, Theorem 14], there exists a polynomial $f_1(x)\in L_0[x]$ of degree $2g_L+1$ or $2g_L+2$ which is not divisible by the square of any non-constant polynomial of $L_0[x]$ such that $L=L_0(x,\sqrt{f_1(x)})$. It follows that $f_1(x)$ and $f(x)$ differ multiplicatively by the square of an element of $L_0(x)$. Using the fact that $f_1(x)$ is square-free over L_0, it can be easily seen that there exists a polynomial $A(x)\in L_0[x]$ such that $f_1(x)A(x)^2=f(x)$. In particular $\deg f(x)\geqq\deg f_1(x)\geqq 2g_L+1$, which is contrary to our supposition. This contradiction proves the desired assertion.

Proof of Theorem 1.3. By assertion (i) of the above lemma, we can write $K=K_0(x_0,\sqrt{h(x_0)})$ where $h(x_0)$ is a polynomial in (a *tr.* elememt) x_0 over K_0 of degree$\leqq 2g_K+2$. In view of Theorem 1.2, the residue field k of v can be expressed as $k=\Delta(u,\sqrt{A(u)})$ where u is *tr.* over Δ and $A(u)\in\Delta[u]$ is a polynomial of degree $\leqq 1+(\deg h(x_0))/IRD$; in fact when $I=1$, one has $\deg A(u)\leqq(\deg h(x_0))/RD$.

If $\deg A(u)=0$, then k is a simple *tr.* extension of Δ and $g_k=0$; thus the desired relations (i) and (ii) are trivially true in this case. Assume that $\deg A(u)\geqq 1$. Applying Lemma 4.1, we see that

$$g_k\leqq\frac{\deg A(u)}{2}-\frac{1}{2}\leqq\frac{\deg h(x_0)}{2IRD}\leqq\frac{g_K+1}{IRD}$$

which proves (i).

In the case $I=R=D=1$, using the estimate $\deg A(u)\leqq\deg h(x_0)$, and arguing as above, it can be easily seen that $g_k\leqq g_K$.

Examples 4.2. We give examples to point out that the estimates on g_k given by both the assertions of Theorem 1.3 are best possible.

(i) Let $K_0=Q(t)$ where Q is the field of rational numbers and t is an indeterminate. Let v_0 be the t-adic valuation of $Q(t)$ trivial on Q (which is characterized by $v_0(t)=1$). Let x be *tr.* over K_0 and set

$$\xi=((x^2+1)^2-3t^2)/t^3.$$

Let v_1 denote the valuation of $K_0(\xi)$ defined on $K_0[\xi]$ by

$$v_1\left(\sum_i a_i\xi^i\right)=\min_i v_0(a_i),\, a_i\in K_0.$$

Extend v_1 arbitrarily to a valuation v' of $K_0(x)$. As shown in [**15**, 4.5] the residue field k' of v' is $\Delta'(\xi^*)$ where the v'-residue ξ^* of ξ is *tr.* over $\Delta'=Q(\sqrt{-1},\sqrt{3})$.

Define a square-free polynomial $F(x)\in K_0[x]$ by

$$F(x)=\xi(\xi+1)(\xi+2)(\xi+3)(\xi+4).$$

Let v be an extension of v' to $K_0(x,\sqrt{F(x)})$. It is easily seen that the residue field k of v is $k'(\sqrt{F(x)^*})=\Delta'(\xi^*,\sqrt{h(\xi^*)})$, where

$$h(\xi^*)=\xi^*(\xi^*+1)(\xi^*+2)(\xi^*+3)(\xi^*+4).$$

Clearly $I=1$, $R=4$, and $D=1$, as the characteristic of the residue field is 0.

By a well-known result (cf. [**16**, p. 44]), the genus of $K=(\deg F(x))/(2)-1=9$, and that of k is 2. A simple calculation gives

$$2=g_k=[(g_K+1)/IRD]<[(g_K-1)/IRD]+1$$

where [ ] denotes the integral part. This together with Remark 1.5 shows that the bound on g_k given by Theorem 1.3(i) is actually attained and is definitely better than the one yielded by (1).

(ii) Let v_0 be the 5-adic valuation of the field $K_0=Q$ characterized by $v_0(5)=1$. Let v' be the valuation of a simple transcendental extension $K_0(x)$ defined on $K_0[x]$ by

$$v'\left(\sum_i a_i x^i\right)=\min_i v_0(a_i), a_i \in Q.$$

The residue field k' of v' is $\Delta'(x^*)$ with the v'-residue x^* of x *tr.* over the field Δ' of 5 elements. Let v be an extension of v' to $K_0(x, \sqrt{F(x)})$, where $F(x)=x(x+1)(x+2)$. Then the residue field k of v is $\Delta'(x^*, \sqrt{F(x)^*})$. Observe that $F(x)^*$ is a square-free polynomial in x^* of degree 3 over Δ'. So $g_K=g_k=1$. In this case, clearly $I=R=1$, and $D=1$ in view of the fact that the extension $K_0(x, \sqrt{F(x)})/K_0(x)$ has henselian defect 1.

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