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ON THE STRUCTURE OF NON-NEGATIVE INTEGER SETS WHICH HAVE IDENTICAL REPRESENTATION FUNCTIONS

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Abstract Let $\mathbb{N}$ be the set of all non-negative integers. For any integer r and m, let $r+m\mathbb{N}=\{r+mk: k\in\mathbb{N}\}$. For $S\subseteq\mathbb{N}$ and $n\in\mathbb{N}$, let $R_S(n)$ denote the number of solutions of the equation $n=s+s'$ with $s,s'\in S$ and $s<s'$. Let r_1,r_2,m be integers with $0<r_1<r_2<m$ and $2\mid r_1$. In this paper, we prove that there exist two sets C and D with $C\cup D=\mathbb{N}$ and $C\cap D=(r_1+m\mathbb{N})\cup(r_2+m\mathbb{N})$ such that $R_C(n)=R_D(n)$ for all $n\in\mathbb{N}$ if and only if there exists a positive integer l such that $r_1=2^{2l+1}-2, r_2=2^{2l+1}-1, m=2^{2l+2}-2$.

Keywords: Sárközy's problem; representation function; Thue–Morse sequence

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1. Introduction

Let $\mathbb{N}$ be the set of all non-negative integers. For any integer r and m, let $r+m\mathbb{N}=\{r+mk: k\in\mathbb{N}\}$. Let A be the set of all non-negative integers which contain an even number of digits 1 in their binary representations and $B=\mathbb{N}\backslash A$. The set A is called Thue–Morse sequence. For any positive integer l, let $A_l=A\cap[0,2^l-1]$ and $B_l=B\cap[0,2^l-1]$. For $S\subseteq\mathbb{N}$ and $n\in\mathbb{N}$, let the representation function $R_S(n)$ denote the number of solutions of the equation $s+s'=n$ with $s,s'\in S$ and $s<s'$. Sárközy asked whether there exist two subsets $C,D\subseteq\mathbb{N}$ with $|(C\cup D)\backslash(C\cap D)|=\infty$ such that $R_C(n)=R_D(n)$ for all sufficiently large integers n. By using the Thue–Morse sequence, Dombi [6] answered Sárközy's problem affirmatively. Later, Lev [10], Sándor [12] and Tang [17] proved this result by different methods. Partitions of non-negative integers and their corresponding representation functions have been extensively studied by many authors. The related results can be found in [4, 5, 7–9, 12–17, 19].

In 2012, Yu and Tang [20] began to focus on partitions of non-negative integers with the intersection not empty. They studied the intersection of two sets is an arithmetic progression and posed the following conjecture:

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Conjecture 1.1. *Let $m \in \mathbb{N}$ and $R \subset \{0, 1, \ldots, m-1\}$. If $C \cup D = \mathbb{N}$ and $C \cap D = \{r + km : k \in \mathbb{N}, r \in R\}$, then $R_C(n) = R_D(n)$ cannot hold for all sufficiently large n.*

In 2016, Tang [18] obtained the following theorem.

Theorem A [18, Theorem 1]. *Let m be an integer with $m \geq 2$. If $C \cup D = \mathbb{N}$ and $C \cap D = m\mathbb{N}$, then $R_C(n) = R_D(n)$ cannot hold for all large enough integers n.*

In 20l6, Chen and Lev [3] disproved Conjecture 1.1 by the following result.

Theorem B [3, Theorem 1]. *Let l be a positive integer. There exist two sets C and D with $C \cup D = \mathbb{N}$ and $C \cap D = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}$ such that $R_C(n) = R_D(n)$ for every positive integer n.*

In [3], Chen and Lev also proposed the following problem.

Problem 1.2. *Given $R_C(n) = R_D(n)$ for every positive integer n, $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$ with $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1, m = 2^{2l+1} - 1$?*

Afterwards, Li and Tang [11], Chen, Tang and Yang [2] solved Problem 1.2 under the condition $0 \leq r < m$. In 2021, Chen and Chen [1] solved Problem 1.2 affirmatively.

Theorem C [1, Theorem 1.1]. *Let $m \geq 2$ and $r \geq 0$ be two integers and let C and D be two sets with $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$ such that $R_C(n) = R_D(n)$ for every positive integer n. Then there exists a positive integer l such that $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$.*

Let r_1, r_2, m be integers with $0 < r_1 < r_2 < m$. In this paper, we focus on partitions of non-negative integers into two sets C, D with $C \cup D = \mathbb{N}$ and $C \cap D = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N})$ such that $R_C(n) = R_D(n)$ for all $n \in \mathbb{N}$ and obtain the following result.

Theorem 1.3. *Let r_1, r_2, m be integers with $0 < r_1 < r_2 < m$ and $2 \mid r_1$. Then there exist two sets C and D with $C \cup D = \mathbb{N}$ and $C \cap D = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N})$ such that $R_C(n) = R_D(n)$ for all $n \in \mathbb{N}$ if and only if there exists a positive integer l such that $r_1 = 2^{2l+1} - 2, r_2 = 2^{2l+1} - 1, m = 2^{2l+2} - 2$.*

Motivated by Theorems B and C, we propose the following conjecture for further research.

Conjecture 1.4. *Let r_1, r_2, m be integers with $0 < r_1 < r_2 < m$ and $2 \nmid r_1$. Then there exist two sets C and D with $C \cup D = \mathbb{N}$ and $C \cap D = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N})$ such that $R_C(n) = R_D(n)$ for all $n \in \mathbb{N}$ if and only if there exists a positive integer l such that $r_1 = 2^{2l} - 1, r_2 = 2^{2l+1} + 2^{2l} - 2, m = 2^{2l+2} - 2$.*

Throughout this paper, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ and for $m \leq n$, define

$$(f(x))_m = a_0 + a_1 x + \cdots + a_m x^m.$$

For $C, D \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_{C,D}(n)$ be the number of solutions of $n = c + d$ with $c \in C$ and $d \in D$. Let $C + D = \{c + d : c \in C, d \in D\}$. Let $C(x)$ be the set of integers in C

which are less than or equal to x. The characteristic function of C is denoted by

$$\chi_C(n) = \begin{cases} 1, & n \in C, \\ 0, & n \notin C. \end{cases}$$

2. Some lemmas

Lemma 2.1. [3, **Lemma 1**]. *Suppose that $C_0, D_0 \subseteq \mathbb{N}$ satisfy $R_{C_0}(n) = R_{D_0}(n)$ for all $n \in \mathbb{N}$, and that m is a non-negative integer with $m \notin (C_0 - D_0) \cup (D_0 - C_0)$. Then, letting*

$$C_1 := C_0 \cup (m + D_0) \text{and } D_1 := D_0 \cup (m + C_0),$$

we have $R_{C_1}(n) = R_{D_1}(n)$ for all $n \in \mathbb{N}$ and furthermore

(i) $C_1 \cup D_1 = (C_0 \cup D_0) \cup (m + C_0 \cup D_0)$;
(ii) $C_1 \cap D_1 \supseteq (C_0 \cap D_0) \cup (m + C_0 \cap D_0)$, *the union being disjoint.*

Moreover, if $m \notin (C_0 - C_0) \cup (D_0 - D_0)$, then also in (i) the union is disjoint, and in (ii) the inclusion is in fact an equality. In particular, if $C_0 \cup D_0 = [0, m-1]$, then $C_1 \cup D_1 = [0, 2m-1]$, and if C_0 and D_0 indeed partition the interval $[0, m-1]$, then C_1 and D_1 partition the interval $[0, 2m-1]$.

Lemma 2.2. [8, **Claim 1**]. *Let $0 < r_1 < \cdots < r_s \leq m$ be integers. Then there exists at most one pair of sets (C, D) such that $C \cup D = [0, m], 0 \in C, C \cap D = \{r_1, \ldots, r_s\}$ and $R_C(n) = R_D(n)$ for every $n \leq m$.*

Lemma 2.3. [8, **Claim 3**]. *If for some positive integer M, the integers $M-1, M-2, M-4, M-8, \ldots, M-2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set A, then $\lceil \log_2 M \rceil$ is odd and $M = 2^{\lceil \log_2 M \rceil} - 1$.*

Lemma 2.4. [8, **Claim 4**]. *If for some positive integer M, the integers $M-1, M-2, M-4, M-8, \ldots, M-2^{\lceil \log_2 M \rceil - 1}$ are all contained in the set B, then $\lceil \log_2 M \rceil$ is even and $M = 2^{\lceil \log_2 M \rceil} - 1$.*

Lemma 2.5. [8, **Theorem 3**]. *Let C and D be sets of non-negative integers such that $C \cup D = [0, m], C \cap D = \emptyset$ and $0 \in C$. Then $R_C(n) = R_D(n)$ for every positive integer n if and only if there exists a positive integer l such that $C = A_l$ and $D = B_l$.*

3. Proofs

Proof of Theorem 1.3. (Sufficiency). For any given positive integer l, let

$$m_i = \begin{cases} 2^{i+1}, & 0 \leq i \leq 2l-1, \\ 2^{i+1} - 2, & i = 2l, \\ 2^{i+1} - 2^{i-2l}, & i \geq 2l+1. \end{cases} \tag{3.1}$$

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For given sets $C_0 = \{0\}, D_0 = \{1\}$, define

$$C_i = C_{i-1} \cup (m_{i-1} + D_{i-1}), \quad D_i = D_{i-1} \cup (m_{i-1} + C_{i-1}), \quad i = 1, 2, \ldots \tag{3.2}$$

and

$$C = \bigcup_{i \in \mathbb{N}} C_i, \qquad D = \bigcup_{i \in \mathbb{N}} D_i. \tag{3.3}$$

It is clear that C_0 and D_0 partition the interval $[0, m_0 - 1]$ and

$$m_0 = 2 \notin (C_0 - D_0) \cup (D_0 - C_0) \cup (C_0 - C_0) \cup (D_0 - D_0)$$

and $R_{C_0}(n) = R_{D_0}(n)$ for all $n \in \mathbb{N}$ (both representation functions are identically equal to 0). Applying Lemma 2.1 inductively $2l-1$ times, we can deduce that for any $i \in [0, 2l-1]$, $R_{C_i}(n) = R_{D_i}(n)$ for all $n \in \mathbb{N}$, the sets C_i and D_i partition the interval $[0, m_i - 1]$ and

$$m_i = 2^{i+1} \notin (C_i - D_i) \cup (D_i - C_i) \cup (C_i - C_i) \cup (D_i - D_i).$$

In particular, $R_{C_{2l-1}}(n) = R_{D_{2l-1}}(n)$ for all $n \in \mathbb{N}$, the sets C_{2l-1} and D_{2l-1} partition the interval $[0, m_{2l-1} - 1] = [0, 2^{2l} - 1]$ and

$$m_{2l-1} = 2^{2l} \notin (C_{2l-1} - D_{2l-1}) \cup (D_{2l-1} - C_{2l-1}) \cup (C_{2l-1} - C_{2l-1}) \cup (D_{2l-1} - D_{2l-1}).$$

By Lemma 2.1, we have $R_{C_{2l}}(n) = R_{D_{2l}}(n)$ for all $n \in \mathbb{N}$, the sets C_{2l} and D_{2l} partition the interval $[0, 2m_{2l-1} - 1] = [0, 2^{2l+1} - 1] = [0, m_{2l} + 1]$. In addition, it is easily seen that $\{0, m_{2l}\} \subseteq C_{2l}$ and $\{1, m_{2l} + 1\} \subseteq D_{2l}$. Then

$$m_{2l} \notin (C_{2l} - D_{2l}) \cup (D_{2l} - C_{2l}), \quad m_{2l} \in (C_{2l} - C_{2l}) \cup (D_{2l} - D_{2l}).$$

By Lemma 2.1, we have $R_{C_{2l+1}}(n) = R_{D_{2l+1}}(n)$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
C_{2l+1} \cup D_{2l+1} &= [0, 2m_{2l} + 1] = [0, m_{2l+1} - 1], \\
C_{2l+1} \cap D_{2l+1} &= (C_{2l} \cup (m_{2l} + D_{2l})) \cap (D_{2l} \cup (m_{2l} + C_{2l})) \\
&= (C_{2l} \cap D_{2l}) \cup (C_{2l} \cap (m_{2l} + C_{2l})) \cup (D_{2l} \cap (m_{2l} + D_{2l})) \\
&\quad \cup (m_{2l} + C_{2l} \cap D_{2l}) \\
&= \{m_{2l}, m_{2l} + 1\}.
\end{aligned}$$

Applying again Lemma 2.1, we can conclude that $R_{C_i}(n) = R_{D_i}(n)$ for all $n \in \mathbb{N}$, $C_i \cup D_i = [0, m_i - 1]$ and $C_i \cap D_i = \{m_{2l}, m_{2l} + 1\} + \{0, m_{2l+1}, \ldots, (2^{i-2l} - 1)m_{2l+1}\}$ for each $i \geq 2l + 1$.

Therefore, by the definitions of C and D in (3.1)–(3.3), we have $R_C(n) = R_D(n)$ for all $n \in \mathbb{N}$, $C \cup D = \mathbb{N}$ and

$$C \cap D = \{m_{2l}, m_{2l} + 1\} + m_{2l+1}\mathbb{N} = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N}).$$

(Necessity). To prove the necessity of Theorem 1.3, we need the following three claims.

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Claim 1. *Given $0 < r_1 < r_2 \le m$, there exists at most one pair of sets (C, D) such that $C \cup D = \mathbb{N}$, $C \cap D = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N})$ and $R_C(n) = R_D(n)$ for all $n \in \mathbb{N}$.*

Proof of Claim 1. Assume that there exist at least two pairs of sets (C, D) and (C', D') which satisfy the conditions

$$C \cup D = \mathbb{N}, C \cap D = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N}), R_C(n) = R_D(n) \text{ for all } n \in \mathbb{N},$$

$$C' \cup D' = \mathbb{N}, C' \cap D' = (r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N}), R_{C'}(n) = R_{D'}(n) \text{ for all } n \in \mathbb{N}.$$

We may assume that $0 \in C \cap C'$. Let k be the smallest positive integer such that $\chi_C(k) \ne \chi_{C'}(k)$. Write

$$\big((r_1 + m\mathbb{N}) \cup (r_2 + m\mathbb{N})\big) \cap [0, k] = \{t_1, \dots, t_s\},$$

$$C_1 = C \cap [0, k], \quad D_1 = D \cap [0, k],$$

$$C_2 = C' \cap [0, k], \quad D_2 = D' \cap [0, k].$$

Then

$$C_1 \cup D_1 = C_2 \cup D_2 = [0, k], \tag{3.4}$$

$$C_1 \cap D_1 = C_2 \cap D_2 = \{t_1, \dots, t_s\}, \tag{3.5}$$

$$\chi_{C_1}(k) \ne \chi_{C_2}(k), \quad 0 \in C_1 \cap C_2. \tag{3.6}$$

For any integer $n \in [0, k]$, by the hypothesis, we have

$$\begin{aligned} R_{C_1}(n) &= |\{(c, c') : c < c' \le n, c, c' \in C_1, c + c' = n\}| \\ &= R_C(n) = R_D(n) = R_{D_1}(n), \end{aligned} \tag{3.7}$$

$$\begin{aligned} R_{C_2}(n) &= |\{(c, c') : c < c' \le n, c, c' \in C_2, c + c' = n\}| \\ &= R_{C'}(n) = R_{D'}(n) = R_{D_2}(n). \end{aligned} \tag{3.8}$$

Thus there exist two pairs of sets (C_1, D_1) and (C_2, D_2) satisfying (3.4)–(3.8). By Lemma 2.2, this is impossible. This completes the proof of Claim 1. □

Claim 2. *Let r_1, r_2, m be integers with $0 < r_1 < r_2 < r_1 + r_2 \le m$ and $2 \mid r_1$. Let C and D be sets of non-negative integers such that $C \cup D = [0, m]$, $C \cap D = \{r_1, r_2\}$ and $0 \in C$. If $R_C(n) = R_D(n)$ for any integer $n \in [0, m]$, then there exists a positive integer l such that $r_1 = 2^{2l+1} - 2, r_2 = 2^{2l+1} - 1$.*

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Proof of Claim 2. Let

$$p_C(x)=\sum_{i=0}^{m}\chi_C(i)x^i,\ \ p_D(x)=\sum_{i=0}^{m}\chi_D(i)x^i. \tag{3.9}$$

Then

$$\frac{1}{2}(p_C(x)^2-p_C(x^2))=\sum_{n=0}^{\infty}R_C(n)x^n,\ \ \frac{1}{2}(p_D(x)^2-p_D(x^2))=\sum_{n=0}^{\infty}R_D(n)x^n. \tag{3.10}$$

Since $R_C(n)=R_D(n)$ for any integer $n\in[0,m]$, we have

$$\left(\sum_{n=0}^{\infty}R_C(n)x^n\right)_m=\left(\sum_{n=0}^{\infty}R_D(n)x^n\right)_m. \tag{3.11}$$

By (3.9)–(3.11), we have

$$\begin{aligned}\left(\frac{1}{2}(p_C(x)^2-p_C(x^2))\right)_m&=\left(\sum_{n=0}^{\infty}R_C(n)x^n\right)_m=\left(\sum_{n=0}^{\infty}R_D(n)x^n\right)_m\\&=\left(\frac{1}{2}(p_D(x)^2-p_D(x^2))\right)_m.\end{aligned}$$

Noting that $C\cup D=[0,m]$, $C\cap D=\{r_1,r_2\}$, we have

$$p_D(x)=\frac{1-x^{m+1}}{1-x}-p_C(x)+x^{r_1}+x^{r_2}.$$

Then

$$\begin{aligned}(p_C(x)^2-p_C(x^2))_m=\Bigg(&\left(\frac{1-x^{m+1}}{1-x}-p_C(x)+x^{r_1}+x^{r_2}\right)^2\\&-\left(\frac{1-x^{2m+2}}{1-x^2}-p_C(x^2)+x^{2r_1}+x^{2r_2}\right)\Bigg)_m.\end{aligned}$$

Thus

$$\begin{aligned}(2p_C(x^2))_m\ =\ &\Bigg(\tfrac{1-x^{2m+2}}{1-x^2}+2p_C(x)x^{r_1}+2p_C(x)x^{r_2}+2p_C(x)\tfrac{1-x^{m+1}}{1-x}\\&-\left(\tfrac{1-x^{m+1}}{1-x}\right)^2-2x^{r_1}\tfrac{1-x^{m+1}}{1-x}-2x^{r_2}\tfrac{1-x^{m+1}}{1-x}-2x^{r_1+r_2}\Bigg)_m.\end{aligned} \tag{3.12}$$

An easy calculation shows that $r_1\ge 6$, $\{0,3,5,6\}\subset C$ and $\{1,2,4,7\}\subset D$.

In order to prove $r_2=r_1+1$, we suppose that $r_2\ge r_1+2$ and we will show that this leads to a contradiction.

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The coefficient of x^{r_1-1} in (3.12) is $0 = 2\sum_{i=0}^{r_1-1}\chi_C(i) - r_1$. Since $r_1 \in C$, we have $\chi_C(r_1) = 1$. Then $2\sum_{i=0}^{r_1}\chi_C(i) = r_1 + 2$. The coefficient of x^{r_1} in (3.12) is $2\chi_C\big(\frac{r_1}{2}\big) = 2\sum_{i=0}^{r_1}\chi_C(i) - r_1 = 2$. Then $\chi_C\big(\frac{r_1}{2}\big) = 1$. The coefficient of x^{r_1+1} in (3.12) is $0 = 2\sum_{i=0}^{r_1+1}\chi_C(i) - r_1 - 4 = 2\chi_C(r_1+1) - 2$. Then $\chi_C(r_1+1) = 1$. The coefficient of x^{r_1+2} in (3.12) is $2\chi_C\big(\frac{r_1+2}{2}\big) = 2\sum_{i=0}^{r_1+2}\chi_C(i) - r_1 - 4$. Then $\chi_C\big(\frac{r_1+2}{2}\big) = \chi_C(r_1+2)$. If $r_2 = r_1 + 2$, then $\chi_C(r_1+2) = 1$. Comparing the coefficients of x^{r_1+s} with $s \in \{3,4,5\}$ on the both sides of (3.12), we have

$$\begin{aligned} 0 &= 2\sum_{i=0}^{r_1+3}\chi_C(i) - r_1 - 6,\\ 2\chi_C\left(\frac{r_1+4}{2}\right) &= 2\sum_{i=0}^{r_1+4}\chi_C(i) - r_1 - 8,\\ 0 &= 2\sum_{i=0}^{r_1+5}\chi_C(i) - r_1 - 6. \end{aligned}$$

Then $\chi_C(r_1+3) = 0$, $\chi_C(r_1+4) = 1$ and $\chi_C(r_1+5) = -1$, a contradiction. Thus $r_2 \ge r_1 + 3$. The coefficient of x^{r_1+3} in (3.12) is

$$0 = 2\sum_{i=0}^{r_1+3}\chi_C(i) - r_1 - 4 = 2\chi_C(r_1+2) + 2\chi_C(r_1+3).$$

Then $\chi_C(r_1+2) = \chi_C(r_1+3) = 0$. Thus $\chi_C\big(\frac{r_1+2}{2}\big) = 0$ and $r_2 \ge r_1 + 4$. The coefficient of x^{r_1+4} in (3.12) is

$$2\chi_C\left(\frac{r_1+4}{2}\right) = 2\sum_{i=0}^{r_1+4}\chi_C(i) - r_1 - 6 = 2\chi_C(r_1+4) - 2.$$

Then $\chi_C(r_1+4) = 1$, $\chi_C\big(\frac{r_1+4}{2}\big) = 0$ and $2\sum_{i=0}^{r_1+4}\chi_C(i) = r_1 + 6$. By Lemma 2.2, we have

$$C \cap [0, r_1 - 1] = A \cap [0, r_1 - 1], \quad D \cap [0, r_1 - 1] = B \cap [0, r_1 - 1]. \tag{3.13}$$

Since $\chi_C\big(\frac{r_1+2}{2}\big) = \chi_C\big(\frac{r_1+4}{2}\big)$ and $\frac{r_1+4}{2} \le r_1 - 1$, by (3.13) and the definition of A, we have $r_1 \equiv 0 \pmod 4$. It follows that $r_1 \ge 8$ and $\chi_C\big(\frac{r_1+6}{2}\big) = 1$.

Let k be a positive even integer such that $r_1 \le k < k + 1 < \min\{r_2, 2r_1\} \le m$. Comparing the coefficients of x^k and x^{k+1} on the both sides of (3.12) respectively, we

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have

$$2\chi_C\left(\frac{k}{2}\right)=2\chi_C(k-r_1)+2\sum_{i=0}^{k}\chi_C(i)-k-2,$$

$$0=2\chi_C(k+1-r_1)+2\sum_{i=0}^{k+1}\chi_C(i)-k-4.$$

Subtracting the above two equalities and dividing by 2 we can get

$$\chi_C\left(\frac{k}{2}\right)=\chi_C(k-r_1)-\chi_C(k+1-r_1)-\chi_C(k+1)+1. \tag{3.14}$$

Since $k+1-r_1<r_1, k-r_1$ is even, by (3.13), we have

$$\chi_C(k-r_1)+\chi_C(k+1-r_1)=1.$$

If $\chi_C(k-r_1)=0$, then $\chi_C(k+1-r_1)=1$. By (3.14), we get $\chi_C(\frac{k}{2})=0$. If $\chi_C(k-r_1)=1$, then $\chi_C(k+1-r_1)=0$. By (3.14), we get $\chi_C(\frac{k}{2})=1$. Thus

$$\chi_C(k-r_1)=\chi_C\left(\frac{k}{2}\right). \tag{3.15}$$

If $\min\{r_2,2r_1\}>2r_1-1$, then choose $k=2r_1-2^{i+1}$ with $i\geq 0$ in (3.15), we have

$$\chi_C(r_1-2^{i+1})=\chi_C(r_1-2^i).$$

Then

$$\chi_C(r_1-1)=\chi_C(r_1-2)=\chi_C(r_1-4)=\cdots=\chi_C(r_1-2^{\lceil\log_2 r_1\rceil-1}).$$

By Lemmas 2.3 and 2.4, we have $r_1=2^{\lceil\log_2 r_1\rceil}-1$, which contradicts $2\mid r_1$.

If $r_2=2r_1-1$, then compare the coefficients of x^{r_2} and x^{r_2-1} on the both sides of (3.12) respectively, we have

$$0=2\chi_C(r_1-1)+2\sum_{i=0}^{r_2}\chi_C(i)-r_2-3=2\chi_C(r_1-1)+2\sum_{i=0}^{r_2-1}\chi_C(i)-r_2-1,$$

$$2\chi_C(r_1-1)=2\chi_C(r_1-2)+2\sum_{i=0}^{r_2-1}\chi_C(i)-r_2-1.$$

Then $2\chi_C(r_1-1)=\chi_C(r_1-2)$. It follows that $\chi_C(r_1-2)=\chi_C(r_1-1)=0$, which contradicts $\chi_C(r_1-2)+\chi_C(r_1-1)=1$. Thus $r_2\leq 2r_1-2$.

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Let k be a non-negative integer such that $2r_1 \le 2r_1 + k < 2r_1 + k + 1 < r_1 + r_2 \le m$. If k is even, then compare the coefficients of x^{2r_1+s} with $s \in \{k, k+1\}$ on the both sides of (3.12), we have

$$2\chi_C\left(\tfrac{2r_1+k}{2}\right) = 2\chi_C(r_1+k) + 2\chi_C(2r_1+k-r_2) + 2\sum_{i=0}^{2r_1+k}\chi_C(i) - 2r_1 - k - 4,$$
$$0 = 2\chi_C(r_1+k+1) + 2\chi_C(2r_1+k+1-r_2) + 2\sum_{i=0}^{2r_1+k+1}\chi_C(i) - 2r_1 - k - 6.$$

Subtracting the above two equalities and dividing by 2 we can get

$$\chi_C\left(\frac{2r_1+k}{2}\right) = \chi_C(r_1+k) + \chi_C(2r_1+k-r_2) - \chi_C(2r_1+k+1) - \chi_C(r_1+k+1) - \chi_C(2r_1+k+1-r_2) + 1. \quad (3.16)$$

If k is odd, then compare the coefficients of x^{2r_1+s} with $s \in \{k, k+1\}$ on the both sides of (3.12), we have

$$0 = 2\chi_C(r_1+k) + 2\chi_C(2r_1+k-r_2) + 2\sum_{i=0}^{2r_1+k}\chi_C(i) - 2r_1 - k - 5,$$

$$2\chi_C\left(\frac{2r_1+k+1}{2}\right) = 2\chi_C(r_1+k+1) + 2\chi_C(2r_1+k+1-r_2) + 2\sum_{i=0}^{2r_1+k+1}\chi_C(i) - 2r_1 - k - 5.$$

Subtracting the above two equalities and dividing by 2 we can get

$$\chi_C\left(\frac{2r_1+k+1}{2}\right) = \chi_C(r_1+k+1) + \chi_C(2r_1+k+1-r_2) + \chi_C(2r_1+k+1) - \chi_C(r_1+k) - \chi_C(2r_1+k-r_2). \quad (3.17)$$

If r_2 is even, then choose $k=0$ and $k=2$ in (3.16) respectively, we have

$$\chi_C(r_1) = \chi_C(r_1) + \chi_C(2r_1-r_2) - \chi_C(2r_1+1) - \chi_C(r_1+1) - \chi_C(2r_1+1-r_2) + 1,$$

$$\chi_C(r_1+1) = \chi_C(r_1+2) + \chi_C(2r_1+2-r_2) - \chi_C(2r_1+3) - \chi_C(r_1+3) - \chi_C(2r_1+3-r_2) + 1.$$

Then

$$\chi_C(2r_1-r_2) - \chi_C(2r_1+1) - \chi_C(2r_1+1-r_2) = 0,$$

$$\chi_C(2r_1+2-r_2) - \chi_C(2r_1+3) - \chi_C(2r_1+3-r_2) = 0.$$

By (3.13), we have $\chi_C(2r_1-r_2) + \chi_C(2r_1+1-r_2) = 1$ and $\chi_C(2r_1+2-r_2) + \chi_C(2r_1+3-r_2) = 1$. Then $\chi_C(2r_1-r_2) = 1$ and $\chi_C(2r_1+2-r_2) = 1$. It follows that $r_2 \equiv 2$

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(mod 4). The coefficient of x^{r_2} in (3.12) is

$$2\chi_C\left(\frac{r_2}{2}\right) = 2\chi_C(r_2 - r_1) + 2\sum_{i=0}^{r_2} \chi_C(i) - r_2 - 2$$

and the coefficient of x^{r_2+1} in (3.12) is

$$0 = 2\chi_C(r_2 + 1 - r_1) + 2\sum_{i=0}^{r_2+1} \chi_C(i) - r_2 - 6.$$

Then

$$\chi_C\left(\frac{r_2}{2}\right) = \chi_C(r_2 - r_1) - \chi_C(r_2 + 1 - r_1) - \chi_C(r_2 + 1) + 2.$$

By $\chi_C(r_2 - r_1) + \chi_C(r_2 + 1 - r_1) = 1$, we have $\chi_C(r_2 - r_1) = 0$, $\chi_C(r_2 + 1 - r_1) = 1$. By $r_1 \equiv 0 \pmod 4$ and $r_2 \equiv 2 \pmod 4$, we have $\chi_C(r_2 - 1 - r_1) = 0$, $\chi_C(r_2 - 2 - r_1) = 1$. The coefficient of x^{r_2-1} in (3.12) is

$$0 = 2\chi_C(r_2 - 1 - r_1) + 2\sum_{i=0}^{r_2-1} \chi_C(i) - r_2 - 2.$$

Then $2\sum_{i=0}^{r_2-1} \chi_C(i) = r_2 + 2$. It follows that $2\sum_{i=0}^{r_2} \chi_C(i) = r_2 + 4$ and $\chi_C(\frac{r_2}{2}) = 1$. The coefficient of x^{r_2-2} in (3.12) is

$$2\chi_C\left(\frac{r_2 - 2}{2}\right) = 2\chi_C(r_2 - 2 - r_1) + 2\sum_{i=0}^{r_2-2} \chi_C(i) - r_2.$$

Then $\chi_C(\frac{r_2-2}{2}) = 2 - \chi_C(r_2 - 1)$. Thus $\chi_C(\frac{r_2-2}{2}) = \chi_C(r_2 - 1) = 1$. By (3.13) and $\chi_C(\frac{r_2-2}{2}) = \chi_C(\frac{r_2}{2}) = 1$, we have $r_2 \equiv 0 \pmod 4$, a contradiction.

If r_2 is odd, then $r_1 + 5 \leq r_2 \leq 2r_1 - 3$. The coefficient of x^{r_1+5} in (3.12) is

$$0 = 2\sum_{i=0}^{r_1+5} \chi_C(i) - r_1 - 6 = 2\chi_C(r_1 + 5).$$

Then $\chi_C(r_1 + 5) = 0$ and so $r_2 \geq r_1 + 7$. The coefficient of x^{r_1+6} in (3.12) is

$$2\chi_C(\frac{r_1 + 6}{2}) = 2\sum_{i=0}^{r_1+6} \chi_C(i) - r_1 - 6 = 2\chi_C(r_1 + 6).$$

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Then $\chi_C(r_1+6)=\chi_C(\frac{r_1+6}{2})=1$. By choosing $k=3$ and $k=5$ in (3.17) respectively, we have

$$\chi_C(r_1+2)=\chi_C(r_1+4)+\chi_C(2r_1+4-r_2)+\chi_C(2r_1+4)-\chi_C(r_1+3)-\chi_C(2r_1+3-r_2),$$
$$\chi_C(r_1+3)=\chi_C(r_1+6)+\chi_C(2r_1+6-r_2)+\chi_C(2r_1+6)-\chi_C(r_1+5)-\chi_C(2r_1+5-r_2).$$

Then

$$\chi_C(2r_1+4-r_2)+\chi_C(2r_1+4)-\chi_C(2r_1+3-r_2)+1=0,$$
$$\chi_C(2r_1+6-r_2)+\chi_C(2r_1+6)-\chi_C(2r_1+5-r_2)+1=0.$$

By (3.13), we have $\chi_C(2r_1+4-r_2)+\chi_C(2r_1+3-r_2)=1$ and $\chi_C(2r_1+6-r_2)+\chi_C(2r_1+5-r_2)=1$. Then $\chi_C(2r_1+3-r_2)=\chi_C(2r_1+5-r_2)=1$. Applying again (3.13), we have $r_2\equiv 1 \pmod 4$. The coefficient of x^{2r_1-2} in (3.12) is

$$2\chi_C(r_1-1)=2\chi_C(r_1-2)+2\chi_C(2r_1-2-r_2)+2\sum_{i=0}^{2r_1-2}\chi_C(i)-2r_1-2$$

and the coefficient of x^{2r_1-1} in (3.12) is

$$0=2\chi_C(r_1-1)+2\chi_C(2r_1-1-r_2)+2\sum_{i=0}^{2r_1-1}\chi_C(i)-2r_1-4.$$

Subtracting the above two equalities and dividing by 2 we can obtain

$$2\chi_C(r_1-1)=1+\chi_C(r_1-2)+\chi_C(2r_1-2-r_2)-\chi_C(2r_1-1)-\chi_C(2r_1-1-r_2).$$

Noting that $\chi_C(r_1-2)+\chi_C(r_1-1)=1$ and $\chi_C(2r_1-2-r_2)=\chi_C(2r_1-1-r_2)$, we have $3\chi_C(r_1-1)=2-\chi_C(2r_1-1)$. However, it is impossible. Therefore $r_2=r_1+1$.

The remainder of the proof is similar to the proof of [13, Theorem 1.1]. For the sake of completeness we give the details.

Let k be a positive even integer with $r_2<k<k+1<2r_1<r_1+r_2\le m$. Comparing the coefficients of x^{k-1}, x^k and x^{k+1} on the both sides of (3.12) respectively, we have

$$0=2\chi_C(k-1-r_1)+2\chi_C(k-1-r_2)+2\sum_{i=0}^{k-1}\chi_C(i)-k-4,$$

$$2\chi_C\left(\frac{k}{2}\right)=2\chi_C(k-r_1)+2\chi_C(k-r_2)+2\sum_{i=0}^{k}\chi_C(i)-k-4,$$

$$0=2\chi_C(k+1-r_1)+2\chi_C(k+1-r_2)+2\sum_{i=0}^{k+1}\chi_C(i)-k-6.$$

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Calculating the above three equalities, we have

$$\chi_C\left(\frac{k}{2}\right) = \chi_C(k-r_1) - \chi_C(k-1-r_2) + \chi_C(k), \tag{3.18}$$

$$\chi_C\left(\frac{k}{2}\right) = \chi_C(k-r_2) - \chi_C(k+1-r_1) - \chi_C(k+1) + 1. \tag{3.19}$$

By choosing $k = 2r_1 - 2$ in (3.19), we have

$$2\chi_C(r_1-1) = \chi_C(r_1-3) - \chi_C(2r_1-1) + 1.$$

Then $\chi_C(r_1-1) = \chi_C(r_1-3)$. Thus $r_1 \equiv 2 \pmod 4$ and $r_2 \equiv 3 \pmod 4$.

If $k-1-r_2 \equiv 0 \pmod 4$, then $k - r_1 \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 4$. Thus

$$\chi_C\left(\frac{k-1-r_2}{2}\right) + \chi_C\left(\frac{k-r_1}{2}\right) = 1.$$

Hence

$$\chi_C(k-1-r_2) + \chi_C(k-r_1) = 1.$$

If $\chi_C(k-1-r_2) = 0$, then $\chi_C(k-r_1) = 1$. By (3.18), we have $\chi_C(\frac{k}{2}) = 1$. If $\chi_C(k-1-r_2) = 1$, then $\chi_C(k-r_1) = 0$. By (3.18), we have $\chi_C(\frac{k}{2}) = 0$. Thus $\chi_C(\frac{k}{2}) = \chi_C(k-r_1)$ and $\chi_C(\frac{k}{2}) + \chi_C(k-1-r_2) = 1$. Noting that $\chi_C(k-1-r_2) + \chi_C(k-r_2) = 1$, we have $\chi_C(\frac{k}{2}) = \chi_C(k-r_2)$.

If $k-1-r_2 \equiv 2 \pmod 4$, then $k-r_1 \equiv 0 \pmod 4$ and $k \equiv 2 \pmod 4$. By (3.18), we have

$$\chi_C\left(\frac{k-2}{2}\right) = \chi_C(k-2-r_1) - \chi_C(k-3-r_2) + \chi_C(k-2).$$

Then $\chi_C(\frac{k-2}{2}) = \chi_C(k-2-r_1)$. Noting that $\chi_C(\frac{k-2}{2}) + \chi_C(\frac{k}{2}) = 1$ and $\chi_C(k-1-r_2) + \chi_C(k-r_2) = 1$, we have $\chi_C(\frac{k}{2}) = \chi_C(k-r_2)$.

As a result, we can obtain $\chi_C(\frac{k}{2}) = \chi_C(k-r_2)$. Put $k = 2r_2 - 2^{i+1}$ with $i \geq 0$. Then $\chi_C(r_2 - 2^i) = \chi_C(r_2 - 2^{i+1})$. Thus

$$1 = \chi_C(r_1) = \chi_C(r_2-1) = \chi_C(r_2-2) = \chi_C(r_2-4) = \cdots = \chi_C(r_2 - 2^{\lceil \log_2 r_2 \rceil - 1}).$$

By Lemma 2.3, we have $r_1 = 2^{2l+1} - 2$ and $r_2 = 2^{2l+1} - 1$ for some positive integer l.

This completes the proof of Claim 2. □

Claim 3. *Let l be a positive integer and let E, F be two sets of non-negative integers with $E \cup F = [0, 3 \cdot 2^{2l+1} - 4], 0 \in E$ and $E \cap F = \{2^{2l+1} - 2, 2^{2l+1} - 1\}$. Then $R_E(n) =$*

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$R_F(n)$ for any integer $n \in [0, 3\cdot 2^{2l+1}-4]$ if and only if

$$E = A_{2l+1} \cup (2^{2l+1}-2+B_{2l+1}) \cup (2^{2l+2}-2+(B_{2l+1}\cap[0,2^{2l+1}-3])) \cup \{3\cdot 2^{2l+1}-4\},$$
$$F = B_{2l+1} \cup (2^{2l+1}-2+A_{2l+1}) \cup (2^{2l+2}-2+(A_{2l+1}\cap[0,2^{2l+1}-3])).$$

Proof of Claim 3. We first prove the sufficiency of Claim 3. It is easy to verify that $E\cup F = [0, 3\cdot 2^{2l+1}-4]$, $0\in E$ and $E\cap F = \{2^{2l+1}-2, 2^{2l+1}-1\}$.

If $n \in [0, 2^{2l+2}-3]$, then

$$\begin{aligned} R_E(n) &= R_{A_{2l+1}}(n) + R_{A_{2l+1}, 2^{2l+1}-2+B_{2l+1}}(n) + R_{2^{2l+1}-2+B_{2l+1}}(n) \\ &= R_{A_{2l+1}}(n) + R_{A_{2l+1},B_{2l+1}}(n-(2^{2l+1}-2)) + R_{B_{2l+1}}(n-2(2^{2l+1}-2)) \end{aligned}$$

and

$$\begin{aligned} R_F(n) &= R_{B_{2l+1}}(n) + R_{2^{2l+1}-2+A_{2l+1}, B_{2l+1}}(n) + R_{2^{2l+1}-2+A_{2l+1}}(n) \\ &= R_{B_{2l+1}}(n) + R_{A_{2l+1},B_{2l+1}}(n-(2^{2l+1}-2)) + R_{A_{2l+1}}(n-2(2^{2l+1}-2)). \end{aligned}$$

By Lemma 2.5, for all $k\in\mathbb{N}$, we have $R_{A_{2l+1}}(k) = R_{B_{2l+1}}(k)$. Then $R_E(n) = R_F(n)$.

If $n \in [2^{2l+2}-2, 3\cdot 2^{2l+1}-5]$, then

$$\begin{aligned} R_E(n) &= R_{A_{2l+1}, 2^{2l+1}-2+B_{2l+1}}(n) + R_{2^{2l+1}-2+B_{2l+1}}(n) \\ &\quad + R_{A_{2l+1}, 2^{2l+2}-2+(B_{2l+1}\cap[0,2^{2l+1}-3])}(n) \\ &= R_{A_{2l+1},B_{2l+1}}(n-(2^{2l+1}-2)) + R_{B_{2l+1}}(n-2(2^{2l+1}-2)) \\ &\quad + R_{A_{2l+1},B_{2l+1}}(n-(2^{2l+2}-2)) \end{aligned}$$

and

$$\begin{aligned} R_F(n) &= R_{B_{2l+1}, 2^{2l+1}-2+A_{2l+1}}(n) + R_{2^{2l+1}-2+A_{2l+1}}(n) \\ &\quad + R_{B_{2l+1}, 2^{2l+2}-2+(A_{2l+1}\cap[0,2^{2l+1}-3])}(n) \\ &= R_{B_{2l+1},A_{2l+1}}(n-(2^{2l+1}-2)) + R_{A_{2l+1}}(n-2(2^{2l+1}-2)) \\ &\quad + R_{B_{2l+1},A_{2l+1}}(n-(2^{2l+2}-2)). \end{aligned}$$

By Lemma 2.5, $R_{A_{2l+1}}(k) = R_{B_{2l+1}}(k)$ holds for all $k\in\mathbb{N}$ and then $R_E(n) = R_F(n)$.

By $3\cdot 2^{2l+1}-4 = (2^{2l+1}-2)+(2^{2l+2}-2)$ in D, we have

$$R_C(3\cdot 2^{2l+1}-4) = 1 + R_{B_{2l+1}}(2^{2l+1}) + R_{A_{2l+1},B_{2l+1}}(2^{2l+1}-2)$$

and

$$R_D(3\cdot 2^{2l+1}-4) = 1 + R_{A_{2l+1}}(2^{2l+1}) + R_{B_{2l+1},A_{2l+1}}(2^{2l+1}-2).$$

Thus $R_C(3\cdot 2^{2l+1}-4) = R_D(3\cdot 2^{2l+1}-4)$.

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The necessity of Claim 3 follows from Lemma 2.2 and the sufficiency of Claim 3. This completes the proof of Claim 3. □

Now let

$$C_1 = C \cap [0, m-1+r_1], \quad D_1 = D \cap [0, m-1+r_1].$$

Then

$$C_1 \cup D_1 = [0, m-1+r_1], \quad C_1 \cap D_1 = \{r_1, r_2\}.$$

Moreover, for any integer $n \in [0, m-1+r_1]$, we have

$$\begin{aligned} R_{C_1}(n) &= |\{(c,c') : c < c' \leq n, c, c' \in C_1, c+c' = n\}| \\ &= |\{(c,c') : c < c' \leq n, c, c' \in C, c+c' = n\}| \\ &= R_C(n), \end{aligned}$$

$$\begin{aligned} R_{D_1}(n) &= |\{(d,d') : d < d' \leq n, d, d' \in D_1, d+d' = n\}| \\ &= |\{(d,d') : d < d' \leq n, d, d' \in D, d+d' = n\}| \\ &= R_D(n). \end{aligned}$$

Thus for any integer $n \in [0, m-1+r_1]$, we have

$$R_{C_1}(n) = R_C(n) = R_D(n) = R_{D_1}(n).$$

Noting that $r_2 \leq m-1$, we see that $r_1 + r_2 \leq m-1+r_1$. By Claim 2, there exists a positive integer l such that $r_1 = 2^{2l+1}-2, r_2 = 2^{2l+1}-1$.

Let E and F be as in Claim 3. If $m \geq 2^{2l+2}-1$ and $0 \in C$, then $m-1+r_1 \geq 3\cdot 2^{2l+1}-4$ and

$$C(3\cdot 2^{2l+1}-4) \cup D(3\cdot 2^{2l+1}-4) = [0, 3\cdot 2^{2l+1}-4],$$

$$C(3\cdot 2^{2l+1}-4) \cap D(3\cdot 2^{2l+1}-4) = \{2^{2l+1}-2, 2^{2l+1}-1\}.$$

Moreover, $R_{C(3\cdot 2^{2l+1}-4)}(n) = R_C(n) = R_D(n) = R_{D(3\cdot 2^{2l+1}-4)}(n)$ for all $n \in [0, 3\cdot 2^{2l+1}-4]$. By Lemma 2.2, we have

$$C(3\cdot 2^{2l+1}-4) = E, \quad D(3\cdot 2^{2l+1}-4) = F.$$

By

$$R_C(3\cdot 2^{2l+1}-3) = \chi_C(3\cdot 2^{2l+1}-3) + R_{A_{2l+1},B_{2l+1}}(2^{2l+1}-1) + R_{B_{2l+1}}(2^{2l+1}+1) - 1,$$

$$R_D(3\cdot 2^{2l+1}-3) = R_{B_{2l+1},A_{2l+1}}(2^{2l+1}-1) + R_{A_{2l+1}}(2^{2l+1}+1) - 1,$$

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we know that $R_C(3\cdot 2^{2l+1}-3)=R_D(3\cdot 2^{2l+1}-3)$ if and only if $\chi_C(3\cdot 2^{2l+1}-3)=0$, that is, $3\cdot 2^{2l+1}-3\in D$. Noting that $2^{2l+1}-2\in A_{2l+1}, 2^{2l+1}-1\in B_{2l+1}$, $3\cdot 2^{2l+1}-2=(2^{2l+1}-1)+(2^{2l+2}-1)$ in C and $3\cdot 2^{2l+1}-2=1+(3\cdot 2^{2l+1}-3)$ in D, we obtain

$$\begin{aligned}R_C(3\cdot 2^{2l+1}-2)&=1+\chi_C(3\cdot 2^{2l+1}-2)+R_{A_{2l+1},B_{2l+1}}(2^{2l+1})+R_{B_{2l+1}}(2^{2l+1}+2)\\&\quad-\chi_{A_{2l+1}}(3\cdot 2^{2l+1}-2-(2^{2l+2}-2+2^{2l+1}-1))\\&=1+\chi_C(3\cdot 2^{2l+1}-2)+R_{A_{2l+1},B_{2l+1}}(2^{2l+1})+R_{B_{2l+1}}(2^{2l+1}+2)\end{aligned}$$

and

$$\begin{aligned}R_D(3\cdot 2^{2l+1}-2)&=1+R_{B_{2l+1},A_{2l+1}}(2^{2l+1})+R_{A_{2l+1}}(2^{2l+1}+2)\\&\quad-\chi_{B_{2l+1}}(3\cdot 2^{2l+1}-2-(2^{2l+2}-2+2^{2l+1}-2))\\&=R_{B_{2l+1},A_{2l+1}}(2^{2l+1})+R_{A_{2l+1}}(2^{2l+1}+2).\end{aligned}$$

Thus by Lemma 2.5, we have $R_C(3\cdot 2^{2l+1}-2)>R_D(3\cdot 2^{2l+1}-2)$, which is impossible. Therefore $m\le 2^{2l+2}-2$.

Now we assume that $2^{2l+1}\le m\le 2^{2l+2}-3$ and $0\in C$. Let

$$M=r_1+m=2^{2l+1}-2+m.$$

Since $2^{2l+2}-2\le M\le 3\cdot 2^{2l+1}-5$, by Lemma 2.2, we have

$$E(M)\cup F(M)=[0,M],\quad E(M)\cap F(M)=\{2^{2l+1}-2,2^{2l+1}-1\},\tag{3.20}$$

$$R_{E(M)}(n)=R_E(n)=R_F(n)=R_{F(M)}(n)\text{ for any integer } n\in[0,M].\tag{3.21}$$

Moreover,

$$C(M)\cup D(M-1)=[0,M],\quad C(M)\cap D(M-1)=\{2^{2l+1}-2,2^{2l+1}-1\}.\tag{3.22}$$

Since $R_C(n)=R_D(n)$ for all $n\in\mathbb{N}$ and $0\notin D$, we have

$$R_{C(M)}(n)=R_C(n)=R_D(n)=R_{D(M-1)}(n)\tag{3.23}$$

for any integer $n\in[0,M]$. By (3.20)–(3.23) and Lemma 2.2, we have

$$C(M)=E(M),\quad D(M-1)=F(M).\tag{3.24}$$

Then $\chi_E(M)=1$, $\chi_F(M)=0$.

By $2^{2l+1}-3\in A_{2l+1}$, we have $3\cdot 2^{2l+1}-5\in F$. Then $M<3\cdot 2^{2l+1}-5$. If $\chi_E(M+1)=1$, then $\chi_F(M+1)=0$ and $C(M+1)=E(M+1), D(M+1)=F(M+1)\cup\{M,M+1\}$.

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Thus

$$\begin{aligned} R_C(M+1) &= |\{(c,c') : 0 \le c < c' \le M+1, c, c' \in C, c+c' = M+1\}| \\ &= |\{(c,c') : 0 \le c < c' \le M+1, c, c' \in C(M+1), c+c' = M+1\}| \\ &= R_{E(M+1)}(M+1) \end{aligned}$$

and

$$\begin{aligned} R_D(M+1) &= |\{(d,d') : 1 \le d < d' \le M+1, d, d' \in D, d+d' = M+1\}| \\ &= |\{(d,d') : 1 \le d < d' \le M+1, d, d' \in D(M+1), d+d' = M+1\}| \\ &= 1 + |\{(d,d') : 1 \le d < d' \le M+1, d, d' \in F(M+1), d+d' = M+1\}| \\ &= 1 + R_{F(M+1)}(M+1). \end{aligned}$$

By Claim 3, we have $R_{E(M+1)}(M+1) = R_{F(M+1)}(M+1)$. Then $R_C(M+1) \neq R_D(M+1)$, a contradiction. Thus $\chi_E(M+1) = 0$ and $\chi_F(M+1) = 1$.

Let t be an arbitrary positive integer such that $M < M+t < M+t+1 \le 3 \cdot 2^{2l+1} - 4$. Then $1 \le t \le 2^{2l+1} - 3$. Define the sets S and T by

$$S = (E \cap C)(M+t) \cup (F(M+t) \backslash D(M+t)),$$

$$T = (F \cap D)(M+t) \cup (E(M+t) \backslash C(M+t)).$$

Noting that

$$E(M+t) \cup F(M+t) = [0, M+t] = (C(M+t) \backslash \{M+1\}) \cup (D(M+t) \backslash \{M\}),$$

$$E(M+t) \cap F(M+t) = \{2^{2l+1} - 2, 2^{2l+1} - 1\},$$

we have

$$S \subseteq C(M+t) \backslash \{M+1\}, \quad T \subseteq D(M+t) \backslash \{M\},$$

$$S \cup T = (C(M+t) \backslash \{M+1\}) \cup (D(M+t) \backslash \{M\}),$$

$$S \cap T = \{2^{2l+1} - 2, 2^{2l+1} - 1\} = (C(M+t) \backslash \{M+1\}) \cap (D(M+t) \backslash \{M\}).$$

Then

$$|S| + |T| = |S \cup T| + |S \cap T| = |C(M+t) \backslash \{M+1\}| + |D(M+t) \backslash \{M\}|.$$

It follows that

$$S = C(M+t) \backslash \{M+1\}, \quad T = D(M+t) \backslash \{M\}. \tag{3.25}$$

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For $M+t \le n \le 3\cdot 2^{2l+1}-4$, let

$$\begin{aligned} N_1(t,n) &= R_{E(2^{2l+1}-3),E(M+t)\setminus C(M+t)}(n),\\ N_2(t,n) &= R_{F(2^{2l+1}-3),E(M+t)\setminus C(M+t)}(n),\\ N_3(t,n) &= R_{E(2^{2l+1}-3),F(M+t)\setminus D(M+t)}(n),\\ N_4(t,n) &= R_{F(2^{2l+1}-3),F(M+t)\setminus D(M+t)}(n). \end{aligned}$$

We claim that

$$|E(M+t)\setminus C(M+t)| = N_1(t,n)+N_2(t,n), \tag{3.26}$$

$$|F(M+t)\setminus D(M+t)| = N_3(t,n)+N_4(t,n). \tag{3.27}$$

In fact, if $E(M+t)\setminus C(M+t)=\emptyset$, then $N_1(t,n)=N_2(t,n)=0$; if

$$E(M+t)\setminus C(M+t)=\{c_1,\dots,c_u\}$$

for some positive integer u, then by (3.24), we have $c_i \ge M+1$ and so $0\le n-c_i \le 2^{2l+1}-3$ for $i\in[1,u]$. In view of

$$E(2^{2l+1}-3)\cup F(2^{2l+1}-3)=[0,2^{2l+1}-3],\quad E(2^{2l+1}-3)\cap F(2^{2l+1}-3)=\emptyset,$$

we have

$$N_1(t,n)+N_2(t,n)=\sum_{i=1}^{u}\chi_{E(2^{2l+1}-3)}(n-c_i)+\sum_{i=1}^{u}\chi_{F(2^{2l+1}-3)}(n-c_i)=u.$$

Thus (3.26) holds. Similarly, we can deduce (3.27) holds.

By $M+t<3\cdot 2^{2l+1}-4<2^{2l+3}-4\le 2M$, we can obtain

$$\begin{aligned} R_{E(M+t)}(n) &= R_{(E\cap C)(M+t)}(n)+R_{E(2^{2l+1}-3),E(M+t)\setminus C(M+t)}(n)\\ &= R_{(E\cap C)(M+t)}(n)+N_1(t,n). \end{aligned}$$

By (3.24) and (3.25), we have

$$\begin{aligned} R_{C(M+t)}(n) &= R_{C(M+t)\setminus\{M+1\}}(n)+\chi_{C(M+t)\setminus\{M+1\}}(n-M-1)\\ &= R_{(E\cap C)(M+t)}(n)+R_{E(2^{2l+1}-3),F(M+t)\setminus D(M+t)}(n)+\chi_E(n-M-1)\\ &= R_{E(M+t)}(n)-N_1(t,n)+N_3(t,n)+\chi_E(n-M-1). \end{aligned} \tag{3.28}$$

Similarly, we can get

$$\begin{aligned} R_{D(M+t)}(n) &= R_{D(M+t)\setminus\{M\}}(n)+\chi_{D(M+t)\setminus\{M\}}(n-M)\\ &= R_{(F\cap D)(M+t)}(n)+R_{F(2^{2l+1}-3),E(M+t)\setminus C(M+t)}(n)+\chi_F(n-M)\\ &= R_{F(M+t)}(n)-N_4(t,n)+N_2(t,n)+\chi_F(n-M). \end{aligned} \tag{3.29}$$

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By choosing $n = M + t$ and $n = M + t + 1$ in (3.28) respectively, we have

$$R_{C(M+t)}(M+t) = R_{E(M+t)}(M+t) - N_1(t, M+t) + N_3(t, M+t) + \chi_E(t-1) \quad (3.30)$$

and

$$\begin{aligned} R_{C(M+t+1)}(M+t+1) &= R_{C(M+t)}(M+t+1) + \chi_C(M+t+1) \\ &= R_{E(M+t+1)}(M+t+1) - \chi_E(M+t+1) - N_1(t, M+t+1) \\ &\quad + N_3(t, M+t+1) + \chi_E(t) + \chi_C(M+t+1). \end{aligned} \quad (3.31)$$

By choosing $n = M + t$ and $n = M + t + 1$ in (3.29) respectively, we have

$$R_{D(M+t)}(M+t) = R_{F(M+t)}(M+t) - N_4(t, M+t) + N_2(t, M+t) + \chi_F(t) \quad (3.32)$$

and

$$\begin{aligned} R_{D(M+t+1)}(M+t+1) &= R_{D(M+t)}(M+t+1) \\ &= R_{F(M+t+1)}(M+t+1) - N_4(t, M+t+1) \\ &\quad + N_2(t, M+t+1) + \chi_F(t+1). \end{aligned} \quad (3.33)$$

Note that $R_{C(n)}(n) = R_{D(n)}(n)$ and $R_{E(n)}(n) = R_{F(n)}(n)$. By (3.30)–(3.33), we have

$$N_1(t, M+t) + N_2(t, M+t) + \chi_F(t) = N_3(t, M+t) + N_4(t, M+t) + \chi_E(t-1)$$

and

$$\begin{aligned} &N_1(t, M+t+1) + N_2(t, M+t+1) + \chi_E(M+t+1) + \chi_F(t+1) \\ &= N_3(t, M+t+1) + N_4(t, M+t+1) + \chi_E(t) + \chi_C(M+t+1). \end{aligned}$$

By (3.26) and (3.27), we have

$$|E(M+t)\backslash C(M+t)| + \chi_F(t) = |F(M+t)\backslash D(M+t)| + \chi_E(t-1)$$

and

$$\begin{aligned} &|E(M+t)\backslash C(M+t)| + \chi_E(M+t+1) + \chi_F(t+1) \\ &= |F(M+t)\backslash D(M+t)| + \chi_E(t) + \chi_C(M+t+1). \end{aligned}$$

Then

$$\chi_F(t) + \chi_E(t) + \chi_C(M+t+1) = \chi_E(t-1) + \chi_E(M+t+1) + \chi_F(t+1). \quad (3.34)$$

If M is even, then we can write

$$M = (2^{2l+2} - 2) + \sum_{i=1}^{2l} b_i 2^i,$$

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where $b_i \in \{0,1\}$. It follows from $\chi_F(M)=0$ that $\chi_{B_{2l+1}}\big(\sum_{i=1}^{2l} b_i 2^i\big)=1$. By choosing $M+t+1=3\cdot 2^{2l+1}-4$ in (3.34), we see that t is odd and

$$\chi_F(t+1)=\chi_F\Big(2^{21+1}-2-\sum_{i=1}^{2l} b_i 2^i\Big)=\chi_{B_{2l+1}}\Big(\sum_{i=1}^{2l}(1-b_i)2^i\Big)=1.$$

Then $\chi_E(t+1)=0$. It follows from $\chi_F(t)+\chi_E(t)=1$ and $\chi_E(3\cdot 2^{2l+1}-4)=1$ that $\chi_E(t-1)=0$ and $\chi_F(t-1)=1$. Since $\chi_E(t-1)+\chi_E(t)=1$, we have $\chi_E(t)=1$ and $\chi_F(t)=0$. Noting that $\chi_E(t-1)=\chi_E(t+1)$, we have $t\equiv 3 \pmod 4$ and so $t\geq 3$. Then $\chi_E(t-2)=0$. By choosing $M+(t-1)+1=3\cdot 2^{2l+1}-5$ in (3.34), we have

$$\chi_F(t-1)+\chi_E(t-1)+\chi_C(M+(t-1)+1)=\chi_E(t-2)+\chi_E(M+(t-1)+1)+\chi_F(t).$$

It follows from $\chi_E(M+(t-1)+1)=\chi_E(3\cdot 2^{2l+1}-5)=0$ that $\chi_C(M+(t-1)+1)=-1$, which is clearly false.

If M is odd, then we can write

$$M=(2^{2l+2}-2)+\sum_{i=0}^{f}2^i+\sum_{i=f+2}^{2l} b_i 2^i,$$

where $f\in\{0,1,\ldots,2l-1\}$ and $b_i\in\{0,1\}$. It follows from $\chi_E(M+1)=0$ and $\chi_F(M)=0$ that

$$\chi_{A_{2l+1}}\left(2^{f+1}+\sum_{i=f+2}^{2l} b_i 2^i\right)=1,\quad \chi_{B_{2l+1}}\left(\sum_{i=0}^{f}2^i+\sum_{i=f+2}^{2l} b_i 2^i\right)=1.$$

Then f is odd. By choosing $M+t+1=3\cdot 2^{2l+1}-4$ in (3.34), we see that t is even and

$$\chi_F(t+1)=\chi_F\left(2^{21+1}-2-\sum_{i=0}^{f}2^i-\sum_{i=f+2}^{2l} b_i 2^i\right)=\chi_{B_{2l+1}}\Big(2^{f+1}-1+\sum_{i=f+2}^{2l}(1-b_i)2^i\Big)=1.$$

Then $\chi_E(t+1)=0$ and $\chi_F(t)=0$. Thus $\chi_E(t)=1$. It follows from $\chi_E(3\cdot 2^{2l+1}-4)=1$ that $\chi_E(t-1)=0$ and $\chi_F(t-1)=1$. Since $\chi_E(t-1)=\chi_E(t+1)$, we have $t\equiv 0 \pmod 4$ and so $t\geq 4$. Then $\chi_E(t-2)=\chi_E(t-3)=1$ and $\chi_F(t-2)=0$. By choosing $M+(t-2)+1=3\cdot 2^{2l+1}-6$ in (3.34), we have

$$\chi_F(t-2)+\chi_E(t-2)+\chi_C(M+(t-2)+1)=\chi_E(t-3)+\chi_E(M+(t-2)+1)+\chi_F(t-1).$$

It follows from $\chi_E(M+(t-2)+1)=\chi_E(3\cdot 2^{2l+1}-6)=1$ that $\chi_C(M+(t-2)+1)=2$, which is also impossible. Therefore $m=2^{2l+2}-2$.

This completes the proof of Theorem 1.3.

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