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Stability of Almost Periodic Nicholson's Blowflies Model Involving Patch Structure and Mortality Terms

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Abstract. Taking into account the effects of patch structure and nonlinear density-dependent mortality terms, we explore a class of almost periodic Nicholson's blowflies model in this paper. Employing the Lyapunov function method and differential inequality technique, some novel assertions are developed to guarantee the existence and exponential stability of positive almost periodic solutions for the addressed model, which generalize and refine the corresponding results in some recently published literatures. Particularly, an example and its numerical simulations are arranged to support the proposed approach.

1 Introduction

The qualitative theory of differential equations model has been an attractive topic because of its significance and applications in areas such as physics, mathematical biology, and control theory [9,11,21,26]. In population systems, due to factors such as seasonal variation of weather, mating, harvesting and so on, the periodic fluctuations are a widely occurring process and play key roles in modeling [12,13,15]. However, when there are nonintegral multiples periods (also called incommensurable) for different components of the temporally nonuniform environment, more and more scientists realize that assuming the environment has almost periodicity instead of periodicity might be a better candidate [4,7,25,27]. Nowadays, the investigations of almost periodic dynamics systems have been the new world-wide focus (see [5,6,10,14,16,18]). In particular, the existence and global stability of almost periodic solutions for the famous scalar Nicholson's blowflies model with a nonlinear density-dependent mortality term,

$$x'(t) = -a(t) + b(t)e^{-x(t)} + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))}, \tag{1.1}$$

and the Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms,

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$$(1.2)\quad x_i'(t) = -a_{ii}(t) + b_{ii}(t)e^{-x_i(t)} + \sum_{j=1, j\neq i}^{n} (a_{ij}(t) - b_{ij}(t)e^{-x_j(t)}) + \sum_{j=1}^{m} \beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))},\ i \in Q := \{1,2,\dots,n\},$$

have been extensively investigated in previous studies [16, 22] and [3], respectively. Here, the information on the delay and coefficient functions presented in (1.1) and (1.2) can found in [1, 2, 20, 23] and the references cited therein. For the feedback function xe^{-x} and its derivative $\frac{1-x}{e^x}$, the author of [17] pointed out that there exist two fixed positive numbers κ and $\widetilde{\kappa}$ such that

$$\kappa \approx 0.7215355, \qquad \widetilde{\kappa} \approx 1.342276, \qquad \frac{1-\kappa}{e^{\kappa}} = \frac{1}{e^2},$$
$$\sup_{x\geq\kappa} \left|\frac{1-x}{e^x}\right| = \frac{1}{e^2}, \qquad \kappa e^{-\kappa} = \widetilde{\kappa}e^{-\widetilde{\kappa}}.$$

It should be pointed out that the global exponential stability of almost periodic solutions of (1.1) has been shown in [16, 22] under the restriction that the almost periodic solution exists in a small interval $[\kappa, \widetilde{\kappa}]$. The global exponential stability of (1.2) was established in [3], where the authors adopted the restraint that the almost periodic solution exists in a small domain

$$\underbrace{[\kappa, \widetilde{\kappa}] \times [\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n}.$$

Obviously, the above restriction and restraint do not correspond to the biological significance of the population models. In particular, to the best of our knowledge, no research has been conducted on the global stability of almost periodic solutions of Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms when the almost periodic solutions do not belong to the above domain.

According to the above discussions, in this paper, without adopting

$$\underbrace{[\kappa, \widetilde{\kappa}] \times [\kappa, \widetilde{\kappa}] \times \cdots \times [\kappa, \widetilde{\kappa}]}_{n}$$

as the existence domain of almost periodic solutions, we establish the existence and global exponential stability of positive almost periodic solutions for Nicholson's blowflies systems (1.2) involving patch structure and nonlinear density-dependent mortality terms. The proposed criterion improves and complements some existing results in the recent publications [3,16,20,22,24], and its effectiveness is demonstrated by a numerical example.

2 Preliminaries

The following notation will be used throughout the rest of this paper. Let

$$g^{\sup} = \sup_{t\in[t_0,+\infty)} g(t), \qquad g^{\inf} = \inf_{t\in[t_0,+\infty)} g(t),$$

$$\sigma_i = \max_{1\le j\le m} \tau_{ij}^{sup}, \qquad C_+ = \prod_{i=1}^{n} C([-\sigma_i,\ 0],[0,+\infty)).$$

For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, define $|x| = (|x_1|,\dots,|x_n|)$ and $\|x\| = \max_{i\in Q}|x_i|$.

Definition 2.1 (See [7,27]) Let $u(t)\colon \mathbb{R}\to\mathbb{R}^n$ be continuous in t. Then $u(t)$ is said to be almost periodic on $\mathbb{R}$, if for any $\varepsilon > 0$, the set $T(u,\varepsilon) = \{\delta : \|u(t+\delta)-u(t)\| < \varepsilon \text{ for all } t\in\mathbb{R}\}$ is relatively dense, *i.e.*, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, such that for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $\|u(t+\delta)-u(t)\| < \varepsilon$ for all $t\in\mathbb{R}$.

Hereafter, for $i\in Q$, $j\in I = \{1,2,\dots,m\}$, it will be assumed that $a_{ii}, b_{ii}, \gamma_{ij}\colon \mathbb{R}\to(0,+\infty)$, $a_{ij}(i\ne j)$, $b_{ij}(i\ne j)$, $\beta_{ij},\tau_{ij}\colon\mathbb{R}\to[0,+\infty)$ are almost periodic functions and there exist two positive constants S_- and S^+ such that

$$S_- = \min_{i\in Q}\Big\{\liminf_{t\to+\infty}\ln\Big(\frac{b_{ii}(t)-\sum_{j=1,j\ne i}^{n} b_{ij}(t)}{a_{ii}(t)-\sum_{j=1,j\ne i}^{n} a_{ij}(t)}\Big)\Big\},$$

$$S^+ = \max_{i\in Q}\Big\{\limsup_{t\to+\infty}\ln\Big(\frac{b_{ii}(t)-\sum_{j=1,j\ne i}^{n} b_{ij}(t)}{a_{ii}(t)-\sum_{j=1,j\ne i}^{n}\big(a_{ij}(t)+\sum_{j=1}^{m}\frac{1}{e}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\big)}\Big)\Big\}.$$

Furthermore, the following admissible initial conditions will be considered:

$$\begin{aligned} &x_i(t_0+\theta) = \varphi_i(\theta), && \theta\in[-\sigma_i,0],\\ &\varphi = (\varphi_1,\dots,\varphi_n)\in C_+ && \varphi_i(0) > 0, i\in Q. \end{aligned} \tag{2.1}$$

We designate $x(t;t_0,\varphi)$ to be a solution of the initial value problem (1.2) and (2.1), and denote the maximal right-interval of existence of $x(t;t_0,\varphi)$ by $[t_0,\ \eta(\varphi))$.

Lemma 2.2 (see [23, Lemma 2.1) *For any two fixed positive constants ω_1 and ω_2,*

$$(e^{-s}-e^{-t})\,\mathrm{sgn}(s-t) \le -e^{-\omega_2}|s-t|,\ \ \textit{where}\ \ s,t\in[\omega_1,\ \omega_2], \omega_1\le\omega_2,$$

and

$$|se^{-s}-te^{-t}| \le \max\Big\{\frac{1}{e^2},\frac{1-\omega_1}{e^{\omega_1}}\Big\}|s-t|,\ \ \textit{where}\ \ s,t\in[\omega_1,\ +\infty).$$

Lemma 2.3 *Assume that*

$$b_{ii}(t) > a_{ii}(t) - \sum_{j=1,j\ne i}^{n}(a_{ij}(t)-b_{ij}(t)),\ \textit{for all}\ t\in[t_0,\ +\infty),\ i\in Q, \tag{2.2}$$

and

$$\sup_{t\in[t_0,\ +\infty)}\Big\{-a_{ii}(t)\sum_{j=1,j\ne i}^{n} a_{ij}(t) + \sum_{j=1}^{m}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\frac{1}{e}\Big\} < 0,\ i\in Q. \tag{2.3}$$

Then $x(t) = x(t;t_0,\varphi)$ is positive and bounded on $[t_0,+\infty)$, and

$$0 < S_- \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le S^+,\qquad i\in Q. \tag{2.4}$$

Proof First, we claim that

$$x_i(t) > 0\ \ \text{for all}\ \ t\in[t_0,\ \eta(\varphi)),\ i\in Q.$$

Otherwise, we can pick $i_0 \in Q$ and $\bar{t}_{i_0} \in (t_0,\ \eta(\varphi))$ to satisfy

$$x_{i_0}(\bar{t}_{i_0}) = 0, \quad x_j(t) > 0 \quad \text{for all} \quad t \in [t_0,\ \bar{t}_{i_0}),\ j \in Q.$$

Apparently, (1.2) and (2.2) yield

$$\begin{aligned}
0 &\geq x'_{i_0}(\bar{t}_{i_0}) \\
&= -a_{i_0 i_0}(\bar{t}_{i_0}) + b_{i_0 i_0}(\bar{t}_{i_0})e^{-x_{i_0}(\bar{t}_{i_0})} + \sum_{j=1, j\neq i_0}^{n} \left(a_{i_0 j}(\bar{t}_{i_0}) - b_{i_0 j}(\bar{t}_{i_0})e^{-x_j(\bar{t}_{i_0})}\right) \\
&\quad + \sum_{j=1}^{m} \beta_{i_0 j}(\bar{t}_{i_0}) x_{i_0}(\bar{t}_{i_0} - \tau_{i_0 j}(\bar{t}_{i_0})) e^{-\gamma_{i_0 j}(\bar{t}_{i_0}) x_{i_0}(\bar{t}_{i_0} - \tau_{i_0 j}(\bar{t}_{i_0}))} \\
&\geq -a_{i_0 i_0}(\bar{t}_{i_0}) + b_{i_0 i_0}(\bar{t}_{i_0}) + \sum_{j=1, j\neq i_0}^{n} \left(a_{i_0 j}(\bar{t}_{i_0}) - b_{i_0 j}(\bar{t}_{i_0})\right) > 0,
\end{aligned}$$

which is a contradiction and proves the claim proves the claim.

Now, we demonstrate that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. For $t \in [t_0 - \sigma_i, \eta(\varphi))$ and $i \in Q$, we define

$$M_i(t) = \max\left\{ \xi : \xi \leq t, x_i(\xi) = \max_{t_0 - \sigma_i \leq s \leq t} x_i(s) \right\}.$$

Suppose that $x(t)$ is unbounded on $[t_0, \eta(\varphi))$. Then we can choose $i^* \in Q$ and a strictly monotone increasing sequence $\{\zeta_n\}_{n=1}^{+\infty}$ such that

$$\begin{aligned}
&x_{i^*}\left(M_{i^*}(\zeta_n)\right) = \max_{j\in Q}\left\{x_j(M_j(\zeta_n))\right\}, \\
&\lim_{n\to+\infty} x_{i^*}\left(M_{i^*}(\zeta_n)\right) = +\infty, \qquad \lim_{n\to+\infty} \zeta_n = \eta(\varphi),
\end{aligned} \tag{2.5}$$

and then $\lim_{n\to+\infty} M_{i^*}(\zeta_n) = \eta(\varphi)$. According to (1.2) and the fact $\sup_{u\geq 0} ue^{-u} = \frac{1}{e}$, it follows from (2.5) that

$$\begin{aligned}
0 &\leq x'_{i^*}\left(M_{i^*}(\zeta_n)\right) \\
&= -a_{i^* i^*}\left(M_{i^*}(\zeta_n)\right) + b_{i^* i^*}\left(M_{i^*}(\zeta_n)\right) e^{-x_{i^*}(M_{i^*}(\zeta_n))} \\
&\quad + \sum_{j=1, j\neq i^*}^{n} \left(a_{i^* j}\left(M_{i^*}(\zeta_n)\right) - b_{i^* j}\left(M(\zeta_n)\right) e^{-x_j(M_{i^*}(\zeta_n))}\right) \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{i^* j}(M_{i^*}(\zeta_n))}{\gamma_{i^* j}(M_{i^*}(\zeta_n))} \gamma_{i^* j}(M_{i^*}(\zeta_n)) x_{i^*}(M_{i^*}(\zeta_n) - \tau_{i^* j}(M_{i^*}(\zeta_n))) \\
&\qquad \times e^{-\gamma_{i^* j}\left(M_{i^*}(\zeta_n)\right) x_{i^*}\left(M_{i^*}(\zeta_n) - \tau_{i^* j}(M_{i^*}(\zeta_n))\right)} \\
&\leq -a_{i^* i^*}\left(M_{i^*}(\zeta_n)\right) + b_{i^* i^*}\left(M_{i^*}(\zeta_n)\right) e^{-x_{i^*}(M_{i^*}(\zeta_n))} \\
&\quad + \sum_{j=1, j\neq i^*}^{n} \left(a_{i^* j}(M_{i^*}(\zeta_n)) - b_{i^* j}\left(M(\zeta_n)\right) e^{-x_{i^*}\left(M_{i^*}(\zeta_n)\right)}\right) \\
&\quad + \sum_{j=1}^{m} \frac{\beta_{i^* j}(M_{i^*}(\zeta_n))}{\gamma_{i^* j}(M_{i^*}(\zeta_n))} \frac{1}{e}, \quad \text{for all } M_{i^*}(\zeta_n) > t_0.
\end{aligned}$$

Taking $n \to +\infty$ leads to

$$0 \le \lim_{n\to+\infty} \Big[-a_{i^*i^*}(M_{i^*}(\zeta_n)) + \sum_{j=1, j\neq i^*}^{n} a_{i^*j}\big(M_{i^*}(\zeta_n)\big) + \sum_{j=1}^{m} \frac{\beta_{i^*j}(M_{i^*}(\zeta_n))}{\gamma_{i^*j}(M_{i^*}(\zeta_n))}\frac{1}{e}\Big]$$
$$\le \sup_{t\in[t_0,+\infty)} \Big[-a_{i^*i^*}(t) + \sum_{j=1, j\neq i^*}^{n} a_{i^*j}(t) + \sum_{j=1}^{m} \frac{\beta_{i^*j}(t)}{\gamma_{i^*j}(t)}\frac{1}{e}\Big] < 0,$$

which is absurd and suggests that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. By [Theorem 2.3.1][8], we easily show $\eta(\varphi) = +\infty$.

Next, we prove that (2.4) is true. Designate $i^l, i^L \in Q$ such that

$$l = \liminf_{t\to+\infty} x_{i^l}(t) = \min_{i\in Q}\liminf_{t\to+\infty} x_i(t), \ L = \limsup_{t\to+\infty} x_{i^L}(t) = \max_{i\in Q}\limsup_{t\to+\infty} x_i(t).$$

By the fluctuation (by [19, Lemma A.1.]), we can select two sequences $\{t_k^*\}_{k=1}^{+\infty}$ and $\{t_k^{**}\}_{k=1}^{+\infty}$ satisfying

$$\text{(2.6)}\qquad \lim_{k\to+\infty} t_k^* = +\infty, \quad \lim_{k\to+\infty} x_{i^l}(t_k^*) = l = \liminf_{t\to+\infty} x_{i^l}(t), \quad \lim_{k\to+\infty} x'_{i^l}(t_k^*) = 0,$$

and

$$\text{(2.7)}\quad \lim_{k\to+\infty} t_k^{**} = +\infty, \quad \lim_{k\to+\infty} x_{i^L}(t_k^{**}) = L = \limsup_{t\to+\infty} x_{i^L}(t), \quad \lim_{k\to+\infty} x'_{i^L}(t_k^{**}) = 0,$$

respectively. From the almost periodicity of (1.2), we can select a subsequence of $\{k\}_{k\ge1}$, still denoted by $\{k\}_{k\ge1}$, such that

$$\lim_{k\to+\infty} a_{i^lj}(t_k^*), \quad \lim_{k\to+\infty} b_{i^lj}(t_k^*), \quad \lim_{k\to+\infty} \beta_{i^lq}(t_k^*), \quad \lim_{k\to+\infty} \gamma_{i^lq}(t_k^*), \quad \lim_{k\to+\infty} x_j(t_k^*),$$
$$\lim_{k\to+\infty} x_{i^l}(t_k^* - \tau_{i^lq}(t_k^*)), \quad \lim_{k\to+\infty} a_{i^Lj}(t_k^{**}), \quad \lim_{k\to+\infty} b_{i^Lj}(t_k^{**}), \quad \lim_{k\to+\infty} \beta_{i^Lq}(t_k^{**}),$$
$$\lim_{k\to+\infty} \gamma_{i^Lq}(t_k^{**}), \quad \lim_{k\to+\infty} x_j(t_k^{**}), \quad \text{and} \quad \lim_{k\to+\infty} x_{i^L}(t_k^{**} - \tau_{i^Lq}(t_k^{**}))$$

exist for all $j \in Q, q \in I$. Furthermore, by taking limits, we have from (2.6) and (2.7) that

$$\begin{aligned} 0 &= \lim_{k\to+\infty} x'_{i^l}(t_k^*) \\ &\ge -\lim_{k\to+\infty} a_{i^li^l}(t_k^*) + \lim_{k\to+\infty} b_{i^li^l}(t_k^*)e^{-l} \\ &\quad + \sum_{j=1, j\neq i^l}^{n} \Big(\lim_{k\to+\infty} a_{i^lj}(t_k^*) - \lim_{k\to+\infty} b_{i^lj}(t_k^*)e^{-l}\Big), \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{k\to+\infty} x'_{i^L}(t_k^{**}) \\ &= -\lim_{k\to+\infty} a_{i^Li^L}(t_k^{**}) + \lim_{k\to+\infty} b_{i^Li^L}(t_k^{**})e^{-L} \\ &\quad + \sum_{j=1, j\neq i^L}^{n} \Big(\lim_{k\to+\infty} a_{i^Lj}(t_k^{**}) - \lim_{k\to+\infty} b_{i^Lj}(t_k^{**})e^{-\lim_{k\to+\infty} x_j(t_k^{**})}\Big) \\ &\quad + \sum_{j=1}^{m} \lim_{k\to+\infty} \frac{\beta_{i^Lj}(t_k^{**})}{\gamma_{i^Lj}(t_k^{**})} \lim_{k\to+\infty} \gamma_{i^Lj} \end{aligned}$$

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$$\begin{aligned}
&\times(t_k^{**})x_{i^L}(t_k^{**}-\tau_{i^L j}(t_k^{**}))e^{-\lim_{k\to+\infty}\gamma_{i^L j}(t_k^{**})\lim_{k\to+\infty}x_{i^L}(t_k^{**}-\tau_{i^L j}(t_k^{**}))}\\
\le&-\lim_{k\to+\infty}a_{i^L i^L}(t_k^{**})+\lim_{k\to+\infty}b_{i^L i^L}(t_k^{**})e^{-L}\\
&+\sum_{j=1,j\ne i^L}^{n}\Big(\lim_{k\to+\infty}a_{i^L j}(t_k^{**})-\lim_{k\to+\infty}b_{i^L j}(t_k^{**})e^{-L}\Big)\\
&+\sum_{j=1}^{m}\lim_{k\to+\infty}\frac{\beta_{i^L j}(t_k^{**})}{\gamma_{i^L j}(t_k^{**})}\frac{1}{e},
\end{aligned}$$

which entail that

$$S_-\le\liminf_{t\to+\infty}\ln\Big(\frac{b_{i^l i^l}(t)-\sum_{j=1,j\ne i^l}^{n}b_{i^l j}(t)}{a_{i^l i^l}(t)-\sum_{j=1,j\ne i^l}^{n}a_{i^l j}(t)}\Big)\le\liminf_{t\to+\infty}x_{i^l}(t)\le\liminf_{t\to+\infty}x_i(t)$$

and

$$\begin{aligned}
\limsup_{t\to+\infty}x_i(t)&\le\limsup_{t\to+\infty}x_{i^L}(t)\\
&\le\limsup_{t\to+\infty}\ln\Bigg(\frac{b_{i^L i^L}(t)-\sum_{j=1,j\ne i^L}^{n}b_{i^L j}(t)}{a_{i^L i^L}(t)-\sum_{j=1,j\ne i^L}^{n}\big(a_{i^L j}(t)-\sum_{j=1}^{m}\frac{1}{e}\frac{\beta_{i^L j}(t)}{\gamma_{i^L j}(t)}\big)}\Bigg)\\
&\le S^+,
\end{aligned}$$

for all $i\in Q$. This completes the proof of Lemma 2.3. ∎

Lemma 2.4 *Assume that* (2.2), (2.3) *hold, and for* $i\in Q$,

$$\limsup_{t\to+\infty}\Big\{-b_{ii}(t)e^{-S^+}+\sum_{j=1,j\ne i}^{n}b_{ij}(t)e^{-S_-}+\sum_{j=1}^{m}\beta_{ij}(t)\max\Big\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{\inf}S_-}{e^{\gamma_{ij}^{\inf}S_-}}\Big\}\Big\}<0. \tag{2.8}$$

Moreover, suppose that $x(t)=x(t;t_0,\varphi)$. *Then for any* $\varepsilon>0$, *there exists* $l=l(\varepsilon)>0$, *such that each interval* $[\alpha,\ \alpha+l]$ *includes at least one number* δ *for which there exists* $\widehat{\Lambda}>0$ *that satisfies*

$$\|x(t+\delta)-x(t)\|\le\varepsilon,\ \textit{for all}\ \ t>\widehat{\Lambda}. \tag{2.9}$$

Proof According to (2.8), for all $i\in Q$ it is easy to see that there exists $t_0^*\ge t_0$ such that

$$\sup_{t\ge t_0^*}\Big\{-b_{ii}(t)e^{-S^+}+\sum_{j=1,j\ne i}^{n}b_{ij}(t)e^{-S_-}+\sum_{j=1}^{m}\beta_{ij}(t)\max\Big\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{\inf}S_-}{e^{\gamma_{ij}^{\inf}S_-}}\Big\}\Big\}<0.$$

Set

$$H_i(u,v)=\sup_{t\ge t_0^*}\Big\{-[b_{ii}(t)e^{-(S^++v)}-u]+\sum_{j=1,j\ne i}^{n}b_{ij}(t)e^{-(S_--v)}$$

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$$+\sum_{j=1}^{m}\beta_{ij}(t)\max\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{\inf}(S_- - v)}{e^{\gamma_{ij}^{\inf}(S_- - v)}}\}e^{u\sigma_i}\Big\},u,v\in[0,\ 1],\ i\in Q.$$

Furthermore, let $B=\frac{1}{2}\min_{i\in Q}|H_i(0,0)|$; then $B<0$. According to the continuity of $H_i(u,v)$, one can pick a sufficiently small constant $0<\eta<1$ such that

$$H_i(u,v)<-B \text{ for all } (u,v)\in[0,\ \eta]\times[0,\ \eta],\ i\in Q,$$

and then for fixed $\lambda\in[0,\ \eta]$, we have

$$H_i(\lambda,\varepsilon)=\sup_{t\geq t_0^*}\Big\{-[b_{ii}(t)e^{-(S^+ +\varepsilon)}-\lambda]+\sum_{j=1,j\neq i}^{n}b_{ij}(t)e^{-(S_- -\varepsilon)}$$
$$+\sum_{j=1}^{m}\beta_{ij}(t)\max\Big\{\frac{1}{e^2},\frac{1-\gamma_{ij}^{\inf}(S_- -\varepsilon)}{e^{\gamma_{ij}^{\inf}(S_- -\varepsilon)}}\Big\}e^{\lambda\sigma_i}\Big\}<0, \tag{2.10}$$

for all $\varepsilon\in(0,\eta],\ i\in Q$, and

$$\max_{i\in Q}\{\sup_{\varepsilon\in[0,\eta]}H_i(\lambda,\varepsilon)\}=-B<0.$$

Without loss of generality, to prove Lemma 2.4, we only need to show that (2.9) holds for $\varepsilon\in(0,\min\{\eta,S_-\})$. For $i\in Q$, $t\in(-\infty,t_0-\sigma_i]$, we add the definition of $x_i(t)$ with $x_i(t)\equiv x_i(t_0-\sigma_i)$. Set

$$A_i(\delta,t)=[b_{ii}(t+\delta)-b_{ii}(t)]e^{-x_i(t+\delta)}-\sum_{j=1,j\neq i}^{n}[b_{ij}(t+\delta)-b_{ij}(t)]e^{-x_j(t+\delta)}$$
$$+\sum_{j=1}^{m}\big[\beta_{ij}(t+\delta)-\beta_{ij}(t)\big]x_i\big(t+\delta-\tau_{ij}(t+\delta)\big)e^{-\gamma_{ij}(t+\delta)x_i(t+\delta-\tau_{ij}(t+\delta))}$$
$$+\sum_{j=1}^{m}\beta_{ij}(t)[x_i(t+\delta-\tau_{ij}(t+\delta))e^{-\gamma_{ij}(t+\delta)x_i(t+\delta-\tau_{ij}(t+\delta))}$$
$$-x_i(t-\tau_{ij}(t)+\delta)e^{-\gamma_{ij}(t+\delta)x_i(t-\tau_{ij}(t)+\delta)}]$$
$$+\sum_{j=1}^{m}\beta_{ij}(t)\Big[x_i(t-\tau_{ij}(t)+\delta)e^{-\gamma_{ij}(t+\delta)x_i(t-\tau_{ij}(t)+\delta)}$$
$$-x_i(t-\tau_{ij}(t)+\delta)e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t)+\delta)}\Big]$$
$$-[a_{ii}(t+\delta)-a_{ii}(t)]$$
$$+\sum_{j=1,j\neq i}^{n}\big[a_{ij}(t+\delta)-a_{ij}(t)\big],\quad t\in\mathbb{R}.$$

For any $\varepsilon\in(0,\min\{\eta,S_-\})$, it follows from Lemma 2.2 that there exists $T_\varphi>t_0^*$ such that

$$S_- -\varepsilon<x_i(t)<S^+ +\varepsilon,\ \text{ for all }\ t\in[T_\varphi-\sigma_i,\ +\infty),\ i\in Q, \tag{2.11}$$

which implies that the right side of (1.1) is also bounded, and $x_i'(t)$ is a bounded function on $[t_0,+\infty)$. Thus, with the help of the fact that $x_i(t)\equiv x_i(t_0-\sigma_i)$ for

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$t \in (-\infty, t_0 - \sigma_i]$, we gain that $x_i(t)$ is uniformly continuous on $\mathbb{R}$. From uniformly almost periodic family theory in [7, Corollary 2.3, p. 19], for each $\varepsilon \in (0, \min\{\eta, S_{-Let}\})$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l] \subseteq \mathbb{R}$, includes a δ for which

$$|A_i(\delta, t)| \le \frac{1}{2} B\varepsilon, \quad \text{for all} \quad t \in \mathbb{R},\ i \in Q.$$

Let $\Lambda_0 \ge \max\{T_\varphi + \max_{i\in Q} \sigma_i,\ T_\varphi + \max_{i\in Q} \sigma_i - \delta\}$. For $t \in \mathbb{R}$, denote

$$u(t) = (u_1(t), u_2(t), \dots, u_n(t)), \qquad u_i(t) = x_i(t+\delta) - x_i(t),$$
$$U(t) = (U_1(t), U_2(t), \dots, U_n(t)), \qquad U_i(t) = e^{\lambda t} u_i(t),$$

where $i \in Q$. Let i_t be such an index that

$$|U_{i_t}(t)| = \|U(t)\|. \tag{2.12}$$

Then, for all $t \ge \Lambda_0$, we have

$$\begin{aligned} u_i'(t) = {} & b_{ii}(t)[e^{-x_i(t+\delta)} - e^{-x_i(t)}] \\ & - \sum_{j=1, j\ne i}^{n} b_{ij}(t)\big[e^{-x_j(t+\delta)} - e^{-x_j(t)}\big] \\ & + \sum_{j=1}^{m} \beta_{ij}(t)\big[x_i(t - \tau_{ij}(t) + \delta) e^{-\gamma_{ij}(t) x_i(t - \tau_{ij}(t) + \delta)} \\ & \qquad - x_i(t - \tau_{ij}(t)) e^{-\gamma_{ij}(t) x_i(t - \tau_{ij}(t))}\big] + A_i(\delta, t). \end{aligned} \tag{2.13}$$

With the help of Lemma 2.2, one can show the following inequalities:

$$\begin{gathered} \gamma_{ij}^{\inf}(S_- - \varepsilon) \le \gamma_{ij}(t) x(t - \tau_{ij}(t)),\ \gamma_{ij}(t) x(t - \tau_{ij}(t) + \delta), \\ i \in Q, j \in I,\ t \ge \Lambda_0, \\ S_- - \varepsilon \le x_i(t),\ i \in Q, t \ge \Lambda_0, \\ (e^{-s} - e^{-t}) \operatorname{sgn}(s - t) \le -e^{-(S^+ + \varepsilon)}|s - t|,\ |e^{-s} - e^{-t}| \le e^{-(S_- - \varepsilon)}|s - t| \\ \text{for } s, t \in [S_- - \varepsilon, S^+ + \varepsilon], \\ |se^{-s} - te^{-t}| \le \max\Big\{\frac{1}{e^2}, \frac{1 - \gamma_{ij}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{ij}^{\inf}(S_- - \varepsilon)}}\Big\}|s - t| \\ \text{for } s, t \in [\gamma_{ij}^{\inf}(S_- - \varepsilon), +\infty), i \in Q, j \in I. \end{gathered}$$

This, together with (2.11), (2.12), and (2.13), gives us we get

(2.14)

$$\begin{aligned} & D^-(|U_{i_s}(s)|)|_{s=t} \\ & \quad \le \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\{b_{i_t i_t}(t)[e^{-x_{i_t}(t+\delta)} - e^{-x_{i_t}(t)}] \operatorname{sgn}\big(x_{i_t}(t+\delta) - x_{i_t}(t)\big) \\ & \qquad + \sum_{j=1, j\ne i_t}^{n} b_{i_t j}(t)|e^{-x_j(t+\delta)} - e^{-x_j(t)}| + \sum_{j=1}^{m} \beta_{i_t j}(t) \\ & \qquad \times \Big| x_{i_t}(t - \tau_{i_t j}(t) + \delta) e^{-\gamma_{i_t j}(t) x_{i_t}(t - \tau_{i_t j}(t) + \delta)} \end{aligned}$$

$$
\begin{aligned}
&\quad - x_{i_t}(t - \tau_{i_t j}(t))e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t))}\Big| \\
&\quad + |A_{i_t}(\delta, t)|\} \\
&= \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\{b_{i_t i_t}(t)[e^{-x_{i_t}(t+\delta)} - e^{-x_{i_t}(t)}]\,\mathrm{sgn}(x_{i_t}(t+\delta) - x_{i_t}(t)) \\
&\quad + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)|e^{-x_j(t+\delta)} - e^{-x_j(t)}| + \sum_{j=1}^{m} \frac{\beta_{i_t j}(t)}{\gamma_{i_t j}(t)} \\
&\quad \times \Big|\gamma_{i_t j}(t)x_{i_t}(t - \tau_{i_t j}(t) + \delta)e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t)+\delta)} \\
&\qquad - \gamma_{i_t j}(t)x_{i_t}(t - \tau_{i_t j}(t))e^{-\gamma_{i_t j}(t)x_{i_t}(t-\tau_{i_t j}(t))}\Big| + |A_{i_t}(\delta, t)|\} \\
&\le \lambda e^{\lambda t}|u_{i_t}(t)| + e^{\lambda t}\Bigg\{ - b_{i_t i_t}(t)e^{-(S^+ + \varepsilon)}|u_{i_t}(t)| + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)e^{-(S_- - \varepsilon)}|u_j(t)| \\
&\quad + \sum_{j=1}^{m}\beta_{i_t j}(t)\max\left\{\frac{1}{e^2}, \frac{1 - \gamma_{i_t j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{i_t j}^{\inf}(S_- - \varepsilon)}}\right\}|u_{i_t}(t - \tau_{i_t j}(t))| + |A_{i_t}(\delta, t)|\Bigg\} \\
&= -[b_{i_t i_t}(t)e^{-(S^+ + \varepsilon)} - \lambda]|U_{i_t}(t)| \\
&\quad + \sum_{j=1, j\neq i_t}^{n} b_{i_t j}(t)e^{-(S_- - \varepsilon)}|U_j(t)| \\
&\quad + \sum_{j=1}^{m}\beta_{i_t j}(t)\max\left\{\frac{1}{e^2}, \frac{1 - \gamma_{i_t j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{i_t j}^{\inf}(S_- - \varepsilon)}}\right\}e^{\lambda\tau_{i_t j}(t)}|U_{i_t}(t - \tau_{i_t j}(t))| \\
&\quad + e^{\lambda t}|A_{i_t}(\delta, t)| \text{ for all } t \ge \Lambda_0.
\end{aligned}
$$

Let

$$E(t) = \sup_{-\infty < s \le t}\{e^{\lambda s}\|u(s)\|\}.$$

It is obvious that $e^{\lambda t}\|u(t)\| \le E(t)$ and $E(t)$ is non-decreasing.

Now the remaining proof will be divided into two steps.

Step one. If $E(t) > e^{\lambda t}\|u(t)\|$ for all $t \ge \Lambda_0$, we assert that

$$E(t) \equiv \|U(\Lambda_0)\|, \text{ for all } t \ge \Lambda_0.$$

In the contrary case, one can pick $\Lambda_1 > \Lambda_0$ such that $E(\Lambda_1) > E(\Lambda_0)$. Because

$$e^{\lambda t}\|u(t)\| \le E(\Lambda_0) \text{ for all } t \le \Lambda_0,$$

there must exist $\beta^* \in (\Lambda_0, \Lambda_1)$ such that

$$e^{\lambda\beta^*}\|u(\beta^*)\| = E(\Lambda_1) \ge E(\beta^*),$$

which contradicts the fact that $E(\beta^*) > e^{\lambda\beta^*}\|u(\beta^*)\|$ and proves the above assertion. Then, we can select $\Lambda_2 > \Lambda_0$ satisfying

$$\|u(t)\| \le e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_0) < \frac{\varepsilon}{2} \text{ for all } t \ge \Lambda_2.$$

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Step two. If there exists $\varsigma \geq \Lambda_0$ such that $E(\varsigma) = e^{\lambda\varsigma}\|u(\varsigma)\|$, we have from (2.14) and the definition of $E(t)$ that

$$\begin{aligned}
0 &\leq D^{-}(|U_{i_s}(s)|)\Big|_{s=\varsigma} \\
&\leq -[b_{i_\varsigma i_\varsigma}(\varsigma)e^{-(S^{+}+\varepsilon)} - \lambda]|U_{i_\varsigma}(\varsigma)| + \sum_{j=1, j\neq i_\varsigma}^{n} b_{i_\varsigma j}(\varsigma)e^{-(S_{-}-\varepsilon)}|U_j(\varsigma)| \\
&\quad + \sum_{j=1}^{m}\beta_{i_\varsigma j}(\varsigma)\max\left\{\frac{1}{e^2}, \frac{1-\gamma_{i_\varsigma j}^{\inf}(S_{-}-\varepsilon)}{e^{\gamma_{i_\varsigma j}^{\inf}(S_{-}-\varepsilon)}}\right\}e^{\lambda\tau_{i_\varsigma j}(\varsigma)}|U_{i_\varsigma}(\varsigma - \tau_{i_\varsigma j}(\varsigma))| \\
&\quad + e^{\lambda\varsigma}|A_{i_\varsigma}(\delta, \varsigma)| \\
&\leq \Bigg\{ -[b_{i_\varsigma i_\varsigma}(\varsigma)e^{-(S^{+}+\varepsilon)} - \lambda] + \sum_{j=1, j\neq i_\varsigma}^{n} b_{i_\varsigma j}(\varsigma)e^{-(S_{-}-\varepsilon)} \\
&\quad + \sum_{j=1}^{m}\beta_{i_\varsigma j}(\varsigma)\max\left\{\frac{1}{e^2}, \frac{1-\gamma_{i_\varsigma j}^{\inf}(S_{-}-\varepsilon)}{e^{\gamma_{i_\varsigma j}^{\inf}(S_{-}-\varepsilon)}}\right\}e^{\lambda\tau_{i_\varsigma j}(\varsigma)}\Bigg\}E(\varsigma) + \frac{1}{2}B\varepsilon e^{\lambda\varsigma} \\
&< -BE(\varsigma) + \frac{1}{2}B\varepsilon e^{\lambda\varsigma},
\end{aligned}$$

which leads to

$$e^{\lambda\varsigma}\|u(\varsigma)\| = E(\varsigma) < \frac{\varepsilon}{2}e^{\lambda\varsigma} \quad \text{and} \quad \|u(\varsigma)\| < \frac{\varepsilon}{2}. \tag{2.15}$$

For any $t > \varsigma$ satisfying $E(t) = e^{\lambda t}\|u(t)\|$, by the same method as in the derivation of (2.15), we can show

$$e^{\lambda t}\|u(t)\| < \frac{\varepsilon}{2}e^{\lambda t} \quad \text{and} \quad \|u(t)\| < \frac{\varepsilon}{2}. \tag{2.16}$$

Furthermore, if $E(t) > e^{\lambda t}\|u(t)\|$ and $t > \varsigma$, one can pick $\Lambda_3 \in [\varsigma, t)$ such that

$$E(\Lambda_3) = e^{\lambda\Lambda_3}\,|u(\Lambda_3)\| \quad \text{and} \quad E(s) > e^{\lambda s}\|u(s)\| \quad \text{for all} \quad s \in (\Lambda_3, t],$$

which, together with (2.15) and (2.16), suggests that

$$\|u(\Lambda_3)\| < \frac{\varepsilon}{2}. \tag{2.17}$$

With similar reasoning to that in the proof of step one, we can infer that

$$E(s) \equiv E(\Lambda_3) \text{ is a constant for all } s \in (\Lambda_3, t],$$

which, together with (2.17), implies that

$$\|u(t)\| < e^{-\lambda t}E(t) = e^{-\lambda t}E(\Lambda_3) = \|u(\Lambda_3)\|e^{-\lambda(t-\Lambda_3)} < \frac{\varepsilon}{2}.$$

Finally, the above discussion infers that there exists $\widehat{\Lambda} > \max\{\varsigma, \Lambda_0, \Lambda_2\}$ such that

$$\|u(t)\| \leq \frac{\varepsilon}{2} < \varepsilon \text{ for all } t > \widehat{\Lambda}, \tag{2.18}$$

which finishes the proof of Lemma 2.4. ∎

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3 Global Exponential Stability of Almost Periodic Solutions

Combining Lemma 2.2 with Lemma 2.3, we have the following theorem.

Theorem 3.1 *Assume that all assumptions of Lemma 2.4 are satisfied. Then (1.2) has a globally exponentially stable, positive, almost periodic solution $x^*(t)$. Moreover, there exist constants K_{φ,x^*} and t_{φ,x^*} such that*

$$\|x(t;t_0,\varphi)-x^*(t)\|<K_{\varphi,x^*}e^{-\lambda t}\ \text{for all}\ t>t_{\varphi,x^*}.$$

Proof Let $v(t)=v(t;t_0,\varphi^v)$ be a solution of equation (1.2) with initial conditions satisfying the assumptions in Lemma 2.4. We also define $v_i(t)\equiv v_i(t_0-\sigma_i)$ for all $t\in(-\infty,t_0-\sigma_i]$, $i\in Q$. Define

$$\begin{aligned}
\Pi_{i,k}(t)&=[b_{ii}(t+t_k)-b_{ii}(t)]e^{-v_i(t+t_k)}\\
&\quad-\sum_{j=1,j\neq i}^{n}[b_{ij}(t+t_k)-b_{ij}(t)]e^{-v_j(t+t_k)}\\
&\quad+\sum_{j=1}^{m}\big[\beta_{ij}(t+t_k)-\beta_{ij}(t)\big]v_i\\
&\qquad\times(t+t_k-\tau_{ij}(t+t_k))e^{-\gamma_{ij}(t+t_k)v_i(t+t_k-\tau_{ij}(t+t_k))}\\
&\quad+\sum_{j=1}^{m}\beta_{ij}(t)[v_i(t+t_k-\tau_{ij}(t+t_k))e^{-\gamma_{ij}(t+t_k)v_i(t+t_k-\tau_{ij}(t+t_k))}\\
&\quad-v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t+t_k)v_i(t-\tau_{ij}(t)+t_k)}]\\
&\quad+\sum_{j=1}^{m}\beta_{ij}(t)[v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t+t_k)v_i(t-\tau_{ij}(t)+t_k)}\\
&\quad-v_i(t-\tau_{ij}(t)+t_k)e^{-\gamma_{ij}(t)v_i(t-\tau_{ij}(t)+t_k)}]-[a_{ii}(t+t_k)-a_{ii}(t)]\\
&\quad+\sum_{j=1,j\neq i}^{n}[a_{ij}(t+t_k)-a_{ij}(t)],\quad t\in\mathbb{R},\ i\in Q,
\end{aligned}$$

where $\{t_k\}$ is any sequence of real numbers. For any $\varepsilon\in(0,\min\{\eta,S_-\})$, by Lemma 2.3, we can choose $t_{\varphi^v}>t_0$ such that

$$S_--\varepsilon<v_i(t)<S^++\varepsilon,\ \text{for all}\ t\ge t_{\varphi^v},\ i\in Q,$$

which, together with the boundedness of $v_i'(t)$ and the fact that $v_i(t)\equiv v_i(t_0-\sigma_i)$ for $t\in(-\infty,t_0-\sigma_i]$, entails that $v(t)$ is uniformly continuous on $\mathbb{R}$. Then, from the almost periodicity of a_{ij}, b_{ij}, τ_{ij}, γ_{ij}, and β_{ij}, we can select a sequence $\{t_k\}\to+\infty$ such that

$$\begin{aligned}
&|a_{ij}(t+t_k)-a_{ij}(t)|\le\frac{1}{k}, && |b_{ij}(t+t_k)-b_{ij}(t)|\le\frac{1}{k},\\
&|\tau_{ij}(t+t_k)-\tau_{ij}(t)|\le\frac{1}{k} && |\beta_{ij}(t+t_k)-\beta_{ij}(t)|\le\frac{1}{k},\\
&|\gamma_{ij}(t+t_k)-\gamma_{ij}(t)|\le\frac{1}{k}, && |\epsilon(k,t)|\le\frac{1}{k},
\end{aligned}\tag{3.1}$$

for all i,j,t.

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Since $\{v(t+t_k)\}_{k=1}^{+\infty}$ is uniformly bounded and equi-uniformly continuous, from Arzala–Ascoli Lemma and the diagonal selection principle, we can select a subsequence $\{t_{k_q}\}$ of $\{t_k\}$, such that $v(t+t_{k_q})$ (which for convenience we still designate by $v(t+t_k)$) uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$ on any compact set of $\mathbb{R}$, and

$$S_- - \varepsilon \le x_i^*(t) \le S^+ + \varepsilon, \qquad \text{for all } t \in \mathbb{R},\ i \in Q. \tag{3.2}$$

Now, we prove that $x^*(t)$ is a solution of (1.2). In fact, for any $t \ge t_0$ and $\Delta t \in \mathbb{R}$, from (3.1), we have

$$\begin{aligned}
(3.3)\quad & x_i^*(t+\Delta t) - x_i^*(t) \\
&= \lim_{k\to+\infty} [v_i(t+\Delta t+t_k) - v_i(t+t_k)] \\
&= \lim_{k\to+\infty} \int_t^{t+\Delta t} \Big[-a_{ii}(s) + b_{ii}(s)e^{-v_i(s+t_k)} \\
&\qquad + \sum_{j=1, j\ne i}^{n} (a_{ij}(s) - b_{ij}(s)e^{-v_j(s+t_k)}) \\
&\qquad + \sum_{j=1}^{m} \beta_{ij}(s) v_i(s+t_k-\tau_{ij}(s)) e^{-\gamma_{ij}(s) v_i(s+t_k-\tau_{ij}(s))} + \Pi_{i,k}(s) \Big] ds \\
&= \int_t^{t+\Delta t} \Big[-a_{ii}(s) + b_{ii}(s)e^{-x_i^*(s)} + \sum_{j=1, j\ne i}^{n} \big(a_{ij}(s) - b_{ij}(s)e^{-x_j^*(s)} \big) \\
&\qquad + \sum_{j=1}^{m} \beta_{ij}(s) x_i^*(s-\tau_{ij}(s)) e^{-\gamma_{ij}(s) x_i^*(s-\tau_{ij}(s))} \Big] ds,
\end{aligned}$$

where $t,\ t+\Delta t \ge t_0,\ i \in Q$. Consequently, (3.3) suggests that

$$\begin{aligned}
\frac{d}{dt}\{x_i^*(t)\} &= -a_{ii}(t) + b_{ii}(t)e^{-x_i^*(t)} + \sum_{j=1, j\ne i}^{n} (a_{ij}(t) - b_{ij}(t)e^{-x_j^*(t)}) \\
&\qquad + \sum_{j=1}^{m} \beta_{ij}(t) x_i^*(t-\tau_{ij}(t)) e^{-\gamma_{ij}(t) x_i^*(t-\tau_{ij}(t))},\ i \in Q.
\end{aligned}$$

Hence, $x^*(t)$ is a solution of (1.2).

Furthermore, from Lemma 2.4 and (2.18), for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $\widehat{\Lambda} > 0$ obeying

$$\|v(t+\delta) - v(t)\| \le \frac{\varepsilon}{2} < \varepsilon, \text{ for all } t > \widehat{\Lambda}.$$

Given $s \in \mathbb{R}$, one can pick a sufficiently large positive integer $N_1 > \widehat{\Lambda}$ such that, for any $k > N_1$,

$$s + t_k > \widehat{\Lambda}, \qquad \|v(s+t_k+\delta) - v(s+t_k)\| \le \frac{\varepsilon}{2} < \varepsilon.$$

Letting $k \to +\infty$ gives us

$$\|x^*(s+\delta) - x^*(s)\| < \varepsilon,$$

which suggests that $x^*(t)$ is a positive almost periodic solution of (1.2).

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Next, we validate that $x^*(t)$ is globally exponentially stable. Let $x(t) = x(t; t_0, \varphi)$, and

$$z_i(t) = x_i(t) - x_i^*(t), \quad W_i(t) = |z_i(t)|e^{\lambda t} \text{ for all } t \in [t_0 - \sigma_i, +\infty).$$

Clearly,

$$(3.4)\quad z_i'(t) = b_{ii}(t)[e^{-x_i(t)} - e^{-x_i^*(t)}] - \sum_{j=1, j\neq i}^{n} b_{ij}(t)[e^{-x_j(t)} - e^{-x_j^*(t)}] + \sum_{j=1}^{m} \beta_{ij}(t)\big[x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}\big].$$

For any $\varepsilon \in (0, \min\{\eta, S_-\})$, it follows from Lemma 2.4 that there exists $T_{\varphi,x^*} > t_0$ such that

$$(3.5)\qquad S_- - \varepsilon \le x_i(t) \le S^+ + \varepsilon, \text{ for all } t \in [T_{\varphi,x^*} - \sigma_i,\ +\infty),\ i \in Q.$$

By (3.4) and calculating the upper-right Dini derivative of $W_i(t)$, we obtain

$$(3.6)\qquad \begin{aligned} D^-(W_i(t)) &\le b_{ii}(t)[e^{-x_i(t)} - e^{-x_i^*(t)}]\operatorname{sgn}(x_i(t) - x_i^*(t))e^{\lambda t} \\ &+ \sum_{j=1, j\neq i}^{n} b_{ij}(t)|e^{-x_j(t)} - e^{-x_j^*(t)}|e^{\lambda t} \\ &+ \sum_{j=1}^{m} \beta_{ij}(t)\big|x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))} \\ &\qquad - x_i^*(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i^*(t-\tau_{ij}(t))}\big|e^{\lambda t} \\ &+ \lambda|z_i(t)|e^{\lambda t}, \text{ for all } t > T_{\varphi,\varphi^*},\ i \in Q. \end{aligned}$$

Now we assert that

$$W_i(t) < e^{\lambda T_{\varphi,x^*}}\Big(\max_{j\in Q}\Big\{\max_{t\in[t_0-\sigma_j,\ T_{\varphi,x^*}]}|x_j(t) - x_j^*(t)|\Big\} + 1\Big) := M_{\varphi,x^*}$$

for all $t > T_{\varphi,\varphi^*}$, $i \in Q$. Otherwise, we can choose $\bar{i} \in Q$ and $T_*^{\bar{i}} > T_{\varphi,x^*}$ such that

$$(3.7)\qquad W_{\bar{i}}(T_*^{\bar{i}}) = M_{\varphi,x^*} \quad \text{and} \quad W_j(t) < M_{\varphi,x^*}$$

for all $t \in [t_0 - \sigma_j, T_*^{\bar{i}})$, $j \in Q$. With the help of (3.2), (3.5), and Lemma 2.2, one can show the following inequalities:

$$\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon) \le \gamma_{\bar{i}j}(T_*^{\bar{i}})x(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}})), \qquad \gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}^*(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}})),\ j \in I,$$

$$(e^{-s} - e^{-t})\operatorname{sgn}(s-t) \le -e^{-(S^+ + \varepsilon)}|s-t|, \quad |e^{-s} - e^{-t}| \le e^{-(S_- - \varepsilon)}|s-t|, \quad \text{where } s, t \in [S_- - \varepsilon, S^+ + \varepsilon],$$

$$|se^{-s} - te^{-t}| \le \max\Big\{\frac{1}{e^2}, \frac{1 - \gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon)}}\Big\}|s-t|, \quad \text{where } s, t \in [\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon),\ +\infty),\ j \in I.$$

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This, together with (3.6) and (3.7), gives us that

$$
\begin{aligned}
0 &\le D^-(W_{\bar{i}}(T_*^{\bar{i}}))\\
&\le b_{\bar{i}\,\bar{i}}(T_*^{\bar{i}})[e^{-x_{\bar{i}}(T_*^{\bar{i}})} - e^{-x_{\bar{i}}^*(T_*^{\bar{i}})}]\operatorname{sgn}(x_{\bar{i}}(T_*^{\bar{i}}) - x_{\bar{i}}^*(T_*^{\bar{i}}))e^{\lambda T_*^{\bar{i}}}\\
&\quad + \sum_{j=1, j\neq \bar{i}}^{n} b_{\bar{i}j}(T_*^{\bar{i}})|e^{-x_j(T_*^{\bar{i}})} - e^{-x_j^*(T_*^{\bar{i}})}|e^{\lambda T_*^{\bar{i}}}\\
&\quad + \sum_{j=1}^{m}\beta_{\bar{i}j}(T_*^{\bar{i}})|x_{\bar{i}}(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}}))e^{-\gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}(T_*^{\bar{i}}-\tau_{\bar{i}j}(T_*^{\bar{i}}))}\\
&\quad - x_{\bar{i}}^*(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}}))e^{-\gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}^*(T_*^{\bar{i}}-\tau_{\bar{i}j}(T_*^{\bar{i}}))}|e^{\lambda T_*^{\bar{i}}} + \lambda|z_{\bar{i}}(T_*^{\bar{i}})|e^{\lambda T_*^{\bar{i}}}\\
&\le -[b_{\bar{i}\,\bar{i}}(T_*^{\bar{i}})e^{-(S^+ +\varepsilon)} - \lambda]|z_{\bar{i}}(T_*^{\bar{i}})|e^{\lambda T_*^{\bar{i}}} + \sum_{j=1, j\neq \bar{i}}^{n} b_{\bar{i}j}(T_*^{\bar{i}})e^{-(S_- -\varepsilon)}|z_j(T_*^{\bar{i}})|e^{\lambda T_*^{\bar{i}}}\\
&\quad + \sum_{j=1}^{m}\frac{\beta_{\bar{i}j}(T_*^{\bar{i}})}{\gamma_{\bar{i}j}(T_*^{\bar{i}})}|\gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}}))e^{-\gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}(T_*^{\bar{i}}-\tau_{\bar{i}j}(T_*^{\bar{i}}))}\\
&\quad - \gamma_{\bar{i}j}(T_*^{\bar{i}})x_{\bar{i}}^*(T_*^{\bar{i}} - \tau_{\bar{i}j}(T_*^{\bar{i}}))e^{-\gamma_{\bar{i}j}(T_*^{i})x_i^*(T_*^{\bar{i}}-\tau_{\bar{i}j}(T_*^{\bar{i}}))}|e^{\lambda T_*^{\bar{i}}}\\
&\le -[b_{\bar{i}\,\bar{i}}(T_*^{\bar{i}})e^{-(S^+ +\varepsilon)} - \lambda]|z_{\bar{i}}(T_*^{\bar{i}})|e^{\lambda T_*^{\bar{i}}} + \sum_{j=1, j\neq \bar{i}}^{n} b_{\bar{i}j}(T_*^{\bar{i}})e^{-(S_- -\varepsilon)}|z_j(T_*^{\bar{i}})|e^{\lambda T_*^{\bar{i}}}\\
&\quad + \sum_{j=1}^{m}\beta_{\bar{i}j}(T_*^{\bar{i}})\max\Big\{\frac{1}{e^2}, \frac{1-\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{\bar{i}j}^{\inf}(S_- -\varepsilon)}}\Big\}|z_{\bar{i}}(T_* - \tau_{\bar{i}j}(T_*^{\bar{i}}))|e^{\lambda(T_*^{\bar{i}}-\tau_{\bar{i}j}(T_*^{\bar{i}}))}e^{\lambda\tau_{\bar{i}j}(T_*^{\bar{i}})}\\
&\le \Big\{-[b_{\bar{i}\,\bar{i}}(T_*^{\bar{i}})e^{-(S^+ +\varepsilon)} - \lambda] + \sum_{j=1, j\neq \bar{i}}^{n} b_{\bar{i}j}(T_*^{\bar{i}})e^{-(S_- -\varepsilon)}\\
&\quad + \sum_{j=1}^{m}\beta_{\bar{i}j}(T_*^{\bar{i}})\max\Big\{\frac{1}{e^2}, \frac{1-\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{\bar{i}j}^{\inf}(S_- -\varepsilon)}}\Big\}e^{\lambda\sigma_{\bar{i}}}\Big\}M_{\varphi,\varphi^*},
\end{aligned}
$$

from which, together with (2.10), we derive that

$$
\begin{aligned}
0 \le -[b_{\bar{i}\,\bar{i}}(T_*^{\bar{i}})e^{-(S^+ +\varepsilon)} - \lambda] &+ \sum_{j=1, j\neq \bar{i}}^{n} b_{\bar{i}j}(T_*^{\bar{i}})e^{-(S_- -\varepsilon)}\\
&+ \sum_{j=1}^{m}\beta_{\bar{i}j}(T_*^{\bar{i}})\max\Big\{\frac{1}{e^2}, \frac{1-\gamma_{\bar{i}j}^{\inf}(S_- - \varepsilon)}{e^{\gamma_{\bar{i}j}^{\inf}(S_- -\varepsilon)}}\Big\}e^{\lambda\sigma_{\bar{i}}} < 0.
\end{aligned}
$$

This is a clear contradiction and proves the above assertion. Hence,

$$|z_i(t)| < M_{\varphi,x^*}e^{-\lambda t} \quad \text{for all } t > T_{\varphi,x^*},\ i \in Q,$$

which finishes the proof of Theorem 3.1. ∎

4 A Numerical Example

Example 4.1 Let us consider system (1.2) involving the following parameters:

$$
(4.1)\qquad
\begin{aligned}
&n = m = 2, a_{11}(t) = e^{-(2+|\cos\sqrt{2}t|)},\ b_{11}(t) = 10.1 + 10.1\cos^2 t,\\
&a_{12}(t) = (0.2 + 0.2\cos t)e^{-(2+|\cos t|)},\ b_{12}(t) = 0.01 + 0.01\cos^2 t,\\
&\beta_{11}(t) = \tfrac{1+\cos t}{1000}, \beta_{12}(t) = \tfrac{1+\sin t}{2000},\ \gamma_{11}(t) = \gamma_{12}(t) = 0.5,\\
&\tau_{11}(t) = 2|\sin\sqrt{5}t|, \tau_{12}(t) = 3|\sin\sqrt{5}t|,\\
&a_{22}(t) = e^{-(2+|\sin\sqrt{3}t|)},\ b_{22}(t) = 20.2 + 20.2\sin^2 t,\\
&a_{21}(t) = (0.2 + 0.2\sin t)e^{-(2+|\sin t|)},\ b_{21}(t) = 0.02 + 0.02\sin^2 t,\\
&\gamma_{21}(t) = \gamma_{22}(t) = 0.5, \beta_{21}(t) = \tfrac{1+\cos t}{2000}, \beta_{22}(t) = \tfrac{1+\sin t}{3000},\\
&\tau_{21}(t) = 2|\cos\sqrt{7}t|, \tau_{22}(t) = 3|\cos\sqrt{7}t|.
\end{aligned}
$$

Obviously, it is observed that

$$
S_- = \min_{1\le i\le 2}\left\{\liminf_{t\to+\infty}\ln\left(\frac{b_{ii}(t) - \sum_{j=1,j\ne i}^n b_{ij}(t)}{a_{ii}(t) - \sum_{j=1,j\ne i}^n a_{ij}(t)}\right)\right\} \approx 4.1,
$$

$$
S^+ = \max_{1\le i\le 2}\left\{\limsup_{t\to+\infty}\ln\left(\frac{b_{ii}(t) - \sum_{j=1,j\ne i}^n b_{ij}(t)}{a_{ii}(t) - \sum_{j=1,j\ne i}^n \left(a_{ij}(t) + \sum_{j=1}^m \frac{1}{e}\frac{\beta_{ij}(t)}{\gamma_{ij}(t)}\right)}\right)\right\} \approx 7.1,
$$

$$
\max_{t\in\mathbb{R}}\left\{ -b_{ii}(t)e^{-S^+} + \sum_{j=1,j\ne i}^{2} b_{ij}(t)e^{-S_-} + \sum_{j=1}^{2}\beta_{ij}(t)\max\left\{\frac{1}{e^2}, \frac{1-\gamma_{ij}^{\inf}S_-}{e^{\gamma_{ij}^{\inf}S_-}}\right\}\right\}
\approx -0.05 < 0,\ \ i = 1, 2,
$$

which suggest that (4.1) satisfies all assumptions adopted in Theorem 3.1. Thus, by Theorem 3.1, we know that system (1.2) with parameters (4.1) has a unique almost periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))$ that is globally exponentially stable (see Figure 1), and $x_i^*(t) \ge S_- > 4$ for all $t \in \mathbb{R}$ and $i = 1, 2$.

Remark 4.2 It should be mentioned that the assumptions

$$
\gamma_{ij}(t) \ge 1,\ \text{for all } t \in \mathbb{R}, i \in Q, j \in I,
$$

and

$$
\inf_{t\ge 0}\{1 - \tau'_{ij}(t)\} = \mu > 0,\ \text{for all } t \in \mathbb{R}, i \in Q, j \in I,
$$

have been adopted as fundamental to showing the stability of periodic and almost periodic solutions for Nicholson's blowflies models in [3,16,17,22,23,25] and [20], respectively. In particular, the results on periodic scalar Nicholson's blowflies model in [24] give no opinions about the problem of almost periodic solutions of Nicholson's blowflies systems involving patch structure and nonlinear density-dependent mortality terms. Clearly, the parameters $\gamma_{ij}(t) = \frac{1}{2},\ i, j = 1, 2$, and $\tau_{11}(t) = 2|\sin\sqrt{5}t|$,

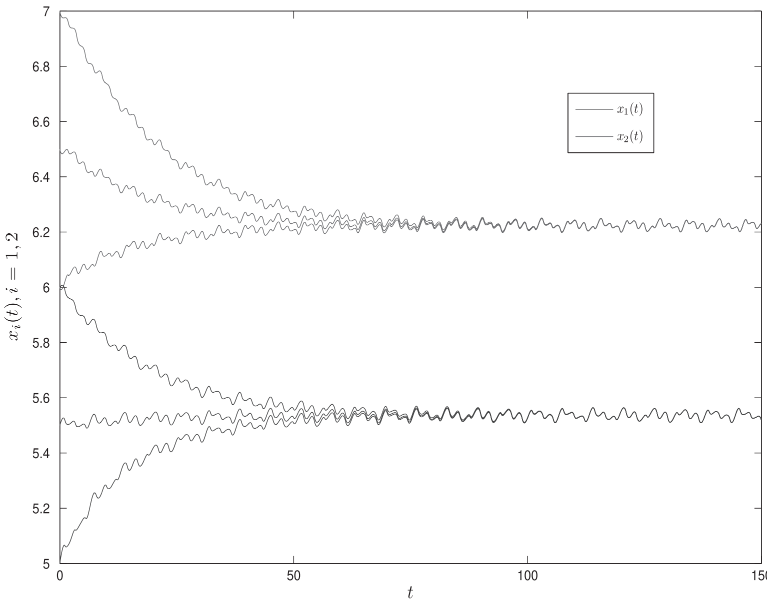

Figure 1: Numerical solutions of (4.1) for initial value $(\varphi_1(s), \varphi_2(s)) = (5,6), (6,7), (5.5,6.5)$, $s \in [-3,0]$.

$\tau_{12}(t) = 3|\sin\sqrt{5}t|$, $\tau_{21}(t) = 2|\cos\sqrt{7}t|$, $\tau_{22}(t) = 3|\cos\sqrt{7}t|$ do not satisfy the above assumptions. Moreover, the fact that

$$x_i^*(t) \geq S_- > 4 > \widetilde{\kappa} \text{ for all } t \in \mathbb{R}, i = 1,2,$$

entails that $x^*(t)$ is out of $[\kappa,\widetilde{\kappa}] \times [\kappa,\widetilde{\kappa}]$. Hence, all the results in [1–4,16,17,20,22–25] cannot be used to show the global exponential stability on the positive almost periodic solution of system (1.1) involving parameters (4.1).

5 Conclusions

In this paper, we combine the Lyapunov function method with the differential inequality method to establish some new criteria ensuring the existence and exponential stability of positive almost periodic solutions for a class of delayed Nicholson's blowflies systems with patch structure and nonlinear density-dependent mortality terms. These criteria are obtained without assuming that

$$\underbrace{[\kappa,\widetilde{\kappa}] \times \cdots \times [\kappa,\widetilde{\kappa}]}_{n} \approx \underbrace{[0.7215355, 1.342276] \times \cdots \times [0.7215355, 1.342276]}_{n}$$

is the existence region of almost periodic solutions, and the homologous results in the recently published literature are summarized and refined. The approach presented in this article can be used as a possible way to study the patch structure population models with nonlinear density-dependent mortality terms, for example, the neoclassical growth model, the Mackey–Glass model, epidemical systems or age-structured population models, and so on.

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