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LARGE DEVIATIONS BOUNDS FOR FACE-HOMOGENEOUS RANDOM WALKS IN THE QUARTER-PLANE

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In this article, we analyze the large deviations bounds for the nonergodic face-homogeneous random walk in the positive quadrant. Under some condition the value of the local rate function for the path identically equal to zero is found, and an explicit expression is derived for it. This makes the computation of its value possible for specific stochastic networks. Some numerical examples are given.

1. INTRODUCTION

The sample path large deviations theorem for random walks with boundaries have been studied in [9,12].

For linear paths

$$\varphi : [0,1] \to \mathbb{R}^2_+ \quad \text{with } \varphi(t) = x + vt,$$

explicit expressions for the local rate function $L(x,v)$ for the case that either $x \neq (0,0)$ or $v \neq (0,0)$ have been derived in [9,12]. Clearly, the local rate function for $x = (0,0)$ and $v = (0,0)$ equals 0 if the process is ergodic. The determination of $L(0,0)$ was left as an open problem in [9] for the transient random walk. In this article, we derive an explicit expression for a lower bound for $L(0,0)$, and under an extra condition, it holds that the lower bound is equal to an upper bound. Hence, in this case, the lower bound gives the local rate function.

In [4], it has been shown for a large class of queueing networks, including the model of this article, that the large deviations principle holds for local rate function expressed in terms of a stochastic optimal control problem. The large deviations bounds for a Jackson network are obtained in [7]. In a recent article [8], the local rate function is connected with the convergence parameter of associated local transform

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matrices. Both approaches are quite general and hold for a large class of models. The specific model we consider in this article allows for an expression in terms of the logarithmic moment-generating functions on the different faces of $\mathbb{R}_+^2$; this expression makes an easy numerical calculation possible. For specific queueing networks, explicit expressions are obtained.

We provide a self-contained proof based on the change of measure and the analysis of the logarithmic moment-generating functions. Almost closed sets play an essential role in the proof. In this article we restrict ourselves to the local rate function for a path identically equal to zero. The proof for linear paths of the type $\varphi(t) = x + vt$ with $v \neq (0,0)$ can be done with the same type of analysis except that the analysis of almost closed sets is not needed. The analysis for the paths not identical to zero can be found in [9,12]. The combination of these results with ours gives a complete solution of the sample path large deviations problem in the positive quadrant.

Also, in a recent article [2], an extensive study has been made for the asymptotic behavior of large deviations for Markov chains in the positive quadrant. Precise asymptotics are obtained for the logarithm of the transition probabilities.

The outline of this article is as follows. In Section 2 we give the model description and we state our main result, including the expression for the local rate function for a linear path identically equal to zero. In Section 3 we first give a complete description of the local rate function for all linear paths. In that section we also summarize the classification of ergodicity, null recurrence, and transience and the results on almost closed sets we need for our analysis. For completeness, we briefly introduce the twisted process and the change-of-measure lemma used in Section 4. The main part of Section 4 is devoted to the analysis of the logarithmic moment-generating functions on the different faces of $\mathbb{R}_+^2$ and to the proof of the large deviations lower bound. With our condition, the proof of the large deviations upper bound is straightforward. In Section 5 we illustrate how the expression for the local rate function can be used to compute the large deviations bounds for a specific random walk in the quarter-plane. In a companion article [6], applications will be made to stochastic networks that model queueing networks with coupled processors.

2. MODEL DESCRIPTION AND MAIN RESULT

2.1. Model Description

We consider an irreducible and aperiodic Markov chain (MC) $\mathcal{M} = \{S_t, t = 0,1,\dots\}$ on the state space $\mathbb{Z}_+^2 = \{i = (i_1, i_2) : i_1, i_2 \in \mathbb{N}\}$, where $\mathbb{N} = \{0,1,2,\dots\}$.

We assume that the following conditions are satisfied for $\mathcal{M}$.

Condition A: The transition probabilities are

$$p_{ij} = \mathsf{P}\{i \to j\} = \begin{cases} p_{j-i}^0 & \text{if } i_1 = 0, i_2 = 0 \\ p_{j-i}^1 & \text{if } i_1 > 0, i_2 = 0 \\ p_{j-i}^2 & \text{if } i_1 = 0, i_2 > 0 \\ p_{j-i}^3 & \text{if } i_1 > 0, i_2 > 0. \end{cases} \tag{1}$$

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Therefore, there are four faces of homogeneity:

$$\Lambda^1 = \{i \in \mathbb{Z}^2 : i_1 > 0, i_2 = 0\}, \qquad \Lambda^2 = \{i \in \mathbb{Z}^2 : i_1 = 0, i_2 > 0\},$$
$$\Lambda^3 = \{i \in \mathbb{Z}^2 : i_1 > 0, i_2 > 0\}, \qquad \text{and } \Lambda^0 = \{0\}.$$

By $\Lambda(i)$ we denote the face to which state i belongs. Then we can write

$$\mathsf{P}\{i \to j\} = p_{j-i}^{\Lambda(i)}.$$

Condition B (Lower Boundedness of Jumps):

$$\begin{aligned} p_{j-i}^0 &= 0 \quad \text{if } j_1 - i_1 < 0 \text{ or } j_2 - i_2 < 0, \\ p_{j-i}^1 &= 0 \quad \text{if } j_1 - i_1 < -1 \text{ or } j_2 - i_2 < 0, \\ p_{j-i}^2 &= 0 \quad \text{if } j_1 - i_1 < 0 \text{ or } j_2 - i_2 < -1, \\ p_{j-i}^3 &= 0 \quad \text{if } j_1 - i_1 < -1 \text{ or } j_2 - i_2 < -1. \end{aligned} \tag{2}$$

Condition C (Upper Boundedness of Jumps): For a face Λ we have

$$p_{j-i}^\Lambda = 0 \quad \text{if } j_1 - i_1 > d_1^+ \quad \text{or} \quad j_2 - i_2 > d_2^+ \quad \text{for some integer } d_1^+ \geq 1, d_2^+ \geq 1.$$

The special case where $d^+ = d^- = 1$ is depicted in Figure 1.

Condition D (Local Irreducibility): Let $\mathcal{M}^3$ be the Markov chain (MC) on $\mathbb{Z}^2$ with transition probabilities $\mathsf{P}\{i \to j\} = p_{j-i}^3$. We assume that $\mathcal{M}^3$ is irreducible.

Condition E:

$$\mathsf{P}\{\Lambda^1 \to \Lambda^3\} \triangleq \sum_{k \in \mathbb{Z}^2 : k_2 > 0} p_k^1 > 0 \quad \text{and} \quad \mathsf{P}\{\Lambda^2 \to \Lambda^3\} \triangleq \sum_{k \in \mathbb{Z}^2 : k_1 > 0} p_k^2 > 0. \tag{3}$$

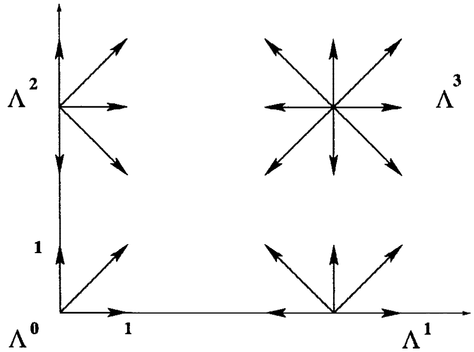

FIGURE 1. Transition probabilities on the faces.

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2.2. Main Result

To any face Λ there corresponds a jump variable $\xi^\Lambda = (\xi_1^\Lambda, \xi_2^\Lambda) \in \mathbb{Z}^2$ having the distribution $\mathsf{P}\{\xi^\Lambda = k\} = p_k^\Lambda, k \in \mathbb{Z}^2$ (see (1)). For any face Λ we also use the logarithmic moment-generating function $H^\Lambda : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H^\Lambda(\alpha) = \log(\mathsf{E}\exp\{\alpha_1 \xi_1^\Lambda + \alpha_2 \xi_2^\Lambda\}),$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and E denotes the expectation. We have that

$$H^\Lambda(\alpha) = \log\left(\sum_{k\in\mathbb{Z}^2} p_k^\Lambda \exp\{\alpha_1 k_1 + \alpha_2 k_2\}\right).$$

We define two important points in $\mathbb{R}^2$:

$$\alpha^1 = \arg\min_\alpha \max\{H^1(\alpha), H^3(\alpha)\},$$

$$\alpha^2 = \arg\min_\alpha \max\{H^2(\alpha), H^3(\alpha)\}. \tag{4}$$

Since $H^1(0) = H^2(0) = H^3(0)$, it follows from the continuity of $H^\Lambda(\alpha)$ in α that α^1 and α^2 exist. In Section 4, we show that they are finite. Let

$$\hat{\alpha} = \arg\max\{H^3(\alpha^1), H^3(\alpha^2)\}. \tag{5}$$

By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{Z}_+^2$ (i.e., $\|i\| = \sqrt{i_1^2 + i_2^2}$).

THEOREM 2.1: *If for any face $\Lambda = 0,1,2$, we have*

$$H^\Lambda(\hat{\alpha}) \le H^3(\hat{\alpha}). \tag{6}$$

Then the random walk S_t satisfies the large deviations (LD) theorem with local rate function

$$L^0 = H^3(\hat{\alpha}),$$

for a path identically equal to zero.

Indeed, the following LD bounds are satisfied:

LD upper bound: *For any $\delta > 0$ there exists $N(\delta)$ such that for all $N > N(\delta)$,*

$$\mathsf{P}\left\{S_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|S_t\| < \delta N\right\} \le \exp\{+\delta N + N\tau L^0\}. \tag{7}$$

LD lower bound: *For any $\delta > 0$ and $\delta' > 0$ there exists $N(\delta, \delta')$ such that*

$$\mathsf{P}\left\{S_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|S_t\| < \delta N\right\} \ge \exp\{-\delta' N + N\tau L^0\} \tag{8}$$

for all $N > N(\delta, \delta')$.

Remark 2.1: In fact, we prove that the LD lower bound always holds with $L^0 = H^3(\hat{\alpha})$. However, for the proof of the LD upper bound we need (6).

Remark 2.2: As we will point out in Section 4, (6) is always satisfied for null recurrent MC, but it is not always fulfilled for transient MC. For an ergodic MC, (6) does not hold, but for ergodic MC the local rate function is known to be equal to zero.

3. RELATED RESULTS AND MAIN DEFINITIONS

3.1. Large Deviations Theorem

We will not repeat the definition of the LD theorem in this article since it is rather standard and it can conveniently be found in the literature (see e.g., [4,9,12]). Here we only give the necessary notations to formulate the LD theorem for our model.

For any $\tau > 0$ consider the metric space $C([0;\tau],\mathbb{R}_+^2)$ of all continuous functions

$$\varphi : [0;\tau] \to \mathbb{R}_+^2.$$

It has been shown in [4,12] that the LD theorem holds with a good rate function

$$\mathcal{L}_\tau : C([0;\tau],\mathbb{R}_+^2) \to [0,+\infty], \qquad \tau > 0.$$

In [9] it has been proved, under the assumption that S_t is ergodic, that the good rate function has the following form:

$$\mathcal{L}_\tau(\varphi) = \begin{cases} \int_0^\tau L(\varphi(t),\varphi'(t))\,dt & \text{if the path } \varphi \text{ is absolutely continuous} \\ +\infty & \text{otherwise,} \end{cases} \tag{9}$$

where the local rate function

$$L(\cdot,\cdot) : \mathbb{Z}_+^2 \times \mathbb{R}^2 \to \mathbb{R}_+ \cup \{+\infty\}$$

is defined by

$$L(i,v) = \begin{cases} L^3(v_1,v_2) & \text{if } i_1 > 0, i_2 > 0 \\ L^2(v_2) & \text{if } i_1 = 0, i_2 > 0 \\ L^1(v_1) & \text{if } i_1 > 0, i_2 = 0 \\ L^0 & \text{if } i_1 = 0, i_2 = 0, \end{cases} \tag{10}$$

where $L^0 = 0$ and L^1, L^2, and L^3 are the following Legendre transforms:

$$L^3(v) = \sup_\alpha \{(\alpha,v) - H^3(\alpha)\},$$

$$L^2(v_2) = \sup_{\alpha_2} \{\alpha_2 v_2 - H^3(\alpha_1(\alpha_2),\alpha_2)\},$$

$$L^1(v_1) = \sup_{\alpha_1} \{\alpha_1 v_1 - H^3(\alpha_2(\alpha_1),\alpha_1)\},$$

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and the functions $\alpha_1(\cdot), \alpha_2(\cdot) : \mathbb{R} \to \mathbb{R}$ are defined as

$$\begin{aligned} \alpha_1(\alpha_2) &\triangleq \arg\min_{\alpha_1} \max\{H^3(\alpha_1,\alpha_2), H^2(\alpha_1,\alpha_2)\}, \\ \alpha_2(\alpha_1) &\triangleq \arg\min_{\alpha_2} \max\{H^3(\alpha_1,\alpha_2), H^1(\alpha_1,\alpha_2)\}. \end{aligned} \tag{11}$$

In Section 4 we will prove that $\alpha_1(\alpha_2)$ and $\alpha_2(\alpha_1)$ are finite. As a consequence of the results in [9] or in [12] and the analysis of this article, we have, under the conditions of Section 2, the following theorem.

THEOREM 3.1: *If the random walk S_t is nonergodic, then the LD theorem remains true with the same L^1, L^2, and L^3, and L^0 as in Theorem 2.1.*

In this article we focus on the derivation of L^0 and the proof of the bounds (7) and (8) given (6).

3.2. Classification: Ergodicity, Null Recurrence, and Transience

Here we recall the criteria for the MC $\mathcal{M}$ to be ergodic, null recurrent, or transient in terms of the mean drift on the faces. We shall especially use these criteria for the twisted processes. Define the vector

$$M^\Lambda = (M_1^\Lambda, M_2^\Lambda) \in \mathbb{R}^2, \qquad \Lambda = 0,1,2,3, \tag{12}$$

as the one-step mean drift from a point, which is an element of Λ, by

$$\begin{aligned} M_1^\Lambda &= \mathsf{E}(\xi_1^\Lambda) = \sum_{k_1} k_1 \mathsf{P}\{\xi_1^\Lambda = k_1\} = \sum_{k\in\mathbb{Z}^2} k_1 p_k^\Lambda, \\ M_2^\Lambda &= \mathsf{E}(\xi_2^\Lambda) = \sum_{k_2} k_2 \mathsf{P}\{\xi_2^\Lambda = k_2\} = \sum_{k\in\mathbb{Z}^2} k_2 p_k^\Lambda. \end{aligned}$$

The following lemma is a consequence of Theorems 3.3.1 and 3.3.2 in [5].

LEMMA 3.2: *Assume for the MC $\mathcal{M}$ that conditions A, B, and C are satisfied. With the mean drift vectors (12) we define two constants:*

$$V^1 = M_1^3 M_2^1 - M_2^3 M_1^1 \quad \textit{and} \quad V^2 = M_2^3 M_1^2 - M_1^3 M_2^2. \tag{13}$$

Then we have the following:

(a) If $M_1^3 < 0$ and $M_2^3 < 0$, then the MC $\mathcal{M}$ is
  (i) ergodic iff $V^1 < 0$ and $V^2 < 0$
  (ii) null recurrent iff

$$\textit{either} \quad \begin{matrix} V^2 = 0 \\ V^1 \le 0 \end{matrix} \quad \textit{or} \quad \begin{matrix} V^1 = 0 \\ V^2 \le 0; \end{matrix} \tag{14}$$

  (iii) transient iff $V^1 > 0$ or $V^2 > 0$.

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(b) If $M_1^3 \geq 0$ *and* $M_2^3 < 0$*, then the MC* $\mathcal{M}$ *is*

$$\textit{ergodic iff } V^1 < 0;\ \textit{null recurrent iff } V^1 = 0;\ \textit{transient iff } V^1 > 0.$$

(c) If $M_1^3 < 0$ *and* $M_2^3 \geq 0$*, then the MC* $\mathcal{M}$ *is*

$$\textit{ergodic iff } V^2 < 0;\ \textit{null recurrent iff } V^2 = 0;\ \textit{transient iff } V^2 > 0.$$

(d) If $M_1^3 \geq 0, M_2^3 \geq 0$*, and* $M_1^3 + M_2^3 > 0$*, then the MC* $\mathcal{M}$ *is transient.*

Remark 3.1: If $M_1^3 = M_2^3 = 0$, then the MC can be ergodic, null recurrent, or transient (see [5]).

3.3. Almost Closed Sets

For our analysis we need results on almost closed sets. We refer to [3] for an introduction to the theory of almost closed sets.

Consider an irreducible aperiodic and transient MC $\{\zeta_n, n = 0, 1, \ldots\}$ on a countable state space in discrete time.

Definition 3.1: *A subset A of the states is called almost closed if*

$$\mathsf{P}(\cup_{m>0} \cap_{t>m} \{\zeta_n \in A\}) = \mathsf{P}(\cap_{m>0} \cup_{t>m} \{\zeta_n \in A\})$$

and this probability is positive.

The following lemma is a consequence of the results in [10].

Lemma 3.3: *Let*

$$M_1^3 > 0, \qquad M_2^3 > 0,$$

then the set Λ^3 *is almost closed.*

Let

$$M_2^3 < 0, \qquad V^1 > 0,$$

then there exists a set $A^1 \subseteq \Lambda^3$ *such that the set* $\Lambda^1 \cup A^1$ *is almost closed.*

Let

$$M_1^3 < 0, \qquad V^2 > 0,$$

then there exists a set $A^2 \subseteq \Lambda^3$ *such that the set* $\Lambda^2 \cup A^2$ *is almost closed.*

If

$$M_1^3 < 0, \qquad M_2^3 < 0, \qquad V^1 > 0, \qquad V^2 > 0, \tag{15}$$

then the sets A^1 *and* A^2 *can be taken disjoint.*

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4. LARGE DEVIATIONS BOUNDS FOR THE PATH IDENTICALLY EQUAL TO ZERO

This section is devoted to the proof of our main result (i.e., the LD bounds as stated in Theorem 2.1). First, we introduce the twisted process and the change of measure and we analyze the H-functions. Then in Sections 4.4 and 4.5 we prove the LD lower and upper bounds.

4.1. Twisted Process

First, we recall the well-known twisted MC. For any $\alpha \in \mathbb{R}^2$ we define a MC

$$\mathcal{M}(\alpha) = \{S_t^\alpha, t = 0,1,2,\dots\}$$

on the state space $\mathbb{Z}_+^2$ with transition probabilities

$$p_{ij}(\alpha) \triangleq \frac{p_{ij}\exp\{(\alpha, j) - (\alpha, i)\}}{\sum_j p_{ij}\exp\{(\alpha, j) - (\alpha, i)\}}, \tag{16}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^2$. The MC $\mathcal{M}(\alpha)$ is said to be a *twisted MC*.

Note that $\mathcal{M}(0) = \mathcal{M}$. Clearly, conditions A, B, C, and D of Section 2.1 hold for the MC $\mathcal{M}(\alpha)$ if they are satisfied for $\mathcal{M}$.

By P_α we denote the probability measure for the twisted MC $\mathcal{M}(\alpha)$. Recall that in Section 2.2 we defined the jumps variable ξ^Λ. Hence, we have that

$$\mathsf{P}_\alpha\{\xi^\Lambda = k\} = p_k^\Lambda(\alpha).$$

By E_α we denote the expectation corresponding to P_α. Similar to (12), define the vector

$$M^\Lambda(\alpha) = (M_1^\Lambda(\alpha), M_2^\Lambda(\alpha)) \triangleq \mathsf{E}_\alpha(\xi^\Lambda) = \sum_k k p_k^\Lambda(\alpha). \tag{17}$$

By $\text{Cov}_\alpha\{\xi_1^\Lambda, \xi_2^\Lambda\}$ we denote the covariance of the random variables ξ_1^Λ and ξ_2^Λ with respect to P_α; that is,

$$\text{Cov}_\alpha\{\xi_1^\Lambda, \xi_2^\Lambda\} \triangleq \mathsf{E}_\alpha(\xi_1^\Lambda \xi_2^\Lambda) - \mathsf{E}_\alpha(\xi_1^\Lambda)\,\mathsf{E}_\alpha(\xi_2^\Lambda).$$

By $\text{Var}_\alpha\{\xi_1^\Lambda\}$ and $\text{Var}_\alpha\{\xi_2^\Lambda\}$ we denote the variance of ξ_1^Λ and ξ_2^Λ, respectively, with respect to P_α. For completeness, we include the following lemma, which might be well known, but we have no reference for it.

LEMMA 4.1: *For any face* Λ *we have*

$$M^\Lambda(\alpha) = \left(\frac{\partial}{\partial\alpha_1}H^\Lambda(\alpha), \frac{\partial}{\partial\alpha_2}H^\Lambda(\alpha)\right), \qquad \text{Cov}_\alpha\{\xi_1^\Lambda, \xi_2^\Lambda\} = \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}H^\Lambda(\alpha), \tag{18}$$

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$$\mathrm{Var}_\alpha\{\xi_1^\Lambda\} = \frac{\partial^2}{\partial^2\alpha_1} H^\Lambda(\alpha), \qquad \mathrm{Var}_\alpha\{\xi_2^\Lambda\} = \frac{\partial^2}{\partial^2\alpha_2} H^\Lambda(\alpha). \tag{19}$$

PROOF: First, note that from (16) it follows that

$$p_k^\Lambda(\alpha) = p_k^\Lambda \exp\{(\alpha, k) - H^\Lambda(\alpha)\}. \tag{20}$$

Let us prove first that $M_2^\Lambda(\alpha) = (\partial/\partial\alpha_2)H^\Lambda(\alpha)$. Clearly, the function

$$H^\Lambda(\alpha) = \log\left(\sum_{k\in\mathbb{Z}^2} p_k^\Lambda \exp\{\alpha_1 k_1 + \alpha_2 k_2\}\right)$$

is differentiable at any point $\alpha \in \mathbb{R}^2$, and

$$\frac{\partial}{\partial\alpha_2} H^\Lambda(\alpha) = \sum_k k_2 \frac{p_k^\Lambda \exp\{\alpha_1 k_1 + \alpha_2 k_2\}}{\sum_k p_k^\Lambda \exp\{\alpha_1 k_1 + \alpha_2 k_2\}} = \sum_k k_2 p_k^\Lambda(\alpha) = M_2^\Lambda(\alpha).$$

Now we calculate $\mathrm{Cov}_\alpha\{\xi_1^\Lambda, \xi_2^\Lambda\}$. We have that

$$\begin{aligned}\frac{\partial}{\partial\alpha_1} p_k^\Lambda(\alpha) &= p_k^\Lambda \exp\{(\alpha, k) - H^\Lambda(\alpha)\}\left(k_1 - \frac{\partial}{\partial\alpha_1} H^\Lambda(\alpha)\right)\\ &= k_1 p_k^\Lambda(\alpha) - p_k^\Lambda(\alpha) M_1^\Lambda(\alpha).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial^2}{\partial\alpha_1\partial\alpha_2} H^\Lambda(\alpha) = \frac{\partial}{\partial\alpha_1} M_2^\Lambda(\alpha) &= \sum_k k_2 \frac{\partial}{\partial\alpha_1} p_k^\Lambda(\alpha)\\ &= \sum_k k_1 k_2 p_k^\Lambda(\alpha) - M_1^\Lambda(\alpha) \sum_k k_2 p_k^\Lambda(\alpha)\\ &= \mathsf{E}_\alpha(\xi_1^\Lambda \xi_2^\Lambda) - M_1^\Lambda(\alpha) M_2^\Lambda(\alpha).\end{aligned}$$

Assertion (19) is a consequence of assertion (18). ■

4.2. The Change of Measure

Let $\delta > 0$ and $\tau > 0$. By $I_{A_{\delta N}}$ we denote the indicator of the event

$$A_{\delta N} = \left\{\omega : \sup_{0\le t\le[\tau N]} \|S_t(\omega)\| < \delta N\right\}.$$

The following lemma is well known; for completeness, we include a proof. We denote $H_i(\alpha)$ for $H^\Lambda(\alpha)$ when $i \in \Lambda$.

LEMMA 4.2: *For any α and any Borel set Ω we have*

$$\mathsf{P}\{A_{\delta N} \cap \Omega\} = \mathsf{E}_\alpha\{I_{A_{\delta N}\cap\Omega}\} \exp\left\{-(\alpha, S_{[\tau N]}) + (\alpha, S_0) + \sum_{t=0}^{[\tau N]-1} H_{S_t}(\alpha)\right\}. \tag{21}$$

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PROOF: Relation (20) using a different notation is

$$\mathsf{P}\{i \to j\} = \mathsf{P}_\alpha\{i \to j\}\exp\{-(\alpha, j) + (\alpha, i) + H_i(\alpha)\}. \tag{22}$$

Let $m = [\tau N]$. Then for any $\omega \in A_{\delta N} \cap \Omega$ we have

$$-(\alpha, S_m(\omega)) + (\alpha, S_0(\omega)) = -\sum_{t=0}^{m-1} (\alpha, S_{t+1}(\omega)) - (\alpha, S_t(\omega)).$$

Taking (22) in account we get

$$\begin{aligned}
\mathsf{P}\{A_{\delta N} \cap \Omega\} &= \sum_{\omega \in A_{\delta N} \cap \Omega} \prod_{t=0}^{m-1} \mathsf{P}\{S_t(\omega) \to S_{t+1}(\omega)\} \\
&= \sum_{\omega \in A_{\delta N} \cap \Omega} \prod_{t=0}^{m-1} \exp\{-(\alpha, S_{t+1}(\omega)) - (\alpha, S_t(\omega)) + H_{S_t(\omega)}(\alpha)\} \\
&\quad \times \mathsf{P}_\alpha\{S_t(\omega) \to S_{t+1}(\omega)\} \\
&= \sum_{\omega \in A_{\delta N} \cap \Omega} \exp\left\{-(\alpha, S_m(\omega)) + (\alpha, S_0(\omega)) + \sum_{t=0}^{m-1} H_{S_t(\omega)}(\alpha)\right\} \\
&\quad \times \mathsf{P}_\alpha\{S_t(\omega) \to S_{t+1}(\omega)\} \\
&= \mathsf{E}_\alpha\{I_{A_{\delta N} \cap \Omega}\} \exp\left\{-(\alpha, S_{[\tau N]}) + (\alpha, S_0) + \sum_{t=0}^{[\tau N]-1} H_{S_t}(\alpha)\right\}. \qquad \blacksquare
\end{aligned}$$

4.3. Analysis of the *H*-Functions

LEMMA 4.3: *The functions H^Λ are convex. The functions H^Λ are strictly convex iff the probability mass of $\xi^\Lambda = (\xi_1^\Lambda, \xi_2^\Lambda)$ is not concentrated on a line.*

PROOF: For any $t_1, t_2 \in \mathbb{R}$, consider the stochastic variable

$$t_1 \xi_1^\Lambda + t_2 \xi_2^\Lambda.$$

For any α we have that

$$\mathrm{Var}_\alpha\{t_1 \xi_1^\Lambda + t_2 \xi_2^\Lambda\} = t_1^2 \,\mathrm{Var}_\alpha\{\xi_1^\Lambda\} + 2 t_1 t_2 \,\mathrm{Cov}_\alpha\{\xi_1^\Lambda, \xi_2^\Lambda\} + t_2^2 \,\mathrm{Var}_\alpha\{\xi_2^\Lambda\}.$$

Then, from Lemma 4.1, it follows for any α that

$$\mathrm{Var}_\alpha\{t_1 \xi_1^\Lambda + t_2 \xi_2^\Lambda\} = t_1^2 \frac{\partial^2}{\partial^2 \alpha_1} H^\Lambda(\alpha) + 2 t_1 t_2 \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} H^\Lambda(\alpha) + t_2^2 \frac{\partial^2}{\partial^2 \alpha_2} H^\Lambda(\alpha).$$

Since the variance is nonnegative, it follows that the Hessian matrix of $H^{\Lambda}(\alpha)$ (i.e.,

$$\begin{pmatrix} \frac{\partial^2}{\partial\alpha_1\partial\alpha_1} H^{\Lambda}(\alpha) & \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} H^{\Lambda}(\alpha) \\ \frac{\partial^2}{\partial\alpha_2\partial\alpha_1} H^{\Lambda}(\alpha) & \frac{\partial^2}{\partial\alpha_2\partial\alpha_2} H^{\Lambda}(\alpha) \end{pmatrix}$$

is positive semidefinite. Hence, the function $H^{\Lambda}(\alpha)$ is convex (see, e.g., [11, p. 448]).

The variance of $t_1\xi_1^{\Lambda} + t_2\xi_2^{\Lambda}$ is equal to 0 if and only if for some constant c,

$$\mathsf{P}\{t_1\xi_1^{\Lambda} + t_2\xi_2^{\Lambda} = c\} = 1.$$

This means that the probability mass of $t_1\xi_1^{\Lambda} + t_2\xi_2^{\Lambda}$ is concentrated on the line $t_1x_1 + t_2x_2 = c$. Since the Hessian matrix of H^{Λ} is positive definite if and only if $H^{\Lambda}(\alpha)$ is strictly convex (see, e.g., [11, p. 448]), the assumption follows. ■

Corollary 4.4: *The function $H^3(\alpha)$ is strictly convex.*

Proof: By condition D, the MC $\mathcal{M}^3$ is irreducible. Therefore, the probability mass of ξ^3 is not concentrated on a line. Hence, by Lemma 4.3, the function $H^3(\alpha)$ is strictly convex. ■

By α^3 we denote the point where the function $H^3(\alpha)$ has its global minimum; that is,

$$\alpha^3 = \arg\min_{\alpha} H^3(\alpha).$$

Lemma 4.5: *The point α^3 is finite iff condition D holds.*

Proof: By Corollary 4.4, the function $H^3(\alpha)$ is strictly convex. Then, it has its minimum at a finite point iff for any fixed $\alpha \neq 0$ the function $f_\alpha(t) \triangleq H^3(t\alpha)$ has a minimum at a finite point. We have that f_α is strictly convex and

$$f_\alpha(t) = \log\left(\sum_{k:(\alpha,k)<0} p_k^3 \exp\{t(\alpha,k)\} + \sum_{k:(\alpha,k)\geq 0} p_k^3 \exp\{t(\alpha,k)\}\right).$$

Clearly, the function $f_\alpha(t)$ has its minimum at a finite point iff

$$\sum_{k:(\alpha,k)<0} p_k^3 > 0 \quad \text{and} \quad \sum_{k:(\alpha,k)\geq 0} p_k^3 > 0. \tag{23}$$

By $\cos(\alpha,k)$ we denote the cosine between vectors $\alpha, k \in \mathbb{R}^2$; then,

$$(\alpha,k) = \|\alpha\|\|k\|\cos(\alpha,k).$$

Hence (23) holds iff there exist $k, l \in \mathbb{R}^2$ such that

$$p_k^3 > 0, \cos(\alpha,k) > 0 \quad \text{and} \quad p_l^3 > 0, \cos(\alpha,l) < 0. \tag{24}$$

Now we will show that $\mathcal{M}^3$ is irreducible iff for any $\alpha \neq 0$ there exist $k, l \in \mathbb{Z}^2$ such that (24) is satisfied. Recall that a MC is called irreducible iff every state can be

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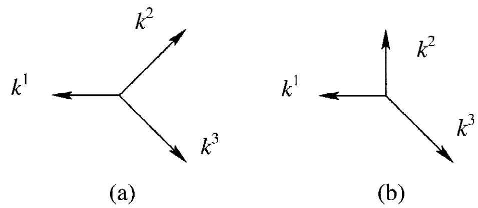

Figure 2. The probability mass related to the irreducible Markov chain $\mathcal{M}^3$.

reached from any other state with positive probability. Clearly, $\mathcal{M}^3$ is not irreducible if the probability mass is concentrated only in two points.

Let us consider the case where the probability mass is concentrated only at three points (vectors k^1, k^2, and k^3). One can easily check that in Figure 2, the vectors k^1, k^2, and k^3 correspond to the irreducible MC $\mathcal{M}^3$, and for any $\alpha \neq 0$, (24) is satisfied.

By rotating Figures 2a and 2b over 90°, 180°, and 270°, one can get all possible cases, which correspond to irreducible $\mathcal{M}^3$ with probability mass concentrated only in three points.

Similarly one can consider the other cases, where the probability mass is concentrated in more than three points. ■

In analogy to (13), we define the functions $V^1(\alpha)$ and $V^2(\alpha)$ as follows:

$$V^1(\alpha) \triangleq M_1^3(\alpha)M_2^1(\alpha) - M_2^3(\alpha)M_1^1(\alpha),$$

$$V^2(\alpha) \triangleq M_2^3(\alpha)M_1^2(\alpha) - M_1^3(\alpha)M_2^2(\alpha).$$

Note that $V^1(0) = V^1$ and $V^2(0) = V^2$.

Lemma 4.6: *The points α^1 and α^2 are finite.*

1. *Either $H^1(\alpha^1) \leq H^3(\alpha^1), M^3(\alpha^1) = 0$ or α^1 is the unique solution of the system*

$$H^1(\alpha) = H^3(\alpha), \qquad V^1(\alpha) = 0, \qquad M_2^3(\alpha) < 0. \tag{25}$$

2. *Either $H^2(\alpha^2) \leq H^3(\alpha^2), M^3(\alpha^2) = 0$ or α^2 is the unique solution of the system*

$$H^2(\alpha) = H^3(\alpha), \qquad V^2(\alpha) = 0, \qquad M_1^3(\alpha) < 0. \tag{26}$$

Proof: By Corollary 4.4, the function $H^3(\alpha)$ is strictly convex, and by Lemma 4.5, it has its minimum at a finite point. Hence, the set $\{\alpha : H^3(\alpha) \leq C\}$ with $C > H^3(\alpha^3)$ is a compact set. Since $H^1(0) = H^2(0) = H^3(0) = 0$, we have that

$$\alpha^1, \alpha^2 \in \{\alpha : H^3(\alpha) \leq 0\}.$$

Hence, α^1 and α^2 are finite.

Let us analyze the point α^2. For any α_2, define

$$\alpha_1^0 = \alpha_1^0(\alpha_2) \triangleq \arg\min_{\alpha_1} H^3(\alpha_1, \alpha_2).$$

From condition D it follows that

$$\sum_{k_1<0} \sum_{k_2} p_k^3 > 0 \quad \text{and} \quad \sum_{k_1>0} \sum_{k_2} p_k^3 > 0.$$

Then, for any fixed α_2, the function

$$H^3(\alpha_1, \alpha_2) = \log\left\{ \sum_{k_1<0} e^{\alpha_1 k_1} \left(\sum_{k_2} p_k^3 e^{\alpha_2 k_2} \right) + \sum_{k_1\geq 0} e^{\alpha_1 k_1} \left(\sum_{k_2} p_k^3 e^{\alpha_2 k_2} \right) \right\}$$

is strictly monotone decreasing in α_1 on the interval $(-\infty, \alpha_1^0)$ and it is strictly monotone increasing in α_1 on the interval $(\alpha_1^0, +\infty)$. From condition E, it follows for any fixed α_2 that the function

$$H^2(\alpha_1, \alpha_2) = \log\left\{ \sum_{k_1\geq 0} e^{\alpha_1 k_1} \sum_{k_2} p_k^2 e^{\alpha_2 k_2} \right\},$$

is strictly monotone increasing in α_1. Hence, for each fixed α_2, we have two cases.

Case 1:

$$H^2(\alpha_1^0(\alpha_2), \alpha_2) > H^3(\alpha_1^0(\alpha_2), \alpha_2).$$

Then, H^2 and H^3 as functions of α_1 intersect on $(-\infty, \alpha_1^0(\alpha_2))$ at

$$\alpha_1(\alpha_2) = \arg\min_{\alpha_1} \max\{H^2(\alpha_1, \alpha_2), H^3(\alpha_1, \alpha_2)\}.$$

This case is depicted in Figure 3a. Since $H^3(\alpha_1, \alpha_2)$ is strictly monotone decreasing in α_1 on $(-\infty, \alpha_1^0]$, then $M_1^3(\alpha_1(\alpha_2), \alpha_2) < 0$.

Case 2:

$$H^2(\alpha_1^0(\alpha_2), \alpha_2) \leq H^3(\alpha_1^0(\alpha_2), \alpha_2).$$

Then $\alpha_1^0(\alpha_2) = \alpha_1(\alpha_2)$, and we have that $M_1^3(\alpha_1^0(\alpha_2), \alpha_2) = 0$. This case is depicted in Figure 3b.

Since either Case 1 or Case 2 holds for each α_2, then either

$$H^2(\alpha^2) = H^3(\alpha^2), M_1^3(\alpha^2) < 0 \quad \text{or} \quad H^2(\alpha^2) \leq H^3(\alpha^2), M_1^3(\alpha^2) = 0.$$

From Case 1 and Case 2 it also follows that

$$H^3(\alpha^2) = \min_{\alpha_2} H^3(\alpha_1(\alpha_2), \alpha_2). \tag{27}$$

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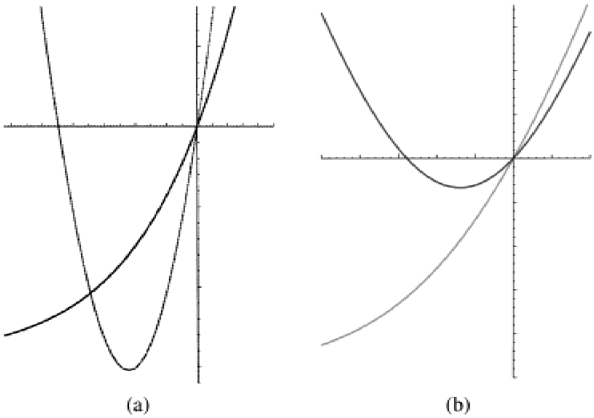

FIGURE 3. Two cases for each fixed α_2: (a) Case 1 and (b) Case 2.

Now let us show that either $V^2(\alpha^2) = 0$ or $M_2^3(\alpha^2) = 0$. Since (27) holds, it follows that

$$\frac{d}{d\alpha_2} H^3(\alpha_1(\alpha_2), \alpha_2) = 0 \quad \text{iff } V^2(\alpha_1(\alpha_2), \alpha_2) = 0 \text{ or } M^3(\alpha_1(\alpha_2), \alpha_2) = 0.$$

Here we give a geometrical proof. If $H^3(\alpha^3) \geq H^2(\alpha^3)$, then $\alpha^2 = \alpha^3$ and, clearly, $M^3(\alpha^3) = 0$. Suppose that $H^3(\alpha^3) < H^2(\alpha^3)$. Then, for any $C > 0$, let us consider the set

$$K_C = \{\alpha : H^3(\alpha) \leq C\} \cap \{\alpha : H^2(\alpha) \leq C\}. \tag{28}$$

Since the H-functions are convex, we have that the sets

$$\{\alpha : H^3(\alpha) \leq C\} \quad \text{and} \quad \{\alpha : H^2(\alpha) \leq C\}$$

are convex as well. If K_C is not empty, then K_C is a compact convex set and $\alpha^2 \in K_C$. Hence,

$$\alpha^2 \in \bigcap_{C : K_C \neq \varnothing} K_C.$$

Moreover, for any two levels $C_1 < C_2$, we have $K_{C_1} \subset K_{C_2}$. Therefore, the intersection of all nonempty K_C sets is compact. Since $\max\{H^2(\alpha), H^3(\alpha)\}$ is strictly convex, this intersection is a single point; hence, α^2 is uniquely defined. Moreover, it is the point where the functions H^2 and H^3 have a point of contact.

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Note that $M^2(\alpha^2)$ is a normal to the level line $\{\alpha : H^2(\alpha) = H^2(\alpha^2)\}$, and $M^3(\alpha^2)$ is a normal to the level line $\{\alpha : H^3(\alpha) = H^3(\alpha^2)\}$. This means that M^2 and M^3 have opposite directions at α^2; that is,

$$\frac{M_1^2(\alpha^2)}{M_2^2(\alpha^2)} = \frac{M_1^3(\alpha^2)}{M_2^3(\alpha^2)}$$

and so

$$V^2(\alpha^2) = 0.$$

This completes the proof of (26). The proof of (25) is similar. ■

LEMMA 4.7: *If $M^3(\hat{\alpha}) \neq 0$ then the twisted MC $\mathcal{M}(\hat{\alpha})$ is null recurrent or transient.*

PROOF: Suppose that $\hat{\alpha} = \alpha^1$. If $M^3(\alpha^1) \neq 0$ then by Lemma 4.6 we obtain

$$M_2^3(\alpha^1) < 0 \quad \text{and} \quad V^1(\alpha^1) = 0. \tag{29}$$

If in addition to (29) we have that $M_1^3(\alpha^1) \geq 0$, then by Lemma 3.2 the twisted MC $\mathcal{M}(\alpha^1)$ is null recurrent. Suppose that $M_1^3(\alpha^1) < 0$ in addition to (29); then, by the same lemma, the twisted MC $\mathcal{M}(\alpha^1)$ is null recurrent if $V^2(\alpha^1) \leq 0$; it is transient if $V^2(\alpha^1) > 0$.

The same analysis holds for the case $\hat{\alpha} = \alpha^2$. ■

LEMMA 4.8: *If the MC $\mathcal{M}$ is null recurrent, then $\hat{\alpha} = 0$.*

PROOF: Let the MC $\mathcal{M}$ be null recurrent. Suppose that $M^3(0) = 0$; then $\alpha^3 = 0$. Since $H^\Lambda(0) = 0$ for any face Λ, $\alpha^3 = \hat{\alpha}$ (i.e., $\hat{\alpha} = 0$).

Suppose that $M^3(0) \neq 0$. From Lemma 3.2, it follows that one of the following cases is satisfied:

$$V^1(0) = 0, \qquad M_2^3(0) < 0 \qquad \text{(see parts a and b of Lemma 3.2)}, \tag{30}$$

$$V^2(0) = 0, \qquad M_1^3(0) < 0 \qquad \text{(see parts a and c of Lemma 3.2)}. \tag{31}$$

Then by Lemma 4.6 we have that $\alpha^1 = 0$ if (30) holds, and $\alpha^2 = 0$ if (31) holds. Since $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$, we get that $\hat{\alpha} = 0$. ■

Remark 4.1: From Lemma 4.8, it follows for a null recurrent $\mathcal{M}$ that $H^\Lambda(\hat{\alpha}) = H^\Lambda(0) = 0$ for any Λ. Hence, in this case, (6) is fulfilled. In Section 5 we present some transient models for which (6) is satisfied, but also a transient MC for which it is not true. In [6] we show that coupled processor models do satisfy the condition. In general, an ergodic MC does not satisfy it, but for these models, the local rate function $L^0 = 0$.

4.4. Proof of the LD Lower Bound

The proof of the LD lower bound is rather involved. The assertion of Lemma 4.9 is the relation (8) in adequate notation. For the proof of Lemma 4.9 we need two more lemmas (4.10 and 4.11). In these lemmas, almost closed sets play an essential role.

LEMMA 4.9: *For any* $\delta > 0$ *and* $\delta' > 0$ *we have that*

$$\mathsf{P}\{A_{\delta N}\} \geq \exp\{N\tau H^3(\hat{\alpha}) - \delta' N\} \tag{32}$$

for all sufficiently large N.

PROOF:

Case 1: Let $M^3(\hat{\alpha}) = 0$. It means that

$$H^3(\hat{\alpha}) = \min_{\alpha} H^3(\alpha). \tag{33}$$

We have that

$$\mathsf{P}\{A_{\delta N}\} > \mathsf{P}\{A_{\delta N} \cap \Omega_m^3\},$$

where

$$\Omega_m^3 = \{\omega : S_t(\omega) \in \Lambda^3 \text{ for all } t > m\}.$$

From Lemma 4.2, by the change of measure, it follows for any α that

$$\mathsf{P}\{A_{\delta N} \cap \Omega_m^3\} = \mathsf{E}_\alpha \left\{ \exp\left\{ -(\alpha, S_{[\tau N]}) + (\alpha, S_0) + \sum_{t=0}^{[\tau N]-1} H^{S_t}(\alpha) \right\} I_{A_{\delta N} \cap \Omega_m^3} \right\}. \tag{34}$$

First, note that $\|S_{[\tau N]}(\omega)\| \leq \delta N$ for any $\omega \in A_{\delta N}$. Hence, $|(\alpha, S_{\tau N})| \leq \|\alpha\| \delta N$. Second, $S_t(\omega) \in \Lambda^3, t > m$, for any $\omega \in \Omega_m^3$. Then, for any α, we get

$$\sum_{t=0}^{[\tau N]-1} H^{S_t(\omega)}(\alpha) = [\tau N] H^3(\alpha) + \sum_{t=0}^{m} (H^{S_t(\omega)}(\alpha) - H^3(\alpha)). \tag{35}$$

Clearly, for any given α, $m > 0$, and $\delta' > 0$, there exists $N(\alpha, m, \delta')$ such that

$$\sum_{t=0}^{m} (H^{S_t(\omega)}(\alpha) - H^3(\alpha)) > -\delta' N \tag{36}$$

for all $N > N(\alpha, m, \delta')$. Recall that $H^3(\alpha) \geq H^3(\hat{\alpha})$ by (33). Then, from (35) and (36), it follows for any $\omega \in \Omega_m^3$ and all large N that

$$\sum_{t=0}^{[\tau N]-1} H^{S_t(\omega)}(\alpha) \geq [\tau N] H^3(\hat{\alpha}) - \delta' N.$$

Now, using (34), we obtain that

$$\mathsf{P}\{A_{\delta N} \cap \Omega_m^3\} \geq \exp\{N\tau H^3(\alpha^3) - \|\alpha\| \delta N - \delta' N\} \mathsf{P}_\alpha\{A_{\delta N} \cap \Omega_m^3\} \tag{37}$$

for any α, δ, and δ' and for all sufficiently large N.

Recall that $M^3(\hat{\alpha}) = 0$. Since the function $H^3(\alpha)$ is strictly convex, then for any $\delta > 0$, we can find α_δ close to $\hat{\alpha}$ such that

$$M_1^3(\alpha_\delta) > 0, \qquad M_2^3(\alpha_\delta) > 0, \quad \text{and} \quad \tau\|M^3(\alpha_\delta)\| < \delta. \tag{38}$$

We conclude the proof of Lemma 4.9 for Case 1 by proving the following lemma. Indeed, (37) together with (39) implies (32).

LEMMA 4.10: *There exists a positive constant $q > 0$ such that*

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N} \cap \Omega_m^3\} > q \tag{39}$$

for all sufficiently large N.

PROOF OF LEMMA 4.10: First, note that

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N} \cap \Omega_m^3\} \geq \mathsf{P}_{\alpha_\delta}\{A_{\delta N} \,|\, \Omega_m^3\}\mathsf{P}_{\alpha_\delta}\{\Omega_m^3\}.$$

Since $M_1^3(\alpha_\delta) > 0$ and $M_2^3(\alpha_\delta) > 0$ by (38), then by Lemma 3.3 the set Λ^3 is almost closed and so $\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega_m^3\} > 0$. Recall that

$$\Omega_m^3 \subset \Omega_{m+1}^3 \subset \cup_{m>0}\Omega_m^3$$

and, therefore, there exists a positive q_m such that

$$\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega^3\} > \mathsf{P}_{\alpha_\delta}\{\Omega_m^3\} > q_m$$

for all sufficiently large m.

On other hand,

$$M^3(\alpha_\delta) = E_{\alpha_\delta}\{S_{t+1} - S_t \,|\, S_t \in \Lambda^3\}$$

and, therefore, it follows from Kolmogorov's inequality that for any $\epsilon > 0$,

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N} \,|\, \Omega_m^3\} = \mathsf{P}_{\alpha_\delta}\left\{\sup_{0\le t\le[\tau N]}\|S_t - tM^3(\alpha_\delta)\| < \epsilon N \,|\, \Omega_m^3\right\} \to 1.$$

By the same relation (38), we can take ϵ small enough such that

$$\tau\|M^3(\alpha_\delta)\| < \delta - \epsilon.$$

Then

$$\left\{\sup_{0\le t\le[\tau N]}\|S_t^{\alpha_\delta} - tM^3(\alpha_\delta)\| < \epsilon N\right\} \subset A_{\delta N} = \left\{\sup_{0\le t\le[\tau N]}\|S_t^{\alpha_\delta}\| < \delta N\right\}.$$

It implies that

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N} \,|\, \Omega_m^3\} \to 1,$$

which completes the proof of (39). ■

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Case 2: Suppose that $\alpha^1 = \hat{\alpha}$ and $M^3(\hat{\alpha}) \neq 0$. We have that

$$\mathsf{P}\{A_{\delta N}\} > \mathsf{P}\{A_{\delta N} \cap \Omega_m^1\},$$

where

$$\Omega_m^1 = \{\omega : S_t(\omega) \in \Lambda^1 \cup \Lambda^3 \text{ for all } t > m\}.$$

From Lemma 4.2 by the change of measure, it follows for any α that

$$\mathsf{P}\{A_{\delta N} \cap \Omega_m^1\} = \mathsf{E}_\alpha \left\{ \exp \left\{ -(\alpha, S_{[\tau N]}) + (\alpha, S_0) + \sum_{t=0}^{[\tau N]-1} H^{S_t}(\alpha) \right\} I_{A_{\delta N} \cap \Omega_m^1} \right\}. \tag{40}$$

By Lemma 4.6 we have

$$H^1(\alpha^1) = H^3(\alpha^1), \qquad V^1(\alpha^1) = 0, \qquad M_2^3(\alpha^1) < 0.$$

Recall that $M_2^1(\alpha) > 0$ for all α by condition E. Hence, for any $\delta > 0$ and $\delta' > 0$ we can find α_δ close to α^1 such that

$$M_2^3(\alpha_\delta) < 0, \qquad 0 < \frac{V^1(\alpha_\delta)}{M_2^1(\alpha_\delta) - M_2^3(\alpha_\delta)} < \delta \tag{41}$$

and

$$H^1(\alpha_\delta) > H^3(\alpha^1) - \delta', \qquad H^3(\alpha_\delta) > H^3(\alpha^1) - \delta'. \tag{42}$$

Note that conditions D and E exclude the case that $V^1 \equiv 0$.

We have $S_t(\omega) \in \Lambda^1 \cup \Lambda^3$ for any $\omega \in \Omega_m^1$ and all $t > m$. This means that for any $t > m$

$$\text{either } H^{S_t(\omega)}(\alpha_\delta) = H^1(\alpha_\delta) \quad \text{or} \quad H^{S_t(\omega)}(\alpha_\delta) = H^3(\alpha_\delta).$$

Then, using (42), we get

$$\sum_{t>m}^{[\tau N]-1} H^{S_t(\omega)}(\alpha_\delta) > \sum_{t>m}^{[\tau N]-1} (H^3(\alpha^1) - \delta').$$

Therefore, for any given $m > 0$, $\delta > 0$, and $\delta' > 0$, there exists $N(m, \delta, \delta')$ such that

$$\sum_{t=0}^{[\tau N]-1} H^{S_t(\omega)}(\alpha_\delta) \geq [\tau N] H^3(\alpha^1) - \delta' N$$

for all $N > N(m, \delta, \delta')$. Then, using (40), we get

$$\mathsf{P}\{A_{\delta N} \cap \Omega_m^1\} \geq \exp\{N\tau H^3(\alpha^1) - \|\alpha_\delta\|\delta N - \delta' N\} \mathsf{P}_{\alpha_\delta}\{A_{\delta N} \cap \Omega_m^1\}.$$

In order to complete the proof of Lemma 4.9 for Case 2, we need to show the following lemma.

LEMMA 4.11: *There exists a positive constant q such that*

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N} \cap \Omega_m^1\} \geq q$$

for all sufficiently large N.

PROOF: Since $V^1(\alpha_\delta) > 0$ and $M_2^3(\alpha_\delta) < 0$ by (41), then from Lemma 3.3 we find that the set $\Lambda^1 \cup \Lambda^3$ is almost closed. Hence, $\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega_m^1\} > 0$, and since

$$\Omega_m^1 \subset \Omega_{m+1}^1 \subset \cup_{m>0}\Omega_m^1,$$

there exist positive q_m such that

$$\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega_m^1\} > \mathsf{P}_{\alpha_\delta}\{\Omega_m^1\} > q_m \tag{43}$$

and $q_m < q_{m+1}$.

Suppose, for simplicity of notation, that $\tau = 1$. Let

$$v(\alpha) = \left(\frac{V^1(\alpha)}{M_2^1(\alpha) - M_2^3(\alpha)}, 0\right).$$

It follows from (41) that $\|v(\alpha_\delta)\| < \delta$. Then, for any positive ϵ with

$$2\epsilon < \delta - \|v(\alpha_\delta)\|,$$

we have that

$$A_{\delta N} = \left\{S_0 = 0, \sup_{0\leq t\leq N}\|S_t\| < \delta N\right\} \supset \left\{\sup_{0\leq t\leq N}\|S_t - tv(\alpha_\delta)\| < 2\epsilon N\right\}. \tag{44}$$

By $M_t(\alpha)$ we denote the mean drift of S_t^α at time t; that is,

$$M_t(\alpha) = \mathsf{E}_\alpha\{S_{t+1} - S_t \,|\, S_l, 0 \leq l \leq t\}.$$

We introduce two events:

$$A_{\delta N}^1 = \left\{\sup_{0\leq t\leq N}\left\|S_t - \sum_{l=0}^{t-1} M_l(\alpha_\delta)\right\| < \epsilon N\right\},$$

$$A_{\delta N}^2 = \left\{\sup_{0\leq t\leq N}\left\|tv(\alpha_\delta) - \sum_{l=0}^{t-1} M_l(\alpha_\delta)\right\| < \epsilon N\right\}.$$

Then from (44) it follows for any N that

$$A_{\delta N} \supset A_{\delta N}^1 \cap A_{\delta N}^2. \tag{45}$$

Now we will give lower bounds for the probabilities $\mathsf{P}_{\alpha_\delta}\{A_{\delta N}^1\}$ and $\mathsf{P}_{\alpha_\delta}\{A_{\delta N}^2 \cap \Omega_m^1\}$. Let

$$\eta_t \triangleq S_t - S_0 - \sum_{l=0}^{t-1} M_l, \quad \eta_0 = 0.$$

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We have that

$$\eta_{t+1} - \eta_t = S_{t+1} - S_t - M_t \quad \text{for all } t > 0.$$

Since S_t has bounded jumps, η_t also has bounded jumps; say for some constant $D > 0$ that we have $\|\eta_{t+1} - \eta_t\| \le D$. Moreover, for any α,

$$\begin{aligned} \mathsf{E}_\alpha\{\eta_{t+1} - \eta_t \,|\, \eta_l, 0 \le l \le t\} &= \mathsf{E}_\alpha\{S_{t+1} - S_t \,|\, S_l, 0 \le l \le t\} \\ &\quad - \mathsf{E}_\alpha\{M_t \,|\, S_l, 0 \le l \le t\} = 0. \end{aligned}$$

This means that η_t is a zero-mean martingale with $\eta_0 = 0$. From the Azuma–Hoeffding inequality (see [13, p. 237]), it follows for any $\epsilon > 0$ and α that

$$\mathsf{P}_\alpha\{A^1_{\delta N}\} = \mathsf{P}_\alpha\left\{\sup_{0\le t\le N} \|\eta_t\| \le \epsilon N\right\} \ge 1 - \exp\left\{-\frac{1}{2D^2}\,\epsilon N\right\}.$$

Hence,

$$\mathsf{P}_\alpha\{A^1_{\delta N}\} \to 1 \quad \text{as } N \to +\infty. \tag{46}$$

Now we estimate the probability $\mathsf{P}_{\alpha_\delta}\{A^2_{\delta N} \,|\, \Omega^1_m\}$. Let

$$B_t = \left\{\left\|\sum_{l>0}^{t} (v(\alpha) - M_l(\alpha))\right\| > \epsilon t\right\}, \qquad \epsilon > 0.$$

Then for any α we have

$$\mathsf{P}_\alpha\{A^2_{\delta N} \,|\, \Omega^1_m\} \ge 1 - \mathsf{P}_\alpha\{\cup_{t>0}^N B_t \,|\, \Omega^1_m\}.$$

Clearly, for any sufficiently large N, there exists $t_N = o(N)$ such that for any ω,

$$\sup_{0\le t\le t_N} \left\| tv(\alpha) - \sum_{l=0}^{t-1} M_l(\alpha)\right\| < \epsilon N.$$

Then

$$\mathsf{P}_\alpha\{A^2_{\delta N} \,|\, \Omega^1_m\} > 1 - \sum_{t>t_N}^{N} \mathsf{P}_\alpha\{B_t\}. \tag{47}$$

For any $\omega \in \Omega^1_m$, either $M_t(\alpha) = M^1(\alpha)$ or $M_t(\alpha) = M^3(\alpha)$ for all $t > m$. Note that

$$v(\alpha) = \pi^0(\alpha)M^1(\alpha) + (1 - \pi^0(\alpha))M^3(\alpha), \qquad \pi^0(\alpha) = \frac{-M^3_2(\alpha)}{M^1_2(\alpha) - M^3_2(\alpha)},$$

where $\pi^0(\alpha)$ is the stationary probability that the first component of the twisted process S^α_t is equal to zero, conditioned on the sample path set Ω^1, where $\Omega^1 = \cup_m \Omega^1_m$.

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If $M_2^3(\alpha) < 0$, then $0 < \pi^0(\alpha) < 1$, and from the results of [1] it follows that for any $\epsilon > 0$ there exists $C > 0$ and $b > 0$ such that

$$\mathsf{P}_\alpha\{B_t \,|\, \Omega_m^1\} < C \exp\{-bt\}, \tag{48}$$

for all sufficiently large t. We have that $M_2^3(\alpha_\delta) < 0$. So (48) holds for $\alpha = \alpha_\delta$ as well. Then it follows from (47) and (48) that

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N}^2 \,|\, \Omega_m^1\} > 1 - C \sum_{t > t_N}^{+\infty} \exp\{-bt\}.$$

Since the series $\sum_t \exp\{-bt\}$ is convergent and $t_N \to +\infty$, we get that

$$\mathsf{P}_{\alpha_\delta}\{A_{\delta N}^2 \,|\, \Omega_m^1\} \to 1 \quad \text{as } N \to +\infty. \tag{49}$$

Now, the assertion of Lemma 4.11 follows from (43), (45), (46), and (49). ■

The proof of the case $\alpha^2 = \hat{\alpha}$ is similar. This completes the proof of Lemma 4.9. ■

4.5. Proof of the LD Upper Bound

The proof of the LD upper bound is a straightforward consequence of (6). It is proved in the following lemma.

LEMMA 4.12: *Let (6) be satisfied. Then, for any $\delta > 0$, we have*

$$\mathsf{P}\{A_{\delta N}\} \le \exp\{N\tau H^3(\hat{\alpha}) + \delta N\}$$

for all sufficiently large N.

PROOF: By (6), we have that

$$\sum_{t=0}^{[\tau N]-1} H^{S_t(\omega)}(\hat{\alpha}) \le [\tau N] H^3(\hat{\alpha}).$$

Therefore, from (21), it follows that

$$\mathsf{P}\{A_{\delta N}\} \le \mathsf{E}_{\hat{\alpha}}\{\exp\{-(\hat{\alpha}, S_{[\tau N]}) + (\hat{\alpha}, S_0) + [\tau N] H^3(\hat{\alpha})\}\}. \tag{50}$$

Since $S_0 = 0$ and $\|S_t(\omega)\| \le \delta N$ for any $\omega \in A_{\delta N}$ and $t \le [\tau N]$, then

$$\|(\hat{\alpha}, S_{[\tau N]})\| \le \|\hat{\alpha}\| \delta N.$$

It implies immediately that

$$\mathsf{P}\{A_{\delta N}\} \le \exp\{\|\hat{\alpha}\| \delta N + \tau N H^3(\hat{\alpha})\},$$

which proves the lemma. ■

5. APPLICATIONS AND NUMERICAL EXAMPLES

In this section we demonstrate how the local rate function L^0 can be computed for specific models (the computations have been done with Maple). In the subsequent article [6] we show how the large deviation theorems 2.1 and 3.1 can be applied in a coupled processors system.

5.1. Numerical Examples

In this subsection, we present a couple of models for which (6) is satisfied (i.e., the LD bounds hold). First, recall that

$$p^{\Lambda}_{k_1 k_2} = \mathsf{P}\{(i_1, i_2) \to (i_1, i_2) + (k_1, k_2)\} \quad \text{with } (i_1, i_2) \in \Lambda.$$

Model 1: Let the transition probabilities p^{Λ}_k have the values shown in Figure 4.

Figure 5 shows the level lines of the H-functions at level 0. We have

$$M_1^3 > 0, M_2^3 < 0 \quad \text{and} \quad M_1^1 > 0, M_2^1 > 0.$$

Then $V^1 > 0$ and, therefore, by Lemma 3.2, the MC $\mathcal{M}$ (at $\alpha = 0$) is transient.

In Figure 6 we have the H-functions level lines at the level equal to $H^3(\hat{\alpha})$. Here,

$$H^3(\hat{\alpha}) = H^2(\hat{\alpha}) \quad \text{with } \hat{\alpha} = \alpha^2 \approx (-0.27, 0.1).$$

We have

$$M_1^3(\hat{\alpha}) < 0, M_2^3(\hat{\alpha}) < 0 \quad \text{and} \quad V^1(\hat{\alpha}) > 0, V^2(\hat{\alpha}) = 0.$$

Then by Lemma 3.2, the MC $\mathcal{M}(\hat{\alpha})$ is transient.

Note that $H^0(\hat{\alpha}) \le H^3(\hat{\alpha})$ and $H^1(\hat{\alpha}) \le H^3(\hat{\alpha})$ (i.e., (6) is satisfied). Hence, by Theorem 2.1, the LD bounds hold with $L^0 = H^3(\hat{\alpha}) \approx -0.009$.

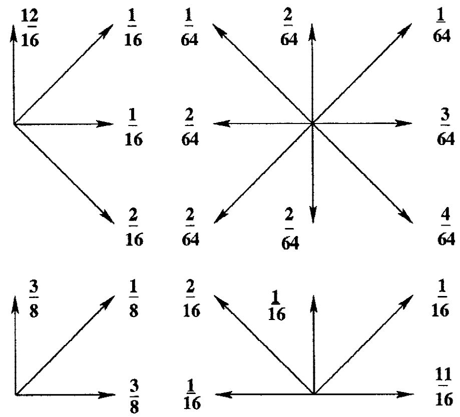

FIGURE 4. Model 1 transition probabilities.

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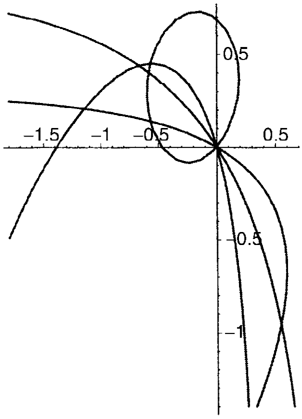

FIGURE 5. The H-function level lines at level 0.

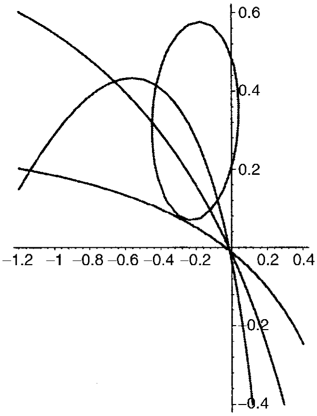

FIGURE 6. The H-function level lines at level $H^3(\hat{\alpha})$.

In Figure 7, the lines $V^1(\alpha) = 0$ and $V^2(\alpha) = 0$ intersect each other in two points:

$$\alpha^3 \approx (-0.21, 0.3) \quad \text{and} \quad \bar{\alpha} \approx (-0.7, -0.7).$$

Starting from the point $\bar{\alpha}$ and forward in the southwest direction, these lines form a cone-shaped region. We call this region the *ergodic* region. The interior of this region contains the points α for which $\mathcal{M}(\alpha)$ is ergodic. Outside of the ergodic region, we have the points for which $\mathcal{M}(\alpha)$ is transient. The boundary (the lines $V^1(\alpha) = 0$ and $V^2(\alpha) = 0$) correspond to null recurrent MC $\mathcal{M}(\alpha)$. In Figure 7 also, the level line $H^3(\alpha) = H^3(\hat{\alpha})$ is depicted; the intersection with $V^2(\alpha) = 0$ gives $\hat{\alpha}$.

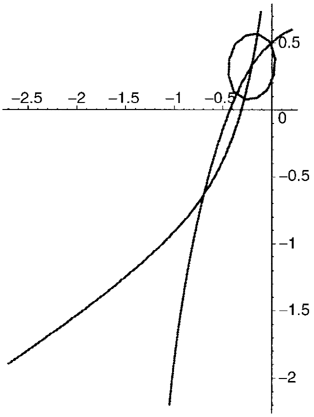

FIGURE 7. The lines $V^1 = 0$, $V^2 = 0$, $H^3 = 0$.

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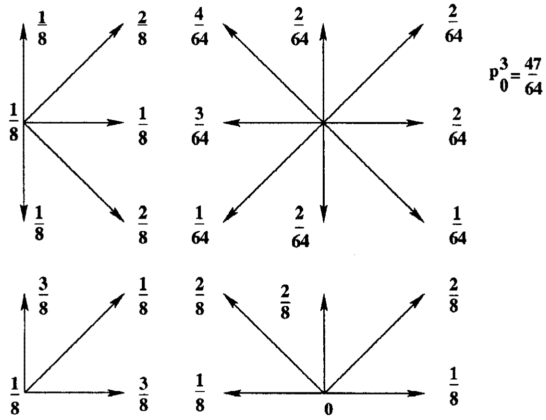

FIGURE 8. Model 2 transition probabilities.

Model 2: Next, we give a model with $\hat{\alpha} = \alpha^2$ and MC $\mathcal{M}(\hat{\alpha})$ is null recurrent. Let the transition probabilities have the values as shown in Figure 8.

Figure 9 shows the level lines of the H-functions at level 0. We have

$$M_1^3 < 0, M_2^3 > 0 \quad \text{and} \quad M_2^2 = 0, M_1^2 > 0.$$

Then, $V^2 > 0$ and, therefore, by Lemma 3.2, the MC $\mathcal{M}$ (at $\alpha = 0$) is transient.

In Figure 10 we have the H-functions level lines at the level equal to $H^3(\hat{\alpha})$ with $\hat{\alpha} = \alpha^2 \approx (-0.05, -0.27)$. We have that

$$M_1^3(\hat{\alpha}) < 0, M_2^3(\hat{\alpha}) > 0, \quad \text{and} \quad V^2(\hat{\alpha}) = 0.$$

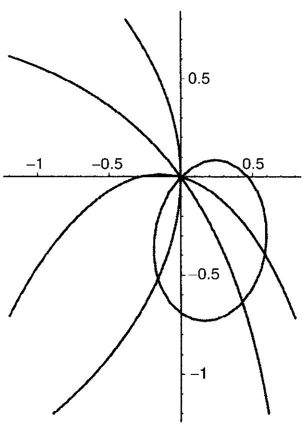

FIGURE 9. The H-function level lines at level 0.

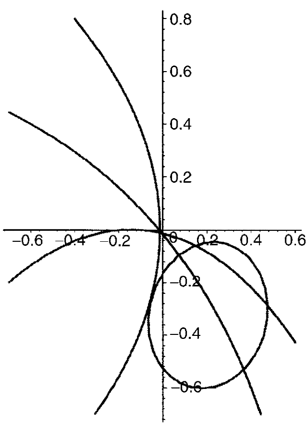

FIGURE 10. The H-function level lines at level $H^3(\hat{\alpha})$.

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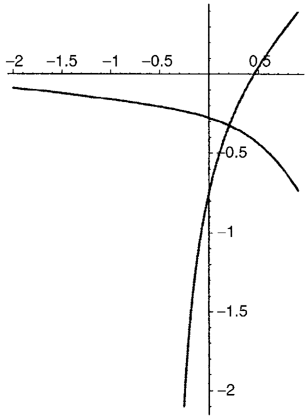

FIGURE 11. The level lines $V^1 = 0$, $V^2 = 0$

By Lemma 3.2 the MC $\mathcal{M}(\hat{\alpha})$ is null recurrent. It is easily checked that $H^0(\hat{\alpha}) < H^3(\hat{\alpha})$ and $H^1(\hat{\alpha}) \leq H^3(\hat{\alpha})$ (i.e., (6) of Theorem 2.1 is satisfied). Hence, also for this model, the LD theorem holds with $L^0 = H^3(\hat{\alpha}) \approx -0.0078$.

Figure 11 shows the ergodic region. Here the level lines $V^1(\alpha) = 0$ and $V^2(\alpha) = 0$ intersect at $\alpha^3 \approx (0.21, -0.32)$.

5.2. Open Problem

The numerical examples of Section 5.1 satisfy (6) of Theorem 2.1 and, hence, the local rate function L^0 is found. In model 3, $\mathcal{M}$ is transient and (6) is not satisfied. It is our conjecture that also in this case the LD lower bound is tight (i.e., the local rate function is equal to $H^3(\hat{\alpha})$).

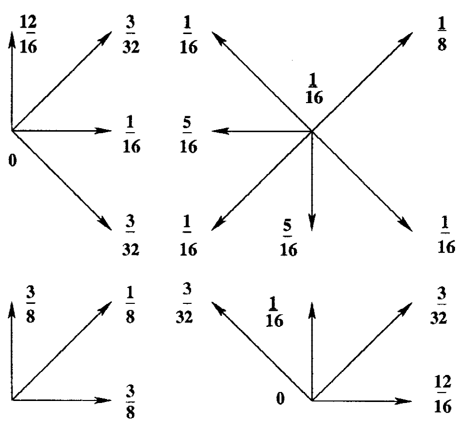

FIGURE 12. Model 3 transition probabilities.

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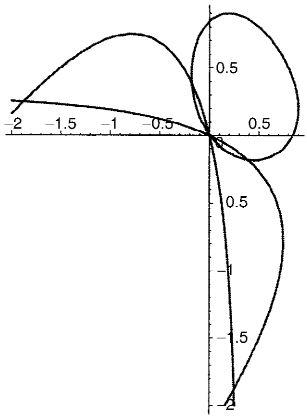

FIGURE 13. The lines $H^1 = H^2 = H^3 = 0$.

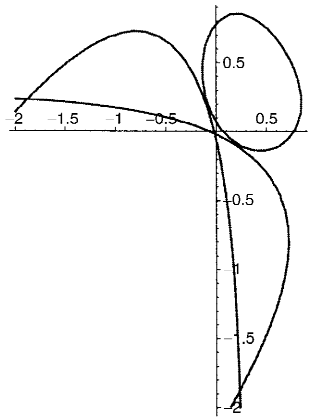

FIGURE 14. The H-function lines at level $H^3(\hat{\alpha})$.

Model 3: Let the transition probabilities p_k^Λ have the values shown in Figure 12. This model is interesting because of the following property: For each $\alpha \neq 0$,

$$\text{either } H^1(\alpha) \leq H^3(\alpha) < H^2(\alpha) \quad \text{or} \quad H^2(\alpha) \leq H^3(\alpha) < H^1(\alpha). \tag{51}$$

Hence, (6) is not satisfied. The level lines $H^1(\alpha) = 0$, $H^2(\alpha) = 0$, and $H^3(\alpha) = 0$ are depicted in Figure 13. In this model,

$$\alpha^1 \approx (-0.1, 0.2) \text{ and } \alpha^2 \approx (0.1, -0.2).$$

Moreover, $H^3(\alpha^1) = H^3(\alpha^2)$ since the transition probabilities are symmetric. Thus, we can take $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$.

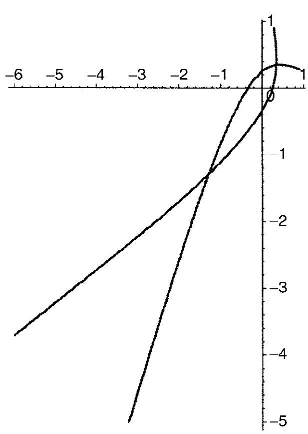

FIGURE 15. The level lines $V^1 = 0, V^2 = 0$.

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Figure 14 shows the level lines of the functions $H^1(\alpha)$, $H^2(\alpha)$, and $H^3(\alpha)$ at level $H^3(\hat{\alpha}) \approx -0.015$. In Figure 15, the level lines $V^1(\alpha) = 0$ and $V^2(\alpha) = 0$ intersect at two points:

$$\bar{\alpha} \approx (-1.3, -1.3) \quad \text{and} \quad \alpha^3 \approx (0.3, 0.3).$$

Starting from the point $\bar{\alpha}$, these lines give the ergodic region. Clearly, the points α^1 and α^2 do not belong to the ergodic region. Hence, the twisted MC $\mathcal{M}(\hat{\alpha})$ is transient, and from (51), we have that (6) is not satisfied.

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