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$2^{\aleph_0}$ PAIRWISE NONISOMORPHIC MAXIMAL-CLOSED SUBGROUPS OF SYM($\mathbb{N}$) VIA THE CLASSIFICATION OF THE REDUCTS OF THE HENSON DIGRAPHS

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Abstract. Given two structures $\mathcal{M}$ and $\mathcal{N}$ on the same domain, we say that $\mathcal{N}$ is a reduct of $\mathcal{M}$ if all $\emptyset$-definable relations of $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$. In this article the reducts of the Henson digraphs are classified. Henson digraphs are homogeneous countable digraphs that omit some set of finite tournaments. As the Henson digraphs are $\aleph_0$-categorical, determining their reducts is equivalent to determining the closed supergroups $G \leq \mathrm{Sym}(\mathbb{N})$ of their automorphism groups.

A consequence of the classification is that there are $2^{\aleph_0}$ pairwise noninterdefinable Henson digraphs which have no proper nontrivial reducts. Taking their automorphisms groups gives a positive answer to a question of Macpherson that asked if there are $2^{\aleph_0}$ pairwise nonconjugate maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$. By the reconstruction results of Rubin, these groups are also nonisomorphic as abstract groups.

This article contributes to the large body of work concerning the two intimately related topics of reducts of countable structures and of closed subgroups of $\mathrm{Sym}(\mathbb{N})$. Motivation for this work comes from both areas.

In the topic of reducts, the reducts of the Henson digraphs are classified up to first order interdefinability. To our knowledge this is the first time the reducts of uncountably many homogeneous structures have been classified. In all cases only finitely many reducts appear. This result supports a conjecture of Thomas in [26] which says that all countable homogeneous structures in a finite relational language have only finitely many reducts. Evidence for this conjecture has been building as there have been numerous classification results, e.g., [1, 6, 11, 16, 23, 25, 26]. On the other hand, recent work gives evidence that the conjecture is false. In [8], it is shown that the countable homogeneous Boolean-algebra has infinitely many reducts. (This is not a counter-example to Thomas' Conjecture because the structure is homogeneous in a functional language, not a relational one.) This conjecture is unresolved and continues to provide motivation for study.

The main tool used in this classification of the reducts of the Henson digraphs is that of the so-called 'canonical functions'. This Ramsey-theoretic tool was developed by Bodirsky and Pinsker to help analyse certain closed clones in relation to constraint satisfaction problems, a topic in theoretical computer science. With further developments [4, 7], canonical functions have become powerful tools in

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studying reducts. The robustness and relative ease of the methodology is becoming more evident as several classifications have been achieved by their use, e.g., [1, 2, 6, 10, 19, 23].

In the topic of permutation groups, the main result of this article positively answers a question of Macpherson, Question 5.10 in [3], which asked whether there are $2^{\aleph_0}$ pairwise nonconjugate maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ with $\mathrm{Sym}(\mathbb{N})$ bearing the pointwise convergence topology. Several related questions have recently been tackled. Independently, [3] and [9] showed that there exist nonoligomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$, the existence of which was asked in [16]. Also, independently, [17] and [9] positively answered Macpherson's question of whether there are maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ of countable cardinality. One question that remains open is whether every proper closed subgroup of $\mathrm{Sym}(\mathbb{N})$ is contained in a maximal-closed subgroup of $\mathrm{Sym}(\mathbb{N})$, (Question 7.7 in [21] and Question 5.9 in [3]).

The description of $2^{\aleph_0}$ maximal-closed subgroups follows from the classification of reducts by taking the automorphism groups of a suitably modified version of Henson's [13] construction of $2^{\aleph_0}$ pairwise nonisomorphic countable homogeneous digraphs. A short argument shows that their automorphism groups are pairwise nonconjugate. However, we can say more: by Rubin's work on reconstruction [24], the automorphism groups will be pairwise nonisomorphic as abstract groups.

We outline the structure of the article. In Section 1, we provide the necessary preliminary definitions and facts on the Henson digraphs, reducts, and canonical functions. We also describe various notational conventions that we use. In Section 2, we prove the classification of the reducts of the Henson digraphs. In Section 2.1 we state the classification. In Section 2.2 we describe the reducts, establishing notation and important lemmas that are used in the rest of the article. In Section 2.3 we carry out the combinatorial analysis of the possible behaviours of canonical functions. 2.4 contains the proof of the classification. Section 3 contains the denouement of the article: the classification is used to show that there exist $2^{\aleph_0}$ pairwise nonisomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

For those who are primarily interested in the construction of the maximal-closed subgroups, from Section 1 you can skip Section 1.5 and from Section 2 you can skip everything after Lemma 2.4.

§1. Preliminaries.

1.1. Conventions. If A is a subset of D, A^c denotes the complement of A in D. We sometimes write 'ab' as an abbreviation for (a, b), e.g., we may write "Let ab be an edge of the digraph D". Structures are denoted by $\mathcal{M}, \mathcal{N}$, and their domains are M and N, respectively. All of the structures that appear in our article are structures in a relational language; constant symbols will implicitly be interpreted as unary, singleton relations. $\mathrm{Sym}(M)$ is the set of all bijections $M \to M$ and $\mathrm{Aut}(\mathcal{M})$ is the set of all automorphisms of $\mathcal{M}$. Given a formula $\phi(x, y)$, we use $\phi^*(x, y)$ to denote the formula $\phi(y, x)$. $S(\mathcal{M})$ denotes the space of types of the theory of $\mathcal{M}$. If f has domain A and $(a_1, \dots, a_n) \in A^n$, then $f(a_1, \dots, a_n) := (f(a_1), \dots, f(a_n))$. For tuples $\bar{a}, \bar{b} \in M^n$, we say $\bar{a}$ and $\bar{b}$ are isomorphic, and write $\bar{a} \cong \bar{b}$, to mean that

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the function $a_i \mapsto b_i$ for all i such that $1 \leq i \leq n$ is an isomorphism. We say $\bar{a}$ is a *proper* tuple if all the elements of $\bar{a}$ are pairwise distinct.

There will be instances where we do not adhere to strictly correct notational usage, however, the meaning should be clear from the context. We highlight some examples. We write '$a \in (a_1, \dots, a_n)$' instead of '$a = a_i$ for some i such that $1 \leq i \leq n$'. We write 'Let $\bar{a} \in A$' instead of 'Let $\bar{a} \in A^n$, where n is the length of $\bar{a}$'. Another example is that we sometimes use c to represent the singleton set $\{c\}$ containing it.

1.2. Henson digraphs. A directed graph (V, E), or digraph for short, is a set V with an irreflexive antisymmetric relation $E \subseteq V^2$. V is the set of vertices, E is the set of edges and we visualise an element $(a, b) \in E$ as being an edge going out of a and into b. We say a digraph is edgeless if $E = \emptyset$. By L_n we denote the linear order on n-elements, regarded as a digraph.

A tournament is a digraph in which there is an edge between every pair of distinct vertices. Throughout this article, $\mathcal{T}$ will denote a set of finite tournaments. We will often refer to elements of $\mathcal{T}$ as forbidden tournaments.

Definition 1.1. (i) A (relational) structure $\mathcal{M}$ is *homogeneous* if every isomorphism $f : A \to B$ between finite substructures A, B of $\mathcal{M}$ can be extended to an automorphism $g \in \mathrm{Aut}(\mathcal{M})$.

(ii) For a (relational) structure $\mathcal{M}$, the *age* of $\mathcal{M}$, $\mathrm{Age}(\mathcal{M})$, is the class of finite structures embeddable in $\mathcal{M}$.

(iii) Let $\mathcal{T}$ be a set of finite tournaments. We let $\mathrm{Forb}(\mathcal{T})$ be the class of finite digraphs D such that for all $T \in \mathcal{T}$, D does not embed T.

(iv) If $\mathcal{T}$ does not contain the 1-element tournament, we let $(D_\mathcal{T}, E_\mathcal{T})$ be the unique (up to isomorphism) countable homogeneous digraph whose age is $\mathrm{Forb}(\mathcal{T})$.

(v) A *Henson digraph* is a digraph isomorphic to $(D_\mathcal{T}, E_\mathcal{T})$ where $\mathcal{T}$ is nonempty and does not contain the 1- or 2-element tournament.

The fact that $(D_\mathcal{T}, E_\mathcal{T})$ exists and is unique follows from the general Fraïssé theory of amalgamation classes, developed by Fraïssé in [12]. This particular construction of digraphs was used by Henson in [13] to show there exists uncountably many countable homogeneous digraphs. An accessible account on the theory of amalgamation classes can be found in [14]; a survey on homogeneous structures can be found in [20].

If $\mathcal{T} = \emptyset$ then $(D_\mathcal{T}, E_\mathcal{T})$ is the generic digraph, the unique countable homogeneous digraph that embeds all finite digraphs. The reducts of the generic digraph are classified in [1]. If $\mathcal{T}$ contains the 1-element tournament, then $\mathrm{Forb}(\mathcal{T}) = \emptyset$. If $\mathcal{T}$ contains the 2-element tournament, then $(D_\mathcal{T}, E_\mathcal{T})$ is the countable edgeless digraph. These are degenerate cases which is why we defined the term Henson digraph to exclude these options.

Lemma 1.2. *Let (D, E) be a Henson digraph.*

(i) *$Th(D, E)$ is $\aleph_0$-categorical.*

(ii) *Let (D', E') be a digraph such that $Age(D', E') \subseteq Age(D, E)$. Then (D', E') is embeddable in (D, E).*

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(iii) (D, E) *is connected*: *for every distinct* $a, b \in D$, *there is a path from* a *to* b *or from* b *to* a. (*In fact, an oriented path of length at most two.*)

PROOF. (i) The theory of any homogeneous structure in a finite relational language is $\aleph_0$-categorical. See [14, Theorem 6.4.1] for a proof.

(ii) This follows by using only the 'forth' part of a back-and-forth argument.

(iii) Let $a, b \in D$ be distinct and without loss of generality suppose that there is no edge between a and b. Consider the finite digraph $\{a', b', c'\}$ such that there is no edge between a' and b', and there are edges from a' to c' and from c' to b'. Observe that $\{a', b', c'\}$ lies in $\mathrm{Forb}(\mathcal{T})$, so is embeddable in (D, E). By the homogeneity of (D, E), we map a' to a and b' to b to obtain a $c \in D$ with $E(a, c)$ and $E(c, b)$. ⊣

In order to use the canonical functions machinery, we need to expand the Henson digraphs to ordered digraphs. This is described in the following definition.

DEFINITION 1.3. (i) An *ordered digraph* is a digraph which is also linearly ordered. Formally, it is a structure $(V, E, <)$ where (V, E) is a digraph and $(V, <)$ is a linear order.

(ii) We let $(D_\mathcal{T}, E_\mathcal{T}, <)$ be the unique (up to isomorphism) countable homogeneous ordered digraph such that a finite ordered digraph $(D, E, <)$ is embeddable in $(D_\mathcal{T}, E_\mathcal{T}, <)$ iff $(D, E) \in \mathrm{Forb}(\mathcal{T})$.

(iii) We say $(D, E, <)$ is a *Henson ordered digraph* if $(D, E, <) \cong (D_\mathcal{T}, E_\mathcal{T}, <)$ for some $\mathcal{T}$.

FACT 1.4. *All Henson ordered digraphs are Ramsey structures.*

This fact follows by applying the main theorem of [22]; additionally, the fact is stated in [15]. For the purposes of this article, it is not necessary to know what it means to be a Ramsey structure. The definition and examples of Ramsey structures can be found in [15] and references therein. The importance of the Ramsey property and of introducing ordered digraphs will become evident in Section 1.5.

1.3. Reducts. Let $\mathcal{M}, \mathcal{N}$ be two structures on the same domain M. We say $\mathcal{N}$ is a *reduct* of $\mathcal{M}$ if all $\emptyset$-definable relations in $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$. We say $\mathcal{N}$ is a proper reduct of $\mathcal{M}$ if $\mathcal{N}$ is a reduct of $\mathcal{M}$ but $\mathcal{M}$ is not a reduct of $\mathcal{N}$. In this article, if two structures $\mathcal{M}$ and $\mathcal{N}$ are both reducts of each other, we consider them to be the same structure.

For any structure $\mathcal{M}$, the reducts of $\mathcal{M}$ form a lattice where $\mathcal{N} \leq \mathcal{N}'$ if $\mathcal{N}$ is a reduct of $\mathcal{N}'$. In addition to classifying the reducts of a Henson digraph, the lattice they form is also determined.

1.4. The topology of $\mathrm{Sym}(\mathbb{N})$ **and** $\mathbb{N}^\mathbb{N}$**.** Let $F \subseteq \mathbb{N}^\mathbb{N}$ and $g \in \mathbb{N}^\mathbb{N}$. We say g is in the *closure* of F, $\mathrm{cl}(F)$, if for all finite $A \subset \mathbb{N}$ there is $f \in F$ such that $f(a) = g(a)$ for all $a \in A$. We say F is *closed* if $F = \mathrm{cl}(F)$. This defines the so-called pointwise convergence topology on $\mathbb{N}^\mathbb{N}$. Equipped with this topology, $\mathbb{N}^\mathbb{N}$ becomes a topological monoid. $\mathrm{Sym}(\mathbb{N})$ inherits this topology, via the subspace topology, and becomes a topological group (in fact, a Polish group).

As a consequence of the theorem of Engeler, Ryll-Nardzewski, and Svenonius (see [14, Theorem 6.3.1]), if $\mathcal{M}$ is a countable $\aleph_0$-categorical structure then the lattice of reducts is antiisomorphic to the lattice of closed groups G such that $\mathrm{Aut}(\mathcal{M}) \leq G \leq \mathrm{Sym}(M)$. This means that determining the lattice of reducts of the

Henson digraphs is equivalent to determining the lattice of closed supergroups of the automorphism groups of the Henson digraphs.

For $F \subseteq \mathbb{N}^{\mathbb{N}}$, we let $\mathrm{cl}_{\mathrm{tm}}(F)$ denote the smallest topologically closed submonoid of $\mathbb{N}^{\mathbb{N}}$ containing F, and we let $\langle F \rangle$ denote the smallest topologically closed subgroup of $\mathrm{Sym}(\mathbb{N})$ containing F.

1.5. Canonical functions.

Definition 1.5. Let $\mathcal{M}, \mathcal{N}$ be any structures. Let $f : M \to N$ be any function between the domains of the structures.

(i) The *behaviour* of f is the relation $\{(p, q) \in S(\mathcal{M}) \times S(\mathcal{N}) : \exists \bar{a} \in M, \bar{b} \in N$ such that $\mathrm{tp}(\bar{a}) = p, \mathrm{tp}(\bar{b}) = q$ and $f(\bar{a}) = \bar{b}\}$.
(ii) If the behaviour of f is a function $S(M) \to S(N)$, then we say f is *canonical*. Rephrased, we say f is canonical if for all $\bar{a}, \bar{a}' \in M$, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}') \Rightarrow \mathrm{tp}(f(\bar{a})) = \mathrm{tp}(f(\bar{a}'))$.
(iii) If f is canonical, we use the same symbol f to denote its behaviour.

For example, for any structure $\mathcal{M}$, every automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is a canonical function, and for all types $p \in S(\mathcal{M})$, $f(p) = p$. A useful property that is used implicitly in several arguments is that the composition of canonical functions is canonical.

The benefit of canonical functions is that they are particularly well-behaved and can be easily manipulated and analysed. Furthermore, the next theorem essentially reduces the task of determining reducts to the task of analysing the behaviours of canonical functions.

Theorem 1.6. *Let $(D, E, <)$ be a Henson ordered digraph. Let $f \in \mathrm{Sym}(D)$ and $c_1, \dots, c_n \in D$ be any vertices. Then there exists a function $g : D \to D$ such that*

(i) $g \in cl_{tm}(\mathrm{Aut}(D, E) \cup \{f\})$.
(ii) $g(c_i) = f(c_i)$ *for* $i = 1, \dots, n$.
(iii) *When regarded as a function from $(D, E, <, \bar{c})$ to (D, E), g is a canonical function.*

The above theorem holds in general for all homogeneous Ramsey structures, and as discussed earlier, Henson ordered digraphs are indeed Ramsey structures. In its general version, Theorem 1.6 was first proved in [7, Lemma 14]. A shorter proof that uses topological dynamics can be found in [5].

§2. Classification of the reducts.

For this section, we fix a Henson ordered digraph $(D, E, <)$ and let $\mathcal{T}$ be its set of forbidden tournaments.

2.1. Statement of the classification.

Definition 2.1. (i) Recall that for $F \subseteq \mathrm{Sym}(D)$, $\langle F \rangle$ denotes the smallest closed subgroup of $\mathrm{Sym}(D)$ containing F. For brevity, when it is clear we are discussing supergroups of $\mathrm{Aut}(D, E)$, we may abuse notation and write $\langle F \rangle$ to mean $\langle F \cup \mathrm{Aut}(D, E) \rangle$.
(ii) We let $\overline{E}(x, y)$ denote the underlying (undirected) graph relation $E(x, y) \vee E(y, x)$. We let $N(x, y)$ denote the nonedge relation $\neg \overline{E}(x, y)$.

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(iii) A *Henson graph* is the Fraïssé limit of the class of all finite K_n-free graphs, for some integer $n \geq 3$.
(iv) Assume (D, E) is isomorphic to the digraph obtained by changing the direction of all its edges. In this case $- \in \mathrm{Sym}(D)$ will denote a bijection $D \to D$ such that for all $x, y \in D$, $E(-(x), -(y))$ iff $E(y, x)$.
(v) Assume (D, E) is isomorphic to the digraph obtained by changing the direction of all the edges adjacent to one particular vertex of D. In this case $\mathrm{sw} \in \mathrm{Sym}(D)$ will denote a bijection $D \to D$ such that for some $a \in D$:

$$E(\mathrm{sw}(x), \mathrm{sw}(y)) \text{ if and only if } \begin{cases} E(x, y) \text{ and } x, y \neq a, \text{ OR}, \\ E(y, x) \text{ and } x = a \vee y = a. \end{cases}$$

In words, $-$ is a function which changes the direction of all the edges of the digraph and sw is a function which changes the direction of those edges adjacent to one particularly vertex. Note that $-$ is not necessarily an involution, however, $-$ can be chosen to be an involution (via a back-and-forth argument). The existence of $-$ or sw depends on which tournaments are forbidden; see Lemma 2.4. This explains the wording of Theorem 2.2(iii): if, for example, $-$ exists but sw does not, then $\max\{\mathrm{Aut}(D, E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\} = \langle - \rangle$. Also, note that the groups $\langle - \rangle$ and $\langle \mathrm{sw} \rangle$ are independent from the choice of the specific functions $-$ or sw; again see Lemma 2.4.

THEOREM 2.2. *Let (D, E) be a Henson digraph and let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of* $\mathrm{Aut}(D, E)$. *Then*

(i) $G \leq \mathrm{Aut}(D, \overline{E})$ *or* $G \geq \mathrm{Aut}(D, \overline{E})$.
(ii) *If* $G < \mathrm{Aut}(D, \overline{E})$ *then* $G = \mathrm{Aut}(D, E), \langle - \rangle, \langle \mathrm{sw} \rangle$ *or* $\langle -, \mathrm{sw} \rangle$.
(iii) $(D, \overline{E})$ *is the random graph,* $(D, \overline{E})$ *is a Henson graph or* $(D, \overline{E})$ *is not homogeneous. In the last case* $\mathrm{Aut}(D, \overline{E})$ *is equal to* $\max\{\mathrm{Aut}(D, E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\}$ *and is a maximal-closed subgroup of* $\mathrm{Sym}(D)$.

The reducts of the random graph and the Henson graphs were classified by Thomas in [25]. If $(D, \overline{E})$ is the random graph, its proper reducts are $\langle -_\Gamma \rangle, \langle \mathrm{sw}_\Gamma \rangle, \langle -_\Gamma, \mathrm{sw}_\Gamma \rangle$ and $\mathrm{Sym}(D)$, where $-_\Gamma \in \mathrm{Sym}(D)$ is a bijection which maps every edge to a nonedge and every nonedge to an edge and sw_Γ is a bijection which does the same but only for those edges adjacent to a particular vertex $a \in D$. A Henson graph has only two reducts, its automorphism group and the full symmetric group. As an immediate consequence we get the following corollary of Theorem 2.2:

COROLLARY 2.3. *Let (D, E) be a Henson digraph. Then its lattice of reducts is a sublattice of the lattice in Figure* 1. *In particular, the lattice of reducts of (D, E) is* (*isomorphic to*) *a sublattice of the lattice of reducts of the generic digraph* ([1]).

2.2. Understanding the reducts. In this section, we establish several important lemmas that play prominent roles in the proof of the main theorem. We omit most of the proofs of the lemmas. This is because they are relatively straightforward and are identical to the lemmas in [1, Section 3]. Before we delve into the lemmas, we describe some terminology.

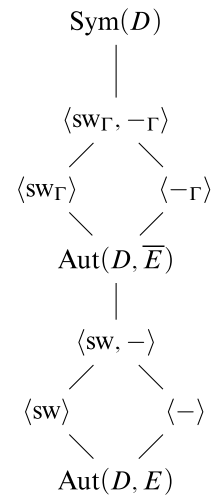

FIGURE 1. Lattice of reducts.

- Let $f, g : D \to D$ and $A \subseteq D$. We say f *behaves like* g *on* A if for all finite tuples $\bar{a} \in A$, $f(\bar{a})$ is isomorphic (as a finite digraph) to $g(\bar{a})$. If $A = D$, we simply say f *behaves like* g.
- Let A, B be disjoint subsets of D. We say f *behaves like* sw *between* A *and* B if f switches the direction of all edges between A and B and preserves all nonedges between A and B.
- Let $A \subseteq D$. We let $\mathrm{sw}_A : D \to D$ denote a function that behaves like id on A and A^c and that behaves like sw between A and A^c. Note that the existence of sw_A will depend on A and on $\mathcal{T}$.
- We overload the symbols $-$ and sw by letting them denote actions on finite tournaments. We say $\mathcal{T}$ *is closed under* $-$ if for every $T \in \mathcal{T}$, the tournament obtained from T by changing the direction of all its edges is in $\mathcal{T}$. We say $\mathcal{T}$ *is closed under* sw if for every $T \in \mathcal{T}$ and $t \in T$, the tournament obtained by changing the direction of those edges adjacent to t is in $\mathcal{T}$.

REMARK. The terminology is somewhat unlucky as the notions of 'behaviour' and 'behaving like' do not match exactly. If f and g are functions $D \to D$ and f behaves like g, then f and g have the same behaviour. The issue is that the converse is not true in general; it is possible that f and g have the same behaviour but f does not behave like g. Note that the converse *is* true if f and g are canonical.

LEMMA 2.4. (i) $- : D \to D$ *exists if and only if* $\mathcal{T}$ *is closed under* $-$.
(ii) $\mathrm{sw} : D \to D$ *exists if and only if* $\mathcal{T}$ *is closed under* sw.
(iii) *If* sw *exists, then for all* $A \subseteq D$, sw_A *exists.*
(iv) $\langle - \rangle \supseteq \{f \in \mathrm{Sym}(D) : f$ *behaves like* $-\}$.
(v) $\langle \mathrm{sw} \rangle \supseteq \{f \in \mathrm{Sym}(D) :$ *there is* $A \subseteq D$ *such that* f *behaves like* $\mathrm{sw}_A\}$.

PROOF. (i) 'LHS $\Rightarrow$ RHS': Suppose $-$ exists. To show $\mathcal{T}$ is closed under $-$, it suffices to show that if $T \notin \mathcal{T}$, then $-(T) \notin \mathcal{T}$. So suppose a finite tournament T

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is not in $\mathcal{T}$. Then T is embeddable in (D, E). Then applying $-$ shows that $-(T)$ is embeddable in (D, E), i.e., that $-(T) \notin \mathcal{T}$.

'RHS $\Rightarrow$ LHS': To show $-$ exists, we need to show that (D, E^*) is isomorphic to (D, E). (Recall that $\phi^*(x, y) := \phi(y, x)$.) By the uniqueness of Fraïssé limits, it suffices to show that (D, E^*) is homogeneous and that $\mathrm{Age}(D, E^*) = \mathrm{Age}(D, E)$. That the ages are equal follows from the assumption that $\mathcal{T}$ is closed under $-$. That (D, E^*) is homogeneous follows from the observation that for all $A, B \subseteq D$ and all $f : A \to B$, $f : (A, E|_A) \to (B, E|_B)$ is an isomorphism if and only if $f : (A, E^*|_A) \to (B, E^*|_B)$ is an isomorphism.

(ii) 'LHS $\Rightarrow$ RHS': Apply the same argument as in (i) to prove this.

'RHS $\Rightarrow$ LHS': Let $a \in D$, $X_{\mathrm{out}} = \{x \in D : E(a, x)\}$ and $X_{\mathrm{in}} = \{x \in D : E(x, a)\}$. Suppose we found an isomorphism $f : (X_{\mathrm{out}}, E) \to (X_{\mathrm{in}}, E)$. Then we can define sw as the function which maps a to a, maps elements of X_{out} using f and maps elements of X_{in} using f^{-1}. Thus to complete this proof, we need to prove that X_{out} and X_{in} are isomorphic digraphs. To do this, we will show that they are homogeneous and have the same age.

First we show that X_{out} is homogeneous. Note in advance that the same argument shows that X_{in} is homogeneous. Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in X_{\mathrm{out}}$ be isomorphic. Then $(a, a_1, \dots, a_n)$ and $(a, b_1, \dots, b_n)$ are isomorphic, so by homogeneity of (D, E) there is $g \in \mathrm{Aut}(D, E)$ mapping $(a, a_1, \dots, a_n)$ to $(a, b_1, \dots, b_n)$. Since g fixes a, g fixes X_{out} setwise. Then the restriction of g to X_{out} is an automorphism of (X_{out}, E) mapping $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$, as required.

Next we show that $\mathrm{Age}(X_{\mathrm{out}}) = \mathrm{Age}(X_{\mathrm{in}})$. Let A be a finite sub-digraph of X_{out}. Then let $A' = A \cup \{a\}$ and note that A' is an element of $\mathrm{Forb}(\mathcal{T})$. Now let A'' be the digraph obtained from A' by changing the direction of all the edges adjacent to a. Since $\mathcal{T}$ is closed under sw and $A' \in \mathrm{Forb}(\mathcal{T})$, A'' is also in $\mathrm{Forb}(\mathcal{T})$, so A'' is embeddable in (D, E). By homogeneity, we may assume that the embedding maps $a \in A''$ to $a \in (D, E)$, so we have embedded A into X_{in}. Thus we have shown that $\mathrm{Age}(X_{\mathrm{out}}) \subseteq \mathrm{Age}(X_{\mathrm{in}})$. A symmetric argument shows that $\mathrm{Age}(X_{\mathrm{in}}) \subseteq \mathrm{Age}(X_{\mathrm{out}})$, so we are done.

(iii) Let $A \subseteq D$. Consider the digraph (D, E') obtained from (D, E) by changing the direction of the edges between A and A^c and leaving all other edges unchanged. If (D, E') is embeddable in (D, E), then sw_A exists as any embedding $(D, E') \to (D, E)$ has the desired property.

We will prove the contrapositive, so suppose sw_A does not exist. This implies that the digraph (D, E') is not embeddable in (D, E), which by Lemma 1.2 implies that $\mathrm{Age}(D, E') \nsubseteq \mathrm{Age}(D, E)$. This implies there exists $T \in \mathcal{T}$ which is embeddable in (D, E'); let g be such an embedding. Let $B = g^{-1}(g(T) \cap A)$, so B is a subset of T. Now consider the tournament T' obtained by applying the switch operation on T about every element of B. By choice of T and B, T' is isomorphic to $(g(T), E|_{g(T)})$. Hence T' is in the age of (D, E) and so $T' \notin \mathcal{T}$. To summarise, we have $T \in \mathcal{T}, T' \notin \mathcal{T}$ and T' is obtained from T by switching. This means $\mathcal{T}$ is not closed under sw and so by (ii) sw does not exist, as required.

(iv) and (v) We omit these proofs for the reasons described at the start of this section. ⊣

DEFINITION 2.5. Let G be a subgroup of $\mathrm{Sym}(D)$ and $n \in \mathbb{N}$. G is *n-transitive* if for all proper tuples $\bar{a}, \bar{b} \in D^n$, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$. G is *n-homogeneous* if for all subsets $A, B \subset D$ of size n, there exists $g \in G$ such that $g(A) = B$.

LEMMA 2.6. *Let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of $\mathrm{Aut}(D, E)$.*

(i) *If G is n-transitive for all $n \in \mathbb{N}$, then $G = \mathrm{Sym}(D)$.*
(ii) *If G is n-homogeneous for all $n \in \mathbb{N}$, then $G = \mathrm{Sym}(D)$.*
(iii) *Suppose that whenever $A \subset D$ is finite and has edges, there exists $g \in G$ such that $g(A)$ has less edges than in A. Then $G = \mathrm{Sym}(D)$.*
(iv) *Suppose that there exists a finite $A \subset D$ and $g \in G$ such that g behaves like* id *on $D \backslash A$, g behaves like* id *between A and $D \backslash A$, and, g deletes at least one edge in A. Then, $G = \mathrm{Sym}(D)$.*

TERMINOLOGY. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in D$. We say $\bar{a}$ and $\bar{b}$ are isomorphic as graphs if $\overline{E}(a_i, a_j) \leftrightarrow \overline{E}(b_i, b_j)$ for all i, j.

LEMMA 2.7. *Let $G \leq \mathrm{Sym}(D)$ be a closed supergroup of $\mathrm{Aut}(D, E)$.*

(i) *Suppose that whenever $\bar{a}$ and $\bar{b}$ are isomorphic as graphs, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$. Then $G \geq \mathrm{Aut}(D, \overline{E})$.*
(ii) *Suppose that for all $A = \{a_1, \ldots, a_n\} \subset D$, there exists $g \in G$ such that for all edges $a_i a_j$ in A, $E(g(a_i), g(a_j))$ if $i < j$ and $E(g(a_j), g(a_i))$ if $i > j$. (Intuitively, such a g is aligning the edges so they all point in the same direction.) Then, $G \geq \mathrm{Aut}(D, \overline{E})$.*
(iii) *Suppose that for all finite $A \subset D$ and all edges $aa' \in A$ there is $g \in G$ such that g changes the direction of aa' and behaves like* id *on all other edges and nonedges of A. Then $G \geq \mathrm{Aut}(D, \overline{E})$.*
(iv) *Suppose there is a finite $A \subset D$ and a $g \in G$ such that g behaves like* id *on $D \backslash A$, g behaves like* id *between A and $D \backslash A$, and g switches the direction of some edge in A. Then, $G \geq \mathrm{Aut}(D, \overline{E})$.*

Furthermore, in all of these cases we can also conclude that the underlying graph $(D, \overline{E})$ is homogeneous.

2.3. Analysis of canonical functions. To help motivate the analysis we are about to undertake, we sketch a part of the proof of the main theorem. One task will be to show that if $G > \mathrm{Aut}(D, E)$ then $G \geq \langle - \rangle$ or $G \geq \langle \mathrm{sw} \rangle$. Since $G > \mathrm{Aut}(D, E)$, G does not preserve the relation E, so there exist $g \in G$ and $c_1, c_2 \in D$ witnessing this. Then by Theorem 1.6, we find a canonical function $f : (D, E, <, c_1, c_2) \to (D, E)$ that agrees with g on (c_1, c_2) and which is generated by G. The behaviour of f will give us information about G. We only have to consider the behaviour of f on the 2-types, since $(D, E, <, c_1, c_2)$ has quantifier elimination and all relations are of arity ≤ 2. Therefore there are only finitely many possibilities for the behaviour of f, so we can check each case and show that G must contain $\langle - \rangle$ or $\langle \mathrm{sw} \rangle$.

2.3.1. Canonical functions from $(D, E, <)$. We start our analysis of the behaviours with the simplest case, which is when no constants are added.

NOTATION AND FACTS.

- *Let $\phi_1(x, y), \ldots, \phi_n(x, y)$ be formulas. We let $p_{\phi_1, \ldots, \phi_n}(x, y)$ denote the (partial) type determined by the formula $\phi_1(x, y) \wedge \cdots \wedge \phi_n(x, y)$.*

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- *There are four 2-types in (D, E): $p_=, p_E, p_{E^*}$ and p_N.*
- *There are seven 2-types in $(D, E, <)$: $p_=, p_{<,E}, p_{<,E^*}, p_{<,N}, p_{>,E}, p_{>,E^*}$ and $p_{>,N}$.*

The following lemma contains a little 'trick' that proves useful during the analysis of the behaviours. Roughly, this lemma allows us to manipulate freely how finitely many elements are ordered, and its benefits will be seen shortly.

LEMMA 2.8. *Let $\{a_1, \dots, a_n\} \in (D, E, <)$ and let $\sigma \in S_n$. Then there exists $g \in \mathrm{Aut}(D, E)$ such that for all i, j, $E(a_i, a_j)$ if and only if $E(g(a_i), g(a_j))$, and for all i, j, $g(a_i) < g(a_j)$ if and only if $\sigma(i) < \sigma(j)$.*

PROOF. Follows straightforwardly from the definition of the age of $(D, E, <)$ and the homogeneity of (D, E). ⊣

LEMMA 2.9. *Let G be a closed supergroup of $\mathrm{Aut}(D, E)$, let $f \in cl_{tm}(G)$, and let f be canonical when considered as a function from $(D, E, <)$ to (D, E).*

(i) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$, then $-$ exists and $- \in G$.*
(ii) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D, \overline{E})$.*
(iii) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D, \overline{E})$.*
(iv) *If $f(p_{<,N}) = p_E$ or $p_{E^*}, f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_N$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq \mathrm{Aut}(D, \overline{E})$.*
(v) *If f has any other nonidentity behaviour, then either we get a contradiction (i.e., that behaviour is not possible) or $G = \mathrm{Sym}(D)$.*

PROOF. (i) By Lemma 2.4, to show $-$ exists, it suffices to show that if $T \notin \mathcal{T}$, then $-(T) \notin \mathcal{T}$. So let T be a finite tournament not in $\mathcal{T}$. This means T is embeddable in (D, E); let $T' \subset (D, E)$ be isomorphic to T. Then the conditions in the lemma tell us that $f(T') \cong -(T)$, so $-(T)$ is embeddable in (D, E), so $-(T) \notin \mathcal{T}$, as required.

Next we show $- \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = -(\bar{a})$. So let $\bar{a} \in D$ be finite. By the conditions in the lemma, $f(\bar{a}) \cong -(\bar{a})$. By homogeneity, there exists $g_1 \in \mathrm{Aut}(D, E)$ mapping $f(\bar{a})$ to $-(\bar{a})$. Since $f \in \mathrm{cl_{tm}}(G)$, there is $g_2 \in G$ such that $g_2(\bar{a}) = f(\bar{a})$. Letting $g = g_1 \circ g_2$ completes the argument.

(ii) We will use Lemma 2.7(ii). Let $(a_1, \dots, a_n) \in D$. By Lemma 2.8, there is $g_1 \in \mathrm{Aut}(D, E)$ such that $g_1(a_1) < g_1(a_2) < \cdots < g_1(a_n)$. Then, due to the conditions in the lemma, applying f aligns the edges of this tuple to point in the same direction. As $f \in \mathrm{cl_{tm}}(G)$, there exists $g_2 \in G$ which agrees with f on $g_1(\bar{a})$. Letting $g = g_2 \circ g_1$ completes the argument.

Note: For the remaining arguments, we will no longer comment explicitly on the fact that $f \in \mathrm{cl_{tm}}(G)$ implies that f can be imitated on a finite set by a function in G.

(iii) Use the same argument as (ii).

(iv) Let $\bar{a}$ be any tuple. Then apply f once to get $f(\bar{a})$. By Lemma 2.8, there is $g \in \mathrm{Aut}(D, E)$ such that $gf(\bar{a})$ is linearly ordered the same way as $\bar{a}$. Now apply f again. Observe that the behaviour of fgf on $\bar{a}$ matches the behaviour of the canonical function in (ii) or (iii). Thus, this case is reduced to one of those.

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Terminology. In future, we use the phrase *applying f twice* to abbreviate the procedure of applying f, re-ordering the elements to match the ordering of the initial tuple, and applying f again.

(v) Case 1: $f(p_{<,N}) = p_N$. We are left with the behaviours where $f(p_{<,E}) = p_N$ or $f(p_{<,E^*}) = p_N$ (or both), as all the other possibilities have been dealt with above. Now for any finite $A \subset D$ that has edges, $f(A)$ has less edges than A does. So by Lemma 2.6(iii), we conclude that $G = \mathrm{Sym}(D)$.

Case 2: $f(p_{<,N}) = p_E$.

Case 2a: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$. For every proper tuple $\bar{a}, \bar{b} \in D^n$, $f(\bar{a}) \cong f(\bar{b}) \cong L_n$ (as digraphs), so G is n-transitive for all n, so $G = \mathrm{Sym}(D)$ by Lemma 2.6(i).

Case 2b: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$. Apply f twice and use the same argument as in Case 2a to show that $G = \mathrm{Sym}(D)$.

Case 2c: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_{E^*}$. We will show that this behaviour is not possible. Let $T \in \mathcal{T}$ be of minimal cardinality. Enumerate T as $T = (t_1, \ldots, t_n)$ so that we have an edge going from t_1 to t_2 (as opposed to t_2 to t_1). Now let $A = (a_1, \ldots, a_n)$ be the ordered digraph constructed as follows: Start with T, delete the edge t_1t_2, and add a linear order so that $a_1 < a_2$. As T was minimal, A can be embedded in $(D, E, <)$, so then $f(A) \subset (D, E)$. But by the construction of A, $f(A) \cong T$, so we have shown that T is embeddable in (D, E). This contradicts that $T \in \mathcal{T}$.

Case 2d: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to a case that is dual to Case 2c.

Case 2e: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_N$. Applying f twice reduces to Case 2a.

Case 2f: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to Case 1.

Case 2g: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_N$. We will show that this behaviour is not possible. Let $T \in \mathcal{T}$ be of minimal cardinality. Observe that f^3 has the identity behaviour, so that $f^3(T) = T$. Now observe that $f^2(T)$ is a digraph that contains nonedges, so by the minimality of T, $f^2(T)$ can be embedded in $(D, E, <)$. But then applying f shows that $f(f^2(T))$ is embeddable in (D, E), i.e., that $f^3(T) = T$ is embeddable in (D, E). This contradicts that $T \in \mathcal{T}$.

Case 2h: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_{E^*}$. Using the same argument as in 2g shows that this case is not possible.

Case 3: $f(p_{<,N}) = p_{E^*}$. This case is symmetric to Case 2. ⊣

Now we have seen an analysis, we provide more detailed intuition. Given some closed supergroup G of $\mathrm{Aut}(D, E)$, we want to know what functions it contains. Since G is closed, this amounts to knowing how G acts on finite tuples in D. But this is exactly the information a canonical function in $\mathrm{cl_{tm}}(G)$ provides! For example, in (i) the canonical function tells us that G can behave like $-$ on any finite tuple, which implies that $G \geq \langle - \rangle$. The role of homogeneity is that it allows us to move between isomorphic tuples, so knowing how G acts on one tuple automatically tells us how G acts on all tuples isomorphic to that one tuple.

2.3.2. Canonical functions from $(D, E, <, \bar{c})$. We now move on to the general situation where we have added constants $\bar{c} \in D$ to the structure. For convenience, we assume that $c_i < c_j$ for all $i < j$. Since (D, E) is $\aleph_0$-categorical, $(D, E, \bar{c})$

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is also $\aleph_0$-categorical, so the n-types of $(D, E, <, \bar{c})$ correspond to the orbits of $\mathrm{Aut}(D, E, <, \bar{c})$ acting on the set of n-tuples of D. For this reason, we often conflate the notion of types and orbits.

We need to describe the 2-types of $(D, E, <, \bar{c})$, and to do that we first need to describe the 1-types. There are two kinds of 1-types, i.e., two kinds of orbits. The first is a singleton, e.g., $\{c_1\}$. The other orbits are infinite and are determined by how their elements are related to the c_i. These infinite orbits are of the form $\{x \in D : \bigwedge_i (\phi_i(x, c_i) \wedge \psi_i(x, c_i))\}$, where $\phi_i \in \{<, >\}$ and $\psi_i \in \{E, E^*, N\}$.

Unlike in the case of the generic digraph, the substructures induced on these orbits will not necessarily be isomorphic to the original structure. For example, let $\mathcal{T} = \{L_3\}$ and $\bar{c} = (c_1)$. Then consider the orbit $X = \{x \in D : x < c_1 \wedge E(x, c_1)\}$. If there was an edge, ab say, in X, then $\{c_1, a, b\}$ would be a copy of L_3. However, L_3 is forbidden. Thus, X contains no edges so in particular X is not isomorphic to $(D_\mathcal{T}, E_\mathcal{T}, <)$.

However, there are always orbits such that the substructures induced on them are isomorphic to the original structure. For example, regardless of $\mathcal{T}$, the orbit $X = \{x \in D : x < c_1 \wedge \bigwedge_i N(x, c_i)\}$ is isomorphic to $(D, E, <)$. These orbits form a central part of the argument so we give them a definition.

Definition 2.10. Let $\bar{c} \in D$ and $X \subset D$ be an orbit of $(D, E, <, \bar{c})$. We say X is *independent* if X is infinite and there are no edges between $\bar{c}$ and X.

The following lemma highlights the key feature of independent orbits that makes them useful.

Lemma 2.11. *Let X be an independent orbit of $(D, E, <, \bar{c})$.*

(i) *Let $v \in D \setminus (X \cup \bar{c})$. Let $A = (a_0, \dots, a_n)$ be a finite digraph in the age of (D, E). Then there are $x_1, \dots, x_n \in X$ such that $(a_0, a_1, \dots, a_n) \cong (v, x_1, \dots, x_n)$ as tuples in $(D, E, <, \bar{c})$.*

(ii) *The substructure induced on X is isomorphic to (D, E).*

Proof. Let k be the length of the tuple $\bar{c}$ and let x be any element of X. Consider the finite ordered digraph A' which is constructed as follows: start with A, add new vertices $c'_1, \dots, c'_k$ and then add edges and an ordering so that we have $(a_0, c'_1, \dots, c'_k) \cong (v, c_1, \dots, c_k)$ and so that $(a_i, c'_1, \dots, c'_k) \cong (x, c_1, \dots, c_k)$ for all $i > 0$.

A' is embeddable in $(D, E, <)$ so let f be such an embedding. By composing with an automorphism of $(D, E, <)$ if necessary, we can assume that $f(c'_j) = c_j$ for $j = 1, \dots, k$. Then letting $x_i = f(a_i)$ for $i = 1, \dots, n$ completes the proof.

(ii) From (i), we know that the age of X equals the age of (D, E), so it suffices to show that X is homogeneous. Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in X$ be isomorphic tuples, as ordered digraphs. Then $(c_1, \dots, c_k, a_1, \dots, a_n) \cong (c_1, \dots, c_k, b_1, \dots, b_n)$. By the homogeneity of $(D, E, <)$, there is $f \in \mathrm{Aut}(D, E, <)$ mapping $(c_1, \dots, c_k, a_1, \dots, a_n)$ to $(c_1, \dots, c_k, b_1, \dots, b_n)$. Since f fixes $\bar{c}$, f fixes X setwise, and so $f|_X$ is an automorphism of X mapping $\bar{a}$ to $\bar{b}$, as required. $\dashv$

Notation. *Let A, B be definable subsets of D and let $\phi_1(x, y), \dots, \phi_n(x, y)$ be formulas. We let $p_{A,B,\phi_1,\dots,\phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \cdots \wedge \phi_n(x, y)$.*

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Using this notation, we can describe the 2-types of $(D, E, <, \bar{c})$. They are all of the form $p_{X,Y,\phi,\psi} = \{(a, b) \in D : a \in X, b \in Y, \phi(a, b) \text{ and } \psi(a, b)\}$, where X and Y are orbits, $\phi \in \{<, =, >\}$ and $\psi \in \{E, E^*, N\}$.

Our task now is to analyse the possibilities for $f(p_{X,Y,\phi,\psi})$, where f is a canonical function. It turns out that it is sufficient to study those cases where we assume X is an independent orbit. The first lemma deals with the situation when $X = Y$.

Lemma 2.12. *Let G be a closed supergroup of* $\mathrm{Aut}(D, E)$, *let $\bar{c} \in D$, let $f \in cl_{tm}(G)$, and let f be canonical when considered as a function from $(D, E, <, \bar{c})$ to (D, E). Let $X \subset D$ be an independent orbit.*

(i) *If $f(p_{X,X,<,N}) = p_N$, $f(p_{X,X,<,E}) = p_{E^*}$ and $f(p_{X,X,<,E^*}) = p_E$, then $-$ exists and $- \in G$.*
(ii) *If $f(p_{X,X,<,N}) = p_N$, $f(p_{X,X,<,E}) = p_E$ and $f(p_{X,X,<,E^*}) = p_E$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq$ $\mathrm{Aut}(D, \overline{E})$.*
(iii) *If $f(p_{X,X,<,N}) = p_N$, $f(p_{X,X,<,E}) = p_{E^*}$ and $f(p_{X,X,<,E^*}) = p_{E^*}$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq$ $\mathrm{Aut}(D, \overline{E})$.*
(iv) *If $f(p_{X,X,<,N}) = p_E$ or p_{E^*}, $f(p_{X,X,<,E}) = p_N$ and $f(p_{X,X,<,E^*}) = p_N$, then $(D, \overline{E})$ is a homogeneous graph and $G \geq$ $\mathrm{Aut}(D, \overline{E})$.*
(v) *If f has any other nonidentity behaviour, then either we get a contradiction or $G =$ $\mathrm{Sym}(D)$.*

Proof. Intuitively, since $X \cong (D, E)$, the canonical functions here provide us the same information as the canonical functions in Lemma 2.9, so we are done. More formally, one can copy the arguments from Lemma 2.9 and add minor adjustments as necessary. We do this for (i) as an example, and leave the rest to be checked by the reader.

(i) First we show $-$ exists, so let T be a tournament not in $\mathcal{T}$. This means T is embeddable in (D, E) and so, by Lemma 2.11, T is embeddable in X; let $T' \subset X$ be isomorphic to T. Then the conditions in the lemma tell us that $f(T') \cong -(T)$, so $-(T)$ is embeddable in (D, E), so $-(T) \notin \mathcal{T}$, as required.

Next we show $- \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = -(\bar{a})$. So let $\bar{a} \in D$ be finite. By Lemma 2.11, there is $\bar{a}' \in X$ isomorphic to $\bar{a}$. By the conditions in the lemma, $f(\bar{a}') \cong -(\bar{a})$. By homogeneity, there exist $g_1 \in \mathrm{Aut}(D, E)$ mapping $\bar{a}$ to $\bar{a}'$ and $g_2 \in \mathrm{Aut}(D, E)$ mapping $f(\bar{a}')$ to $-(\bar{a})$. Since $f \in \mathrm{cl}_{\mathrm{tm}}(G)$, there is $g_3 \in G$ such that $g_3(\bar{a}) = f(\bar{a})$. Letting $g = g_2 \circ g_3 \circ g_1$ completes the argument. ⊣

Next we look at the behaviour of f between an independent orbit X and any other orbit Y. This task is split depending on how X and Y relate with regard to the linear order.

Facts and notation. *There are two ways that two infinite orbits X and Y of* $\mathrm{Aut}(D, E, <, \bar{c})$ *can relate to each other with respect to the linear order $<$:*

- *All of the elements of one orbit, X say, are smaller than all of the elements of Y. This is abbreviated by '$X < Y$'.*
- *X and Y are interdense: $\forall x < x' \in X, \exists y \in Y$ such that $x < y < x'$ and vice versa.*

The next lemma contains the analysis for the case where $X < Y$ or $X > Y$.

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Lemma 2.13. *Let G be a closed supergroup of* $\mathrm{Aut}(D,E)$*, let* $\bar{c} \in D$*, let* $f \in cl_{tm}(G)$*, and let f be canonical when considered as a function from* $(D,E,<,\bar{c})$ *to* (D,E)*. Let $X \subset D$ be an independent orbit on which f behaves like* id *and let Y be an infinite orbit such that $X < Y$ or $X > Y$.*

(i) *If $f(p_{X,Y,N}) = p_N$, $f(p_{X,Y,E}) = p_{E^*}$ and $f(p_{X,Y,E^*}) = p_E$, then* sw *exists and* sw $\in G$.
(ii) *If $f(p_{X,Y,N}) = p_N$, $f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_E$, then $(D,\overline{E})$ is a homogeneous graph and* $G \geq \mathrm{Aut}(D,\overline{E})$.
(iii) *If $f(p_{X,Y,N}) = p_N$, $f(p_{X,Y,E}) = p_{E^*}$ and $f(p_{X,Y,E^*}) = p_{E^*}$, then $(D,\overline{E})$ is a homogeneous graph and* $G \geq \mathrm{Aut}(D,\overline{E})$.
(iv) *If $f(p_{X,Y,N}) = p_E$ or p_{E^*}, $f(p_{X,Y,E}) = p_N$ and $f(p_{X,Y,E^*}) = p_N$, then $(D,\overline{E})$ is a homogeneous graph and* $G \geq \mathrm{Aut}(D,\overline{E})$.
(v) *If f has any other nonidentity behaviour, then either we get a contradiction or* $G = \mathrm{Sym}(D)$.

Remark: We do not need to include $<$ or $>$ in the subscripts of the type because it is automatically determined by how X and Y are related to $\bar{c}$.

Proof. Assume that $X < Y$. The proof for the case $Y < X$ is symmetric. Let $y_0 \in Y$ be any element.

(i) The proof is analogous to that of Case (i) in Lemma 2.9 and is left as an exercise for the reader. Note that Lemma 2.11 is needed for this.

(ii) Using Lemma 2.7(ii), it suffices to show that for any finite $A \subset D$ we can align all its edges by using functions in G. Let $A = \{a_1,\dots,a_n\}$. First we map a_{n-1} to y_0 and the rest of A into X (possible by Lemma 2.11), and then apply f. Then we repeat but with a_{n-2} instead of a_{n-1}, then with a_{n-3}, and so on until a_1.

(iii) Same as (ii).

(iv) The same argument as in (ii) works but with a slight modification: the intuition is that whenever f was applied to some tuple $(a_0,\dots,a_n)$ in those proofs, here we apply f twice to get the same effect. To be more precise, the modification is as follows. Let $(a_0,\dots,a_n) \in D$. We first map this to an isomorphic copy $(y_0,x_1,\dots,x_n)$ for some $x_i \in X$. Then apply f. Then again we map this to an isomorphic tuple $(y_0,x'_1,\dots,x'_n)$ for some $x'_i \in X$. Then apply f a second time. The total effect of this procedure is the same as what the canonical function did in Case (ii) or (iii). Thus we have reduced this case to either (ii) or (iii).

Remark: For the rest of this proof, we will use the phrase "by applying f twice" to refer to the procedure described above.

(v) Case 1: $f(p_{<,N}) = p_N$. By a similar argument as in Case 1 of Lemma 2.9, $G = \mathrm{Sym}(D)$. Note that Lemma 2.11 is needed for this.

Case 2: $f(p_{X,Y,N}) = p_E$.

Case 2a: $f(p_{X,Y,E}) = p_{E^*}$. We will show that this behaviour is not possible, in a similar fashion to Case 2c of Lemma 2.9. Let $T \in \mathcal{T}$ be of minimal size and enumerate T as $(t_0,t_1,\dots,t_n)$ so that t_0 has at least one edge going into it. Construct a digraph $A = (a_0,a_1,\dots,a_n)$ as follows: start with A being equal to T and then replace edges into a_0 with nonedges, replace edges out of a_0 with incoming edges, and leave all other edges of A the same.

Since T was minimal, $A \in \mathrm{Forb}(\mathcal{T})$ so A can be embedded in D. Furthermore, by Lemma 2.11 there are $x_i \in X$ such that $(a_0,a_1,\dots,a_n) \cong (y_0,x_1,\dots,x_n)$.

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Now apply f. By construction of A, $f(y_0, x_1, \ldots, x_n) \cong (t_0, \ldots, t_n)$. Thus, T is embeddable in D, contradicting $T \in \mathcal{T}$.

Case 2b: $f(p_{X,Y,E^*}) = p_{E^*}$. Use the same argument as Case 2a to show this is not possible.

Now there are only three behaviours left to analyse.

Case 2c: $f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_E$. We will show that $G = \mathrm{Sym}(D)$, by showing that every tuple $(a_0, \ldots, a_{n-1}) \in D^n$ can be mapped to L_n using functions in G. We do this by induction on n. The base case $n = 1$ is trivial so let $n > 1$. By the inductive hypothesis we can assume that $(a_1, \ldots, a_{n-1}) \cong L_{n-1}$. By Lemma 2.11 we map $\bar{a}$ to an isomorphic tuple $(y_0, x_1, \ldots, x_{n-1})$ for some $x_i \in X$. Then applying f maps the tuple to a copy of L_n, as required.

Case 2d: $f(p_{X,Y,E}) = p_E$ and $f(p_{X,Y,E^*}) = p_N$. By applying f twice this case is reduced to Case 2c.

Case 2e: $f(p_{X,Y,E}) = p_N$ and $f(p_{X,Y,E^*}) = p_E$. By applying f twice this case is reduced to Case 1.

Case 3: $f(p_{X,Y,N}) = p_{E^*}$. This case is symmetric to Case 2. ⊣

In the proof above we only had to study the behaviour of f on $\{y_0\} \cup X$ for some element $y_0 \in Y$. The key property which allowed this is Lemma 2.11. This feature allows us to use these arguments with minimal modification to prove the subsequent lemmas.

The next lemma deals with the case where X and Y are interdense.

Lemma 2.14. *Let G be a closed supergroup of* $\mathrm{Aut}(D, E)$, *let $\bar{c} \in D$, let $f \in cl_{tm}(G)$, and let f be canonical when considered as a function from $(D, E, <, \bar{c})$ to (D, E). Let $X \subset D$ be an independent orbit on which f behaves like* id *and let Y be an infinite orbit such that X and Y are interdense. Then at least one of the following holds.*

(i) *f preserves all the edges and nonedges between X and Y.*
(ii) *f switches the direction of every edge between X and Y and preserves nonedges between X and Y. In this case* sw *exists.*
(iii) *$G \geq \mathrm{Aut}(D, \overline{E})$ and $(D, \overline{E})$ is a homogeneous graph.*
(iv) *$G = \mathrm{Sym}(D)$.*

Proof. First just consider the increasing tuples from X to Y. With the same arguments as in Lemma 2.13 one can show that either

(a) $f(p_{X,Y,N,<}) = p_N$, $f(p_{X,Y,E,<}) = p_E$ and $f(p_{X,Y,E^*,<}) = p_{E*}$,
(b) $f(p_{X,Y,N,<}) = p_N$, $f(p_{X,Y,E,<}) = p_{E^*}$ and $f(p_{X,Y,E^*,<}) = p_E$,
(c) $G \geq \mathrm{Aut}(D, \overline{E})$ and $(D, \overline{E})$ is a homogeneous graph, or
(d) $G = \mathrm{Sym}(D)$.

If (c) or (d) is true we are done, so assume (a) or (b) is true. Similarly we can assume that f behaves like id or sw between decreasing tuples from X to Y. If the behaviours between increasing and decreasing tuples are the same, then (i) or (ii) will be true so we would be done. Thus it remains to check what happens if f behaves like id on decreasing tuples and sw on increasing tuples. Explicitly we are assuming that

$f(p_{X,Y,N,<}) = p_N$, $f(p_{X,Y,E,<}) = p_{E^*}$, $f(p_{X,Y,E^*,<}) = p_E$ and
$f(p_{X,Y,N,>}) = p_N$, $f(p_{X,Y,E,>}) = p_E$, $f(p_{X,Y,E^*,>}) = p_{E^*}$.

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Let $\bar{a} = (a_0, a_1, \ldots, a_n) \in \mathrm{Forb}(\mathcal{T})$ be a digraph with at least one edge $E(a_0, a_1)$. We can consider $\bar{a}$ as an ordered digraph by setting $a_i < a_j \leftrightarrow i < j$. Then by Lemma 2.11 $\bar{a}$ has an isomorphic copy $\bar{b} = (b_0, b_1, \ldots, b_n)$ such that $b_1 \in Y$ and $b_i \in X$ for $i \neq 1$. All the edges of $\bar{b}$ are preserved under f, except for the edge $E(b_0, b_1)$ whose direction is switched. By Lemma 2.7, we conclude that $G \geq \mathrm{Aut}(D, \overline{E})$ and $(D, \overline{E})$ is a homogeneous graph. ⊣

We end by looking at how f can behave between the constants $\bar{c}$ and the rest of the structure.

LEMMA 2.15. *Let G be a closed supergroup of* $\mathrm{Aut}(D, E)$, *let* $(c_1, \ldots, c_n) \in D$, *let* $f \in cl_{tm}(G)$, *and let f be canonical when considered as a function from* $(D, E, <, \bar{c})$ *to* (D, E). *Suppose that f behaves like* id *on* $D^- := D \backslash \{c_1, \ldots, c_n\}$. *Then at least one of the following holds.*

(i) *For all* $i, 1 \leq i \leq n$, *f behaves like* id *or like* sw *between c_i and D^-.*
(ii) $G \geq \mathrm{Aut}(D, \overline{E})$ *and* $(D, \overline{E})$ *is a homogeneous graph.*
(iii) $G = \mathrm{Sym}(D)$.

PROOF. Fix some i, $1 \leq i \leq n$. Let $X_{\text{out}} = \{x \in D : x < c_1 \wedge E(c_i, x) \wedge \bigwedge_{j \neq i} N(c_j, x)\}$. Define X_{in} and X_N similarly, with $E(c_i, x)$ replaced with $E(x, c_i)$ and $N(x, c_i)$, respectively. Then for any finite digraph $(a_0, a_1, \ldots, a_n)$, there exist $x_1, \ldots, x_n \in X_{\text{out}} \cup X_{\text{in}} \cup X_N$ such that $(a_0, a_1, \ldots, a_n) \cong (c_i, x_1, \ldots, x_n)$. So by replicating the proof of Lemma 2.13 we can assume that f behaves like id or sw between c_i and $X_{\text{out}} \cup X_{\text{in}} \cup X_N$. Without loss of generality, we assume f behaves like id, because we can compose f with sw_{c_i} if necessary.

If f behaves like id between c_i and D^- we are done, so suppose there is an infinite orbit X such that f does not behave like id between c_i and X. Assume that there are edges from c_i into X—the arguments for the other two cases are similar.

Let A be a finite digraph in the age of D which contains an edge, ab say. Then observe that there is an embedding of A into D such that a is mapped to c_i, b is mapped into X, and the rest of A is mapped into $X_{\text{out}} \cup X_{\text{in}} \cup X_N$. Then applying f changes exactly the one edge ab in A, so by Lemma 2.6 or Lemma 2.7 as appropriate, we are done. ⊣

2.4. Proof of the classification. We now piece together these lemmas to prove the classification theorem, Theorem 2.2.

PROOF. (i) Suppose for contradiction that $G \not\geq \mathrm{Aut}(D, \overline{E})$ and $G \not\leq \mathrm{Aut}(D, \overline{E})$. Because of the second assumption G violates the relation $\overline{E}$. By Theorem 1.6 this can be witnessed by a canonical function. Precisely, this means there are $c_1, c_2 \in D$ and $f \in \mathrm{cl_{tm}}(G)$ such that $f : (D, E, <, c_1, c_2) \to (D, E)$ is a canonical function, $\overline{E}(c_1, c_2)$ and $N(f(c_1), f(c_2))$.

Now let X be an independent orbit of $(D, E, <, c_1, c_2)$.

CLAIM 1. We may assume that f behaves like id on X.

By Lemma 2.12 we know that f behaves like id or $-$ on X, otherwise G would contain $\mathrm{Aut}(D, \overline{E})$. If f behaves like $-$ on X, then we continue by replacing f by $- \circ f$.

CLAIM 2. We may assume that f behaves like id between X and every other infinite orbit Y.

Let Y be another infinite orbit. By Lemmas 2.13 and 2.14, f behaves like id or sw between X and Y, as otherwise G would contain $\mathrm{Aut}(D, \overline{E})$. If f behaves like sw between them, then we simply replace f by $\mathrm{sw}_Y \circ f$. Note that one needs to check sw_Y is a legitimate function, but this has been done in Lemma 2.4(iii).

Claim 3. We may assume that f behaves like id on every infinite orbit and between every pair of infinite orbits.

Suppose not, so there are infinite orbits Y_1 and Y_2 (possibly the same) and there are distinct $y_1, y_2 \in Y_1, Y_2$, respectively, such that $(y_1, y_2) \ncong f(y_1, y_2)$. Now for any finite digraph $(a_1, a_2, \ldots, a_n) \in \mathrm{Forb}(\mathcal{T})$ with $(y_1, y_2) \cong (a_1, a_2)$, we can find $x_3, \ldots, x_n \in X$ such that $(y_1, y_2, x_3, \ldots, x_n) \cong (a_1, \ldots, a_n)$ (this statement can be verified analogously to Lemma 2.11). Then f has the effect of only changing what happens between y_1 and y_2, since we know f behaves like id on X and between X and all other infinite orbits. In short, given any finite digraph, we can use f to change what happens between exactly two of the vertices of the digraph.

There are three options. If f creates an edge from a nonedge, then we can use f to introduce a forbidden tournament, which gives a contradiction. If f deletes the edge or changes the direction of the edge, then by Lemma 2.6 or Lemma 2.7, as appropriate, we get that $G \geq \mathrm{Aut}(D, \overline{E})$.

Claim 4. We may assume that f behaves like id between $\{c_1, c_2\}$ and the union of all infinite orbits.

The follows immediately from Lemma 2.15, composing with sw_{c_i} if necessary.

Conclusion. We can assume that f behaves everywhere like the identity, except on (c_1, c_2), where it maps an edge to a nonedge. But then we get that $G = \mathrm{Sym}(D)$ by Lemma 2.6, completing the proof of (i).

(ii) The proof follows exactly the same series of claims as in part (i) but with minor adjustments to how one starts and concludes. We go through one case as an example, leaving the rest to the reader. We will show that if $\mathrm{Aut}(D, E) < G \leq \mathrm{Aut}(D, \overline{E})$, then $G \geq \langle - \rangle$ or $G \geq \langle \mathrm{sw} \rangle$ (if they exist). So suppose $\mathrm{Aut}(D, E) < G \leq \mathrm{Aut}(D, \overline{E})$. Then G preserves nonedges but not the relation E. By Theorem 1.6, there is an edge $c_1c_2 \in D$ and a canonical function $f : (D, E, <, c_1, c_2) \to (D, E)$ which changes the direction of the edge c_1c_2. Suppose for contradiction that $G \ngeq \langle - \rangle$ and $G \ngeq \langle \mathrm{sw} \rangle$.

Let X be an independent orbit. By Lemma 2.12, f must behave like id on X and then by Lemmas 2.13 and 2.14, f must behave like id between X and all other infinite orbits. By repeating the argument of Claim 3 above, f must behave like id on the union of infinite orbits and so by Lemma 2.15 f must behave like id between the constants and the union of infinite orbits. Now we are in the situation of Lemma 2.7(iv), so we conclude that $G \geq \mathrm{Aut}(D, \overline{E})$, so $G \geq \langle - \rangle, \langle \mathrm{sw} \rangle$.

(iii) $(D, \overline{E})$ embeds every finite edgeless graph and is connected (Lemma 1.2(ii)). Hence, if $(D, \overline{E})$ is a homogeneous graph then $(D, \overline{E})$ has to be the random graph or a Henson graph, by the classification of countable homogeneous graphs ([18]).

Thus assume that $(D, \overline{E})$ is not a homogeneous graph. Let $G' := \max\{\mathrm{Aut}(D, E), \langle - \rangle, \langle \mathrm{sw} \rangle, \langle -, \mathrm{sw} \rangle\}$. Now let G be a closed group such that $G' < G \leq \mathrm{Sym}(D)$. We want to show that $G = \mathrm{Sym}(D)$. By Theorem 1.6, there are $\bar{c} \in D$ and a canonical $f : (D, E, <, \bar{c}) \to (D, E)$ such that f cannot be imitated by any function of G' on $\bar{c}$. To be precise, we mean that for all $g \in G'$, $g(\bar{c}) \neq f(\bar{c})$.

Now we continue as in (i), proving that we may assume f behaves like id on the union of all infinite orbits and like id between $\bar{c}$ and the union of infinite orbits.

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In doing so, we may have composed f with $-$ or sw_A for some A. Since $-$ and sw_A are elements of G', these compositions do not change the fact that f could not be imitated by G' on $\bar{c}$. In particular, $f(\bar{c}) \ncong \bar{c}$. Hence, we are in the situation of either Lemma 2.6(iv) or Lemma 2.7(iv). Thus, either $G = \mathrm{Sym}(D)$ and we are done, or $(D, \overline{E})$ is a homogeneous graph-contradiction.

We have shown that there are no closed groups in between G' and $\mathrm{Sym}(D)$. Since $\mathrm{Aut}(D, \overline{E})$ contains G' and is a proper subgroup of $\mathrm{Sym}(D)$, we must conclude that $G' = \mathrm{Aut}(D, \overline{E})$, as required. ⊣

§3. $2^{\aleph_0}$ pairwise nonisomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

Definition 3.1. Let G be a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. We say that G is *maximal-closed* if $G \neq \mathrm{Sym}(\mathbb{N})$ and there are no closed groups G' such that $G < G' < \mathrm{Sym}(\mathbb{N})$.

We construct $2^{\aleph_0}$ pairwise nonisomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ by modifying Henson's construction of $2^{\aleph_0}$ pairwise nonisomorphic homogeneous countable digraphs and taking their automorphism groups. The modification is needed to ensure that the groups are maximal. A short argument will show that the automorphism groups are pairwise nonconjugate. The groups are even pairwise nonisomorphic, since by a result of Rubin [24] automorphism groups of Henson digraphs are conjugate if and only if they are isomorphic as abstract groups.

Henson's construction in [13] centres on finding an infinite antichain, with respect to embeddability, of finite tournaments.

Definition 3.2. Let $n \in \mathbb{N}\backslash\{0\}$. I_n denotes the n-element tournament obtained from the linear order L_n by changing the direction of the edges $(i, i+1)$ for $i = 1, \ldots, n-1$ and of the edge $(1, n)$.

By counting 3-cycles, Henson showed that $\{I_n : n \geq 6\}$ is an antichain. It is a short exercise to show that the 3-cycles in I_n are $(1, 3, n), (1, 4, n), \ldots, (1, n-2, n), (3, 2, 1), (4, 3, 2), \ldots, (n, n-1, n-2)$. In particular, observe that I_n has at most two vertices through which there are more than four 3-cycles, namely the vertices 1 and n; this observation is useful in our modification.

The automorphism groups of the Henson digraphs constructed by forbidding any subset of these I_n's are not maximal: $\langle - \rangle$ and the automorphism group of the random graph are closed supergroups. By forbidding a few extra tournaments, however, we can ensure that the automorphism groups are maximal.

In a digraph, a *source*, respectively *sink*, is a vertex which only has outgoing, respectively incoming, edges adjacent to it. Then let T be a finite tournament that is not embeddable in I_n for any n and that contains a source but no sink. Such a T can be found, for example, by ensuring there are at least three vertices through which there are more than four 3-cycles.

Let $k = |T|$. Let $\mathcal{T} = \{T' : |T| = k+1, T \text{ is embeddable in } T'\}$. Then for $A \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$, let $\mathcal{T}_A = \mathcal{T} \cup \{I_n : n \in A\}$. Then let D_A be the Henson digraph whose set of forbidden tournaments is $\mathcal{T}_A$. The automorphism groups of these D_A is the set of groups we want.

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THEOREM 3.3. $\{\text{Aut}(D_A) : A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}\}$ *is a set of* $2^{\aleph_0}$ *maximal-closed subgroups of* $\text{Sym}(\mathbb{N})$ *which are pairwise nonisomorphic as abstract groups.*

PROOF. CLAIM 1. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under $-$.

Let T' be obtained as follows: Change the direction of all the edges of T and then add a new vertex t which is a sink. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider $-(T')$. By construction, T is embeddable in $-(T')$, so $-(T') \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under $-$.

CLAIM 2. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under sw.

Let T' be obtained as follows: Change the source s in T to a sink, and then add a new vertex which will be a sink of T'. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider switching T' about s, to obtain T''. By construction, T is embeddable in T'', so $T'' \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under sw.

CLAIM 3. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, (D_A, \overline{E})$ is not a Henson graph nor the random graph.

Finite linear orders do not embed any element of $\mathcal{T}_A$, thus are embeddable in D_A. Removing the direction of the edges in a finite linear order gives a complete graph, so $(D_A, \overline{E})$ is not K_n-free for any n, so $(D_A, \overline{E})$ is not a Henson graph.

Now let $U \subset D_A$ be isomorphic to T—this is possible as T has not been forbidden. Then there is no vertex $x \in D$ such that for all $u \in U$, $E(x,u) \vee E(u,x)$, because all tournaments containing T are forbidden. Hence $(D_A, \overline{E})$ does not satisfy the extension property of the random graph and so is not isomorphic to the random graph.

CLAIM 4. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$, $\text{Aut}(D_A)$ is a maximal-closed subgroup of $\text{Sym}(\mathbb{N})$.

This follows from the classification Theorem 2.2 and the previous three claims.

CLAIM 5. Let $A = \mathbb{N}\backslash\{1,\ldots,k+1\}$. Then $\mathcal{T}_A$ is an antichain with respect to embeddability.

Let $T_1, T_2 \in \mathcal{T}_A$ and suppose for contradiction that T_1 is embeddable in T_2. All elements of $\mathcal{T}_A$ have size at least $k+1$ and $|T_2|$ must be bigger than $|T_1|$, so $|T_2| \geq k+2$. Hence, $T_2 \notin \mathcal{T}$, so $T_2 = I_n$ for some $n \in A$. By Henson's arguments, T_1 cannot equal I_m for any $m \in A$. Thus $T_1 \in \mathcal{T}$, which implies that T is embeddable in I_n, contradicting our choice for T.

CLAIM 6. If $A, B \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$ are not equal, then $D_A \ncong D_B$.

Suppose, without loss of generality, that there is some n in A but not in B. Then I_n is not embeddable in D_A. To prove the claim, it suffices to show that I_n *is* embeddable in D_B. Suppose for contradiction that it is not. Hence, $I_n \notin \text{Forb}(\mathcal{T}_B)$ which means that I_n embeds an element of $\mathcal{T}_B$. But this implies that $\mathcal{T}_{B\cup\{n\}}$ is not an antichain, contrary to Claim 5.

CLAIM 7. If $A, B \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$ are not equal, then $\text{Aut}(D_A)$ and $\text{Aut}(D_B)$ are not conjugate.

We prove the contrapositive so suppose that $\text{Aut}(D_A)$ and $\text{Aut}(D_B)$ are conjugate. By the Theorem of Engeler, Ryll-Nardzewski, and Svenonius (see [14, Theorem 6.3.1]), this implies that D_A and D_B are interdefinable up to isomorphism. Let $\phi(x,y)$ be the definition of $E_A(x,y)$ in D_B. By quantifier elimination, $\phi(x,y) = E_B(x,y), E_B^*(x,y)$ or $N_B(x,y)$. N_B is a symmetric relation, but E is antisymmetric, so ϕ cannot equal N_B. If $\phi = E_B^*$, it implies that if a tournament T is embeddable in D_A, then $-(T)$ is embeddable in D_B. This is not possible because we know from (the proof of) Claim 1 that $\mathcal{T}$ is not closed under $-$. Hence, $\phi = E_B$, which implies that D_A and D_B are isomorphic, which by Claim 6 implies that $A = B$.

CLAIM 8. If $A, B \subseteq \mathbb{N}\backslash\{1, \ldots, k+1\}$ are not equal, then $\mathrm{Aut}(D_A)$ and $\mathrm{Aut}(D_B)$ are not isomorphic as pure groups.

This follows from Claim 7 and Rubin's reconstruction results [24].

Together, Claims 4 and 8 prove the theorem. ⊣

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