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Abstract

Let $\Gamma(n,p)$ denote the binomial model of a random triangular group. We show that there exist constants $c, C > 0$ such that if $p \leqslant c/n^2$, then asymptotically almost surely (a.a.s.) $\Gamma(n,p)$ is free, and if $p \geqslant C \log n/n^2$, then a.a.s. $\Gamma(n,p)$ has Kazhdan's property (T). Furthermore, we show that there exist constants $C', c' > 0$ such that if $C'/n^2 \leqslant p \leqslant c' \log n/n^2$, then a.a.s. $\Gamma(n,p)$ is neither free nor has Kazhdan's property (T).

1. Introduction

Let $\langle S|R\rangle$ be a group presentation with a set of generators S and a set of relations R. We consider the following binomial model $\Gamma(n,p)$ of a random group which is very similar to the model introduced by Żuk in [Żuk03] (see also comments in §§ 3 and 4 below). Here S consists of n generators, while R consists of relations taken independently at random with probability p among all cyclically reduced words of length three, i.e. relations are of the form abc, where $a \neq b^{-1}$, $b \neq c^{-1}$ and $c \neq a^{-1}$. Presentations obtained in this way are called *triangular presentations*, and groups induced by them are called *triangular groups*.

In this paper we study the asymptotic behavior of the random triangular group $\Gamma(n,p)$ as $n \to \infty$. For a given function $p = p(n)$ we say that $\Gamma(n,p)$ has a given property asymptotically almost surely (a.a.s.) if the probability of $\Gamma(n,p)$ having this property tends to 1 as $n \to \infty$. From Żuk's result [Żuk03] (see also [KK13]) it follows that for every $\epsilon > 0$ a.a.s. $\Gamma(n,p)$ is free provided $p \leqslant n^{-2-\epsilon}$, and it has a.a.s. Kazhdan's property (T) whenever $p \geqslant n^{-2+\epsilon}$. This phenomenon is known as the phase transition at density $\frac{1}{3}$ for triangular random groups. We study the behavior at density $\frac{1}{3}$ more carefully. In particular, we show that there exist constants $c, C > 0$ such that if $p \leqslant c/n^2$ then a.a.s. the random group in $\Gamma(n,p)$ is a free group, while for $p \geqslant C \log n/n^2$ a.a.s. it has Kazhdan's property (T). What is more interesting, we identify a range for parameter p for which a.a.s. the random group in $\Gamma(n,p)$ is neither free nor has property (T). The main results of this paper are stated as the following three theorems.

Theorem 1. *There exists a constant $c > 0$ such that if $p \leqslant c/n^2$, then a.a.s. $\Gamma(n,p)$ is a free group.*

Theorem 2. *There exist constants $C', c' > 0$ such that if $C'/n^2 \leqslant p \leqslant c' \log n/n^2$, then a.a.s. $\Gamma(n,p)$ neither is a free group nor has Kazhdan's property (T).*

Theorem 3. *There exists a constant $C > 0$ such that if $p \geqslant C \log n/n^2$, then a.a.s. $\Gamma(n,p)$ has Kazhdan's property (T).*

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2. Proof of Theorem 1

We shall deduce Theorem 1 from the following lemma.

Lemma 4. *There exists a constant $c > 0$ such that for $p \leqslant c/n^2$ a.a.s. the random group $\Gamma(n,p) = \langle S|R\rangle$ has the following property: for every non-empty subset $R' \subseteq R$ of relations there exist $a \in S$ and $r \in R'$ such that neither a nor a^{-1} appears in any relation $t \in R' \backslash \{r\}$ and precisely one letter in r belongs to the set $\{a, a^{-1}\}$.*

Before we prove the above result let us observe that it implies Theorem 1.

Proof of Theorem 1. Indeed, if the random group $\Gamma(n,p) = \langle S|R\rangle$ has the property described in the assertion of Lemma 4, then we can eliminate generators one by one, each time decreasing the size of both the set of generators and the set of relations by one. Eventually we end up with a free group with $|S| - |R|$ generators. □

We shall deduce Lemma 4 from the corresponding result on random hypergraphs. To this end let us partition all relations from R into three different types.

- Relations of *type 1* are relations of the form aaa.
- Relations of *type 2* are relations of the form aab, aba and baa, where $a \neq b$. For such a relation b is called a pivotal term.
- Relations of *type 3* are all the remaining relations, that is, relations of the form abc, where $a \neq b$, $b \neq c$ and $c \neq a$. In this case each element of the relation is pivotal.

Now let us introduce an auxiliary random hypergraph (in fact, multi-hypergraph) $\mathcal{H} = \mathcal{H}(\Gamma(n,p))$. The set of vertices of $\mathcal{H}$ consists of all generators of $\Gamma(n,p)$, i.e. it coincides with the set S. Every relation in $\Gamma(n,p)$ generates a hyperedge in $\mathcal{H}$. For relation r of type 3 we generate a 3-edge $E = \{a,b,c\}$ consisting of three generators of S such that r is built from the letters of $E \cup E^{-1}$. If r is a relation of type 2, then we generate a 2-edge $E = \{a,b\}$ such that r is build from the letters of $E \cup E^{-1}$ and mark an appropriate element as the pivotal for this edge. Finally, if r is a relation of type 1, i.e. we have either $r = aaa$ or $r = a^{-1}a^{-1}a^{-1}$ for some $a \in S$, then we put the 1-edge $\{a\}$ in $\mathcal{H}$.

Our aim is to prove the following lemma which immediately implies Lemma 4.

Lemma 5. *There exists a constant $c > 0$ such that for $p \leqslant c/n^2$ the following holds:* a.a.s. *for every non-empty subset F of hyperedges of $\mathcal{H} = \mathcal{H}(\Gamma(n,p))$ there exist a vertex v of $\mathcal{H}$ and a hyperedge $E \in F$ such that E is the only hyperedge from F containing v and, moreover, v is pivotal for E.*

Proof of Lemma 5. It is enough to show that the lemma is true for some $p = c/n^2$, where $c > 0$ is a constant to be chosen later.

Observe first that the probability that there is a type 1 edge in $\mathcal{H}$ tends to 0 as $n \to \infty$. Indeed, let X denote the random variable which counts such edges in $\mathcal{H}$. Then

$$\mathbb{P}(\mathcal{H} \text{ has a 1-edge}) \leqslant \mathrm{E}X = 2np \leqslant 2c/n \to 0.$$

Thus, we may and shall assume that $\mathcal{H}$ contains only edges of type either 2 or 3.

Our further argument is based on the fact that in the 3-uniform random hypergraph in which the expected number of edges is an, where a is a small positive constant, a.a.s. all components are of size $O(\log n)$. More specifically, we shall use the following special case of a theorem of Schmidt-Pruzan and Shamir [SS85].

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THEOREM 6 (Schmidt-Pruzan and Shamir [SS85]). *Let $\mathcal{H}(n,\rho)$ be the random hypergraph with vertex set $[n]=\{1,2,\dots,n\}$, in which each subset of $[n]$ with three vertices appears independently with probability ρ. Moreover, let $2\rho\binom{n-1}{2}\leqslant C$, where C is a constant such that $C<1$. Then, there exists a constant K_C such that with probability $1-o(n^{-2})$ the largest connected component in $\mathcal{H}$ contains fewer than $K_C\log n$ vertices.*

Now let $\mathcal{H}_3$ denote the subgraph of $\mathcal{H}$ which consists of all its edges of size three. For each such edge there are 48 different relations which can generate it. Therefore, due to Theorem 6, each component in $\mathcal{H}_3$ has fewer than $K\log n$ vertices for some constant K depending only on c and provided $c<\frac{1}{48}$. We show first that each connected subgraph of $\mathcal{H}_3$, other than an isolated vertex, contains at least two vertices belonging to exactly one edge of $\mathcal{H}_3$. Indeed, let us assume that this is not the case and denote by X the number of non-trivial connected subgraphs on k vertices, $4\leqslant k\leqslant K\log n$, contained in $\mathcal{H}_3$ in which all but at most one vertex belongs to at least two edges. Each such subgraph has at least $\lceil(2k-1)/3\rceil\geqslant 3k/5$ edges. Thus, instead of the random variable X we consider another random variable Y which counts graphs on k vertices, $4\leqslant k\leqslant K\log n$, with exactly $\lceil 3k/5\rceil$ edges. Let ρ be the probability that there is an edge in $\mathcal{H}_3$ containing any given three vertices. Obviously, $\rho\leqslant 48p$. We take k vertices and we want to estimate the probability that there is a subgraph of H_3 consisting of these k vertices and having exactly $3k/5$ edges. Then this probability can be bounded from above by the expression

$$\binom{\binom{k}{3}}{\lceil 3k/5\rceil}\rho^{\lceil 3k/5\rceil}\leqslant (ek^2/3)^{\lceil 3k/5\rceil}(48p)^{\lceil 3k/5\rceil}\leqslant (16ek^2p)^{3k/5}.$$

Hence for the expectation $\mathrm{E}Y$ of Y we have

$$\begin{aligned}\mathrm{E}Y &\leqslant \sum_{k=4}^{K\log n}\binom{n}{k}(16ek^2p)^{3k/5}\leqslant \sum_{k=4}^{K\log n}\left(\frac{en}{k}\right)^{k}\left(\frac{16ek^3c}{n^2}\right)^{3k/5}\\ &\leqslant \sum_{k=4}^{K\log n}\left(\frac{16^3e^8c^3k^4}{n}\right)^{k/5}\leqslant K\log n\left(\frac{16^3e^8c^3(K\log n)^4}{n}\right)^{4/5}\to 0.\end{aligned}$$

Since clearly

$$\Pr(X>0)\leqslant\Pr(Y>0)\leqslant \mathrm{E}Y,$$

a.a.s. in each non-trivial connected subgraph of $\mathcal{H}_3$ at least two vertices belong to exactly one edge.

As for edges of size two in $\mathcal{H}$, we shall show that a.a.s. each component of $\mathcal{H}_3$ shares a vertex with at most one such edge. To this end let Z count components of $\mathcal{H}_3$ (including trivial ones) for which this is not the case. Then

$$\mathrm{E}Z\leqslant n(K\log n)^2n^2(24p)^2\leqslant 24^2K^2c^2(\log n)^2/n\to 0,$$

and so a.a.s. each component of $\mathcal{H}_3$ intersects at most one edge of $\mathcal{H}$ of size two; in particular, a.a.s. all edges of size two form a matching.

It is easy to see that from the above two statements the assertion of Lemma 5 easily follows. Indeed, take any subset F of hyperedges of $\mathcal{H}$, and let F' be the subhypergraph of F which consists of edges of size three. If F' is not empty, then a.a.s. each of its non-trivial components contains at least two vertices of degree one, and at most one such vertex belongs in F to an edge of size two. Thus, a.a.s. there is at least one vertex in F which belongs to precisely one edge in F and this edge is of size three. Now let us suppose that F' is empty, i.e. all edges of F are of size two. Then they a.a.s. form a matching and it is enough to take the pivotal vertex of one of these edges. □

Proof of Lemma 4. Lemma 4 is a straightforward consequence of Lemma 5. □

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3. Proof of Theorem 2

We recall some rather well-known results concerning random triangular groups. We also briefly comment on the arguments used to verify them, as explicit proofs seem not to be present in the literature.

We start with a few preliminary comments. In his paper [Żuk03], Żuk studied triangle groups $\Gamma(n,t)$, where n denotes the cardinality of a generating set S and t denotes the cardinality of a set R of relations chosen uniformly at random out of all subsets of cardinality t of the set of all cyclically reduced words of length three over the alphabet $S \cup S^{-1}$. Żuk used the so-called density approach to asymptotic properties of random groups, in which we let $n \to \infty$ and we keep $t \sim n^{3d+o(1)}$, where $d \in [0,1]$ is the (constant) density parameter. Let us remark also that, as is well known from the general theory of random structures, if t is close to the expected number of relations in $\Gamma(n,p)$, i.e. $t = (8+o(1))n^3 p$, then the asymptotic behavior of $\Gamma(n,t)$ and $\Gamma(n,p)$ is very similar for a large class of properties, including all those considered in this paper (see, for instance, [JŁR00, ch. 1.4)].

The following result, very useful in our further developments, is essentially due to Ollivier. We follow the notation of [Oll05] concerning van Kampen diagrams D. In particular, $|D|$ denotes the number of 2-cells in D, and $|\partial D|$ is the boundary length of D (i.e. the length of the word for which D is a van Kampen diagram).

Lemma 7. *Let $\Gamma(n,p)$ be the triangle random group such that $p = n^{3(d-1)+o(1)}$ for some $d < \frac{1}{2}$. Then for any $\epsilon > 0$, a.a.s. all reduced van Kampen diagrams D for $\Gamma(n,p)$ satisfy the isoperimetric inequality*

$$|\partial D| \geqslant 3(1-2d-\epsilon)|D|.$$

Proof. The proof consists of two steps. First, one shows that for any fixed number K the required inequality holds a.a.s. for all reduced van Kampen diagrams D with $|D| \leqslant K$. Next, one uses a propagation argument to conclude the full statement.

In [Oll05, Proposition 58] an analog of the first step above is proved in the context of random groups corresponding to Gromov's density model. More precisely, it is shown that a.a.s. any reduced van Kampen diagram with $|D| \leqslant K$ for a density d random group with relations of length $L \to \infty$ satisfies

$$|\partial D| \geqslant L(1-2d-\epsilon)|D|.$$

The same argument applies to density d triangle groups and, in view of the remarks at the beginning of this section, yields the first required step.

The appropriate propagation argument for the second step is [Oll07, Theorem 8] (mentioned also as [Oll05, Theorem 60]). It applies directly to random triangular groups and, in view of the first step, concludes the proof. □

Recall that, given a presentation P of the form $\langle a_1, \ldots, a_n | r_1, \ldots, r_t \rangle$, the *presentation complex* C_P is a two-dimensional cell complex with a single vertex v_0, with edges (being oriented loops attached to v_0) corresponding to the generators $a_1, \ldots, a_n$, and 2-cells corresponding to relations $r_1, \ldots, r_t$ attached to the 1-skeleton accordingly. The group Γ given by the presentation P is (canonically isomorphic to) the fundamental group of C_P. Lemma 7 can be used to show our next result which essentially belongs to Gromov. An outline of its proof (in a slightly different setting) is given in [Oll04, last two paragraphs of § 2]; compare also [Oll05, § I.3.b].

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Lemma 8. *Let $\Gamma = \Gamma(n, p)$ be the random triangular group such that $p = n^{3(d-1)+o(1)}$ for some $d < \frac{1}{2}$, and let P denote its presentation. Then a.a.s. the presentation complex C_P is aspherical. In particular, C_P is a classifying space for Γ.*

Lemma 8 has the following corollary.

Corollary 9. *Let $\Gamma = \Gamma(n, p)$ be the random triangular group such that $p = n^{3(d-1)+o(1)}$ for some $d < \frac{1}{2}$, and let P denote its presentation. Then a.a.s. the Euler characteristic of Γ is given by $\chi(\Gamma) := \chi(C_P) = 1 - n + t$. Moreover, Γ is torsion-free.*

Lemma 7 has also the following less-known consequence.

Corollary 10. *Let $\Gamma = \Gamma(n, p)$ be the random triangular group such that $p = n^{3(d-1)+o(1)}$ for some $d < \frac{4}{9}$, and let $P = \langle S|R\rangle$ denote its presentation. Then a.a.s. every generator $s \in S$ is non-trivial in Γ.*

Proof. Applying Lemma 7 to any density $d < \frac{4}{9}$ (and using sufficiently small ϵ), we get the inequality $|\partial D| > |D|/3$. Now trivialization of a generator implies the existence of a reduced van Kampen diagram D with $|\partial D| = 1$. By the above inequality we get that such a diagram consists of $|D| < 3$ cells. Furthermore, any such diagram D (i.e. with $|\partial D| = 1$) has an odd number of cells. However, this is impossible because relations in R have length three and are cyclically reduced. □

In the proof of Theorem 2 we need the following two simple probabilistic facts.

Lemma 11. *If $p \geqslant 3/n^2$ then a.a.s. $\Gamma(n, p)$ has at least $3n$ relations.*

Proof. It is enough to estimate the number of relations which have three different elements. Let X be the random variable which counts such relations in $\Gamma(n, p)$, where $p = 3/n^2$. There are $N = (8 + o(1))\binom{n}{3}$ different relations of this type and each of them is chosen independently with probability p. Therefore, X has the binomial distribution $B(N, p)$ and so, since $p \to 0$, we have

$$\mathrm{Var}X = Np(1 - p) = (1 - p)\mathrm{E}X = (1 + o(1))\mathrm{E}X,$$

where

$$\mathrm{E}X = Np = (8 + o(1))\binom{n}{3}\frac{3}{n^2} = (4 + o(1))n \to \infty,$$

so the assertion follows from Chebyshev's inequality. □

Lemma 12. *If $p \leqslant \log n/(25n^2)$ then a.a.s. there exists a generator s such that neither s nor s^{-1} belongs to any relation in $\Gamma(n, p) = \langle S|R\rangle$.*

Proof. Let Y count the generators s such that neither s nor s^{-1} belongs to any relation in $\Gamma(n, p)$. For a given generator s there are $48\binom{n-1}{2} + 24(n - 1) + 2 = an^2$ different relations which contain either s or s^{-1}, where $a = 24 + o(1)$. Therefore

$$\mathrm{E}Y = n(1 - p)^{an^2} \geqslant n^{1/50} \to \infty.$$

Furthermore, it is easy to check that $\mathrm{Var}Y = (1 + o(1))\mathrm{E}Y$, so the assertion follows from Chebyshev's inequality. □

Proof of Theorem 2. In view of Corollary 9, it follows from Lemma 11 that for $p \geqslant 3n^{-2}$ a.a.s. the Euler characteristic $\chi(\Gamma(n,p))$ of $\Gamma(n,p)$ is positive. Since any free group has non-positive Euler characteristic, it follows that a.a.s. $\Gamma(n,p)$ is not free.

Now, if $p \leqslant \log n/(25n^2)$, Lemma 12 asserts that a.a.s. there is a generator $s \in S$ such that neither s nor its inverse s^{-1} appears in any relation from R. By Corollary 10, a.a.s. all generators from S are non-trivial in $\Gamma(n,p)$. Thus, a.a.s. $\Gamma(n,p)$ splits non-trivially as the free product $\Gamma(n,p) = \langle s\rangle * \langle S\backslash\{s\}\rangle$. Consequently, a.a.s. $\Gamma(n,p)$ does not have Kazhdan's property (T), which completes the proof. □

4. Proof of Theorem 3

We prove that the random group $\Gamma(n,p)$ has a.a.s. Kazhdan's property (T) using spectral properties of a special random graph associated with $\Gamma(n,p)$. We begin by recalling a few notions from the spectral graph theory.

Let $G = (V,E)$ be a multigraph. We denote by $A = A(G)$ the *adjacency matrix* of G, that is, $A = (a_{vw})_{v,w\in V}$, where a_{vw} is the number of edges between v and w. Next, let $d_G(v)$ denote the degree of vertex v in G and let $\bar{d}$ be the average degree of a vertex in G. In what follows we use the term graph instead of multigraph, keeping in mind that we allow graphs to have multiple edges.

The *normalized Laplacian* of graph G is a symmetric matrix $\mathcal{L}(G) = (b_{vw})_{v,w\in V}$, where

$$b_{vw} = \begin{cases} 1 & \text{if } v = w \text{ and } d_G(v) > 0, \\ -a_{vw}/\sqrt{d_G(v)d_G(w)} & \text{if } \{v,w\} \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ denote the eigenvalues of $\mathcal{L}(G)$. It is easy to see that the Laplacian is positive semidefinite and so all its eigenvalues are non-negative. The eigenvector corresponding to the smallest eigenvalue $\lambda_1 = 0$ has entries $(\sqrt{d_G(v)})_{v\in V}$. Furthermore, the remaining eigenvalues are bounded above by 2. The value of λ_2 is called the *spectral gap* of $\mathcal{L}(G)$.

If G has no isolated vertices, then taking $D = D(G)$ to be the diagonal degree matrix with entries $d_{vv} = d_G(v)$, we can express the normalized Laplacian of G as

$$\mathcal{L}(G) = I - D^{-1/2}AD^{-1/2}.$$

Thus, $1 - \lambda_i$, $i = 1, 2, \ldots, n$, are the eigenvalues of matrix $I - \mathcal{L}(G) = D^{-1/2}AD^{-1/2}$ with the same eigenvectors as the eigenvectors for λ_i for $\mathcal{L}(G)$. Therefore, to find the spectrum of $\mathcal{L}(G)$ it is enough to study the spectrum of $D^{-1/2}AD^{-1/2}$ instead. One way to do this is to use the well-known Courant–Fischer principle (cf. [Cou20, Fis05]).

Theorem 13 (Courant–Fischer formula). *Let M be a $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. For $1 \leqslant k \leqslant n-1$, let R_k denote the span of $v_{k+1}, \ldots, v_n$, $R_n = \{0\}$. Let $R_k^\perp$ denote the orthogonal complement of R_k. Then*

$$\lambda_k = \max_{\substack{\|x\|=1\\ x\in R_k^\perp}} \langle Mx, x\rangle.$$

For a triangular group presentation $P = \langle S|R\rangle$ we define a graph $L = L_P$, called the *link graph* of P, in the following way. The set of vertices of L consists of all generators from S together with their formal inverses S^{-1}. Furthermore, every relation abc present in R generates three edges $\{a, b^{-1}\}$, $\{b, c^{-1}\}$ and $\{c, a^{-1}\}$ in L.

The key ingredient of our argument is the following result of Żuk who showed that if the spectral gap of $\mathcal{L}(L)$ is sufficiently large, then the group has property (T) (see [Żuk03, Proposition 6, p. 661], where $L'(S)$ coincides with our L_P, and where λ_1 denotes the same as our λ_2).

THEOREM 14 (Żuk [Żuk03]). *Let Γ be a group generated by a finite group presentation $P = \langle S|T\rangle$ and let $L = L_P$. If L is connected and $\lambda_2[\mathcal{L}(L)] > \frac{1}{2}$, then Γ has Kazhdan's property (T).*

Recall that Żuk studied the model $\Gamma(n,t)$ of a random triangular group in which we keep $t \sim n^{3d+o(1)}$. To simplify calculations in estimating the spectral gap of the Laplacian he considered the so-called *permutation model* of random triangular groups which is more convenient to work with than the triangular model $\Gamma(n,t)$ because in this model the corresponding link graph is regular. Żuk proved that a.a.s. the Laplacian of the link graph in the permutation model has large spectral gap. He also stated that the same holds for the triangular model $\Gamma(n,t)$ for $d > \frac{1}{3}$. However, he did not fully justify this statement. Recently, Kotowski and Kotowski [KK13] showed how to modify Żuk's argument for the permutation model to make it work for the triangular model $\Gamma(n,t)$ as well, completing the proof of the following theorem.

THEOREM 15 (Żuk [Żuk03], Kotowski and Kotowski [KK13]). *If $d > \frac{1}{3}$, then a.a.s. $\Gamma(n,t)$ has property (T).*

Since property (T) is a monotone property, Theorem 15 implies that for any $\epsilon > 0$ and $p = \Omega(n^{\epsilon-2})$, a.a.s. the random group $\Gamma(n,p)$ has property (T). Here we give a stronger result, namely we show that the Laplacian of the link graph of $\Gamma(n,p)$ has a large spectral gap provided that $p \geqslant C \log n/n^2$ for a sufficiently large constant $C > 0$.

THEOREM 16. *Let L be the link graph of $\Gamma(n,p)$. There exists $C > 0$ such that if $p \geqslant C \log n/n^2$, then a.a.s. $\lambda_2[\mathcal{L}(L)] > \frac{1}{2}$.*

Theorem 3 is an immediate consequence of Theorems 14 and 16. In the remainder of this section we give the proof of Theorem 16.

The main idea of our argument comes from Żuk who used a similar approach in [Żuk03]. First, we divide the graph L into three random graphs L_1, L_2 and L_3 which will behave in a similar way to the Erdős–Rényi random graph $G(2n,\rho)$, for some appropriately chosen ρ. We partition L into graphs L_i in the following way. The three graphs L_i have the same vertex set as L, that is, the set $S \cup S^{-1}$ of all generators together with their formal inverses. For every relation $abc \in R$ we place the edge $\{a, b^{-1}\}$ in L_1, $\{b, c^{-1}\}$ in L_2 and $\{c, a^{-1}\}$ in L_3. Therefore in graphs L_i every edge appears independently of other edges. Note, however, that between any two vertices we can have multiple edges, in particular there can be up to $4(n-1)$ such edges between any pair of vertices a and b, where $a^{-1} \neq b$, and up to $4n-2$ edges between any pair of vertices of the form a and a^{-1}. Furthermore, unlike in Żuk's original proof, our graphs are not regular, which is the main small difficulty in our argument. However, it turns out that adding a small correction to a graph which has the degree sequence concentrated around a particular value does not much affect the size of the spectral gap of the Laplacian.

LEMMA 17. *Let $0 < \epsilon \leqslant 0.01$ and let G be a connected graph on n vertices such that for any vertex v in G, $|d_G(v) - d| \leqslant \epsilon d$. Let H be a graph on the same vertex set and such that $d_H(v) \leqslant \epsilon d$ for any vertex v in H. Then*

$$\lambda_{n-1}[I - \mathcal{L}(G \cup H)] \leqslant \frac{1}{1-\epsilon}\lambda_{n-1}[I - \mathcal{L}(G)] + \frac{4\epsilon}{1-\epsilon}$$

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or equivalently

$$\lambda_2[\mathcal{L}(G \cup H)] \geqslant \frac{1}{1-\epsilon}\lambda_2[\mathcal{L}(G)] - \frac{5\epsilon}{1-\epsilon}.$$

Proof. We begin by establishing upper bounds on the spectral norms of some matrices associated with the graphs G, H and $G \cup H$. Thus, let $A_G = A(G)$, $D_G = D(G)$, $A_H = A(H)$ and $D_H = D(H)$.

Recall that for a symmetric matrix M having all entries non-negative, the spectral norm of M can be bounded from above by the maximum sum of entries in a row. In particular, if M is the adjacency matrix of a given graph, the spectral norm $\|M\|$ can be bounded by the maximum degree of this graph. Therefore we infer that

$$\|D_G^{1/2}\| \leqslant ((1+\epsilon)d)^{1/2},$$
$$\|D^{-1/2}\| \leqslant ((1-\epsilon)d)^{-1/2},$$

and

$$\|A_H\| \leqslant \epsilon d.$$

Notice that $A = A_G + A_H$ is the adjacency matrix of the graph $G \cup H$ and $D = D_G + D_H$ is its degree matrix. Let $d(v)$ denote the degree of a vertex v of $G \cup H$, that is, $d(v) = d_G(v) + d_H(v)$, and a_{ij} denote the entries of the matrix A_G.

Now, for the spectral norm of the diagonal matrix $D_G^{-1/2}D^{1/2}$ we get the upper bound

$$\|D_G^{-1/2}D^{1/2}\| \leqslant \max_i \left(\frac{d_G(i)+d_H(i)}{d_G(i)}\right)^{1/2} \leqslant \left(1+\frac{\epsilon d}{(1-\epsilon)d}\right)^{1/2} = \left(\frac{1}{1-\epsilon}\right)^{1/2}.$$

We also have the estimate

$$\begin{aligned}
\|D^{-1}A_GD^{-1} - D_G^{-1}A_GD_G^{-1}\| &\leqslant \max_i \sum_j \left|\frac{a_{ij}}{d(i)d(j)} - \frac{a_{ij}}{d_G(i)d_G(j)}\right| \\
&\leqslant \max_i \sum_j a_{ij}\left|\frac{1}{(d_G(i)+\epsilon d)(d_G(j)+\epsilon d)} - \frac{1}{d_G(i)d_G(j)}\right| \\
&\leqslant \max_i \sum_j a_{ij}\frac{\epsilon d(d_G(i)+d_G(j)+\epsilon d)}{(d_G(i)+\epsilon d)(d_G(j)+\epsilon d)d_G(i)d_G(j)} \\
&\leqslant \frac{\epsilon d(2d+3\epsilon d)}{(1-\epsilon)^2 d^4}\max_i \sum_j a_{ij} \leqslant \frac{\epsilon(2+3\epsilon)(1+\epsilon)}{(1-\epsilon)^2 d}.
\end{aligned}$$

Let X be the eigenvector of $D^{-1/2}AD^{-1/2}$ corresponding to the largest eigenvalue $\lambda_n[D^{-1/2}AD^{-1/2}] = 1$ and Y be the eigenvector of $D_G^{-1/2}A_GD_G^{-1/2}$ corresponding to the largest eigenvalue $\lambda_n[D_G^{-1/2}A_GD_G^{-1/2}] = 1$. Then

$$X_i = \sqrt{d_G(i)+d_H(i)} \quad \text{and} \quad Y_i = \sqrt{d_G(i)}.$$

Furthermore, notice that since $D_G^{1/2}D^{-1/2}X = Y$, if $x \perp X$ and $y = D_G^{-1/2}D^{1/2}x$, then $y \perp Y$.

We can now estimate $\lambda_{n-1}[I - \mathcal{L}(G \cup H)]$ using the Courant–Fischer formula, the Cauchy–Schwarz inequality, and the fact that for a diagonal matrix M we have $\langle Mx, y\rangle = \langle x, My\rangle$ for any vectors x and y:

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$$\begin{aligned}
\lambda_{n-1}[I-\mathcal{L}(G\cup H)] &= \lambda_{n-1}[D^{-1/2}AD^{-1/2}]\\
&= \max_{x\perp X,\|x\|=1}\langle D^{-1/2}(A_G+A_H)D^{-1/2}x,x\rangle\\
&\leqslant \max_{x\perp X,\|x\|=1}\langle D^{-1/2}A_GD^{-1/2}x,x\rangle+\|D^{-1/2}\|^2\|A_H\|\|x\|^2\\
&\leqslant \max_{y\perp Y,\|y\|\leqslant\|D_G^{-1/2}D^{1/2}\|}\langle D^{-1/2}A_GD^{-1}D_G^{1/2}y,D_G^{1/2}D^{-1/2}y\rangle+\frac{\epsilon}{1-\epsilon}\\
&\leqslant \max_{y\perp Y,\|y\|\leqslant\|D_G^{-1/2}D^{1/2}\|}\{\langle D_G^{-1/2}A_GD_G^{-1/2}y,y\rangle\\
&\quad+\langle(D_G^{1/2}D^{-1}A_GD^{-1}D_G^{1/2}-D_G^{-1/2}A_GD_G^{-1/2})y,y\rangle\}+\frac{\epsilon}{1-\epsilon}\\
&\leqslant \max_{y\perp Y,\|y\|\leqslant\|D_G^{-1/2}D^{1/2}\|}\left\{\frac{\langle D_G^{-1/2}A_GD_G^{-1/2}y,y\rangle}{\langle y,y\rangle}\|D_G^{-1/2}D^{1/2}\|^2\right.\\
&\quad\left.+\|D_G^{1/2}D^{-1}A_GD^{-1}D_G^{1/2}-D_G^{-1/2}A_GD_G^{-1/2}\|\|y\|^2\right\}+\frac{\epsilon}{1-\epsilon}\\
&\leqslant \frac{1}{1-\epsilon}\lambda_{n-1}[D_G^{-1/2}A_GD_G^{-1/2}]\\
&\quad+\|D_G^{1/2}\|^2\|D^{-1}A_GD^{-1}-D_G^{-1}A_GD_G^{-1}\|\frac{1}{1-\epsilon}+\frac{\epsilon}{1-\epsilon}\\
&\leqslant \frac{1}{1-\epsilon}\lambda_{n-1}[I-\mathcal{L}(G)]+\frac{\epsilon(2+3\epsilon)(1+\epsilon)^2}{(1-\epsilon)^3}+\frac{\epsilon}{1-\epsilon}\\
&\leqslant \frac{1}{1-\epsilon}\lambda_{n-1}[I-\mathcal{L}(G)]+\frac{4\epsilon}{1-\epsilon}. \qquad \square
\end{aligned}$$

We also need the fact that in dense random graphs the degree distribution is almost surely concentrated around the average degree. This is stated in the following well-known lemma (cf. [Bol01, Exercise 3.4]), which is a straightforward consequence of Chernoff's inequality.

LEMMA 18. *For every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that if $\rho > C_\epsilon \log m/m$ then a.a.s. for any vertex v in $G(m,\rho)$,*

$$|d(v) - m\rho| < \epsilon m\rho.$$

To estimate the spectral gap of L_i we use the result by Coja-Oghlan who in [Coj07, Theorem 1.2] gave precise bounds on the eigenvalues of the Laplacian of $G(m,\rho)$. In particular, he proved the following theorem.

THEOREM 19 (Coja-Oghlan [Coj07]). *Let $\mathcal{L} = \mathcal{L}(G(m,\rho))$. There exist constants $c_0, c_1 > 0$ such that if $\rho \geqslant c_0 \log m/m$, then a.a.s. we have*

$$0 = \lambda_1[\mathcal{L}] < 1 - c_1(m\rho)^{-1/2} \leqslant \lambda_2[\mathcal{L}] \leqslant \cdots \leqslant \lambda_n[\mathcal{L}] \leqslant 1 + c_1(m\rho)^{-1/2}.$$

Proof of Theorem 16. It is enough to show that Theorem 16 holds for some $p = C\log n/n^2$, where $C > 0$ is a sufficiently large constant.

Note that each graph L_i can be generated in the following way. Take an auxiliary multigraph $\mathcal{K}$ on $2n$ vertices with vertices labeled by the generators and their inverses. Two vertices a, b, where $a \neq b^{-1}$, are joined by $4(n-1)$ edges in $\mathcal{K}$, while vertices a, a^{-1} are joined by $4n-2$ edges. This is because there are $2(n-1)$ irreducible words which start with ab^{-1}, and $2(n-1)$ irreducible words which start with ba^{-1}; on the other hand, there are $2n-1$ irreducible words which start with aa, and $2n-1$ irreducible words which start with $a^{-1}a^{-1}$. Then L_i is obtained from $\mathcal{K}$ by leaving each of its edges independently with probability p. First we shall show that the spectral gap of L_i does not differ significantly from the spectral gap of the random graph $G(2n, \rho)$, in which each two vertices are joined by an edge independently with probability $\rho = 1-(1-p)^{4n-4}$.

To this end let $\hat{\mathcal{K}}$ be obtained from $\mathcal{K}$ by deleting two edges between vertices a and a^{-1}, so that each pair of vertices of $\hat{\mathcal{K}}$ is connected by exactly $4(n-1)$ edges, and let $\hat{L}_i$ be the random (multi)graph obtained from $\hat{\mathcal{K}}$ by selecting its edges with probability p. We show that a.a.s. $\hat{L}_i$ contains no edges with multiplicity larger than two, all double edges of L_i form a matching, and furthermore each pair a, a^{-1} is connected by at most one edge. Indeed, the probability that some edge has multiplicity at least three is bounded above by

$$(2n)^2(4n)^3p^3 \leqslant O(\log^3 n/n) \to 0,$$

while the probability that two double edges share a vertex can be estimated from above by

$$(2n)(2n)^2(4n)^4p^4 \leqslant O(\log^4 n/n) \to 0.$$

Finally, the probability that there is a multiple edge connecting a and a^{-1} for some generator a is bounded from above by

$$n(4n)^2p^2 \leqslant O(\log^2 n/n) \to 0.$$

Thus, L_i can be viewed as obtained from L_i' which is a copy of $G(2n, \rho)$, $\rho = 1-(1-p)^{4n-4} \geqslant C\log n/n$, by adding to it some matching M_i (which takes care of multiple edges) and adding another matching $\bar{M}_i$ (which corresponds to edges deleted from $\mathcal{K}$ in the first place). Since C is large, in particular $C \geqslant c_0$, we infer from Theorem 19 that a.a.s. all eigenvalues of the Laplacian of L_i' but the smallest one are concentrated around 1. In particular, a.a.s.

$$\lambda_2[\mathcal{L}(L_i')] > 1-\epsilon$$

for, say, $\epsilon = 0.01$. Since C is large from Lemma 18 a.a.s. for any vertex $v \in L_i'$ we have

$$|d_{L_i'}(v) - 2n\rho| < 2\epsilon n\rho$$

which means that each L_i' is almost regular. Since a.a.s. L_i can be obtained from L_i' by adding two matchings, using Lemma 17 we infer that a.a.s.

$$\lambda_2[\mathcal{L}(L_i)] \geqslant \frac{1}{1-\epsilon}\lambda_2[\mathcal{L}(L_i')] - \frac{5\epsilon}{1-\epsilon} > 1-6\epsilon,$$

or equivalently

$$\lambda_{2n-1}[I - \mathcal{L}(L_i)] < 6\epsilon.$$

Moreover, a.a.s. $|d_{L_i}(v) - 8C\log n| < 2\epsilon \cdot 8C\log n$. Therefore graphs L_i are also almost regular and the Laplacian of each L_i has a large spectral gap.

Thus, it is enough to show that the sum of three almost regular graphs with Laplacians having large spectral gaps is also a graph which has Laplacian with a large spectral gap. Let A_i be the adjacency matrix of the graph L_i and D_i be the corresponding diagonal degree matrix of L_i. Then $A = A_1 + A_2 + A_3$ is the adjacency matrix of L and $D = D_1 + D_2 + D_3$ is its degree matrix.

It is also easy to see that

$$\|D_i^{1/2}\| \leqslant ((1+2\epsilon)8C\log n)^{1/2}$$

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and

$$\begin{aligned}\|D_i^{-1/2}D^{1/2}\| &\leqslant \max_j\bigg(\frac{d_{L_1}(j)+d_{L_2}(j)+d_{L_3}(j)}{d_{L_i}(j)}\bigg)^{1/2}\\ &\leqslant \bigg(1+\frac{2(1+2\epsilon)8C\log n}{(1-2\epsilon)8C\log n}\bigg)^{1/2}=\bigg(\frac{3+2\epsilon}{1-2\epsilon}\bigg)^{1/2}.\end{aligned}$$

Now, let a^i_{jk} denote the entries of matrix A_i and $d(v)$ be the degree of a vertex v in L. We have the bound

$$\begin{aligned}\|9D^{-1}A_iD^{-1}-D_i^{-1}A_iD_i^{-1}\| &\leqslant \max_j\sum_k\bigg|\frac{9a^i_{jk}}{d(j)d(k)}-\frac{a^i_{jk}}{d_{L_i}(j)d_{L_i}(k)}\bigg|\\ &\leqslant \max_j\sum_k a^i_{jk}\bigg|\frac{9}{(d_{L_1}(j)+d_{L_2}(j)+d_{L_3}(j))(d_{L_1}(k)+d_{L_2}(k)+d_{L_3}(k))}-\frac{1}{d_{L_i}(j)d_{L_i}(k)}\bigg|\\ &\leqslant \max_j\sum_k a^i_{jk}\frac{8((1+2\epsilon)^2-(1-2\epsilon)^2)(8C\log n)^2}{(3(1-2\epsilon)8C\log n)^2((1-2\epsilon)8C\log n)^2}\\ &\leqslant (1+2\epsilon)(8C\log n)\frac{64\epsilon}{9(1-2\epsilon)^4(8C\log n)^2}=\frac{64\epsilon(1+2\epsilon)}{9(1-2\epsilon)^4(8C\log n)}.\end{aligned}$$

Let X be the eigenvector of $D^{-1/2}AD^{-1/2}$ corresponding to the largest eigenvalue $\lambda_{2n}[D^{-1/2}AD^{-1/2}]=1$, and similarly, for $i=1,2,3$, let X_i be the eigenvector of $D_i^{-1/2}A_iD_i^{-1/2}$ corresponding to the largest eigenvalue $\lambda_{2n}[D_i^{-1/2}A_iD_i^{-1/2}]=1$. The entries of vectors X and X_i are square roots of vertex degrees in corresponding graphs and since $D_i^{1/2}D^{-1/2}X=X_i$ it follows that if $x\perp X$ and $y=\frac{1}{3}D_i^{-1/2}D^{1/2}x$, then $y\perp X_i$.

We can now estimate $\lambda_{2n-1}[I-\mathcal{L}(L)]$ as follows:

$$\begin{aligned}\lambda_{2n-1}[I-\mathcal{L}(L)]=\lambda_{2n-1}[D^{-1/2}AD^{-1/2}] &= \max_{x\perp X,\|x\|=1}\langle D^{-1/2}AD^{-1/2}x,x\rangle\\ &\leqslant \sum_{i=1}^3\max_{y\perp X_i,\|y\|\leqslant\frac{1}{3}\|D_i^{-1/2}D^{1/2}\|}\langle 3D^{-1/2}A_iD^{-1}D_i^{1/2}y,3D_i^{1/2}D^{-1/2}y\rangle\\ &\leqslant \sum_{i=1}^3\max_{y\perp X_i,\|y\|\leqslant\frac{1}{3}\|D_i^{-1/2}D^{1/2}\|}\{\langle D_i^{-1/2}A_iD_i^{-1/2}y,y\rangle\\ &\quad+\langle(9D_i^{1/2}D^{-1}A_iD^{-1}D_i^{1/2}-D_i^{-1/2}A_iD_i^{-1/2})y,y\rangle\}\\ &\leqslant \sum_{i=1}^3\max_{y\perp X_i,\|y\|\leqslant\frac{1}{3}\|D_i^{-1/2}D^{1/2}\|}\bigg\{\frac{\langle D_i^{-1/2}A_iD_i^{-1/2}y,y\rangle}{\langle y,y\rangle}\frac{1}{9}\|D_i^{-1/2}D^{1/2}\|^2\\ &\quad+\|9D_i^{1/2}D^{-1}A_iD^{-1}D_i^{1/2}-D_i^{-1/2}A_iD_i^{-1/2}\|\|y\|^2\bigg\}\\ &\leqslant \sum_{i=1}^3\lambda_{2n-1}[D_i^{-1/2}A_iD_i^{-1/2}]\frac{3+2\epsilon}{9(1-2\epsilon)}\\ &\quad+\|D_i^{1/2}\|^2\|9D^{-1}A_iD^{-1}-D_i^{-1}A_iD_i^{-1}\|\frac{3+2\epsilon}{9(1-2\epsilon)}\\ &\leqslant \sum_{i=1}^3\lambda_{2n-1}[I-\mathcal{L}(L_i)]\frac{3+2\epsilon}{9(1-2\epsilon)}\end{aligned}$$

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$$+ (1+2\epsilon)8C\log n \frac{64\epsilon(1+2\epsilon)}{9(1-2\epsilon)^4(8C\log n)} \frac{3+2\epsilon}{9(1-2\epsilon)}$$
$$\leqslant 2\epsilon\frac{3+2\epsilon}{1-2\epsilon} + \frac{64\epsilon(1+2\epsilon)^2(3+2\epsilon)}{27(1-2\epsilon)^5} \leqslant 15\epsilon.$$

Hence, a.a.s. $\lambda_2[\mathcal{L}(L)] = 1 - \lambda_{2n-1}[I - \mathcal{L}(L)] \geqslant 1 - 15\epsilon$ which can be arbitrarily close to 1. In particular, a.a.s. $\lambda_2[\mathcal{L}(L)] > \frac{1}{2}$. □

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