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The convergence rate of the fast signal diffusion limit for a Keller–Segel–Stokes system with large initial data

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In this paper, we investigate the fast signal diffusion limit of solutions of the fully parabolic Keller–Segel–Stokes system to solution of the parabolic–elliptic-fluid counterpart in a two-dimensional or three-dimensional bounded domain with smooth boundary. Under the natural volume-filling assumption, we establish an algebraic convergence rate of the fast signal diffusion limit for general large initial data by developing a series of subtle bootstrap arguments for combinational functionals and using some maximal regularities. In our current setting, in particular, we can remove the restriction to asserting convergence only along some subsequence in Wang–Winkler and the second author (Cal. Var., 2019).

Keywords: Keller–Segel–Stokes system; large initial data; fast signal diffusion limit; convergence rate

2010 *Mathematics subject classification:* 35K55; 35Q92; 92C17

1. Introduction

Keller–Segel system. In their remarkable paper [12], Keller and Segel heuristically derived the following mathematical model (for the short Keller–Segel system):

$$\begin{cases} \partial_t n = \tau_1 \Delta n - \nabla \cdot \big(n \nabla c\big), \\ \partial_t c = \tau_2 \Delta c - c + n \end{cases} \tag{1.1}$$

to describe the growth phenomena mediated by a chemoattractant, that is, the aggregation of *Dictyostelium discoideum* due to an attractive chemical substance, where n and c stand for the cell density and the concentration of the chemical substance, respectively, whereas the positive constants τ_1 and τ_2 denote the diffusivity of the cells and of the chemoattractant, respectively. The Keller–Segel system (1.1), which looks simple at first sight, is a very rich mathematical system and it has been an object of very extensive investigation for the last 50 years. A striking feature of system (1.1) consists of its ability to spontaneously enhance the singularity formation in the sense of finite-time blow-up throughout various ranges of its ingredients.

For instance, all solutions to the homogenous Neumann initial-boundary value problem of system (1.1) in a bounded domain $\Omega \subset \mathbb{R}^d$ remain global and bounded when either $d = 1$, or $d = 2$ and the initial total mass of cells $\int_\Omega n_0 < 4\pi$ (see [**20**, **21**]), whereas it possesses some solutions blowing up in finite time when either $d \geqslant 3$, or when $d = 2$ and the total mass $\int_\Omega n_0$ is large in some sense (see [**10**, **34**]).

Since the blow-up is an extreme case, the early literature also aims to confirm an intuitive idea that the tendency towards blow-up can be weakened if at large cell densities the cross-diffusion is inhibited. In particular, for the prototypical case

$$\begin{cases} \partial_t n = \tau_1 \Delta n - \nabla \cdot \big(nS(n)\nabla c\big), \\ \partial_t c = \tau_2 \Delta c - c + n, \end{cases} \tag{1.2}$$

a complete picture is available: if

$$S(n) \leqslant \frac{C_S}{(1+n)^\alpha} \quad \text{with} \quad \alpha > 1 - \frac{2}{d} \tag{1.3}$$

for some positive constant C_S, then all solutions to the homogeneous Neumann initial-boundary value problem of system (1.2) are global and uniformly bounded, whereas if

$$S(n) \geqslant \frac{C_S}{(1+n)^\alpha} \quad \text{with} \quad \alpha < 1 - \frac{2}{d}$$

and $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a ball, then some solution may blow up in finite time (see, e.g. Horstmann–Winkler [**11**]). Thus $\alpha_c := 1 - 2/d$ is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect.

Chemotaxis-(Navier–)Stokes system. The interaction between populations of chemotactically migrating individuals and viscous fluid environments has been another objective of considerable developments in the mathematical literature during the past decade. This is partially stimulated by the striking experiments revealing spontaneous formation of plume-like aggregates in populations of *Bacillus subtilis* suspended in sessile water drops, in such situations it may be necessary to take into account the mutual interaction of cells and their movement on the one hand, and of the surrounding medium on the other hand (see Tuval *et al.* [**26**]). Accordingly, the following coupled chemotaxis-(Navier–)Stokes system has been proposed by Tuval *et al.* [**26**] to describe the chemotactic movement, signal consumption, transport of both cells and signal through the fluid, and the buoyancy-driven effect of cells on the fluid dynamics:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS\nabla c\big), \\ \partial_t c + u \cdot \nabla c = \Delta c - nf(c), \\ \partial_t u + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \\ \qquad \nabla \cdot u = 0 \end{cases} \tag{1.4}$$

for the cell population density n, the signal concentration c and the fluid variables u and P, where the chemotactic sensitivity S, the signal consumption rate f and the gravitational potential ϕ are given parameter functions. The coefficient $\kappa \geqslant 0$ is related to the strength of nonlinear fluid convection. System (1.4) has been the

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groundwork for many articles concerning the mathematical analysis of chemotaxis–fluid interaction since the first analytical results asserting local existence of weak solutions in Lorz [**16**]. Obtaining results concerning the global existence of solutions is far from trivial. In the two-dimensional (2D) setting, global classical solutions stemming from reasonably smooth initial data have been shown to exist in [**5**, **33**], whereas many results treating variants of system (1.4) in three-dimensional (3D) frameworks are again restricted to weak solutions emanating from small initial data [**4**]. Intense analysis on the latter has confirmed that after some relaxation time, this weak solution enjoys further properties of an eventual energy solution [**37**]. We also refer to more recent studies on system (1.4) for the large time behaviour [**2**, **29**, **35**, **36**, **40**] or for the more realistic boundary conditions on the chemical signal [**22**, **25**, **40**].

Concerning the framework where the chemical is produced by the cells instead of consumed, as in the actual Keller–Segel model, that is

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(n)\nabla c\big), \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \\ \qquad\quad \nabla \cdot u = 0, \end{cases} \tag{1.5}$$

a wide array of studies dedicated to the mathematical analysis under the volume-filling assumption (1.3), where indeed the scalar function S can be replaced by a general matrix-valued sensitivity function $S(x, n, c)$ (see [**27**, **28**, **31**, **32**, **38**]) by following a recent modelling approach in which for chemotactic movements of bacterial populations, the chemotactic sensitivity S is in general a tensor and when the cells are subject to external forces, this tensor need not be symmetric [**41**]. In particular, it has been showed that under the assumption of

$$|S(x, n, c)| \leqslant \frac{C_S}{(1+n)^\alpha},$$

a corresponding homogeneous Neumann–Neumann–Dirichlet initial-boundary value problem of system (1.5) with $\kappa = 0$ admits a global bounded classical solutions for all sufficiently regular initial data when $\alpha > 0$ for $d = 2$ or $\alpha > 1/3$ for $d = 3$ ([**31**, **38**]), which are consistent with the fluid-free system (1.2).

Fast signal diffusion limit. When the small cell diffusion (or the fast signal diffusion) is involved, we may denote

$$\epsilon := \frac{\tau_1}{\tau_2}$$

by the ratio between the diffusivity of the cells and of the chemoattractant, which can be regarded as a relaxation time scale such that ϵ^{-1} is the rate towards equilibrium, and replace $\tau_1 t$ with t in the original model (1.1) to rewrite the

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system as

$$
\begin{cases}
\partial_t n = \Delta n - \dfrac{1}{\tau_1}\nabla\cdot\big(n\nabla c\big),\\
\epsilon\partial_t c = \Delta c - \dfrac{1}{\tau_2}c + \dfrac{1}{\tau_2}n.
\end{cases}
\tag{1.6}
$$

Corresponding to a fast relaxation of the chemical substance c, i.e. $\epsilon \searrow 0$, system (1.6) is formally reduced to a parabolic–elliptic system

$$
\begin{cases}
\partial_t n = \Delta n - \dfrac{1}{\tau_1}\nabla\cdot\big(n\nabla c\big),\\
\quad 0 = \Delta c - \dfrac{1}{\tau_2}c + \dfrac{1}{\tau_2}n.
\end{cases}
\tag{1.7}
$$

The parabolic–elliptic chemotaxis system (1.7) substantially differs from their fully parabolic prototypical system (1.6) due to the circumstance that the cross-diffusive interaction in $(1.7)_1$ involves a certain memory. A comprehensive picture for the 2D parabolic–elliptic system (1.7) was obtained in [**24**], where the Dirac mass formation and finiteness of blow-up points were derived without substantial restrictions. Even in the mass critical case, in which solutions to the Cauchy problem of the parabolic–elliptic system (1.7) in $\mathbb{R}^2$ without the damping term $-(1/\tau_2)c$ on the second equation exist globally but blow up in infinite time, it is known that the spatial profile near the corresponding blow-up time $T = \infty$ is essentially dictated by Dirac distributions (see [**3**, **9**]).

It seems natural to seek for a uniform control of the error made when approximating a fully parabolic Keller–Segel system (1.6) by its parabolic–elliptic simplification (1.7) in terms of the parameter ϵ. Recently, Liu *et al.* [**15**] developed an asymptotic method to numerically show that the limit of the solutions to the Cauchy problem of system (1.6) without the damping term $-(1/\tau_2)c$ is a solution to the corresponding parabolic–elliptic counterpart (1.7). However, the above asymptotic analysis is unclear from a rigorous standpoint, and until quite recently, with [**7**, **19**, **30**] we now also have two theoretical studies linking the two systems with coupled fluid. Precisely, in a smoothly bounded physical domain $\Omega \subset \mathbb{R}^d$ with $d \geqslant 1$ and under appropriate assumptions on the model ingredients, Wang *et al.* [**30**] confirmed that some subsequence of solutions to the fully parabolic Keller–Segel–(Navier–)Stokes system

$$
\begin{cases}
\partial_t n_\epsilon + u_\epsilon\cdot\nabla n_\epsilon = \Delta n_\epsilon - \nabla\cdot\big(n_\epsilon S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\big) + f(x,n_\epsilon,c_\epsilon),\\
\epsilon\partial_t c_\epsilon + u_\epsilon\cdot\nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon,\\
\partial_t u_\epsilon + \kappa(u_\epsilon\cdot\nabla)u_\epsilon + \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon\nabla\phi,\\
\qquad \nabla\cdot u_\epsilon = 0
\end{cases}
\tag{1.8}
$$

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does in fact converge to a solution of the parabolic–elliptic simplification

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot\big(nS(x,n,c)\cdot\nabla c\big) + f(x,n,c),\\ \qquad u\cdot\nabla c = \Delta c - c + n,\\ \partial_t u + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi,\\ \qquad \nabla\cdot u = 0 \end{cases} \tag{1.9}$$

in $\Omega\times(0,T)$ under the presupposed bounds

$$\sup_{\epsilon}\|\nabla c_\epsilon\|_{L^\lambda((0,T);L^q(\Omega))}<\infty \qquad \text{and} \qquad \sup_{\epsilon}\|u_\epsilon\|_{L^\infty((0,T);L^r(\Omega))}<\infty \tag{1.10}$$

with some $\lambda\in(2,\infty]$, $q>d$ and $r>\max\{2,d\}$ satisfying $1/\lambda+d/2q<1/2$. They also concretized this in the framework of two particular examples: for certain small-data solutions to an unforced chemotaxis-Navier–Stokes system, and for arbitrary solutions to a one-dimensional fluid-free logistic Keller–Segel model. As far as the Cauchy problem is concerned, we may refer to [**1**, **13**, **14**, **23**] for the fluid-free case. For instance, Biler–Brandolese [**1**] established some results on convergence, in strong topologies, of solutions of the minimal system (1.6) in the plane to solutions of the corresponding system (1.7). Their proofs relied on some suitable space–time estimates, implying the global existence of slowly decaying solutions for these models, under a suitable smallness assumption on $n_\epsilon(x,0)$.

Main results. In light of the above, it seems natural to further investigate the fast signal diffusion limit in the unforced Keller–Segel–Stokes system (1.8). Our aim is twofold: on the one hand, we will consider the global classical solutions for general initial data in a 2D or 3D setting, which require a natural volume-filling assumption of the form (1.3) due to the possible singularity in the fluid-free case as mentioned before; on the other hand, we will show the convergence containing an algebraic rate for the whole sequence of solutions. In order to make this more precise, we shall accordingly be concerned with the initial-boundary value problem:

$$\begin{cases} \partial_t n_\epsilon + u_\epsilon\cdot\nabla n_\epsilon = \Delta n_\epsilon - \nabla\cdot\big(n_\epsilon S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\big), & x\in\Omega,\ t>0,\\ \epsilon\partial_t c_\epsilon + u_\epsilon\cdot\nabla c_\epsilon = \Delta c_\epsilon - c_\epsilon + n_\epsilon, & x\in\Omega,\ t>0,\\ \partial_t u_\epsilon + \nabla P_\epsilon = \Delta u_\epsilon + n_\epsilon\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u_\epsilon = 0, & x\in\Omega,\ t>0,\\ \big(\nabla n_\epsilon - n_\epsilon S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\big)\cdot\nu = \nabla c_\epsilon\cdot\nu = 0, \quad u_\epsilon = 0, & x\in\partial\Omega,\ t>0,\\ n_\epsilon(x,0)=n_0(x), \quad c_\epsilon(x,0)=c_0(x), \quad u_\epsilon(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.11}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary. The known chemotactic sensitivity function $S=(S_{ij})_{d\times d}$ is assumed to satisfy the requirements of regularity

$$S_{ij}(x,n_\epsilon,c_\epsilon)\in C^2\big(\overline{\Omega}\times[0,\infty)\times[0,\infty)\big) \tag{1.12}$$

and the structural restrictions

$$\big|S(x,n_\epsilon,c_\epsilon)\big| \leqslant \frac{C_S}{(1+n_\epsilon)^\alpha} \tag{1.13}$$

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for some positive constants C_S and α, and the gravitational potential function ϕ satisfies

$$\phi \in W^{2,\infty}(\Omega). \tag{1.14}$$

As for the initial data, our basic regularity assumptions will be that

$$\begin{cases} n_0 \in W^{1,\infty}(\Omega), \quad n_0 \geqslant 0 \quad \text{and} \quad n_0 \not\equiv 0 \quad \text{in} \quad \overline{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega), \quad c_0 \geqslant 0 \quad \text{and} \quad c_0 \not\equiv 0 \quad \text{in} \quad \overline{\Omega}, \\ u_0 \in W^{2,\infty}\big(\Omega; \mathbb{R}^d\big) \quad \text{satisfies} \quad \nabla \cdot u_0 \equiv 0 \text{ in } \Omega \quad \text{and} \quad u_0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.15}$$

It has been shown that for each fixed $\epsilon > 0$, system (1.11) possesses a global bounded classical solution $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ under the condition that $\alpha > 0$ for $d = 2$ [**31**] or $\alpha > 1/3$ for $d = 3$ [**38**]. Accordingly, we shall examine the relationship between these global solutions to system (1.11) and those to

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(x,n,c) \cdot \nabla c\big), & x \in \Omega,\ t > 0, \\ \qquad u \cdot \nabla c = \Delta c - c + n, & x \in \Omega,\ t > 0, \\ \partial_t u + \nabla P = \Delta u + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \qquad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \big(\nabla n - nS(x,n,c) \cdot \nabla c\big) \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x), \qquad u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.16}$$

in a setting as general as possible. We remark that it seems that there is no any result on the well-posedness of the coupled chemotaxis–fluid limit system (1.9) except for Lorz [**17**], where the blow-up delay was numerically showed for the 2D Cauchy problem with $S \equiv 1$.

Our main result asserts temporally uniform convergence of these solutions in the limit $\epsilon \to 0$; more precisely:

Theorem 1.1. *Let $d = 2, 3$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Assume that $\alpha > 0$ for $d = 2$, whereas $\alpha > 1/2$ for $d = 3$. Suppose that (1.12)–(1.15) hold and that for $\epsilon \in (0,1)$, $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ solves system (1.11) classically in $\Omega \times (0,\infty)$ with $n_\epsilon \geqslant 0$ and $c_\epsilon \geqslant 0$ in $\Omega \times (0,\infty)$. Then there exists a unique classical solution (n, c, u, P) to system (1.16) in $\Omega \times (0,\infty)$ with the property that*

$$\begin{cases} \|n_\epsilon(\cdot,t) - n(\cdot,t)\|_{L^2(\Omega)} + \|n_\epsilon(\cdot,s) - n(\cdot,s)\|_{L^2((0,t);H^1(\Omega))} \leqslant C_1 \mathrm{e}^{C_1 t} \epsilon^{1/2}, \\ \|c_\epsilon(\cdot,s) - c(\cdot,s)\|_{L^2((0,t);H^1(\Omega))} \leqslant C_1 \mathrm{e}^{C_1 t} \epsilon^{1/2}, \\ \|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} + \|u_\epsilon(\cdot,s) - u(\cdot,s)\|_{L^2((0,t);H^1(\Omega))} \leqslant C_1 \mathrm{e}^{C_1 t} \epsilon^{1/2} \end{cases}$$

for all $t \in (0,\infty)$ and some uniform positive constant C_1. For each $\theta \in (0,1)$ and $p \geqslant 4$, we also have

$$\begin{cases} \|A^\theta u_\epsilon(\cdot,t) - A^\theta u(\cdot,t)\|_{L^2(\Omega)} \leqslant C_2 \mathrm{e}^{C_2 t} \epsilon^{1/2}, \\ \|n_\epsilon(\cdot,t) - n(\cdot,t)\|_{L^p(\Omega)} \leqslant C_3 \mathrm{e}^{C_3 t} \epsilon^{(p+d-2)/(p(d+2))} \end{cases}$$

for all $t \in (0,\infty)$ and some positive constants $C_2 := C_2(\theta)$ and $C_3 := C_3(p)$.

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REMARK 1.1. Even for the large initial data, theorem 1.1 removes the restriction to asserting convergence only along some subsequence in [**30**] (theorem 1.1 on p. 4) and presents a precise convergence rate.

REMARK 1.2. There is a subsequence of $(n_\epsilon, c_\epsilon, u_\epsilon)$ such that the strong and pointwise convergence properties hold by applying lemmas 3.8 and 4.5 to the convergence criterion in lemma 2.3.

REMARK 1.3. Due to the potential initial layer, it is natural to further analyse the uniform estimate of the correction term for $c_\epsilon - c$. This is our future research.

Key steps in our analysis. In §3 and §4, we shall prove the existence of classical solution (n, c, u, P) to system (1.16) for 3D and 2D cases, respectively, which in particular is the limit of some subsequence of solutions $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ to system (1.11). Here, our basic idea consists of finding a uniform *a priori* estimate for $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$, which needs to satisfy the supposedly present bounds of the form (1.10) on ∇c_ϵ and u_ϵ.

In the 3D case, the evident mass conservation property $\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)}$ and the bound on $\|c_\epsilon(\cdot,t)\|_{L^1(\Omega)}$ (lemma 2.1) is not sufficient to derive some useful regularity information of u_ϵ and c_ϵ. Thus we will first track the time evolution of the combinational functional of the form

$$\|n_\epsilon(\cdot,t)\|^{2\alpha}_{L^{2\alpha}(\Omega)} + K\epsilon\|c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

for some K to obtain a slightly improved uniform $L^{2\alpha}$ bound for n_ϵ with respect to ϵ for the 3D case (lemma 3.1), which does not give the L^2 bound for c_ϵ but ensures the L^{r_1} bound of u_ϵ for some $r_1 > 3$ (corollary 3.1). The latter will further yield an improved L^p bound for n_ϵ with $2\alpha + 1/3 < p < (7/3)\alpha + 1/3$ by analysing an ODI for the functional

$$\|n_\epsilon(\cdot,t)\|^p_{L^p(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}$$

(lemma 3.3) and the L^r bound for u_ϵ with any $r > 1$ (corollary 3.3). Although lacking uniform parabolicity and thus uniform bound for ∇c_ϵ, a key step is that a very subtle bootstrap argument will finally yield the L^2 bound for n_ϵ (corollary 3.4) by investigating the time evolution of the combinational functional

$$\|n_\epsilon(\cdot,t)\|^{s_k}_{L^{s_k}(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}, \qquad (k = 1, 2, 3, \ldots) \tag{1.17}$$

for some sequence $\{s_k\}$ (lemma 3.4), which together with the damping effect of c_ϵ provides the uniform L^2 bound for ∇c_ϵ (lemma 3.5). Then the eventual L^p bound of n_ϵ with $p > 1$ and L^4 bound of ∇c_ϵ will follow from some similar but more complicated calculations (lemma 3.6, corollary 3.5 and lemma 3.7). These bounds together with the convergence criterion (lemma 2.3) imply the convergence of some subsequence of $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$.

We take a similar strategy to deal with the 2D case. Indeed, unlike the 3D case, the 2D mass conservation property $\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)}$ is sufficient to ensure the validity of the initial iterations and indeed the L^4 bound of n_ϵ can be reached by tracking the time evolution of the functional of the form (1.17) with different parameter choices (lemma 4.2 and corollary 4.1), which yields the uniform L^4 bounds of ∇c_ϵ and u_ϵ by following the proof of lemma 3.7. Again combining these bounds with the convergence criterion, we obtain the convergence of some subsequence of $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$.

Finally, in §5, we will attempt to establish the error control for the whole sequence and obtain a algebraic convergence rate by using a new entropy-like evolution estimate for the mixed functional of the form

$$\|n_\epsilon(\cdot,t)-n(\cdot,t)\|^2_{L^2(\Omega)}+\epsilon\|c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}+\|u_\epsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}.$$

Here, a key observation is the linear growth estimate

$$\int_0^t\int_\Omega \partial_t c_\epsilon c \leqslant C(1+t),$$

which follows from a uniform difference quotient estimate and some maximal regularity estimates for parabolic system and Stokes system. With the lower order estimates at hand, the higher convergence estimates of u_ϵ and n_ϵ can be derived from the standard smoothing effect of Stokes semigroup and the energy estimate, respectively.

Notation: In the rest of this paper, we will suppose that $(n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon)$ is a classical solution to system (1.11) in $\Omega\times(0,\infty)$ with $\epsilon\in(0,1)$.

2. Preliminaries

In this section, we will collect some basic results which are valid for the solutions of system (1.11) and have nothing to do with the particular choice of $\epsilon>0$.

LEMMA 2.1. *Suppose that* (1.12)–(1.15) *hold. Then* $n_\epsilon\geqslant 0$ *and* $c_\epsilon\geqslant 0$ *in* $\Omega\times(0,\infty)$, *and*

$$\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)}=\|n_0\|_{L^1(\Omega)} \quad \text{for all} \quad t\in(0,\infty), \tag{2.1}$$

and

$$\|c_\epsilon(\cdot,t)\|_{L^1(\Omega)}\leqslant \max\left\{\|n_0\|_{L^1(\Omega)},\,\|c_0\|_{L^1(\Omega)}\right\} \qquad \text{for all} \quad t\in(0,\infty). \tag{2.2}$$

Proof. The proof is rather standard and thus we omit the details. □

LEMMA 2.2. *Suppose that* (1.12)–(1.15) *hold. If*

$$\|n_\epsilon(\cdot,t)\|_{L^s(\Omega)}\leqslant K \quad \text{for all} \quad t\in(0,\infty), \tag{2.3}$$

for some $s>1$ *and* $K>0$, *there exists a positive constant* C *depending only on* s, K *and* c_0 *such that*

$$\|c_\epsilon(\cdot,t)\|_{L^s(\Omega)}\leqslant C \quad \text{for all} \quad t\in(0,\infty).$$

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Proof. We multiply equation $(1.11)_2$ by c_ϵ^{s-1}, and use the integration by parts and the Young inequality to obtain

$$\frac{\epsilon}{s}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon^s + (s-1)\int_\Omega c_\epsilon^{s-2}|\nabla c_\epsilon|^2 + \int_\Omega c_\epsilon^s = \int_\Omega n_\epsilon c_\epsilon^{s-1} \leqslant \frac{s-1}{s}\int_\Omega c_\epsilon^s + \frac{1}{s}\int_\Omega n_\epsilon^s$$

and thus

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon^s + s(s-1)\int_\Omega c_\epsilon^{s-2}|\nabla c_\epsilon|^2 + \int_\Omega c_\epsilon^s \leqslant \int_\Omega n_\epsilon^s$$

for all $t \in (0,\infty)$. By setting $y_\epsilon(t) := \int_\Omega c_\epsilon^s(\cdot,t)$ and using (2.3), we have

$$y_\epsilon'(t) + \frac{1}{\epsilon}y_\epsilon(t) \leqslant \frac{1}{\epsilon}K^s \quad \text{for all} \quad t \in (0,\infty).$$

Then by a basic calculation, we deduce that $y_\epsilon(t) \leqslant \max\left\{\int_\Omega c_0^s,\, K^s\right\}$ for all $t \in (0,\infty)$. This completes the proof of lemma 2.2. □

LEMMA 2.3 (Convergence criterion). *Let $d \geqslant 1$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary, and suppose that (1.12)–(1.15) hold. Furthermore, suppose that $(\epsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$ is such that $\epsilon_j \searrow 0$ as $j \to \infty$, and that $((n_\epsilon, c_\epsilon, u_\epsilon, P_\epsilon))_{\epsilon\in(\epsilon_j)_{j\in\mathbb{N}}}$ is such that*

$$\sup_{\epsilon\in(\epsilon_j)_{j\in\mathbb{N}}} \|\nabla c_\epsilon\|_{L^\lambda((0,\infty);L^q(\Omega))} < \infty \qquad \text{and} \qquad \sup_{\epsilon\in(\epsilon_j)_{j\in\mathbb{N}}} \|u_\epsilon\|_{L^\infty((0,\infty);L^r(\Omega))} < \infty$$

with some $\lambda \in (2,\infty], q > d$ and $r > \max\{2,d\}$ satisfying

$$\frac{1}{\lambda} + \frac{d}{2q} < \frac{1}{2}.$$

Then there exist a subsequence $(\epsilon_{j_k})_{k\in\mathbb{N}}$ of $(\epsilon_j)_{j\in\mathbb{N}}$ and functions

$$\begin{aligned}
&n \in C^{\theta,\theta/2}\big(\overline{\Omega}\times[0,+\infty)\big) \cap C^{2+\theta,1+\theta/2}\big(\overline{\Omega}\times(0,+\infty)\big),\\
&c \in C^{2+\theta,\theta}\big(\overline{\Omega}\times(0,+\infty)\big),\\
&u \in C^{\theta,\theta/2}\big(\overline{\Omega}\times[0,+\infty);\mathbb{R}^d\big) \cap C^{2,1}\big(\overline{\Omega}\times(0,+\infty);\mathbb{R}^d\big),\\
&P \in C^{1,0}\big(\Omega\times(0,+\infty)\big)
\end{aligned}$$

for some $\theta \in (0,1)$ such that (n,c,u,P) is a classical solution to system (1.16) in $\Omega \times (0,\infty)$ with the properties that

$$\begin{aligned}
n_\epsilon \to n \quad &\text{in} \quad C^0\big(\overline{\Omega}\times[0,\infty)\big),\\
n_\epsilon \rightharpoonup n \quad &\text{in} \quad L^2_{loc}\big((0,\infty);W^{1,2}(\Omega)\big),\\
c_\epsilon \to c \quad &\text{in} \quad L^\infty_{loc}\big((0,\infty);C^0(\overline{\Omega})\big)\bigcap L^2_{loc}\big((0,\infty);W^{1,2}(\Omega)\big),\\
\nabla c_\epsilon \overset{*}{\rightharpoonup} \nabla c \quad &\text{in} \quad \bigcap_{\widehat{q}>d} L^\infty_{loc}\big((0,\infty);W^{1,\widehat{q}}(\Omega)\big)\bigcap L^\infty_{loc}\big(\Omega\times(0,\infty)\big) \quad \text{and}\\
u_\epsilon \to u \quad &\text{in} \quad C^0\big(\overline{\Omega}\times[0,\infty);\mathbb{R}^d\big)\bigcap C^{2,1}_{loc}\big(\overline{\Omega}\times(0,\infty);\mathbb{R}^d\big)
\end{aligned}$$

as $\epsilon = \epsilon_{j_k} \searrow 0$.

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Proof. The desired conclusion can be directly derived from theorem 1.1 in [**30**] by taking $T = \infty$, where the convexity of Ω is required but can actually be removed as pointed out by Remark (i) in [**30**]. □

For the chemotaxis–fluid system, it is highly nontrivial to obtain the global uniform (with respect to ϵ) *a priori* estimates from the basic mass conservation of n_ϵ although we have showed the above convergence criterion. The main reason is that due to the loss of uniform parabolicity in c_ϵ equation, it is usually difficult to derive the temporal boundedness for c_ϵ in some Sobolev spaces.

3. Existence of solution to limit system: the 3D case

In this section, we focus on the existence of classical solution to limit system (1.16) in the 3D setting. The basic methods are a series of subtle coupled functional evolution estimates and a bootstrap argument.

3.1. $L^{2\alpha}$ regularity of n_ϵ

The evident mass conservation property $\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)}$ is not sufficient to derive some useful regularity information for u_ϵ and c_ϵ directly. To overcome this difficulty, we shall first establish a slightly improved bound for n_ϵ by tracking the time evolution of a combinational functional of the form $\|n_\epsilon(\cdot,t)\|^{2\alpha}_{L^{2\alpha}(\Omega)} + K\epsilon\|c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)}$.

Lemma 3.1. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *Then there exists a positive constant* C *depending only on* α, n_0, c_0 *and* u_0 *such that*

$$\|n_\epsilon(\cdot,t)\|_{L^{2\alpha}(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty). \tag{3.1}$$

Proof. We first multiply equation $(1.11)_1$ by $n_\epsilon^{2\alpha-1}$, integrate by parts over Ω and use the solenoidality of u_ϵ, the upper estimate (1.13) for S and the Young inequality to deduce that

$$\begin{aligned}
&\frac{1}{2\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{2\alpha} + (2\alpha-1)\int_\Omega n_\epsilon^{2\alpha-2}|\nabla n_\epsilon|^2 \\
&\quad = (2\alpha-1)\int_\Omega n_\epsilon^{2\alpha-1}\nabla n_\epsilon\cdot\big(S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\big) \\
&\quad \leqslant (2\alpha-1)C_S\int_\Omega n_\epsilon^{\alpha-1}|\nabla n_\epsilon||\nabla c_\epsilon| \\
&\quad \leqslant \frac{2\alpha-1}{2}\int_\Omega n_\epsilon^{2\alpha-2}|\nabla n_\epsilon|^2 + \frac{2\alpha-1}{2}C_S^2\int_\Omega|\nabla c_\epsilon|^2
\end{aligned}$$

for all $t \in (0,\infty)$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{2\alpha} + \frac{2\alpha-1}{\alpha}\int_\Omega|\nabla n_\epsilon^\alpha|^2 \leqslant \alpha(2\alpha-1)C_S^2\int_\Omega|\nabla c_\epsilon|^2 := C_1\int_\Omega|\nabla c_\epsilon|^2 \quad \text{for all} \quad t\in(0,\infty). \tag{3.2}$$

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We next test equation $(1.11)_2$ by c_ϵ and use the Hölder inequality to obtain that

$$\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon^2+\int_\Omega|\nabla c_\epsilon|^2+\int_\Omega c_\epsilon^2 = \int_\Omega n_\epsilon c_\epsilon \leqslant \|c_\epsilon\|_{L^6(\Omega)}\|n_\epsilon\|_{L^{6/5}(\Omega)} \quad \text{for all} \quad t\in(0,\infty). \tag{3.3}$$

Due to the current 3D setting, the Sobolev embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ and the L^1 boundedness of c_ϵ in (2.2) ensure that

$$\begin{aligned}\|c_\epsilon\|_{L^6(\Omega)}^2 &\leqslant C_2\|\nabla c_\epsilon\|_{L^2(\Omega)}^2+C_2\|c_\epsilon\|_{L^1(\Omega)}^2\\ &\leqslant C_2\|\nabla c_\epsilon\|_{L^2(\Omega)}^2+C_3 \qquad \text{for all} \quad t\in(0,\infty)\end{aligned}$$

for some positive constants C_2 and C_3. Thus applying the Young inequality to the right-hand side of (3.3), we have

$$\int_\Omega n_\epsilon c_\epsilon \leqslant \frac{1}{2C_2}\|c_\epsilon\|_{L^6(\Omega)}^2+\frac{C_2}{2}\|n_\epsilon\|_{L^{6/5}(\Omega)}^2 \leqslant \frac{1}{2}\int_\Omega|\nabla c_\epsilon|^2+\frac{C_2}{2}\|n_\epsilon\|_{L^{6/5}(\Omega)}^2+\frac{C_3}{2C_2} \tag{3.4}$$

for all $t\in(0,\infty)$. For the second term on the right-hand side of (3.4), it follows from the Gagliardo–Nirenberg inequality that one can find some positive constant C_4 such that

$$\begin{aligned}\frac{C_2}{2}\|n_\epsilon\|_{L^{6/5}(\Omega)}^2 = \frac{C_2}{2}\|n_\epsilon^\alpha\|_{L^{6/5\alpha}(\Omega)}^{2/\alpha} &\leqslant C_4\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^{2/(6\alpha-1)}\|n_\epsilon^\alpha\|_{L^{1/\alpha}(\Omega)}^{2(5\alpha-1)/\alpha(6\alpha-1)}\\ &\quad + C_4\|n_\epsilon^\alpha\|_{L^{1/\alpha}(\Omega)}^{2/\alpha}\end{aligned}$$

for all $t\in(0,\infty)$. Then due to the mass conservation (2.1) and the fact $2/(6\alpha-1)<2$ provided that $\alpha>1/3$, we see from the Young inequality that there exist positive constants C_5 and C_6 such that

$$\frac{C_2}{2}\|n_\epsilon\|_{L^{6/5}(\Omega)}^2 \leqslant C_5\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^{2/(6\alpha-1)}+C_5 \leqslant \frac{2\alpha-1}{8C_1\alpha}\int_\Omega|\nabla n_\epsilon^\alpha|^2+C_6$$

for all $t\in(0,\infty)$. This together with (3.3) and (3.4) entails that

$$\begin{aligned}&\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon^2+\int_\Omega|\nabla c_\epsilon|^2+2\int_\Omega c_\epsilon^2\\ &\quad\leqslant \frac{2\alpha-1}{4C_1\alpha}\int_\Omega|\nabla n_\epsilon^\alpha|^2+2C_6+\frac{C_3}{C_2} \qquad \text{for all} \quad t\in(0,\infty).\end{aligned} \tag{3.5}$$

Thus we can derive from an appropriate linear combination of (3.2) and (3.5) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^{2\alpha}+2C_1\epsilon\int_\Omega c_\epsilon^2\right)+\left(\frac{2\alpha-1}{2\alpha}\int_\Omega|\nabla n_\epsilon^\alpha|^2+C_1\int_\Omega|\nabla c_\epsilon|^2+4C_1\int_\Omega c_\epsilon^2\right)\\ &\quad\leqslant 4C_1C_6+\frac{2C_1C_3}{C_2}\end{aligned} \tag{3.6}$$

for all $t\in(0,\infty)$. In order to establish the uniform bound for the functional $\int_\Omega n_\epsilon^{2\alpha}+2C_1\epsilon\int_\Omega c_\epsilon^2$, we will use it to bound the dissipation from below. Precisely, we first employ the Gagliardo–Nirenberg inequality and the mass conservation (2.1) to estimate

$$\int_\Omega n_\epsilon^{2\alpha} = \|n_\epsilon^\alpha\|_{L^2(\Omega)}^2 \leqslant C_7\Big(\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^{6(2\alpha-1)/(6\alpha-1)}\|n_\epsilon^\alpha\|_{L^{1/\alpha}(\Omega)}^{4/(6\alpha-1)} + \|n_\epsilon^\alpha\|_{L^{1/\alpha}(\Omega)}^2\Big)$$
$$\leqslant C_8\Big(\|\nabla n_\epsilon^\alpha\|_{L^2(\Omega)}^{6(2\alpha-1)/(6\alpha-1)} + 1\Big)$$

for all $t \in (0,\infty)$, where C_7 and C_8 are some positive constants. Since $6(2\alpha-1)/(6\alpha-1) \in (0,2)$ by $\alpha > 1/2$, there exists a positive constant C_9 such that

$$\int_\Omega n_\epsilon^{2\alpha} \leqslant \frac{2\alpha-1}{2\alpha}\int_\Omega |\nabla n_\epsilon^\alpha|^2 + C_9 \tag{3.7}$$

for all $t \in (0,\infty)$. On the other hand, it is clear that

$$2C_1\epsilon\int_\Omega c_\epsilon^2 \leqslant 2C_1\int_\Omega c_\epsilon^2 \quad \text{for all} \quad t \in (0,\infty) \tag{3.8}$$

due to $\epsilon \in (0,1)$. Then by setting

$$y_\epsilon(t) := \int_\Omega n_\epsilon^{2\alpha}(\cdot,t) + 2C_1\epsilon\int_\Omega c_\epsilon^2(\cdot,t) \quad \text{for each} \quad t \in (0,\infty)$$

and substituting (3.7) and (3.8) into (3.6), we can deduce that

$$y_\epsilon'(t) + y_\epsilon(t) \leqslant 4C_1C_6 + \frac{2C_1C_3}{C_2} + C_9 := C_{10}$$

and thus that

$$y_\epsilon(t) \leqslant \max\Big\{y_\epsilon(0),\ C_{10}\Big\} \leqslant \max\Big\{\int_\Omega n_0^{2\alpha} + 2C_1\int_\Omega c_0^2,\ C_{10}\Big\} \quad \text{for all} \quad t \in (0,\infty).$$

Thereupon, we have established the slightly improved integrability estimate (3.1). □

3.2. Low L^p regularity of u_ϵ

The following basic and essentially well-known property is the foundation of our bootstrap argument, which shows the gain of regularity of u_ϵ from the *a priori* regularity of n_ϵ.

Lemma 3.2. *Let* $p \in [1, +\infty)$ *and* $r \in [1, +\infty]$ *be such that*

$$\begin{cases} r < \dfrac{3p}{3-2p} & \text{if} \quad p \leqslant \dfrac{3}{2}, \\ r \leqslant \infty & \text{if} \quad p > \dfrac{3}{2}. \end{cases}$$

Then for all $K > 0$ *there exists* $C = C(p, r, K, u_0, \phi)$ *such that if*

$$\|n_\epsilon(\cdot, t)\|_{L^p(\Omega)} \leqslant K \qquad \text{for all} \quad t \in (0, \infty),$$

then we have

$$\|u_\epsilon(\cdot, t)\|_{L^r(\Omega)} \leqslant C \qquad \text{for all} \quad t \in (0, \infty).$$

Proof. It is a direct result of applying the smoothing effect and decay estimates of the Stokes semigroup to the variant-of-constant representation of u_ϵ. We may refer to lemma 2.5 in [**32**] for details. □

Relying on the slightly improved bound on n_ϵ provided by lemma 3.1, we can now prove the following low regularity for u_ϵ, which is the first step towards its eventual L^p regularity.

Corollary 3.1. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *Then there exist* $r_1 > 3$ *and a positive constant* $C = C(r_1, \alpha, u_0, \phi)$ *such that*

$$\|u_\epsilon(\cdot, t)\|_{L^{r_1}(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty). \tag{3.9}$$

Proof. By setting $p := 2\alpha$, we can see from lemma 3.1 that $\|n_\epsilon(\cdot, t)\|_{L^p(\Omega)}$ is bounded for $t \in (0, \infty)$. If $1/2 < \alpha \leqslant 3/4$ and thus $1 < p \leqslant 3/2$, we have $3p/(3-2p) > 3$. Therefore, in this case, we can choose a $r_1 > 3$ such that (3.9) holds by lemma 3.2. On the other hand, if $\alpha > 3/4$ and thus $p > 3/2$, then the estimate (3.9) holds for any $r_1 \in [1, +\infty]$. This completes the proof of corollary 3.1. □

3.3. Improved L^p regularity of n_ϵ and c_ϵ, and eventual L^p regularity of u_ϵ

In order to derive the eventual L^p regularity of u_ϵ, then we will need to proceed to establish the more milder L^p regularity of n_ϵ. By constructing a combinational functional, we now further increase the regularity of n_ϵ from $\|n_\epsilon(\cdot, t)\|_{L^{2\alpha}(\Omega)}$ to $\|n_\epsilon(\cdot, t)\|_{L^{2\alpha+1/3}(\Omega)}$, which is the base of our bootstrap argument.

Lemma 3.3. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *If* $p \in (2\alpha + 1/3, (7/3)\alpha + 1/3)$, *then there exists a positive constant* C *such that*

$$\|n_\epsilon(\cdot, t)\|_{L^p(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty).$$

Proof. We will deduce our desired result by analysing the time evolution of the combinational functional

$$\|n_\epsilon(\cdot, t)\|^p_{L^p(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot, t)\|^2_{L^2(\Omega)}.$$

For this purpose, we first multiply equation $(1.11)_1$ by n_ϵ^{p-1} with $p \geqslant 1$, integrate by parts over Ω, and use the Young inequality and the upper estimate (1.13) for S

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to obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + (p-1)\int_\Omega n_\epsilon^{p-2}|\nabla n_\epsilon|^2 \\ = (p-1)\int_\Omega n_\epsilon^{p-1}\nabla n_\epsilon\cdot\big(S(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big) \\ \leqslant \frac{(p-1)}{4}\int_\Omega n_\epsilon^{p-2}|\nabla n_\epsilon|^2 + (p-1)C_{\mathcal{S}}^2\int_\Omega n_\epsilon^{p-2\alpha}|\nabla c_\epsilon|^2\end{aligned}$$

and thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 \leqslant p(p-1)C_S^2\int_\Omega n_\epsilon^{p-2\alpha}|\nabla c_\epsilon|^2 \tag{3.10}$$

for all $t\in(0,\infty)$. It then follows from the Hölder inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 \leqslant p(p-1)C_S^2\left(\int_\Omega n_\epsilon^{3(p-2\alpha)}\right)^{1/3}\left(\int_\Omega |\nabla c_\epsilon|^3\right)^{2/3} \tag{3.11}$$

for all $t\in(0,\infty)$. For the first integral on the right-hand side of (3.11), the Gagliardo–Nirenberg inequality and the mass conservation (2.1) ensure that

$$\begin{aligned}\left(\int_\Omega n_\epsilon^{3(p-2\alpha)}\right)^{1/3} &= \|n_\epsilon^{p/2}\|_{L^{6(p-2\alpha)/p}(\Omega)}^{2(p-2\alpha)/p} \\ &\leqslant C_1\left(\|\nabla n_\epsilon^{p/2}\|_{L^2(\Omega)}^{\theta_1}\|n_\epsilon^{p/2}\|_{L^{2/p}(\Omega)}^{1-\theta_1} + \|n_\epsilon^{p/2}\|_{L^{2/p}(\Omega)}\right)^{2(p-2\alpha)/p} \\ &\leqslant C_2\|\nabla n_\epsilon^{p/2}\|_{L^2(\Omega)}^{2(p-2\alpha)/p\theta_1} + C_2 \qquad \text{for all} \quad t\in(0,\infty)\end{aligned}$$

with $\theta_1 := (3p(p-2\alpha)-p)/((3p-1)(p-2\alpha))\in(0,1)$ due to $p>2\alpha+1/3$ for some positive constants C_1 and C_2 , and thus that

$$\left(\int_\Omega n_\epsilon^{3(p-2\alpha)}\right)^{1/3} \leqslant C_3\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3(p-2\alpha)-1)/(3p-1)} + C_3 \quad \text{for all } t\in(0,\infty) \tag{3.12}$$

for some positive constant C_3. On the other hand, noticing that

$$\int_\Omega |D^2c_\epsilon|^2 = \frac{1}{2}\int_{\partial\Omega}\nabla|\nabla c_\epsilon|^2\cdot\nu - \int_\Omega \nabla\Delta c_\epsilon\cdot\nabla c_\epsilon = \frac{1}{2}\int_{\partial\Omega}\nabla|\nabla c_\epsilon|^2\cdot\nu + \int_\Omega(\Delta c_\epsilon)^2$$

by the integration by parts and $\nabla c_\epsilon\cdot\nu=0$, we can apply the geometric property

$$\nabla|\nabla c_\epsilon|^2\cdot\nu \leqslant 2C_\Omega|\nabla c_\epsilon|^2 \tag{3.13}$$

with C_Ω an upper bound for the curvatures of $\partial\Omega$ (see lemma 4.2 in [**18**]), the trace theorem and the Gagliardo–Nirenberg inequality to find two positive constants C_4

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and C_5 such that

$$\begin{aligned}\|D^2 c_\epsilon\|^2_{L^2(\Omega)} &\leqslant C_\Omega \|\nabla c_\epsilon\|^2_{L^2(\partial\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\leqslant C_\Omega \|\nabla c_\epsilon\|^2_{W^{3/4,2}(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\leqslant C_4\Big(\|D^2 c_\epsilon\|^{13/7}_{L^2(\Omega)}\|c_\epsilon\|^{1/7}_{L^1(\Omega)} + \|c_\epsilon\|^2_{L^1(\Omega)}\Big) + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\\ &\leqslant \frac{1}{2}\|D^2 c_\epsilon\|^2_{L^2(\Omega)} + \frac{C_5}{2}\Big(\|c_\epsilon\|^2_{L^1(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\Big) \quad \text{for all } t\in(0,\infty)\end{aligned}$$

and thus that

$$\|D^2 c_\epsilon\|^2_{L^2(\Omega)} \leqslant C_5\Big(\|c_\epsilon\|^2_{L^1(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\Big) \qquad \text{for all} \quad t\in(0,\infty). \tag{3.14}$$

Therefore, for the second integral on the right-hand side of (3.11), again it follows from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}\left(\int_\Omega |\nabla c_\epsilon|^3\right)^{2/3} &= \|\nabla c_\epsilon\|^2_{L^3(\Omega)}\\ &\leqslant C_6\Big(\|D^2 c_\epsilon\|^{12/7}_{L^2(\Omega)}\|c_\epsilon\|^{2/7}_{L^1(\Omega)} + \|c_\epsilon\|^2_{L^1(\Omega)}\Big)\\ &\leqslant C_6\Big(C_5\Big(\|c_\epsilon\|^2_{L^1(\Omega)} + \|\Delta c_\epsilon\|^2_{L^2(\Omega)}\Big)^{6/7}\|c_\epsilon\|^{2/7}_{L^1(\Omega)} + \|c_\epsilon\|^2_{L^1(\Omega)}\Big)\\ &\leqslant C_7\Big(\|\Delta c_\epsilon\|^{12/7}_{L^2(\Omega)}\|c_\epsilon\|^{2/7}_{L^1(\Omega)} + \|c_\epsilon\|^2_{L^1(\Omega)}\Big)\end{aligned}$$

for some positive constants C_6 and C_7, which together with (2.2) gives that

$$\left(\int_\Omega |\nabla c_\epsilon|^3\right)^{2/3} \leqslant C_8\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{6/7} + C_8 \quad \text{for all} \quad t\in(0,\infty) \tag{3.15}$$

with some positive constant C_8. Substituting (3.12) and (3.15) into (3.11), we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2\\ &\leqslant C_9\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3(p-2\alpha)-1)/(3p-1)}\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{6/7}\\ &\quad + C_9\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3(p-2\alpha)-1)/(3p-1)} + C_9\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{6/7} + C_9\\ &\leqslant \frac{(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 + C_{10}\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{(3p-1)/7\alpha}\\ &\quad + C_{10}\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{6/7} + C_{10} \qquad \text{for all} \quad t\in(0,\infty)\end{aligned}$$

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with $C_9 := p(p-1)C_S^2 C_3 C_8$ and $C_{10} > 0$. Due to $(3p-1)/7\alpha \in (0,1)$ by $1/3 < p < (7/3)\alpha + 1/3$, we can use the Young inequality again to infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + \frac{2(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 \leqslant \int_\Omega |\Delta c_\epsilon|^2 + C_{11} \quad \text{for all} \quad t \in (0,\infty) \tag{3.16}$$

for some $C_{11} > 0$.

To absorb the integral on the right-hand side of (3.16), we multiply equation $(1.11)_2$ by $-\Delta c_\epsilon$ and integrate on Ω to get that

$$\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^2 = -\int_\Omega n_\epsilon \Delta c_\epsilon + \int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)\Delta c_\epsilon \tag{3.17}$$

for all $t \in (0,\infty)$. For the first integral on the right-hand side of (3.17), it is clear that

$$-\int_\Omega n_\epsilon \Delta c_\epsilon \leqslant \frac{1}{4}\int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega n_\epsilon^2 \quad \text{for all} \quad t \in (0,\infty). \tag{3.18}$$

It follows from the Gagliardo–Nirenberg inequality and the mass conservation (2.1) that

$$\begin{aligned}\int_\Omega n_\epsilon^2 = \|n_\epsilon^{p/2}\|_{L^{4/p}(\Omega)}^{4/p} &\leqslant C_{12}\|\nabla n_\epsilon^{p/2}\|_{L^2(\Omega)}^{4\theta_2/p}\|n_\epsilon^{p/2}\|_{L^{2/p}(\Omega)}^{(4(1-\theta_2))/p} + C_{12}\|n_\epsilon^{p/2}\|_{L^{2/p}(\Omega)}^{4/p} \\ &\leqslant C_{13}\|\nabla n_\epsilon^{p/2}\|_{L^2(\Omega)}^{4\theta_2/p} + C_{13}\end{aligned}$$

with $\theta_2 := 3p/(2(3p-1)) \in (0,1)$ and some positive constants C_{12} and C_{13}. Then due to $p > 2\alpha + 1/3 > 4/3$, we see $4\theta_2/p = 6/(3p-1) < 2$, which ensures us to apply the Young inequality to find $C_{14} > 0$ such that

$$\int_\Omega n_\epsilon^2 \leqslant \frac{p-1}{2p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 + C_{14} \qquad \text{for all} \quad t \in (0,\infty),$$

which together with (3.18) yields that

$$-\int_\Omega n_\epsilon \Delta c_\epsilon \leqslant \frac{1}{4}\int_\Omega |\Delta c_\epsilon|^2 + \frac{p-1}{2p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 + C_{14} \qquad \text{for all} \quad t \in (0,\infty). \tag{3.19}$$

On the other hand, for the second integral on the right-hand side of (3.17), we use the Hölder inequality and corollary 3.1 to obtain that

$$\begin{aligned}\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)\Delta c_\epsilon &\leqslant \|\Delta c_\epsilon\|_{L^2(\Omega)}\|u_\epsilon\|_{L^{r_1}(\Omega)}\|\nabla c_\epsilon\|_{L^{2r_1/(r_1-2)}(\Omega)} \\ &\leqslant C_{15}\|\Delta c_\epsilon\|_{L^2(\Omega)}\|\nabla c_\epsilon\|_{L^{2r_1/(r_1-2)}(\Omega)}\end{aligned} \tag{3.20}$$

for all $t \in (0,\infty)$ and some $C_{15} > 0$, where r_1 is taken from corollary 3.1. For the last factor in (3.20), we employ the Gagliardo–Nirenberg inequality and the estimate

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(3.14) to get

$$
\begin{aligned}
\|\nabla c_\epsilon\|_{L^{2r_1/(r_1-2)}(\Omega)} &\leqslant C_{16}\|D^2 c_\epsilon\|_{L^2(\Omega)}^{\theta_3}\|c_\epsilon\|_{L^1(\Omega)}^{1-\theta_3} + C_{16}\|c_\epsilon\|_{L^1(\Omega)} \\
&\leqslant C_{16}C_5^{\theta_3/2}\Big(\|c_\epsilon\|_{L^1(\Omega)}^2 + \|\Delta c_\epsilon\|_{L^2(\Omega)}^2\Big)^{\theta_3/2}\|c_\epsilon\|_{L^1(\Omega)}^{1-\theta_3} \\
&\quad + C_{16}\|c_\epsilon\|_{L^1(\Omega)} \quad \text{for all} \quad t\in(0,\infty),
\end{aligned}
$$

with $\theta_3 := (5r_1+6)/7r_1 \in (0,1)$ due to $r_1 > 3$ and a positive constant C_{16}, which together with the estimate (2.2) implies that

$$
\|\nabla c_\epsilon\|_{L^{2r_1/(r_1-2)}(\Omega)} \leqslant C_{17}\|\Delta c_\epsilon\|_{L^2(\Omega)}^{\theta_3} + C_{17} \qquad \text{for all} \quad t\in(0,\infty) \tag{3.21}
$$

for some $C_{17} > 0$. Substituting (3.21) into (3.20) and using the Young inequality, we obtain

$$
\int_\Omega (u_\epsilon\cdot\nabla c_\epsilon)\Delta c_\epsilon \leqslant C_{15}C_{17}\|\Delta c_\epsilon\|_{L^2(\Omega)}^{1+\theta_3} + C_{15}C_{17}\|\Delta c_\epsilon\|_{L^2(\Omega)} \leqslant \frac{1}{4}\int_\Omega|\Delta c_\epsilon|^2 + C_{18} \tag{3.22}
$$

for all $t\in(0,\infty)$ and some $C_{18} > 0$. Combining (3.19), (3.22) and (3.17), we can deduce that

$$
\begin{aligned}
&\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla c_\epsilon|^2 + \int_\Omega|\Delta c_\epsilon|^2 + 2\int_\Omega|\nabla c_\epsilon|^2 \\
&\quad\leqslant \frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2 + 2(C_{14}+C_{18}) \qquad \text{for all} \quad t\in(0,\infty).
\end{aligned} \tag{3.23}
$$

By (3.16) and (3.23), we conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^p + \epsilon\int_\Omega|\nabla c_\epsilon|^2\right) + \frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2 + 2\int_\Omega|\nabla c_\epsilon|^2 \leqslant C_{19}
$$

for all $t\in(0,\infty)$ with $C_{19} := C_{11} + 2(C_{14}+C_{18})$. Following the same process as (3.7) and (3.8), we have

$$
\int_\Omega n_\epsilon^p \leqslant \frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2 + C_{20}, \quad \text{and} \quad \epsilon\int_\Omega|\nabla c_\epsilon|^2 \leqslant 2\int_\Omega|\nabla c_\epsilon|^2 \tag{3.24}
$$

for all $t\in(0,\infty)$ and some $C_{20} > 0$, and thus obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^p + \epsilon\int_\Omega|\nabla c_\epsilon|^2\right) + \left(\int_\Omega n_\epsilon^p + \epsilon\int_\Omega|\nabla c_\epsilon|^2\right) \\
&\quad\leqslant C_{19} + C_{20} \quad \text{for all} \quad t\in(0,\infty).
\end{aligned}
$$

By a basic calculation, we deduce that

$$
\int_\Omega n_\epsilon^p(\cdot,t) + \epsilon\int_\Omega|\nabla c_\epsilon(\cdot,t)|^2 \leqslant \max\left\{\int_\Omega n_0^p + \int_\Omega|\nabla c_0|^2,\, C_{19}+C_{20}\right\}
$$

for all $t\in(0,\infty)$. This completes the proof of lemma 3.3. □

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COROLLARY 3.2. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *If* $s \in (2\alpha + 1/3, (7/3)\alpha + 1/3)$, *then there exists a positive constant* C *such that*

$$\|c_\epsilon(\cdot,t)\|_{L^s(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty).$$

Proof. A direct application of lemmas 3.3 and 2.2 yields the desired result. □

COROLLARY 3.3. *Suppose that* (1.12)-(1.15) *hold with* $\alpha > 1/2$. *Then for any* $r > 1$, *there exists a positive constant* $C = C(r, n_0, c_0, u_0, \phi)$ *such that for each* $\epsilon \in (0,1)$, *we have*

$$\|u_\epsilon(\cdot,t)\|_{L^r(\Omega)} \leqslant C \qquad \text{for all} \quad t \in (0,\infty).$$

Proof. Since $\alpha > 1/2$ is equivalent to $(7/3)\alpha + 1/3 > 3/2$, we can fix a p_0 satisfying

$$\frac{7}{3}\alpha + \frac{1}{3} > p_0 > \max\left\{\frac{3}{2}, 2\alpha + \frac{1}{3}\right\},$$

and then deduce the boundedness of $\|n_\epsilon(\cdot,t)\|_{L^{p_0}(\Omega)}$ for all $t \in (0,\infty)$ by lemma 3.3. Thus for any $r > 1$, we can infer the L^r boundedness of u_ϵ from lemma 3.2 due to $p_0 > 3/2$. This completes the proof of corollary 3.3. □

3.4. L^2 regularity of n_ϵ and ∇c_ϵ

On the basis of lemma 3.3 and corollary 3.2, we next plan to improve our knowledge on the spatial regularity of n_ϵ by utilizing a very subtle induction argument for n_ϵ, which together with the damping effect of c_ϵ will provide the key uniform L^2 bound for ∇c_ϵ.

LEMMA 3.4. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *For any fixed* $s_0 \in (2\alpha + 1/3, (7/3)\alpha + 1/3)$, *define the sequence* $\{s_k\}_{k=1}^{\infty}$ *by fixing*

$$s_k \in \left(2\alpha + \frac{s_{k-1}}{2},\ 4\alpha + \min\left\{\frac{2}{3}s_{k-1},\ \frac{2s_{k-1}^2 + 18s_{k-1} - 72\alpha}{3(s_{k-1}+12)}\right\}\right), \quad (k = 1,2,3,\ldots).$$

Then for every $k = 1,2,3,\ldots$, *there exists a positive constant* C_k *such that*

$$\|n_\epsilon(\cdot,t)\|_{L^{s_k}(\Omega)} \leqslant C_k \quad \text{for all} \quad t \in (0,\infty).$$

Proof. We intend to prove our conclusion by induction on k and to analyse the time evolution of the combinational functional

$$\|n_\epsilon(\cdot,t)\|_{L^{s_k}(\Omega)}^{s_k} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|_{L^2(\Omega)}^2.$$

To this end, we first focus on the case $k = 1$ and use a similar process as the proof of (3.10) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{s_1} + \frac{3(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 \leqslant s_1(s_1-1)C_S^2\int_\Omega n_\epsilon^{s_1-2\alpha}|\nabla c_\epsilon|^2,$$

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which together with the Hölder inequality gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{s_1} + \frac{3(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 \leqslant s_1(s_1-1)C_S^2\left(\int_\Omega n_\epsilon^{2(s_1-2\alpha)}\right)^{1/2}\left(\int_\Omega |\nabla c_\epsilon|^4\right)^{1/2} \tag{3.25}$$

for all $t \in (0,\infty)$. For the first integral on the right-hand side of (3.25), we can set $\theta_1^1 := (3s_1(2(s_1-2\alpha)-s_0))/(2(s_1-2\alpha)(3s_1-s_0))$, which belongs to $(0,1)$ due to $s_1 > 2\alpha + s_0/2$, and use the Gagliardo–Nirenberg inequality, lemma 3.3 and the Young inequality to find two positive constants C_1 and C_2 such that

$$\begin{aligned}\left(\int_\Omega n_\epsilon^{2(s_1-2\alpha)}\right)^{1/2} &= \|n_\epsilon^{s_1/2}\|_{L^{4(s_1-2\alpha)/s_1}(\Omega)}^{2(s_1-2\alpha)/s_1}\\ &\leqslant C_1\left(\|\nabla n_\epsilon^{s_1/2}\|_{L^2(\Omega)}^{\theta_1^1}\|n_\epsilon^{s_1/2}\|_{L^{2s_0/s_1}(\Omega)}^{1-\theta_1^1} + \|n_\epsilon^{s_1/2}\|_{L^{2s_0/s_1}(\Omega)}\right)^{2(s_1-2\alpha)/s_1}\\ &\leqslant C_2\|\nabla n_\epsilon^{s_1/2}\|_{L^2(\Omega)}^{(2(s_1-2\alpha))/s_1\theta_1^1} + C_2\\ &= C_2\left(\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2\right)^{(6(s_1-2\alpha)-3s_0)/(2(3s_1-s_0))} + C_2\end{aligned} \tag{3.26}$$

for all $t \in (0,\infty)$. On the other hand, we apply the Gagliardo–Nirenberg inequality to the second integral on the right-hand side of (3.25) and use (3.14) to find two positive constants C_3 and C_4 such that

$$\begin{aligned}\left(\int_\Omega |\nabla c_\epsilon|^4\right)^{1/2} &= \|\nabla c_\epsilon\|_{L^4(\Omega)}^2\\ &\leqslant C_3\left(\|D^2 c_\epsilon\|_{L^2(\Omega)}^{\theta_2^1}\|c_\epsilon\|_{L^{s_0}(\Omega)}^{1-\theta_2^1} + \|c_\epsilon\|_{L^{s_0}(\Omega)}\right)^2\\ &\leqslant C_4\left(\left(\|c_\epsilon\|_{L^1(\Omega)}^2 + \|\Delta c_\epsilon\|_{L^2(\Omega)}^2\right)^{\theta_2^1/2}\|c_\epsilon\|_{L^{s_0}(\Omega)}^{1-\theta_2^1} + \|c_\epsilon\|_{L^{s_0}(\Omega)}\right)^2\end{aligned}$$

for all $t \in (0,\infty)$ with $\theta_2^1 := (s_0+12)/(2s_0+12) \in (0,1)$, which together with (2.2) and corollary 3.2 gives that

$$\begin{aligned}\left(\int_\Omega |\nabla c_\epsilon|^4\right)^{1/2} &\leqslant C_5\left(\left(1 + \|\Delta c_\epsilon\|_{L^2(\Omega)}^2\right)^{\theta_2^1/2}\|c_\epsilon\|_{L^{s_0}(\Omega)}^{1-\theta_2^1} + \|c_\epsilon\|_{L^{s_0}(\Omega)}\right)^2\\ &\leqslant C_6\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{\theta_2^1} + C_6\end{aligned} \tag{3.27}$$

for all $t \in (0,\infty)$ and some positive constant C_6. Since $(6(s_1-2\alpha)-3s_0)/(2(3s_1-s_0)) < (6(s_1-2\alpha)-3s_0)/(3s_1-s_0)$ due to $s_1 > 2\alpha + s_0/2$, and by substituting (3.26) and (3.27) into (3.25), we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{s_1} &+ \frac{3(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 \\ &\leqslant C_7\left(\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2\right)^{(6(s_1-2\alpha)-3s_0)/(2(3s_1-s_0))}\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{\theta_2^1} \\ &\quad + C_7\left(\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2\right)^{(6(s_1-2\alpha)-3s_0)/(3s_1-s_0)} \\ &\quad + C_7\Big(\int_\Omega |\Delta c_\epsilon|^2\Big)^{\theta_2^1} + C_7\end{aligned}$$

for all $t \in (0,\infty)$ with some positive constant C_7. Since

$$0 < \frac{6(s_1-2\alpha)-3s_0}{3s_1-s_0} < 1 \qquad \text{and} \qquad \frac{2(3s_1-s_0)\theta_2^1}{s_0+12\alpha} < 1$$

by $2\alpha + s_0/2 < s_1 < 4\alpha + \min\Big\{(2/3)s_0, (2s_0^2+18s_0-72\alpha)/(3(s_0+12))\Big\}$, we use the Young inequality twice to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^{s_1} &+ \frac{3(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 \\ &\leqslant \frac{(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 + C_8\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{(2(3s_1-s_0)\theta_2^1)/(s_0+12\alpha)} \\ &\quad + C_8\left(\int_\Omega |\Delta c_\epsilon|^2\right)^{\theta_2^1} + C_8 \\ &\leqslant \frac{(s_1-1)}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 + \int_\Omega |\Delta c_\epsilon|^2 + C_9 \end{aligned} \tag{3.28}$$

for all $t \in (0,\infty)$ and some positive constants C_8 and C_9.

Multiplying equation $(1.11)_2$ by $-\Delta c_\epsilon$, integrating on Ω and following proof of lemma 3.3, we also have

$$\begin{aligned}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^2 &+ \int_\Omega |\Delta c_\epsilon|^2 + 2\int_\Omega |\nabla c_\epsilon|^2 \\ &\leqslant \frac{s_1-1}{s_1}\int_\Omega |\nabla n_\epsilon^{s_1/2}|^2 + C_{10} \qquad \text{for all} \quad t \in (0,\infty)\end{aligned} \tag{3.29}$$

with some positive constant C_{10}. Combining (3.28) and (3.29), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^{s_1} + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right) + \frac{s_1-1}{s_1}\int_\Omega |\nabla n_\epsilon^{\frac{s_1}{2}}|^2 + 2\int_\Omega |\nabla c_\epsilon|^2 \leqslant C_{11}$$

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for all $t \in (0,\infty)$ with $C_{11} := C_9 + C_{10}$ and then use a similar process as (3.24) to get

$$\frac{d}{dt}\left(\int_\Omega n_\epsilon^{s_1} + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right) + \left(\int_\Omega n_\epsilon^{s_1} + \epsilon\int_\Omega |\nabla c_\epsilon|^2\right) \leqslant C_{12}$$

for all $t \in (0,\infty)$ and some $C_{12} > 0$. Thus the proof for the case $k = 1$ is completed by a direct calculation.

Assume now the lemma is valid for some positive constant k. Then by lemma 2.2, we see that

$$\|c_\epsilon(\cdot,t)\|_{L^{s_k}(\Omega)} \leqslant C_{13} \quad \text{for all} \quad t \in (0,\infty)$$

for some $C_{13} > 0$, and deduce

$$\|n_\epsilon(\cdot,t)\|_{L^{s_{k+1}}(\Omega)} \leqslant C_{14} \quad \text{for all} \quad t \in (0,\infty)$$

for some positive constant C_{14} by following the proof of the case $k = 1$. This implies the lemma is true for the case $k+1$ and completes the proof of lemma 3.4 by induction. □

Corollary 3.4. *Suppose that (1.12)–(1.15) hold with $\alpha > 1/2$. Then there exists $C > 0$ such that*

$$\|n_\epsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty).$$

Proof. Let $\{s_k\}_{k=0}^{\infty}$ be defined by lemma 3.4. Then $s_k > 2\alpha + (1/2)s_{k-1}$, $(k = 1,2,3,\ldots)$, implies that

$$\begin{aligned} s_k &> 2\alpha + \frac{1}{2}\left(2\alpha + \frac{1}{2}s_{k-2}\right) > \cdots > 2\alpha\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) + \frac{1}{2^k}s_0 \\ &= 4\alpha + \frac{1}{2^k}(s_0 - 4\alpha) \end{aligned}$$

for $k = 1,2,3,\ldots$. Thus due to $\alpha > 1/2$, we have $s_k \geqslant 2$ for k large enough and obtain the desired result by using lemma 3.4 (and the Hölder inequality if necessary). This completes the proof of corollary 3.4. □

With the boundedness of $\|n_\epsilon(\cdot,t)\|_{L^2(\Omega)}$ at hand, we now turn back the proof of lemma 3.3 to establish the key boundedness of $\|\nabla c_\epsilon(\cdot,t)\|_{L^2(\Omega)}$.

Lemma 3.5. *Suppose that (1.12)–(1.15) hold with $\alpha > 1/2$. Then there exists $C > 0$ such that*

$$\|\nabla c_\epsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty).$$

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Proof. By repeating the proof of (3.17), (3.18) and (3.22) in lemma 3.3 and using corollary 3.4, we can deduce that

$$\begin{aligned}\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^2 + \int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^2 \\ \leqslant \frac{1}{2}\int_\Omega |\Delta c_\epsilon|^2 + \int_\Omega n_\epsilon^2 + C_1 \leqslant \frac{1}{2}\int_\Omega |\Delta c_\epsilon|^2 + C_2\end{aligned}$$

for all $t \in (0,\infty)$ and some positive constants C_1 and C_2. Then a direct calculation shows that

$$\int_\Omega |\nabla c_\epsilon(\cdot,t)|^2 \leqslant C_2 + \left(\int_\Omega |\nabla c_0|^2 - C_2\right)\mathrm{e}^{-(2t/\epsilon)} \leqslant \max\left\{C_2, \int_\Omega |\nabla c_0|^2\right\}$$

for all $t \in (0,\infty)$. This completes the proof of lemma 3.5. □

3.5. Eventual L^p regularity of n_ϵ and L^4 regularity of ∇c_ϵ

By making use of the regularity information obtained so far and developing a new coupled estimate, we are now in the position to establish the regularity of n_ϵ in arbitrary L^p space, from which we will eventually deduce the desired L^4 regularity for ∇c_ϵ.

LEMMA 3.6. *Suppose that (1.12)–(1.15) hold with $\alpha > 1/2$. Let $q \geqslant 2$ and $p > 1$ satisfy that*

$$\max\left\{\frac{4}{3}, \frac{6\alpha}{3q-1}\right\} < \frac{3p-1}{3q-1} < 6\alpha. \tag{3.30}$$

Then there exists $C > 0$ such that

$$\|n_\epsilon(\cdot,t)\|_{L^p(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|_{L^{2q}(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty). \tag{3.31}$$

Proof. Our strategy is to investigate the combinational functional of the form

$$\|n_\epsilon(\cdot,t)\|^p_{L^p(\Omega)} + \epsilon\|\nabla c_\epsilon(\cdot,t)\|^{2q}_{L^{2q}(\Omega)}$$

with p and q satisfying (3.30). Firstly, it is clear from the proof of (3.10) and the Hölder inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p + \frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 &\leqslant p(p-1)C_S^2\int_\Omega n_\epsilon^{p-2\alpha}|\nabla c_\epsilon|^2 \\ &\leqslant p(p-1)C_S^2\left(\int_\Omega n_\epsilon^{3(p-2\alpha)}\right)^{1/3}\left(\int_\Omega |\nabla c_\epsilon|^3\right)^{2/3}\end{aligned} \tag{3.32}$$

for all $t \in (0,\infty)$. Since the first inequality in (3.30) implies that $p > 2\alpha + 1/3$, we know from the proof of (3.12) that

$$p(p-1)C_S^2\left(\int_\Omega n_\epsilon^{3(p-2\alpha)}\right)^{1/3} \leqslant C_1\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3p-6\alpha-1)/(3p-1)} + C_1 \tag{3.33}$$

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for all $t \in (0,\infty)$ and some positive constant C_1, while the Gagliardo–Nirenberg inequality and lemma 3.5 give that

$$\begin{aligned}\left(\int_\Omega |\nabla c_\epsilon|^3\right)^{2/3} &= \big\||\nabla c_\epsilon|^q\big\|_{L^{3/q}(\Omega)}^{2/q}\\ &\leqslant C_2(\big\|\nabla|\nabla c_\epsilon|^q\big\|_{L^2(\Omega)}^{q/(3q-1)}\big\||\nabla c_\epsilon|^q\big\|_{L^{2/q}(\Omega)}^{(2q-1)/(3q-1)}+\big\||\nabla c_\epsilon|^q\big\|_{L^{2/q}(\Omega)})^{2/q}\\ &\leqslant C_3\left(\int_\Omega \big|\nabla|\nabla c_\epsilon|^q\big|^2\right)^{1/(3q-1)}+C_3 \end{aligned}\tag{3.34}$$

for all $t \in (0,\infty)$ and some positive constants C_2 and C_3. Submitting (3.33) and (3.34) into (3.32), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p+\frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2\\ &\leqslant C_1C_3\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3p-6\alpha-1)/(3p-1)}\left(\int_\Omega \big|\nabla|\nabla c_\epsilon|^q\big|^2\right)^{1/(3q-1)}\\ &\quad+C_1C_3\left(\int_\Omega |\nabla n_\epsilon^{p/2}|^2\right)^{(3p-6\alpha-1)/(3p-1)}\\ &\quad+C_1C_3\left(\int_\Omega \big|\nabla|\nabla c_\epsilon|^q\big|^2\right)^{1/(3q-1)}+C_1C_3\end{aligned}$$

for all $t \in (0,\infty)$. Noticing that $(3p-6\alpha-1)/(3p-1)+1/(3q-1)<1$ due to $(3p-1)/(3q-1)<6\alpha$, the Young inequality entails that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p+\frac{3(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 \leqslant \frac{p-1}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2+\frac{q-1}{4q}\int_\Omega \big|\nabla|\nabla c_\epsilon|^q\big|^2+C_4$$

and thus that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n_\epsilon^p+\frac{2(p-1)}{p}\int_\Omega |\nabla n_\epsilon^{p/2}|^2 \leqslant \frac{q-1}{4q}\int_\Omega \big|\nabla|\nabla c_\epsilon|^q\big|^2+C_4 \tag{3.35}$$

for all $t \in (0,\infty)$ and some positive constant C_4.

On the other hand, we apply ∇ to equation $(1.11)_2$ and multiply the resulting equation by $|\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon$ to obtain

$$\begin{aligned}&\frac{\epsilon}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^{2q}-\int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\Delta\nabla c_\epsilon+\int_\Omega |\nabla c_\epsilon|^{2q}\\ &\quad=\int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\nabla n_\epsilon-\int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\nabla\big(u_\epsilon\cdot\nabla c_\epsilon\big),\end{aligned}$$

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which together with the pointwise identity $2\nabla c_\epsilon \cdot \nabla \Delta c_\epsilon = \Delta|\nabla c_\epsilon|^2 - 2|D^2 c_\epsilon|^2$ and the integration by parts ensures that

$$\begin{aligned}
&\frac{\epsilon}{2q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^{2q} + \frac{q-1}{2}\int_\Omega |\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2 \\
&\quad + \int_\Omega |\nabla c_\epsilon|^{2(q-1)}|D^2 c_\epsilon|^2 + \int_\Omega |\nabla c_\epsilon|^{2q} \\
&\qquad = \int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla n_\epsilon \cdot \nabla c_\epsilon + (q-1)\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-2)}\nabla c_\epsilon \cdot \nabla|\nabla c_\epsilon|^2 \\
&\qquad\quad + \int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-1)}\Delta c_\epsilon + \frac{1}{2}\int_{\partial\Omega} |\nabla c_\epsilon|^{2(q-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu}
\end{aligned} \tag{3.36}$$

for all $t \in (0,\infty)$. For the first term on the right-hand side of (3.36), we use the pointwise inequality $|\Delta c_\epsilon|^2 \leqslant 3|D^2 c_\epsilon|^2$, the integration by parts and the Young inequality to obtain

$$\begin{aligned}
&\int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla n_\epsilon \cdot \nabla c_\epsilon \\
&\quad = -\int_\Omega |\nabla c_\epsilon|^{2(q-1)} n_\epsilon \Delta c_\epsilon - (q-1)\int_\Omega |\nabla c_\epsilon|^{2(q-2)} n_\epsilon \nabla c_\epsilon \cdot \nabla|\nabla c_\epsilon|^2 \\
&\quad \leqslant \sqrt{3}\int_\Omega |\nabla c_\epsilon|^{2(q-1)} n_\epsilon |D^2 c_\epsilon| + (q-1)\int_\Omega |\nabla c_\epsilon|^{2q-3} n_\epsilon \left|\nabla|\nabla c_\epsilon|^2\right| \\
&\quad \leqslant \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2(q-1)}|D^2 c_\epsilon|^2 + \frac{q-1}{4}\int_\Omega |\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2 \\
&\qquad + \left(q + \frac{1}{2}\right)\int_\Omega n_\epsilon^2 |\nabla c_\epsilon|^{2(q-1)},
\end{aligned} \tag{3.37}$$

while for the second and third terms, we have

$$\begin{aligned}
&(q-1)\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-2)}\nabla c_\epsilon \cdot \nabla|\nabla c_\epsilon|^2 \\
&\quad \leqslant \frac{q-1}{8}\int_\Omega |\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2 + 2(q-1)\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2q}
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
\int_\Omega (u_\epsilon \cdot \nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-1)}\Delta c_\epsilon &\leqslant \sqrt{3}\int_\Omega |u_\epsilon||\nabla c_\epsilon|^{2q-1}|D^2 c_\epsilon| \\
&\leqslant \frac{1}{2}\int_\Omega |\nabla c_\epsilon|^{2(q-1)}|D^2 c_\epsilon|^2 + \frac{3}{2}\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^{2q}.
\end{aligned} \tag{3.39}$$

For the last term on the right-hand side of (3.36), we know from the geometry property (3.13), the trace theorem and the Gagliardo–Nirenberg inequality that

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there exist positive constants C_5, C_6 and C_7 such that

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}|\nabla c_\epsilon|^{2q-2}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu} &\leqslant C_5\int_{\partial\Omega}|\nabla c_\epsilon|^{2q} = C_5\big\||\nabla c_\epsilon|^q\big\|^2_{L^2(\partial\Omega)}\\ &\leqslant C_6\big\||\nabla c_\epsilon|^q\big\|^2_{W^{3/4,2}(\Omega)}\\ &\leqslant C_7\big\|\nabla|\nabla c_\epsilon|^q\big\|^{(3(2q-1))/(3q-1)}_{L^2(\Omega)}\big\||\nabla c_\epsilon|^q\big\|^{1/(3q-1)}_{L^{2/q}(\Omega)}\\ &\quad + C_7\big\||\nabla c_\epsilon|^q\big\|^2_{L^{2/q}(\Omega)},\end{aligned}$$

which together with lemma 3.5 and the Young inequality ensures the existence of a positive constant C_8 satisfying

$$\frac{1}{2}\int_{\partial\Omega}|\nabla c_\epsilon|^{2(q-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu} \leqslant \frac{q-1}{4q^2}\int_\Omega|\nabla|\nabla c_\epsilon|^q|^2 + C_8 \quad \text{for all} \quad t\in(0,\infty). \tag{3.40}$$

Substituting (3.37)–(3.40) into (3.36), we deduce that

$$\begin{aligned}&\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla c_\epsilon|^{2q} + \frac{q-1}{2q}\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2 + 2q\int_\Omega|\nabla c_\epsilon|^{2q}\\ &\quad\leqslant C_9\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2q-2} + C_9\int_\Omega|u_\epsilon|^2|\nabla c_\epsilon|^{2q} + C_9\end{aligned} \tag{3.41}$$

for all $t\in(0,\infty)$ and some positive constant C_9, which together with (3.35) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^p + \epsilon\int_\Omega|\nabla c_\epsilon|^{2q}\right) + \frac{2(p-1)}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2 + \frac{q-1}{4q}\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2\\ &\quad + 2q\int_\Omega|\nabla c_\epsilon|^{2q} \leqslant C_9\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2q-2} + C_9\int_\Omega|u_\epsilon|^2|\nabla c_\epsilon|^{2q} + C_{10}\end{aligned} \tag{3.42}$$

for all $t\in(0,\infty)$ and some $C_{10}>0$.

We now show that the integrals on the right hand of (3.42) can be controlled by the dissipation on the left hand of (3.42). Indeed, for the first one, we first use the Hölder inequality to show that

$$\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(q-1)} \leqslant \left(\int_\Omega n_\epsilon^3\right)^{2/3}\left(\int_\Omega|\nabla c_\epsilon|^{6(q-1)}\right)^{1/3} \quad \text{for all} \quad t\in(0,\infty). \tag{3.43}$$

The two integrals on the right-hand side of (3.43) can be estimated as (3.33) and (3.34), respectively, and thus we can deduce that

$$\begin{aligned}\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(q-1)} &\leqslant C_{11}\left(\int_\Omega|\nabla n_\epsilon^{p/2}|^2\right)^{4/(3p-1)}\left(\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2\right)^{(3q-4)/(3q-1)}\\ &\quad + C_{11}\left(\int_\Omega|\nabla n_\epsilon^{p/2}|^2\right)^{4/(3p-1)}\\ &\quad + C_{11}\left(\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2\right)^{(3q-4)/(3q-1)} + C_{11}\end{aligned}$$

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for all $t \in (0,\infty)$ with some positive constant C_{11}. Then since $4/(3p-1)+(3q-4)/(3q-1)<1$ due to $(3p-1)/(3q-1)>4/3$, the Young inequality entails that

$$C_9\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^{2(q-1)} \leqslant \frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2+\frac{q-1}{8q}\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2+C_{12} \tag{3.44}$$

for all $t \in (0,\infty)$ and some positive constant C_{12}. For the second integral on the right of (3.42), we first fix $r \in (1,3)$ and then use the Hölder inequality and corollary 3.3 to obtain

$$\int_\Omega|\nabla c_\epsilon|^{2q}|u_\epsilon|^2 \leqslant \|u_\epsilon^2\|_{L^{r/(r-1)}(\Omega)}\big\||\nabla c_\epsilon|^{2q}\big\|_{L^r(\Omega)} \leqslant C_{13}\big\||\nabla c_\epsilon|^{2q}\big\|_{L^r(\Omega)}$$

for all $t \in (0,\infty)$ and some $C_{13}>0$. Since the interpolation implies that

$$\begin{aligned}\big\||\nabla c_\epsilon|^{2q}\big\|_{L^r(\Omega)} &= \big\||\nabla c_\epsilon|^q\big\|_{L^{2r}(\Omega)}^2\\ &\leqslant C_{14}\big\|\nabla|\nabla c_\epsilon|^q\big\|_{L^2(\Omega)}^{(6(rq-1))/(r(3q-1))}\big\||\nabla c_\epsilon|^q\big\|_{L^{2/q}(\Omega)}^{(2(3-r))/(r(3q-1))}\\ &\quad + C_{14}\big\||\nabla c_\epsilon|^q\big\|_{L^{2/q}(\Omega)}^2\end{aligned}$$

for all $t \in (0,\infty)$ and some positive constant C_{14}, we can deduce from lemma 3.5 and the Young inequality that

$$\begin{aligned}C_9\int_\Omega|\nabla c_\epsilon|^{2q}|u_\epsilon|^2 &\leqslant C_{15}\big\|\nabla|\nabla c_\epsilon|^q\big\|_{L^2(\Omega)}^{(6(rq-1))/(r(3q-1))}+C_{15}\\ &\leqslant \frac{q-1}{8q}\int_\Omega\big|\nabla|\nabla c_\epsilon|^q\big|^2+C_{16}\end{aligned} \tag{3.45}$$

for all $t \in (0,\infty)$ and some positive constants C_{15} and C_{16}.

Thereupon, by substituting (3.44) and (3.45) into (3.42), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^p+\epsilon\int_\Omega|\nabla c_\epsilon|^{2q}\right)+\frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2+2q\int_\Omega|\nabla c_\epsilon|^{2q} \leqslant C_{17}$$

for all $t \in (0,\infty)$ with $C_{17}:=C_{10}+C_{12}+C_{16}$. Following a similar process as (3.24), we have

$$\int_\Omega n_\epsilon^p \leqslant \frac{p-1}{p}\int_\Omega|\nabla n_\epsilon^{p/2}|^2+C_{18} \qquad \text{and} \qquad \epsilon\int_\Omega|\nabla c_\epsilon|^{2q} \leqslant 2q\int_\Omega|\nabla c_\epsilon|^{2q}$$

for all $t \in (0,\infty)$ and some positive constant C_{18}. Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega n_\epsilon^p+\epsilon\int_\Omega|\nabla c_\epsilon|^{2q}\right)+\left(\int_\Omega n_\epsilon^p+\epsilon\int_\Omega|\nabla c_\epsilon|^{2q}\right) \leqslant C_{19}$$

for all $t \in (0,\infty)$ with $C_{19}:=C_{17}+C_{18}$, which implies (3.31) by a direct calculation. This completes the proof of lemma 3.6. □

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Corollary 3.5. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *Then for any* $p > 1$, *there exists* $C > 0$ *such that*

$$\|n_\epsilon(\cdot,t)\|_{L^p(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty).$$

Proof. Without loss of generality, we may assume $p > 2\alpha + 1/3$ and then take q such that (3.30) holds. Then corollary 3.5 is a direct result of lemma 3.6. □

Now along with our regularity in corollary 3.3 and lemma 3.5, the boundedness in corollary 3.5 asserts the following.

Lemma 3.7. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 1/2$. *Then there exists* $C > 0$ *such that*

$$\|\nabla c_\epsilon(\cdot,t)\|_{L^4(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0,\infty).$$

Proof. We can take $q = 2$ in (3.41) by following the proof of Lemma 3.6 to obtain

$$\begin{aligned}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla c_\epsilon|^4 + \frac{1}{4}\int_\Omega \big|\nabla|\nabla c_\epsilon|^2\big|^2 + 4\int_\Omega |\nabla c_\epsilon|^4 \\ \leqslant C_1\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^2 + C_1\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^4 + C_1\end{aligned}$$

for all $t \in (0,\infty)$ and some positive constant C_1. Noticing that

$$C_1\int_\Omega n_\epsilon^2|\nabla c_\epsilon|^2 \leqslant 2\|\nabla c_\epsilon\|_{L^4}^4 + C_2\|n_\epsilon\|_{L^4}^4 \leqslant 2\|\nabla c_\epsilon\|_{L^4}^4 + C_3$$

by corollary 3.5, and that

$$\begin{aligned}C_1\int_\Omega |u_\epsilon|^2|\nabla c_\epsilon|^4 &\leqslant C_1\|u_\epsilon^2\|_{L^2}\big\||\nabla c_\epsilon|^4\big\|_{L^2} = C_1\|u_\epsilon\|_{L^4}^2\big\||\nabla c_\epsilon|^4\big\|_{L^2} \\ &\leqslant C_4\big\||\nabla c_\epsilon|^4\big\|_{L^2} = C_4\big\||\nabla c_\epsilon|^2\big\|_{L^4}^2 \\ &\leqslant C_5\Big(\big\|\nabla|\nabla c_\epsilon|^2\big\|_{L^2}^{9/5}\big\||\nabla c_\epsilon|^2\big\|_{L^1}^{1/5} + \big\||\nabla c_\epsilon|^2\big\|_{L^1}^2\Big) \\ &\leqslant \frac{1}{4}\big\|\nabla|\nabla c_\epsilon|^2\big\|_{L^2}^2 + C_6\end{aligned}$$

by corollary 3.3 and lemma 3.5, we have

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c_\epsilon\|_{L^4}^4 + 2\|\nabla c_\epsilon\|_{L^4}^4 \leqslant C_7$$

for all $t \in (0,\infty)$, which yields

$$\int_\Omega |\nabla c_\epsilon(\cdot,t)|^4 \leqslant \frac{C_7}{2} + \left(\int_\Omega |\nabla c_0|^4 - \frac{C_7}{2}\right)\mathrm{e}^{-(2t/\epsilon)} \leqslant \max\left\{\frac{C_7}{2}, \int_\Omega |\nabla c_0|^4\right\}$$

for all $t \in (0,\infty)$ by a direct calculation. This completes the proof of lemma 3.7. □

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We are now in the position to extract a suitable subsequence of $(n_\epsilon, c_\epsilon, u_\epsilon)$ along which the respective solutions approach a limit in convenient topologies.

LEMMA 3.8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let $\alpha > 1/2$ and (1.12)–(1.15) hold. Then there exist a sequence $\{\epsilon_j\}_{j=1}^{\infty}$ and a unique classical solution (n, c, u, P) to system (1.16) in $\Omega \times (0, \infty)$ with the properties that*

$$
\begin{aligned}
n_{\epsilon_j} &\to n && \text{in} \quad C^0\big(\overline{\Omega} \times [0,\infty)\big),\\
n_{\epsilon_j} &\rightharpoonup n && \text{in} \quad L^2_{loc}\big((0,\infty); W^{1,2}(\Omega)\big),\\
c_{\epsilon_j} &\to c && \text{in} \quad L^\infty_{loc}\big((0,\infty); C^0(\overline{\Omega})\big) \bigcap L^2_{loc}\big((0,\infty); W^{1,2}(\Omega)\big),\\
\nabla c_{\epsilon_j} &\overset{*}{\rightharpoonup} \nabla c && \text{in} \quad \bigcap_{\hat{q}>3} L^\infty_{loc}\big((0,\infty); W^{1,\hat{q}}(\Omega)\big) \bigcap L^\infty_{loc}\big(\Omega \times (0,\infty)\big) \quad \text{and}\\
u_{\epsilon_j} &\to u && \text{in} \quad C^0\big(\overline{\Omega} \times [0,\infty); \mathbb{R}^3\big) \bigcap C^{2,1}_{loc}\big(\overline{\Omega} \times (0,\infty); \mathbb{R}^3\big)
\end{aligned}
$$

as $j \to \infty$.

Proof. Firstly, it follows from lemma 3.7 and corollary 3.3 that

$$
\sup_{\epsilon\in(0,1)} \|\nabla c_\epsilon\|_{L^\infty((0,\infty);L^4(\Omega))} \leqslant C \qquad \text{and} \qquad \sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty((0,\infty);L^4(\Omega))} \leqslant C
$$

for some positive constant C. Then by taking $\lambda := \infty$, $q := 4$, $r := 4$ and $d = 3$ in lemma 2.3, we complete the proof of lemma 3.8.

The uniqueness of solution (n, c, u, P) to system (1.16) can be showed by a standard energy method together with a bootstrap argument and thus we omit the details here. □

4. Existence of solution to limit system: the 2D case

In this section, we deal with the existence of classical solution to limit system (1.16) in the 2D case. Our basic strategy is similar to the 3D setting in §3 and thus we will just give a sketch for completeness.

Firstly, we can show the gain of regularity for u_ϵ from the mass conservation of n_ϵ by using the smoothing effect and decay estimates of the Stokes semigroup.

LEMMA 4.1. *Suppose that (1.12)–(1.15) hold with $\alpha > 0$. Then for any $r > 1$, there exists a positive constant $C = C(r, n_0, c_0, u_0, \phi)$ such that for each $\epsilon \in (0, 1)$, we have*

$$
\|u_\epsilon(\cdot, t)\|_{L^r(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty).
$$

Then similar to lemma 3.4, we can improve our knowledge on the spatial regularity of n_ϵ by utilizing a very subtle induction argument for n_ϵ.

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LEMMA 4.2. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 0$. *For any fixed* $\widehat{r} > 1$ *satisfying* $(1-\alpha/2)\widehat{r} < 1$, *define the sequence* $\{s_k\}_{k=0}^{\infty}$ *by fixing*

$$s_0 = 1, \quad s_k \in \left(\max\left\{1,\, 2\alpha + \frac{s_{k-1}}{\widehat{r}}\right\}, 2\alpha + \frac{s_{k-1}(s_{k-1} + 2\alpha\widehat{r} + 2 - 2\alpha)}{s_{k-1} + 2\widehat{r}}\right),$$
$$(k = 1, 2, 3, \ldots).$$

Then for every $k = 1, 2, 3, \ldots$, *there exists a positive constant* C_k *such that*

$$\|n_\epsilon(\cdot, t)\|_{L^{s_k}(\Omega)} \leqslant C_k \quad \text{for all} \quad t \in (0, \infty), \quad (k = 1, 2, 3, \ldots).$$

COROLLARY 4.1. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 0$. *Then there exists* $C > 0$ *such that*

$$\|n_\epsilon(\cdot, t)\|_{L^4(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty).$$

Proof. Let $\{s_k\}_{k=0}^{\infty}$ and $\widehat{r}$ be defined by lemma 4.2. Then $s_k > 2\alpha + (1/\widehat{r})s_{k-1}$ $(k = 1, 2, 3, \ldots)$, implies that

$$s_k > 2\alpha + \frac{1}{\widehat{r}}\left(2\alpha + \frac{1}{\widehat{r}}s_{k-2}\right) > \cdots > 2\alpha\left(1 + \frac{1}{\widehat{r}} + \frac{1}{\widehat{r}^2} + \cdots + \frac{1}{\widehat{r}^{k-1}}\right)$$
$$+ \frac{1}{\widehat{r}^k}s_0 = \frac{2\alpha\widehat{r}}{\widehat{r}-1} + \frac{1}{\widehat{r}^k}\left(s_0 - \frac{2\alpha\widehat{r}}{\widehat{r}-1}\right)$$

for $k = 1, 2, 3, \ldots$. Noticing that $2\alpha\widehat{r}/(\widehat{r}-1) > 4$ due to $(1-\alpha/2)\widehat{r} < 1$, we can deduce from $\widehat{r} > 1$ that $s_k \geqslant 4$ for k large enough. It then follows from lemma 4.2 (and the Hölder inequality if necessary) that $\|n_\epsilon(\cdot, t)\|_{L^4(\Omega)}$ is bounded in $(0, \infty)$. This completes the proof of corollary 4.1. □

Thus similar to lemma 3.5, the spatial regularity of n_ϵ together with the damping effect of c_ϵ will provide the key uniform L^2 bound for ∇c_ϵ.

LEMMA 4.3. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 0$. *Then there exists* $C > 0$ *such that*

$$\|\nabla c_\epsilon(\cdot, t)\|_{L^2(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty).$$

Therefore the L^4 regularity of ∇c_ϵ follows by an argument quite similar to that used in lemma 3.7.

LEMMA 4.4. *Suppose that* (1.12)–(1.15) *hold with* $\alpha > 0$. *Then there exists* $C > 0$ *such that*

$$\|\nabla c_\epsilon(\cdot, t)\|_{L^4(\Omega)} \leqslant C \quad \text{for all} \quad t \in (0, \infty).$$

Now based on lemmas 4.4 and 4.1, we can take $\lambda := \infty$, $q := 4$, $r := 4$ and $d = 2$ in lemma 2.3 to deduce the existence of solution to the 2D limit system.

LEMMA 4.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\alpha > 0$ and (1.12)–(1.15) hold. Then there exist a sequence $\{\epsilon_j\}_{j=1}^{\infty}$ and a unique classical solution (n, c, u, P) to system (1.16) in $\Omega \times (0, \infty)$ with the properties that*

$$
\begin{aligned}
n_{\epsilon_j} &\to n && \text{in} \quad C^0\big(\overline{\Omega} \times [0,\infty)\big),\\
n_{\epsilon_j} &\rightharpoonup n && \text{in} \quad L^2_{loc}\big((0,\infty); W^{1,2}(\Omega)\big),\\
c_{\epsilon_j} &\to c && \text{in} \quad L^\infty_{loc}\big((0,\infty); C^0(\overline{\Omega})\big) \bigcap L^2_{loc}\big((0,\infty); W^{1,2}(\Omega)\big),\\
\nabla c_{\epsilon_j} &\overset{*}{\rightharpoonup} \nabla c && \text{in} \quad \bigcap_{\hat{q}>2} L^\infty_{loc}\big((0,\infty); W^{1,\hat{q}}(\Omega)\big) \bigcap L^\infty_{loc}\big(\Omega \times (0,\infty)\big) \quad \text{and}\\
u_{\epsilon_j} &\to u && \text{in} \quad C^0\big(\overline{\Omega} \times [0,\infty); \mathbb{R}^2\big) \bigcap C^{2,1}_{loc}\big(\overline{\Omega} \times (0,\infty); \mathbb{R}^2\big)
\end{aligned}
$$

as $j \to \infty$.

5. Convergence rate: final proof of theorem 1.1

In the section, we will show the convergence for the whole sequence $(n_\epsilon, c_\epsilon, u_\epsilon)$ and derive an algebraic convergence rate with respect to ϵ. The key idea is to analyse a new entropy-like evolution estimate for a mixed functional and then to use the standard smoothing effect of Stokes semigroup and the energy estimate.

Throughout this section, we let $(n, , c, u, P)$ be solutions of system (1.16) obtained in lemmas 3.8 and 4.5, and set

$$
\widehat{n} := n_\epsilon - n, \qquad \widehat{c} := c_\epsilon - c, \qquad \widehat{u} := u_\epsilon - u, \quad \text{and} \quad \widehat{P} := P_\epsilon - P
$$

for simplicity. Then $(\widehat{n}, \widehat{c}, \widehat{u})$ will be a solution to the following system

$$
\begin{cases}
\partial_t \widehat{n} = \Delta \widehat{n} - u_\epsilon \cdot \nabla \widehat{n} - \widehat{u} \cdot \nabla n - \nabla \\
\qquad \cdot \Big(\widehat{n} S(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon + n S(x, n_\epsilon, c_\epsilon) \cdot \nabla \widehat{c} \\
\qquad + n\big(S(x, n_\epsilon, c_\epsilon) - S(x, n, c)\big) \cdot \nabla c \Big), & x \in \Omega,\ t > 0,\\
\epsilon \partial_t c_\epsilon = \Delta \widehat{c} - u_\epsilon \cdot \nabla \widehat{c} - \widehat{u} \cdot \nabla c - \widehat{c} + \widehat{n}, & x \in \Omega,\ t > 0,\\
\partial_t \widehat{u} = \Delta \widehat{u} - \nabla \widehat{P} + \widehat{n} \nabla \phi, & x \in \Omega,\ t > 0,\\
\nabla \cdot \widehat{u} = 0, & x \in \Omega,\ t > 0
\end{cases}
\tag{5.1}
$$

with the initial-boundary values

$$
\begin{cases}
\Big(\nabla \widehat{n} - \widehat{n} S(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon - n S(x, n_\epsilon, c_\epsilon) \cdot \nabla \widehat{c} \\
\quad - n\big(S(x, n_\epsilon, c_\epsilon) - S(x, n, c)\big) \cdot \nabla c \Big) \cdot \nu = 0, \\
\quad \nabla \widehat{c} \cdot \nu = 0, \qquad \widehat{u} = 0, & x \in \partial\Omega,\ t > 0,\\
\widehat{n}(x, 0) = 0, \qquad \widehat{u}(x, 0) = 0, & x \in \Omega.
\end{cases}
$$

As a last preparation for the proof of theorem 1.1, let us draw a linear growth estimate for the spatio-temporal integral of $\partial_t c_\epsilon c$.

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Lemma 5.1. *There exists a positive constant C such that for each $\epsilon \in (0,1)$, we have*

$$\int_0^t \int_\Omega \partial_t c_\epsilon c \leqslant C(1+t) \qquad \text{for all} \quad t \in (0,\infty).$$

Proof. Noticing that

$$\int_\Omega \partial_t c_\epsilon c = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon c - \int_\Omega c_\epsilon \partial_t c \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c_\epsilon c + \frac{1}{2}\left(\int_\Omega c_\epsilon^2 + \int_\Omega (\partial_t c)^2\right)$$

for all $t \in (0,\infty)$, we see from the boundedness of c_ϵ and c, which can be derived from lemmas 3.7 and 4.4 together with the Sobolev embedding, that

$$\begin{aligned}\int_0^t \int_\Omega \partial_t c_\epsilon c &\leqslant \int_\Omega c_\epsilon(\cdot,t)c(\cdot,t) - \int_\Omega c_0(\cdot)c(\cdot,0) + \frac{1}{2}\left(\int_0^t\int_\Omega c_\epsilon^2 + \int_0^t\int_\Omega (\partial_t c)^2\right) \\ &\leqslant \|c_\epsilon\|_{L^\infty(\Omega\times(0,\infty))}\|c\|_{L^\infty(\Omega\times(0,\infty))}|\Omega| \\ &\quad + \frac{1}{2}\left(\|c_\epsilon\|^2_{L^\infty(\Omega\times(0,\infty))}|\Omega|t + \int_0^t\int_\Omega (\partial_t c)^2\right) \\ &\leqslant C_1(1+t) + \frac{1}{2}\int_0^t\int_\Omega (\partial_t c)^2 \end{aligned} \tag{5.2}$$

for all $t \in (0,\infty)$ with some positive constant C_1. It remains to deal with the L^2 space–time estimate of $\partial_t c$.

To this end, we first establish the uniform estimates for the difference quotient

$$c_h(x,t) := \frac{c(x,t+h) - c(x,t)}{h}$$

for any $t \in (\tau,\infty)$ with $\tau \in (0,\infty)$ and $h \in (-\tau,\infty)$. Then due to the classical regularity $c(\cdot,t) \in C^{2+\theta}(\overline{\Omega})$ for some $\theta > 0$ and all $t \in (0,\infty)$, we see that for each $t \in (\tau,\infty)$, $c_h(\cdot,t) \in C^2(\overline{\Omega})$ is a classical solution of the homogeneous Neumann boundary-value problem for

$$\begin{aligned}-\Delta c_h(\cdot,t) + c_h(\cdot,t) &= \frac{n(\cdot,t+h) - u(\cdot,t+h)\cdot\nabla c(\cdot,t+h)}{h} \\ &\quad - \frac{n(\cdot,t) - u(\cdot,t)\cdot\nabla c(\cdot,t)}{h}\end{aligned} \tag{5.3}$$

in Ω. Setting

$$n_h(\cdot,t) := \frac{n(x,t+h) - n(x,t)}{h} \quad \text{and} \quad u_h(\cdot,t) := \frac{u(x,t+h) - u(x,t)}{h}$$

and rearranging the right-hand side of (5.3), we obtain

$$-\Delta c_h(\cdot,t) + c_h(\cdot,t) = -u_h(\cdot,t)\cdot\nabla c(\cdot,t+h) - u(\cdot,t)\cdot\nabla c_h(\cdot,t) + n_h(\cdot,t)$$

in Ω. Testing the above equation by $c_h(\cdot,t)$ and utilizing the integration by parts and the Hölder inequality, we have

$$\begin{aligned}
&\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)}\\
&\quad = -\int_\Omega c_h(\cdot,t)u_h(\cdot,t)\cdot\nabla c(\cdot,t+h) + \int_\Omega c_h(\cdot,t)n_h(\cdot,t)\\
&\quad \leqslant \|c_h(\cdot,t)\|_{L^4(\Omega)}\|u_h(\cdot,t)\|_{L^2(\Omega)}\|\nabla c(\cdot,t+h)\|_{L^4(\Omega)}\\
&\qquad + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)}
\end{aligned}$$

for all $t\in(\tau,\infty)$, which together with the uniform boundedness of $\|\nabla c(\cdot,t+h)\|_{L^4(\Omega)}$, the Gagliardo–Nirenberg inequality and the Young inequality yields that

$$\begin{aligned}
&\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)}\\
&\quad \leqslant C_2\|c_h(\cdot,t)\|_{L^4(\Omega)}\|u_h(\cdot,t)\|_{L^2(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)}\\
&\quad \leqslant C_3\Big(\|\nabla c_h(\cdot,t)\|^{d/4}_{L^2(\Omega)}\|c_h(\cdot,t)\|^{(4-d)/4}_{L^2(\Omega)} + \|c_h(\cdot,t)\|_{L^2(\Omega)}\Big)\|u_h(\cdot,t)\|_{L^2(\Omega)}\\
&\qquad + \|c_h(\cdot,t)\|_{L^2(\Omega)}\|n_h(\cdot,t)\|_{L^2(\Omega)}\\
&\quad \leqslant \frac{1}{2}\Big(\|\nabla c_h(\cdot,t)\|^2_{L^2(\Omega)} + \|c_h(\cdot,t)\|^2_{L^2(\Omega)}\Big) + C_4\Big(\|u_h(\cdot,t)\|^2_{L^2(\Omega)} + \|n_h(\cdot,t)\|^2_{L^2(\Omega)}\Big)
\end{aligned}$$

for all $t\in(\tau,\infty)$ and some uniform positive constants C_2, C_3 and C_4, and thus that

$$\begin{aligned}
\int_\tau^t\int_\Omega|\nabla c_h|^2 + \int_\tau^t\int_\Omega c_h^2 &\leqslant 2C_4\left(\int_\tau^t\int_\Omega n_h^2 + \int_\tau^t\int_\Omega|u_h|^2\right)\\
&\leqslant C_5\left(\int_0^{t+1}\int_\Omega(\partial_t n)^2 + \int_0^{t+1}\int_\Omega|\partial_t u|^2\right) \qquad (5.4)
\end{aligned}$$

for all $t\in(\tau,\infty)$ and some uniform constant $C_5>0$. Here for the last inequality, we used the temporal version of theorem 3(i) in §5.8.2 [**6**].

To deal with the first integral on the right-hand side of (5.4), we first use the maximal regularity of parabolic equations (theorem 2.3 in [**8**]) and the trace theorem to obtain

$$\begin{aligned}
&\|\partial_t n\|_{L^2((0,t+1);L^2(\Omega))}\\
&\quad\leqslant C_6\Big(\|u\cdot\nabla n\|_{L^2((0,t+1);L^2(\Omega))} + \|\nabla\cdot(nS(x,n,c)\cdot\nabla c)\|_{L^2((0,t+1);L^2(\Omega))}\\
&\qquad + \|nS(x,n,c)\cdot\nabla c\|_{L^2((0,t+1);W^{1/2,2}(\partial\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big)\\
&\quad\leqslant C_7\Big(\|u\cdot\nabla n\|_{L^2((0,t+1);L^2(\Omega))} + \|nS(x,n,c)\\
&\qquad \cdot\nabla c\|_{L^2((0,t+1);W^{1,2}(\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big)
\end{aligned}$$

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for some positive constants C_6 and C_7, which together with the assumption on S yields that

$$\begin{aligned}\|\partial_t n\|_{L^2((0,t+1);L^2(\Omega))} &\leqslant C_8\Big(\|u\|_{L^\infty(\Omega\times(0,\infty))}\|\nabla n\|_{L^2((0,t+1);L^2(\Omega))}\\ &\quad + \|\nabla n\|_{L^2((0,t+1);L^2(\Omega))}\|\nabla c\|_{L^\infty(\Omega\times(0,\infty))}\\ &\quad + \|n\|_{L^\infty(\Omega\times(0,\infty))}\|\nabla c\|_{L^2((0,t+1);W^{1,2}(\Omega))} + \|n_0\|_{W^{1,2}(\Omega)}\Big)\\ &\leqslant C_9(1+t)^{1/2} \end{aligned} \tag{5.5}$$

for all $t \in (0,\infty)$ with positive constants C_8 and C_9. Here in the last inequality, we used the boundedness of u, n and ∇c, which follows from lemma 3.2 with $r = \infty$, the proofs of lemmas 2.2 and 5.3 in [**30**], respectively, and the growth estimates $\|\nabla n\|^2_{L^2((0,t+1);L^2(\Omega))} \leqslant C(1+t)$ and $\|D^2 c\|^2_{L^2((0,t+1);L^2(\Omega))} \leqslant C(1+t)$ obtained by following the proof of lemma 3.4 in [**30**] and by a direct integral in equation $(1.16)_2$, respectively. On the other hand, for the second integral on the right-hand side of (5.4), we utilize maximal regularity of Stokes equations to obtain

$$\begin{aligned}\|\partial_t u\|_{L^2((0,t+1);L^2(\Omega))} &\leqslant C_{10}\Big(\|Au_0\|_{L^2(\Omega)} + \|\mathcal{P}(n\nabla\phi)\|_{L^2((0,t+1);L^2(\Omega))}\Big)\\ &\leqslant C_{11}\Big(\|u_0\|_{W^{2,2}(\Omega)} + \|n\|_{L^2((0,t+1);L^2(\Omega))}\|\nabla\phi\|_{L^\infty(\Omega)}\Big)\\ &\leqslant C_{12}(1+t)^{1/2} \end{aligned} \tag{5.6}$$

for all $t \in (0,\infty)$ and some positive constants C_{10}, C_{11} and C_{12}.

Inserting (5.5) and (5.6) into (5.4), we see that

$$\int_\tau^t\int_\Omega |\nabla c_h|^2 + \int_\tau^t\int_\Omega c_h^2 \leqslant C_{13}(1+t) := C_5\big(C_9^2 + C_{12}^2\big)(1+t)$$

for all $t \in (\tau,\infty)$ and some uniform positive constant C_{13}. Consequently, there exists $(h_i)_{i\in\mathbb{N}} \subset (-\tau,\infty)$ such that $h_i \to 0$ and $c_{h_i} \to \partial_t c$ in $L^2(\Omega\times(\tau,t))$ with the same bound $\int_\tau^t\int_\Omega(\partial_t c)^2 \leqslant C_{13}(1+t)$ as $i \to \infty$. Noticing that C_{13} is independent of τ, we may take $\tau \searrow 0$ to find $\int_0^t\int_\Omega(\partial_t c)^2 \leqslant C_{13}(1+t)$, which together with (5.2) yields that $\int_0^t\int_\Omega \partial_t c_\epsilon c \leqslant C_{14}(1+t)$ for all $t \in (0,\infty)$ with $C_{14} = C_1 + (1/2)C_{13}$. This completes the proof of lemma 5.1. □

We can now apply a subtle energy estimate to gain the convergence rate of fast signal diffusion limit.

LEMMA 5.2. *There exists $C > 0$ such that for each $\epsilon \in (0,1)$, we have*

$$\|\widehat{n}(\cdot,t)\|_{L^2(\Omega)} + \|\widehat{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant C\mathrm{e}^{Ct}\epsilon^{1/2} \quad \text{for all} \quad t \in (0,\infty)$$

and

$$\begin{aligned}&\|\widehat{n}(\cdot,s)\|_{L^2((0,t);H^1(\Omega))} + \|\widehat{c}(\cdot,s)\|_{L^2((0,t);H^1(\Omega))} + \|\widehat{u}(\cdot,s)\|_{L^2((0,t);H^1(\Omega))}\\ &\quad\leqslant C\mathrm{e}^{Ct}\epsilon^{1/2} \quad \text{for all} \quad t \in (0,\infty).\end{aligned}$$

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Proof. Our proof is based on a entropy-like evolution estimate for the mixed functional involving $\widehat{n}$, $\widehat{u}$ and c_ϵ of the form

$$\|\widehat{n}(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\|c_\epsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|\widehat{u}(\cdot,t)\|^2_{L^2(\Omega)}.$$

To this end, we first multiply equation $(5.1)_1$ by $\widehat{n}$ and integrate by parts over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{n}\|^2_{L^2(\Omega)} + \|\nabla\widehat{n}\|^2_{L^2(\Omega)} &= \int_\Omega n\widehat{u}\cdot\nabla\widehat{n} + \int_\Omega \widehat{n}S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\cdot\nabla\widehat{n}\\ &\quad + \int_\Omega nS(x,n_\epsilon,c_\epsilon)\cdot\nabla\widehat{c}\cdot\nabla\widehat{n}\\ &\quad + \int_\Omega n\big(S(x,n_\epsilon,c_\epsilon) - S(x,n,c)\big)\cdot\nabla c\cdot\nabla\widehat{n}\\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.7}$$

For I_1 and I_3, we can see from the boundedness of n, the upper estimate (1.13) for S, the Hölder inequality and the Young inequality that

$$I_1 \leqslant \|n\|_{L^\infty(\Omega)}\|\widehat{u}\|_{L^2(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)} \leqslant \frac{1}{8}\|\nabla\widehat{n}\|^2_{L^2(\Omega)} + C_1\|\widehat{u}\|^2_{L^2(\Omega)} \tag{5.8}$$

and

$$I_3 \leqslant C_S\|n\|_{L^\infty(\Omega)}\|\nabla\widehat{c}\|_{L^2(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)} \leqslant \frac{1}{8}\|\nabla\widehat{n}\|^2_{L^2(\Omega)} + C_2\|\nabla\widehat{c}\|^2_{L^2(\Omega)} \tag{5.9}$$

for all $t \in (0,\infty)$ with some positive constants C_1 and C_2. For I_2, we first use the Hölder inequality, the upper estimate (1.13) for S and the boundedness of $\|\nabla c_\epsilon(\cdot,t)\|_{L^4(\Omega)}$ obtained in lemmas 3.7 and 4.4 to deduce that

$$I_2 \leqslant C_S\|\widehat{n}\|_{L^4(\Omega)}\|\nabla c_\epsilon\|_{L^4(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)} \leqslant C_3\|\widehat{n}\|_{L^4(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)}$$

for all $t \in (0,\infty)$ and some positive constant C_3, which together with the Gagliardo–Nirenberg inequality and the Young inequality yields that

$$\begin{aligned} I_2 &\leqslant C_4\Big(\|\nabla\widehat{n}\|^{d/4}_{L^2(\Omega)}\|\widehat{n}\|^{(4-d)/4}_{L^2(\Omega)} + \|\widehat{n}\|_{L^2(\Omega)}\Big)\|\nabla\widehat{n}\|_{L^2(\Omega)}\\ &\leqslant C_5\|\nabla\widehat{n}\|^{(4+d)/4}_{L^2(\Omega)}\|\widehat{n}\|^{(4-d)/4}_{L^2(\Omega)} + C_5\|\widehat{n}\|_{L^2(\Omega)}\|\nabla\widehat{n}\|_{L^2(\Omega)}\\ &\leqslant \frac{1}{8}\|\nabla\widehat{n}\|^2_{L^2(\Omega)} + C_6\|\widehat{n}\|^2_{L^2(\Omega)} \end{aligned} \tag{5.10}$$

for all $t \in (0,\infty)$ with some positive constants C_4, C_5 and C_6. Finally, for I_4, it follows from the differential mean value theorem, (1.12) and the boundedness of

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$n_\epsilon, n, c_\epsilon, c$ and ∇c that

$$\begin{aligned} I_4 &\leqslant \int_\Omega n\big|S(x,n_\epsilon,c_\epsilon)-S(x,n,c)\big||\nabla c||\nabla \widehat{n}| \\ &\leqslant \int_\Omega n\Big(\big|S(x,n_\epsilon,c_\epsilon)-S(x,n,c_\epsilon)\big|+\big|S(x,n,c_\epsilon)-S(x,n,c)\big|\Big)|\nabla c||\nabla \widehat{n}| \\ &\leqslant \int_\Omega n\Big(\big|\nabla S(x,\xi,c_\epsilon)\big||n_\epsilon-n|+\big|\nabla S(x,n,\eta)\big||c_\epsilon-c|\Big)|\nabla c||\nabla \widehat{n}| \\ &\leqslant C_7\|n\|_{L^\infty(\Omega)}\|\nabla c\|_{L^\infty(\Omega)}\int_\Omega\big(|\widehat{n}|+|\widehat{c}|\big)|\nabla \widehat{n}| \\ &\leqslant \frac{1}{8}\|\nabla \widehat{n}\|^2_{L^2(\Omega)}+C_8\big(\|\widehat{n}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)}\big) \end{aligned} \tag{5.11}$$

for all $t \in (0,\infty)$ with positive constants C_7 and C_8, where ξ and η depend on n_ϵ, n and c_ϵ, c, respectively. Substituting the estimates of (5.8)–(5.11) into (5.7), we have

$$\frac{d}{dt}\|\widehat{n}\|^2_{L^2(\Omega)}+\|\nabla \widehat{n}\|^2_{L^2(\Omega)} \leqslant C_9\Big(\|\widehat{n}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)}+\|\nabla \widehat{c}\|^2_{L^2(\Omega)}+\|\widehat{u}\|^2_{L^2(\Omega)}\Big) \tag{5.12}$$

for all $t \in (0,\infty)$ with some positive constant C_9.

Next, testing equation $(5.1)_2$ by $\widehat{c}$ and using the integration by parts over Ω, we can obtain

$$\begin{aligned} &\frac{\epsilon}{2}\frac{d}{dt}\|c_\epsilon\|^2_{L^2(\Omega)}+\|\nabla \widehat{c}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)} \\ &\quad= \epsilon\int_\Omega \partial_t c_\epsilon \widehat{c}+\epsilon\int_\Omega \partial_t c_\epsilon c+\|\nabla \widehat{c}\|^2_{L^2(\Omega)}+\|\widehat{c}\|^2_{L^2(\Omega)} \\ &\quad= \epsilon\int_\Omega \partial_t c_\epsilon c-\int_\Omega \widehat{c}\,\widehat{u}\cdot\nabla c+\int_\Omega \widehat{n}\widehat{c} := \epsilon\int_\Omega \partial_t c_\epsilon c+I_5+I_6. \end{aligned} \tag{5.13}$$

For I_5 and I_6, it is clear from the Hölder inequality, the Young inequality and the boundedness of ∇c that

$$I_5 \leqslant \|\nabla c\|_{L^\infty(\Omega)}\|\widehat{u}\|_{L^2(\Omega)}\|\widehat{c}\|_{L^2(\Omega)} \leqslant \frac{1}{8}\|\widehat{c}\|^2_{L^2(\Omega)}+C_{10}\|\widehat{u}\|^2_{L^2(\Omega)}$$

and

$$I_6 \leqslant \frac{1}{8}\|\widehat{c}\|^2_{L^2(\Omega)}+C_{11}\|\widehat{n}\|^2_{L^2(\Omega)}$$

for all $t \in (0,\infty)$ with some positive constants C_{10} and C_{11}. Substituting them into (5.13), we obtain that

$$\begin{aligned} &\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\|c_\epsilon\|^2_{L^2(\Omega)}+2\|\nabla \widehat{c}\|^2_{L^2(\Omega)}+\frac{3}{2}\|\widehat{c}\|^2_{L^2(\Omega)} \\ &\qquad \leqslant 2\epsilon\int_\Omega \partial_t c_\epsilon c+2C_{11}\|\widehat{n}\|^2_{L^2(\Omega)}+2C_{10}\|\widehat{u}\|^2_{L^2(\Omega)} \end{aligned} \tag{5.14}$$

for all $t \in (0,\infty)$.

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Similarly, multiplying equation $(5.1)_3$ by $\widehat{u}$ and making use of the Hölder inequality, the Poincaré inequality due to $\widehat{u}=0$ on $\partial\Omega$ and the Young inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{u}\|^2_{L^2(\Omega)} + \|\nabla\widehat{u}\|^2_{L^2(\Omega)} = \int_\Omega \widehat{n}\,\widehat{u}\cdot\nabla\phi &\leqslant \|\nabla\phi\|_{L^\infty(\Omega)}\|\widehat{u}\|_{L^2(\Omega)}\|\widehat{n}\|_{L^2(\Omega)} \\ &\leqslant C_{12}\|\nabla\phi\|_{L^\infty(\Omega)}\|\nabla\widehat{u}\|_{L^2(\Omega)}\|\widehat{n}\|_{L^2(\Omega)} \\ &\leqslant \frac{1}{2}\|\nabla\widehat{u}\|^2_{L^2(\Omega)} + C_{13}\|\widehat{n}\|^2_{L^2(\Omega)}\end{aligned}$$

for all $t\in(0,\infty)$ with some positive constants C_{12} and C_{13} and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{u}\|^2_{L^2(\Omega)} + \|\nabla\widehat{u}\|^2_{L^2(\Omega)} \leqslant 2C_{13}\|\widehat{n}\|^2_{L^2(\Omega)}. \tag{5.15}$$

Summarily, we multiply (5.14) by C_9 and combine the resulted inequality with (5.12) and (5.15) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\widehat{n}\|^2_{L^2(\Omega)} + \epsilon C_9\|c_\epsilon\|^2_{L^2(\Omega)} + \|\widehat{u}\|^2_{L^2(\Omega)}\Big) + \|\nabla\widehat{n}\|^2_{L^2(\Omega)} \\ &\quad + C_9\left(\|\nabla\widehat{c}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widehat{c}\|^2_{L^2(\Omega)}\right) + \|\nabla\widehat{u}\|^2_{L^2(\Omega)} \\ &\qquad \leqslant 2\epsilon C_9\int_\Omega \partial_t c_\epsilon c + \big(C_9 + 2C_9C_{11} + 2C_{13}\big)\|\widehat{n}\|^2_{L^2(\Omega)} + \big(C_9 + 2C_9C_{10}\big)\|\widehat{u}\|^2_{L^2(\Omega)}\end{aligned} \tag{5.16}$$

for all $t\in(0,\infty)$. Thus by setting

$$y(t) := \|\widehat{n}\|^2_{L^2(\Omega)} + \epsilon C_9\|c_\epsilon\|^2_{L^2(\Omega)} + \|\widehat{u}\|^2_{L^2(\Omega)}$$

and

$$g(t) := \|\nabla\widehat{n}\|^2_{L^2(\Omega)} + C_9\Big(\|\nabla\widehat{c}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widehat{c}\|^2_{L^2(\Omega)}\Big) + \|\nabla\widehat{u}\|^2_{L^2(\Omega)}$$

as well as

$$h(t) := 2C_9\int_\Omega \partial_t c_\epsilon c,$$

we can simplify (5.16) as

$$y'(t) + g(t) \leqslant C_{14}y(t) + \epsilon h(t)$$

for all $t\in(0,\infty)$ with $C_{14} := \max\big\{C_9 + 2C_9C_{11} + 2C_{13},\, C_9 + 2C_9C_{10}\big\}$. It then follows from the Gronwall inequality and lemma 5.1 that

$$y(t) \leqslant \mathrm{e}^{C_{14}t}\left(y(0) + \epsilon\int_0^t \mathrm{e}^{-C_{14}s}h(s)\mathrm{d}s\right) \leqslant \mathrm{e}^{C_{14}t}\left(\epsilon C_9\|c_0\|^2_{L^2(\Omega)} + \epsilon C_{15}(1+t)\right)$$

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for all $t \in (0,\infty)$ and some $C_{15} > 0$ due to $\widehat{n}(\cdot,0) = 0$ and $\widehat{u}(\cdot,0) = 0$, which together with the fact $1 + t \leqslant \mathrm{e}^t$ implies that

$$\|\widehat{n}(\cdot,t)\|^2_{L^2(\Omega)} + \|\widehat{u}(\cdot,t)\|^2_{L^2(\Omega)} \leqslant y(t) \leqslant C_{16}\mathrm{e}^{C_{16}t}\epsilon$$

for all $t \in (0,\infty)$ and some $C_{16} > 0$ and thus that

$$\int_0^t g(s)\mathrm{d}s \leqslant y(0) + C_{14}\int_0^t y(s)\mathrm{d}s + \epsilon\int_0^t h(s)\mathrm{d}s \leqslant C_{17}\mathrm{e}^{C_{17}t}\epsilon$$

for all $t \in (0,\infty)$ and some $C_{17} > 0$. Since $\int_\Omega \widehat{n} = 0$ and $\widehat{u} = 0$ on $\partial\Omega$, we can also use the Poincaré inequality to deduce that

$$\begin{aligned}&\|\widehat{n}(\cdot,s)\|^2_{L^2((0,t);W^{1,2}(\Omega))} + \|\widehat{c}(\cdot,s)\|^2_{L^2((0,t);W^{1,2}(\Omega))} + \|\widehat{u}(\cdot,s)\|^2_{L^2((0,t);W^{1,2}(\Omega))}\\ &\quad\leqslant C_{18}\mathrm{e}^{C_{18}t}\epsilon\end{aligned}$$

for all $t \in (0,\infty)$ and some $C_{18} > 0$. This completes the proof of lemma 5.2. □

Lemma 5.3. *For any given $\theta \in (0,1)$, there exists a positive constant $C(\theta)$ such that for each $\epsilon \in (0,1)$, we have*

$$\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant C(\theta)\mathrm{e}^{C(\theta)t}\epsilon^{1/2} \quad \text{for all} \quad t \in (0,\infty).$$

In particular, we have

$$\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C\mathrm{e}^{Ct}\epsilon^{1/2} \quad \text{for all} \quad t \in (0,\infty)$$

with some positive constant C.

Proof. We first apply the Helmholtz projection $\mathcal{P}$ to both sides of equation $(5.1)_3$ and then make use of the variation-of-constants representation of $\widehat{u}$ to obtain that

$$\begin{aligned}\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} &= \left\|\int_0^t A^\theta\mathrm{e}^{-(t-s)A}\mathcal{P}(\widehat{n}\nabla\phi)(\cdot,s)\mathrm{d}s\right\|_{L^2(\Omega)}\\ &\leqslant C_1\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\lambda(t-s)}\|\mathcal{P}(\widehat{n}\nabla\phi)(\cdot,s)\|_{L^2(\Omega)}\mathrm{d}s\end{aligned}$$

for some $\lambda > 0$ and $C_1 > 0$ due to $\widehat{u}(x,0) = 0$. It follows from the Hölder inequality and lemma 5.2 that

$$\|\mathcal{P}(\widehat{n}\nabla\phi)(\cdot,s)\|_{L^2(\Omega)} \leqslant \|\widehat{n}(\cdot,s)\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)} \leqslant C_2\mathrm{e}^{C_2s}\epsilon^{1/2},$$

for some $C_2 > 0$ and thus that

$$\|A^\theta\widehat{u}(\cdot,t)\|_{L^2(\Omega)} \leqslant C_1C_2\epsilon^{\frac{1}{2}}\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\lambda(t-s)+C_2s}\mathrm{d}s \leqslant C_3\mathrm{e}^{C_3t}\epsilon^{1/2}$$

for all $t \in (0,\infty)$ and some $C_3 := C_3(\theta) > 0$. Since $D(A^\theta) \hookrightarrow L^\infty(\Omega,\mathbb{R}^d)$ whenever $\theta > d/4$, we have similar estimate for $\|\widehat{u}(\cdot,t)\|_{L^\infty(\Omega)}$. This completes the proof of lemma 5.3. □

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LEMMA 5.4. *For any $p \geqslant 4$, there exists $C(p,t) > 0$ such that for each $\epsilon \in (0,1)$, we have*

$$\|\widehat{n}(\cdot,t)\|_{L^p(\Omega)} \leqslant C(p,t)\epsilon^{(p+d-2)/(p(d+2))} \quad \text{for all} \quad t \in (0,\infty).$$

Proof. For this purpose, we first test equation $(5.1)_1$ by $\widehat{n}^{p-1}$ with $p \geqslant 4$ and integrate by parts over Ω to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{n}^{p/2}\|^2_{L^2(\Omega)} + \frac{4(p-1)}{p^2}\|\nabla\widehat{n}^{p/2}\|^2_{L^2(\Omega)} \\
&\quad = (p-1)\int_\Omega n\widehat{n}^{p-2}\widehat{u}\cdot\nabla\widehat{n} + (p-1)\int_\Omega \widehat{n}^{p-1}S(x,n_\epsilon,c_\epsilon)\cdot\nabla c_\epsilon\cdot\nabla\widehat{n} \\
&\qquad + (p-1)\int_\Omega n\widehat{n}^{p-2}S(x,n_\epsilon,c_\epsilon)\cdot\nabla\widehat{c}\cdot\nabla\widehat{n} \\
&\qquad + (p-1)\int_\Omega n\widehat{n}^{p-2}\big(S(x,n_\epsilon,c_\epsilon) - S(x,n,c)\big)\nabla c\cdot\nabla\widehat{n} \\
&:= I_7 + I_8 + I_9 + I_{10}.
\end{aligned} \tag{5.17}$$

We now estimate I_7, I_8, I_9 and I_{10} one by one. Indeed, it follows from the Hölder inequality, the Gagliardo–Nirenberg inequality and the Young inequality that

$$\begin{aligned}
I_7 &\leqslant (p-1)\|n\|_{L^\infty(\Omega)}\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}^{(p-2)/2}\|_{L^4(\Omega)}\|\widehat{u}\|_{L^4(\Omega)} \\
&\leqslant \frac{p-1}{8}\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|^2_{L^2(\Omega)} + 2(p-1)\|n\|^2_{L^\infty(\Omega)}\|\widehat{n}^{p/2}\|^{(2(p-2))/p}_{L^{(4(p-2))/p}(\Omega)}\|\widehat{u}\|^2_{L^4(\Omega)} \\
&\leqslant \frac{p-1}{2p^2}\|\nabla\widehat{n}^{p/2}\|^2_{L^2(\Omega)} \\
&\quad + C_1\Big(\|\nabla\widehat{n}^{p/2}\|^{(2d(p-3))/((p-2)d+4)}_{L^2(\Omega)}\|\widehat{n}^{p/2}\|^{(2(4-d)p+8d-16)/(p(p-2)d+4p)}_{L^{4/p}(\Omega)} \\
&\quad + \|\widehat{n}^{p/2}\|^{(2(p-2))/p}_{L^{4/p}(\Omega)}\Big)\|\widehat{u}\|^2_{L^4(\Omega)} \\
&\leqslant \frac{p-1}{p^2}\|\nabla\widehat{n}^{p/2}\|^2_{L^2(\Omega)} + C_2\|\widehat{n}^{p/2}\|^{(2(4-d)p+8d-16)/(p(d+4))}_{L^{4/p}(\Omega)}\|\widehat{u}\|^{(2(p-2)d+8)/(d+4)}_{L^4(\Omega)} \\
&\quad + C_2\|\widehat{n}\|^{p-2}_{L^2(\Omega)}\|\widehat{u}\|^2_{L^4(\Omega)} \\
&= \frac{p-1}{p^2}\|\nabla\widehat{n}^{\frac{p}{2}}\|^2_{L^2(\Omega)} + C_2\|\widehat{n}\|^{((4-d)p+4d-8)/(d+4)}_{L^2(\Omega)}\|\widehat{u}\|^{(2(p-2)d+8)/(d+4)}_{L^4(\Omega)} \\
&\qquad + C_2\|\widehat{n}\|^{p-2}_{L^2(\Omega)}\|\widehat{u}\|^2_{L^4(\Omega)}
\end{aligned} \tag{5.18}$$

for some positive constants C_1 and C_2, and

$$\begin{aligned}
I_8 &\leqslant (p-1)C_S\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}^{p/2}\|_{L^4(\Omega)}\|\nabla c_\epsilon\|_{L^4(\Omega)} \\
&\leqslant \frac{p-1}{8}\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|^2_{L^2(\Omega)} + 2(p-1)C_S^2\|\widehat{n}^{p/2}\|^2_{L^4(\Omega)}\|\nabla c_\epsilon\|^2_{L^4(\Omega)}
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \frac{p-1}{2p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 \\
&\quad + C_3\Big(\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^{(2d(p-1))/((p-2)d+4)}\|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^{(2(4-d))/((p-2)d+4)} + \|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^2\Big) \\
&\leqslant \frac{p-1}{p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 + C_4\|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^2 \\
&= \frac{p-1}{p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 + C_4\|\widehat{n}\|_{L^2(\Omega)}^p
\end{aligned}
\tag{5.19}
$$

for some positive constants C_3 and C_4, as well as

$$
\begin{aligned}
I_9 &\leqslant (p-1)C_S\|n\|_{L^\infty(\Omega)}\|\nabla c_\epsilon - \nabla c\|_{L^\infty(\Omega)}\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}^{(p-2)/2}\|_{L^2(\Omega)} \\
&\leqslant \frac{p-1}{8}\|\widehat{n}^{(p-2)/2}\nabla\widehat{n}\|_{L^2(\Omega)}^2 \\
&\quad + 2(p-1)C_S^2\|n\|_{L^\infty(\Omega)}^2\|\nabla c_\epsilon - \nabla c\|_{L^\infty(\Omega)}^2\|\widehat{n}^{p/2}\|_{L^{(2(p-2))/p}(\Omega)}^{(2(p-2))/p} \\
&\leqslant \frac{p-1}{2p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 \\
&\quad + C_5\Big(\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^{(2d(p-4))/((p-2)d+4)}\|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^{(8(d+p-2))/(p(p-2)d+4p)} \\
&\quad + \|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^{(2(p-2))/p}\Big) \\
&\leqslant \frac{p-1}{p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 + C_6\|\widehat{n}^{p/2}\|_{L^{4/p}(\Omega)}^{(4(d+p-2))/(p(d+2))} + C_6\|\widehat{n}\|_{L^2(\Omega)}^{p-2} \\
&= \frac{p-1}{p^2}\|\nabla\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 + C_6\|\widehat{n}\|_{L^2(\Omega)}^{(2(d+p-2))/(d+2)} + C_6\|\widehat{n}\|_{L^2(\Omega)}^{p-2}
\end{aligned}
\tag{5.20}
$$

for some positive constants C_5 and C_6. Similar to (5.20), we also have

$$
\begin{aligned}
I_{10} &\leqslant (p-1)\int_\Omega n\big|S(x,n_\epsilon,c_\epsilon) - S(x,n,c)\big||\nabla c||\widehat{n}^{p-2}\nabla\widehat{n}| \\
&\leqslant 2(p-1)C_S\|n\|_{L^\infty(\Omega)}\|\nabla c\|_{L^\infty(\Omega)}\|\widehat{n}^{\frac{p-2}{2}}\nabla\widehat{n}\|_{L^2(\Omega)}\|\widehat{n}^{\frac{p-2}{2}}\|_{L^2(\Omega)} \\
&\leqslant \frac{p-1}{8}\|\widehat{n}^{\frac{p-2}{2}}\nabla\widehat{n}\|_{L^2(\Omega)}^2 + 8(p-1)C_S^2\|n\|_{L^\infty(\Omega)}^2\|\nabla c\|_{L^\infty(\Omega)}^2\|\widehat{n}^{\frac{p}{2}}\|_{L^{\frac{2(p-2)}{p}}(\Omega)}^{\frac{2(p-2)}{p}} \\
&\leqslant \frac{p-1}{p^2}\|\nabla\widehat{n}^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + C_7\|\widehat{n}\|_{L^2(\Omega)}^{\frac{2(d+p-2)}{d+2}} + C_7\|\widehat{n}\|_{L^2(\Omega)}^{p-2}
\end{aligned}
\tag{5.21}
$$

for some $C_7 > 0$. Substituting (5.18)–(5.21) into (5.17), we deduce that

$$
\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{n}^{p/2}\|_{L^2(\Omega)}^2 &\leqslant C_2\|\widehat{n}\|_{L^2(\Omega)}^{((4-d)p+4d-8)/(d+4)}\|\widehat{u}\|_{L^4(\Omega)}^{(2(p-2)d+8)/(d+4)} \\
&\quad + C_2\|\widehat{n}\|_{L^2(\Omega)}^{p-2}\|\widehat{u}\|_{L^4(\Omega)}^2 + C_4\|\widehat{n}\|_{L^2(\Omega)}^p \\
&\quad + (C_6+C_7)\Big(\|\widehat{n}\|_{L^2(\Omega)}^{(2(d+p-2))/(d+2)} + \|\widehat{n}\|_{L^2(\Omega)}^{p-2}\Big)
\end{aligned}
$$

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for all $t \in (0, \infty)$, which together with lemmas 5.2, 5.3 and the interpolation yields that

$$\frac{d}{dt}\|\widehat{n}^{\frac{p}{2}}(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C_8 \mathrm{e}^{C_8 t}\left(\epsilon^{\frac{p}{2}} + \epsilon^{\frac{d+p-2}{d+2}} + \epsilon^{\frac{p-2}{2}}\right) \leqslant 3C_8 \mathrm{e}^{C_8 t}\epsilon^{\frac{d+p-2}{d+2}}$$

for all $t \in (0, \infty)$ and some positive constant C_8 due to $\epsilon \in (0, 1)$. Then a direct calculation together with the fact $\widehat{n}(x, 0) = 0$ in Ω implies that

$$\|\widehat{n}(\cdot,t)\|^p_{L^p(\Omega)} = \|\widehat{n}^{p/2}(\cdot,t)\|^2_{L^2(\Omega)} \leqslant 3\mathrm{e}^{C_8 t}\epsilon^{(d+p-2)/(d+2)}$$

for all $t \in (0, \infty)$. This completes the proof of lemma 5.4. □

Proof of theorem 1.1. A direct combination of lemmas 5.2–5.4 yields the desired result. □

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