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COMBINATORIAL PROPERTY OF A SPECIAL POLYNOMIAL SEQUENCE

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1. Leeming [4] has defined a sequence of polynomials $\{Q_{4n}(x)\}$ and a sequence of integers $\{Q_{4n}\}$ by means of

$$\frac{\cosh xz+\cos xz}{\cosh z+\cos z}=\sum_{n=0}^{\infty} Q_{4n}(x)\frac{z^{4n}}{(4n)!} \tag{1}$$

and

$$Q_{4n}=Q_{4n}(0). \tag{2}$$

Thus

$$\frac{2}{\cosh z+\cos z}=\sum_{n=0}^{\infty} Q_{4n}\frac{z^{4n}}{(4n)!}. \tag{3}$$

Leeming showed that the Q_{4n} are all odd and that

$$(-1)^n Q_{4n}>0 \qquad (n=0,1,2,\ldots). \tag{4}$$

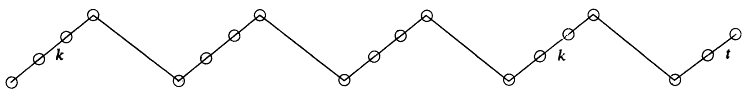
It is proved in [3] that

$$Q_{4n}\equiv 1-2n+8\binom{n}{2} \pmod{16}. \tag{5}$$

Let k and t be fixed integers, $k\geq 2$, $t\geq 0$ and consider permutations $(a_1, a_2, \ldots, a_{kn+t})$ of $Z_{kn+t}=\{1,2,3,\ldots,kn+t\}$ such that

$$\begin{cases} a_{kj+1}<a_{kj+2}<\cdots<a_{kj+k},\ a_{kj+k}>a_{kj+k+1} & (j=0,1,\ldots,n-1)\\ a_{kn+1}<a_{kn+2}<\cdots<a_{kn+t}. \end{cases} \tag{6}$$

This is best indicated by the sketch



Let $A_k(kn+t)$ denote the number of permutations of Z_{kn+t} that satisfy (6). It

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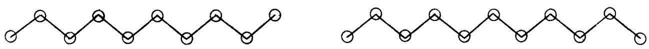
is proved in [1], [2] that

$$\sum_{n=0}^{\infty} A_k(kn)\frac{x^{kn}}{(kn)!}=\frac{1}{\sum_{n=0}^{\infty}(-1)^n\frac{x^{kn}}{(kn)!}} \tag{7}$$

while

$$\sum_{n=0}^{\infty} A_k(kn+t)\frac{x^{kn+t}}{(kn+t)!}=\frac{\sum_{n=0}^{\infty}(-1)^n\frac{x^{kn+t}}{(kn+t)!}}{\sum_{n=0}^{\infty}(-1)^n\frac{x^{kn}}{(kn)!}} \qquad (t\geq 1). \tag{8}$$

In particular, for $k=2$, the permutations that satisfy (6) are so-called up-down permutations:



For this special case (7) and (8) reduce to the well-known result of André [5, 105–112]

$$\sum_{n=0}^{\infty} A_2(n)\frac{z^n}{n!}=\sec z+\tan z. \tag{9}$$

In what follows we take $k=4$, $t=0, 1, 2, 3$. Put

$$\varepsilon=e^{2\pi i/8}=(1+i)/2^{\frac{1}{2}}. \tag{10}$$

Since

$$\tfrac{1}{2}(\cosh z+\cos z)=\sum_{n=0}^{\infty}\frac{z^{4n}}{(4n)!}$$

it follows that

$$\tfrac{1}{2}(\cosh \varepsilon z+\cos \varepsilon z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n}}{(4n)!}. \tag{11}$$

Differentiation yields

$$\begin{cases} \tfrac{1}{2}\varepsilon(\sinh \varepsilon z-\sin \varepsilon z)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{4n+3}}{(4n+3)!} \\ \tfrac{1}{2}\varepsilon^2(\cosh \varepsilon z-\cos \varepsilon z)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{4n+2}}{(4n+2)!} \\ \tfrac{1}{2}\varepsilon^3(\sinh \varepsilon z+\sin \varepsilon z)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{4n+1}}{(4n+1)!}. \end{cases} \tag{12}$$

On the other hand, it follows from (3) that

$$\sum_{n=0}^{\infty}(-1)^n Q_{4n}\frac{z^{4n}}{(4n)!}=\frac{2}{\cosh \varepsilon z+\cos \varepsilon z}, \tag{13}$$

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while differentiation of (1) with respect to x gives

$$\sum_{n=1}^{\infty} Q'_{4n}(x)\frac{z^{4n}}{(4n)!}=z\frac{\sinh xz-\sin xz}{\cosh z+\cos z}$$

$$\sum_{n=1}^{\infty} Q''_{4n}(x)\frac{z^{4n}}{(4n)!}=z^2\frac{\cosh xz-\cos xz}{\cosh z+\cos z}$$

$$\sum_{n=1}^{\infty} Q'''_{4n}(x)\frac{z^{4n}}{(4n)!}=z^3\frac{\sinh xz+\sin xz}{\cosh z+\cos z}.$$

Replacing z by εz and taking $x=1$, the last three formulas become

$$(14)\qquad \begin{cases} \sum_{n=0}^{\infty} (-1)^{n+1} Q'_{4n}(1)\dfrac{z^{4n+4}}{(4n+4)!}=\varepsilon\dfrac{\sinh \varepsilon z-\sin \varepsilon z}{\cosh \varepsilon z+\cos \varepsilon z} \\ \sum_{n=0}^{\infty} (-1)^{n+1} Q''_{4n}(1)\dfrac{z^{4n+4}}{(4n+4)!}=\varepsilon^2\dfrac{\cosh \varepsilon z-\cos \varepsilon z}{\cosh \varepsilon z+\cos \varepsilon z} \\ \sum_{n=0}^{\infty} (-1)^{n+1} Q'''_{4n}(1)\dfrac{z^{4n+4}}{(4n+4)!}=\varepsilon^3\dfrac{\sinh \varepsilon z+\sin \varepsilon z}{\cosh \varepsilon z+\cos \varepsilon z}. \end{cases}$$

Hence by (11), (12), (13), and (14) we get

$$(15)\qquad \begin{cases} (-1)^n Q_{4n}=A_4(4n) \\ (-1)^{n+1}Q'_{4n+4}(1)=(4n+4)A_4(4n+3) \\ (-1)^{n+1}Q''_{4n+4}(1)=(4n+4)(4n+3)A_4(4n+2) \\ (-1)^{n+1}Q'''_{4n+4}(1)=(4n+4)(4n+3)(4n+2)A_4(4n+1). \end{cases}$$

2. Leeming noted that

$$(16)\qquad (-4)^n Q_{4n}=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}^k\binom{4n}{2k}E_{2k}E_{4n-2k},$$

where the E_{2n} are the Euler numbers defined by

$$(17)\qquad \sec z=\sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{z^{2n}}{(2n)!}.$$

Thus, by (9)

$$(18)\qquad A_2(2n)=(-1)^n E_{2n}.$$

Since [6, Ch. 2]

$$(19)\qquad \tan z=\sum_{n=1}^{\infty}(-1)^n C_{2n-1}\frac{z^{2n-1}}{(2n-1)!},$$

where

$$C_{2n-1}=2^{2n}(1-2^{2n})\frac{B_{2n}}{2n}$$

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and the B_n are the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \tag{20}$$

it follows that

$$A_2(2n-1)=(-1)^{n-1}2^{2n}(2^{2n}-1)\frac{B_{2n}}{2n}. \tag{21}$$

Note also that by the first of (15), together with (16), (17), and (18), we have

$$4^n A_4(4n)=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}A_2(2k)A_2(4n-2k). \tag{22}$$

Since

$$\cosh z+\cos z=2\cosh\tfrac{1}{2}(1+i)z\cosh\tfrac{1}{2}(1-i)z$$

$$\cosh z-\cos z=-2\sinh\tfrac{1}{2}(1+i)z\sinh\tfrac{1}{2}(1-i)z,$$

$$\begin{aligned}\frac{\cosh \varepsilon z-\cos \varepsilon z}{\cosh \varepsilon z+\cos \varepsilon z}&=-\tanh\tfrac{1}{2}\varepsilon(1+i)z\tanh\tfrac{1}{2}\varepsilon(1-i)z\\ &=-\tanh\frac{1}{\sqrt{2}}iz\tanh\frac{1}{\sqrt{2}}z\\ &=-i\tan\frac{1}{\sqrt{2}}z\tanh\frac{1}{\sqrt{2}}z.\end{aligned}$$

Hence, by (11) and (12)

$$\tan\frac{1}{\sqrt{2}}z\tanh\frac{1}{\sqrt{2}}z=\frac{\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+2}}{(4n+2)!}}{\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n}}{(4n)!}},$$

so that

$$\sum_{n=0}^{\infty}A_4(4n+2)\frac{z^{4n+2}}{(4n+2)!}=\tan\frac{1}{\sqrt{2}}z\tanh\frac{1}{\sqrt{2}}z,$$

or better

$$\sum_{n=0}^{\infty}A_4(4n+2)\frac{2^{2n+1}z^{4n+2}}{(4n+2)!}=\tan z\tanh z. \tag{23}$$

Since

$$\tan z=\sum_{n=0}^{\infty}A_2(2n+1)\frac{z^{2n+1}}{(2n+1)!},$$

$$\tanh z=\sum_{n=0}^{\infty}(-1)^n A_2(2n+1)\frac{z^{2n+1}}{(2n+1)!},$$

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it follows from (23) that

$$(24)\qquad 2^{2n+1}A_4(4n+2)=\sum_{k=0}^{2n}(-1)^k\binom{4n+2}{2k+1}A_2(2k+1)A_2(4n-2k+1).$$

For example, for $n=1$, this gives

$$8A_4(6)=6A_2(1)A_2(5)-20A_2(3)A_2(3)+6A_2(5)A_2(1).$$

Since $A_2(1)=1$, $A_2(3)=2$, $A_2(5)=16$, $A_4(6)=14$, this is correct.

We shall now show that

$$(25)\qquad A_4(2n+1)=2^{-n}A_2(2n+1).$$

By (8) and (9), (25) is equivalent to

$$2^{1/2}\tan z=\sum_{n=0}^{\infty}A_4(2n+1)\frac{(2^{1/2}z)^{2n+1}}{(2n+1)!}$$
$$=\frac{\sum_{n=0}^{\infty}(-1)^n\frac{(2^{1/2}z)^{4n+1}}{(4n+1)!}+\sum_{n=0}^{\infty}(-1)^n\frac{(2^{1/2}z)^{4n+3}}{(4n+3)!}}{\sum_{n=0}^{\infty}(-1)^n\frac{(2^{1/2}z)^{4n}}{(4n)!}}$$

Replacing z by εz, this becomes

$$2^{1/2}\tan\varepsilon z=\frac{\varepsilon(\sinh z+\sin z)+\varepsilon^3(\sinh z-\sin z)}{\cosh z+\cos z}$$
$$=\varepsilon(1-i)\frac{\sin z+\sin iz}{\cos z+\cos iz}$$
$$=2^{1/2}\frac{\sin\frac{1}{2}(1+i)z\cos\frac{1}{2}(1-i)z}{\cos\frac{1}{2}(1+i)z\cos\frac{1}{2}(1-i)z}$$
$$=2^{1/2}\tan\tfrac{1}{2}(1+i)z.$$

This evidently proves (25).

It is easily verified that

$$A_2(3)=2,\qquad A_2(5)=16,\qquad A_2(7)=272,\qquad A_2(9)=7936,$$

while

$$A_4(3)=1,\qquad A_4(5)=4,\qquad A_4(7)=34,\qquad A_4(9)=496,$$

in agreement with (25).

It would be of interest to find a direct, combinatorial proof of (25).

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