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THE COHOMOLOGY RING OF A COMBINATORIALLY ASPHERICAL GROUP

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Abstract

A presentation is given for the cohomology ring of a finitely presented combinatorially aspherical group with trivial coefficients in an integral domain. Cohomological periodicity is characterized in terms of the cup product.

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1. Introduction

The cohomology ring has not been computed directly for many groups because of the complexity of the calculations required. Using spectral sequences, Evens [5, 6.2] proved that the cohomology ring of any finite group is finitely generated as a ring over any noetherian ring of trivial coefficients, and Lewis [10] gave presentations of the integral cohomology rings of groups with prime-cubed order, and surveyed earlier results for finite cyclic and symmetric groups. Considerable work has been done on the classical groups using K-theory (for example Quillen [12]) and on metacyclic groups (for example Diethelm [4], Huebschmann [9], Rusin [14]). Ratcliffe [13] determined the cup product in the cohomology ring of a one-relator group over selected coefficient rings, using geometric methods.

This paper gives a presentation of the cohomology ring $H^*(G;\Gamma)$ for any finitely presented combinatorially aspherical (CA) group G with trivial coefficients in an integral domain Γ. This presentation is derived from the author's calculation [7] of the cup product in $H^*(G;\Gamma)$. The cohomology ring differs from that of the associated free product $\overline{G}$ of finite cyclic groups only in dimensions 1 and 2, and consequently exhibits similar cohomological properties.

Notation throughout will be as follows. Let $G = F/\langle R\rangle^F$, where F is the free group of free generating set X and $\langle R\rangle^F$ is the normal subgroup of F generated by R. For each r in R, let t_r be the *root* of r; that is, $r = t_r^{n_r}$ where $n_r \geq 1$ is maximal, and let $R^0 = \{r \in R : n_r \geq 2\}$. If $\partial f/\partial x$ is the Fox derivative [6] of the word f in the integral group ring $\mathbf{Z}F$ with respect to the generator x of F, let $\langle f : x\rangle$ and $\langle f : x, y\rangle$ be the images in $\mathbf{Z}$ of $\partial f/\partial x$ and $\partial^2 f/\partial x \partial y$, respectively, under the augmentation map, and let $(\partial R/\partial X)$ be the matrix of exponent sums $(\partial R/\partial X)_{rx} = \langle r : x\rangle$. Finally for any commutative ring Γ with 1, considered as a trivial G-module, define $\Gamma_m = \Gamma/m\Gamma$ and ${}^m\Gamma = \{\gamma \in \Gamma : m\gamma = 0\}$ for any non-negative integer m, and let $R^* = \{r \in R : \operatorname{char}\Gamma \text{ divides } n_r\}$.

A presentation is CA if there are no nontrivial identities among the relators [2, 1.4], and *concise* if no relator is conjugate to another or its inverse [2, page 4].

Examples are one-relator groups, small cancellation groups [3], soluble groups of cohomological dimension 2, and fundamental groups of certain 3-manifolds (including knot groups) [2, Section 5].

If G is a concise CA presentation the cohomology groups $H^m(G;\Gamma)$ are well known (see [7, pages 42–43; 8, Theorem 2]) to be

$$\begin{aligned}
&H^0(G;\Gamma) \cong \Gamma,\\
&H^1(G;\Gamma) \cong \operatorname{Hom}_{\mathbf{Z}}(G_{ab}, \Gamma),\\
&H^2(G;\Gamma) \cong (\Gamma^R)/(\partial R/\partial X)(\Gamma^X),\\
&H^m(G;\Gamma) \cong \begin{cases} \prod_{r\in R^0} {}^{n_r}\Gamma, & m = 2n+1,\ n \geq 1,\\ \prod_{r \in R^0} \Gamma_{n_r}, & m = 2n,\ n \geq 2.\end{cases}
\end{aligned}$$

With each CA presentation $G = \langle X : R\rangle$ there is associated a free product of finite cyclic groups $\overline{G} = \langle t_r, r \in R : t_r^{n_r}, r \in R\rangle$, a homomorphism $\iota : \overline{G} \to G$ mapping $\overline{G}$ onto the subgroup of G generated by $\{t_r : r \in R\}$, and an induced ring homomorphism $H^*(G:\Gamma) \xrightarrow{\iota^*} H^*(\overline{G};\Gamma)$ for which ι_n^* is an isomorphism if $n \geq 3$ and an epimorphism if $n = 2$.

2. The cohomology ring

The cohomology ring of $\overline{G}$ is easy to describe, either from the known cohomology of cyclic groups (for example [1, V.1. Exercise 1, V.3. Exercise 3]) or from [7, 4.1] since $\overline{G}$ is itself a CA concise presentation.

PROPOSITION 2.1. *If R is finite and Γ is an integral domain, then $H^*(\overline{G};\Gamma)$ has the following presentation.*

$$\textit{Generators: } \alpha_r,\ r \in R^*;\ \beta_r,\ r \in R^0;\qquad \deg \alpha_r = 1,\ \deg \beta_r = 2.$$

$$\textit{Relators: } \alpha_r\alpha_s = \beta_r\alpha_s = \alpha_s\beta_r = \beta_r\beta_s = 0,\ r \neq s,$$
$$n_r\beta_r = 0,\ r \in R^0 \backslash R^*,$$
$$\alpha_r^2 = \binom{n_r}{2}\beta_r,\ r \in R^0 \cap R^*,$$
$$\alpha_r\beta_r = \beta_r\alpha_r.$$

In [7] the author calculated a diagonal approximation for the Lyndon resolution of G. From this we can obtain a complete description of the structure of $H^*(G;\Gamma)$ which, however, may be more simply stated in terms of the known structure of $H^*(\overline{G};\Gamma)$.

Clearly the cohomological periodicity of G may be characterised in terms of the cup product.

LEMMA 2.2. *Let β be the 2-cocycle in the Lyndon resolution* [7, *page* 42] *defined by $\beta(r) = 1$, $r \in R$. Then $\beta \cup - : H^n(G;-) \to H^{n+2}(G;-)$ is a natural isomorphism for all $n \geq 3$ and an epimorphism for $n = 2$.*

PROOF. The same result holds for $\iota^*[\beta]$ in $H^*(\overline{G};-)$.

COROLLARY 2.3. *If G is a CA group, $H^*(G;\Gamma)$ is generated as a ring by elements of degree at most 3.*

When Γ is an integral domain, and G is finitely presented, $H^1(G;\Gamma)$ is free of rank $L = |X| - \operatorname{rank}_\Gamma S(\partial R/\partial X)$, where $S(\partial R/\partial X)$ is the Smith normal form of $(\partial R/\partial X)$ [11, Theorem II.9]. Module generators of higher degree are β_r, $r \in R$ and γ_r, $r \in R^*$, where β_r is the 2-cocycle in the Lyndon resolution defined by $\beta_r(s) = \delta_{rs}$ (Kronecker delta), and $\gamma_r = (\iota^*)^{-1}(\alpha_r\beta_r)$, from (2.1), respectively. It is straightforward to show that for $f \in H^1(G;\Gamma)$, $i^*(f) =$

$\sum_{r\in R^*}(\sum_{x\in X}\langle t_r : x\rangle f(x))\alpha_r$. For degree reasons, it is now only necessary to compute the product of degree 1 terms, since in higher degrees for generators $h_n \in H^n(G;\Gamma)$ the product is derived from ι^*:

$$h_n h_m = (\iota^*)^{-1}(\iota^*(h_n)\iota^*(h_m)), \qquad n+m \geq 3.$$

If α_i and α_j are 1-cocycles in the Lyndon resolution then by [7, 3.1, 3.3] their cup product is given by $(\alpha_i\alpha_j)(r) = \sum_{x\in X}\sum_{y\in X}\langle r : x, y\rangle\alpha_i(x)\alpha_j(y)$, $r \in R$ and hence, for free generators α_i and α_j, $1 \leq i, j \leq L$ of $H^1(G;\Gamma)$, $\alpha_i\alpha_j = \sum_{r\in R}\sum_{x\in X}\sum_{y\in X}\langle r : x, y\rangle\alpha_i(x)\alpha_j(y)\beta_r$.

THEOREM 2.4. *If G has a finite CA presentation $\langle X : R\rangle$ and Γ is an integral domain, then $H^*(G;\Gamma)$ has the following presentation.*

Generators:

$$\alpha_l, 1 \leq l \leq L;\ \beta_r,\ r \in R;\ \gamma_r,\ r \in R^*;$$
$$\deg \alpha_l = 1,\ \deg \beta_r = 2,\ \deg \gamma_r = 3.$$

Module relators: $n_r\beta_r^2 = 0$.
For each $x \in X$, a relation $\sum_{r\in R}\langle r : x\rangle\beta_r = 0$.

Multiplication: $\alpha_p\alpha_l = -\alpha_l\alpha_p = \sum_{r\in R}(\sum_{x\in X}\sum_{y\in X}\langle r : x, y\rangle\alpha_p(x)\alpha_l(y))\beta_r$,
$\alpha_l\beta_r = (\iota^*)^{-1}(\iota^*(\alpha_l)\iota^*(\beta_r))$ *etc.*

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