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# $U$-NUMBERS IN FIELDS OF FORMAL POWER SERIES OVER FINITE FIELDS

GÜLCAN KEKEÇ

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**Abstract**

In the field $\mathbb{K}$ of formal power series over a finite field $K$, we consider some lacunary power series with algebraic coefficients in a finite extension of $K(x)$. We show that the values of these series at nonzero algebraic arguments in $\mathbb{K}$ are $U$-numbers.

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## 1. Introduction

Let $K$ be a finite field with $q = p^k$ elements, where $p$ is a prime number and $k$ is a positive rational integer. Denote by $K[x]$ the ring of polynomials in $x$ with coefficients in $K$, by $K(x)$ the quotient field of $K[x]$ and by $\deg(a)$ the degree of a nonzero polynomial $a(x)$ in $K[x]$. By setting

$$|0| = 0 \quad \text{and} \quad \left|\frac{a(x)}{b(x)}\right| = q^{\deg(a)-\deg(b)},$$

where $a(x)$ and $b(x)$ are nonzero polynomials in $K[x]$, a non-Archimedean absolute value $|\cdot|$ is defined on $K(x)$. The completion of $K(x)$ with respect to $|\cdot|$ is called the field of formal power series over $K$ and is denoted by $\mathbb{K}$. We denote the unique extension of $|\cdot|$ to the field $\mathbb{K}$ by the same notation $|\cdot|$.

In 1978, Bundschuh [2] introduced a classification in $\mathbb{K}$, similar to the classification of real transcendental numbers introduced by Mahler [4]. He divided the transcendental formal power series into three disjoint classes and called the transcendental formal power series in these classes $S$-, $T$- and $U$-numbers defined as follows.

Let $P(y) = a_0 + a_1 y + \cdots + a_n y^n$ be a nonzero polynomial in $y$ with coefficients $a_i \in K[x]$ $(i = 0, \ldots, n)$. We denote the degree of $P(y)$ with respect to $y$ by $\deg(P)$. The height $H(P)$ of $P(y)$ is defined by $H(P) = \max\{|a_0|, \ldots, |a_n|\}$. Let $\alpha \in \overline{K(x)}$, where $\overline{K(x)}$

denotes the algebraic closure of $K(x)$, and let $P(y)$ be the minimal polynomial of $\alpha$ over $K[x]$. Then the height $H(\alpha)$ of $\alpha$ and the degree $\deg(\alpha)$ of $\alpha$ are defined as the height and the degree of $P(y)$, respectively. Further, the roots of $P(y)$ are called the conjugates of $\alpha$ over $K(x)$.

Given a transcendental formal power series $\xi$, that is, $\xi \in \mathbb{K}$ and $\xi$ is transcendental over $K(x)$, and positive integers $n$ and $H$, define the quantities

$$w_n(H,\xi) = \min\{|P(\xi)| : P(y) \in K[x][y] - \{0\}, \deg(P) \le n \text{ and } H(P) \le H\},$$
$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log w_n(H,\xi)}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

From [2],

$$w_n(H,\xi) < H^{-n} q^n (\max\{1, |\xi|\})^n.$$

It follows that $w_n(\xi) \ge n$ $(n = 1, 2, 3, \ldots)$ and so $w(\xi) \ge 1$. If $w_n(\xi) = \infty$ for some integers $n$, then $\mu(\xi)$ is defined as the smallest such integer. If $w_n(\xi) < \infty$ for every $n$, put $\mu(\xi) = \infty$. Then $\xi$ is called:

- an $S$-number if $1 \le w(\xi) < \infty$ and $\mu(\xi) = \infty$;
- a $T$-number if $w(\xi) = \infty$ and $\mu(\xi) = \infty$;
- a $U$-number if $w(\xi) = \infty$ and $\mu(\xi) < \infty$.

Moreover, a $U$-number $\xi$ is called a $U_m$-number if $\mu(\xi) = m$.

In 1980, Oryan [6] presented the first explicit constructions of $U_m$-numbers. He considered, in [6, Satz 5], some gap series with coefficients from a finite extension of degree $m$ over $K(x)$ and he showed that the values of these series for nonzero arguments from $K(x)$ are $U_m$-numbers.

Theorem 1.1 (Oryan [6, Satz 5]). *Let $L$ ($L \subset \mathbb{K}$) be a finite extension of degree $m$ over $K(x)$ and*

$$f(y) = \sum_{i=0}^{\infty} \gamma_i y^{n_i}$$

*be a gap series in $\mathbb{K}$, where $\gamma_i$ $(i = 0, 1, 2, \ldots)$ is a nonzero element in $L$, $\deg(\gamma_i) = m$ for infinitely many $i$ and $\{n_i\}_{i=0}^{\infty}$ is a strictly increasing sequence of nonnegative rational integers. Suppose that the following conditions hold:*

$$\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = \infty,$$
$$\limsup_{i\to\infty} \frac{\log H(\gamma_i)}{n_i} < \infty,$$

*and*

$$\limsup_{i\to\infty} \sqrt[n_i]{|\gamma_i^{(j)}|} < \infty \quad (j = 1, \ldots, m),$$

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*where* $\gamma_i^{(1)}, \ldots, \gamma_i^{(m)}$ *denote the field conjugates of* $\gamma_i$ *for* $L$ *(for* $i = 0, 1, 2, \ldots$*). (These are the conjugates of* $\gamma_i$ *over* $K(x)$*, each repeated* $m/\deg(\gamma_i)$ *times.) Let* $\alpha$ *be an element of* $K(x)$ *with* $0 < |\alpha| < R$*, where*

$$R = \min_{j=1,\ldots,m}\Big\{1/\limsup_{i\to\infty} \sqrt[n_i]{|\gamma_i^{(j)}|}\Big\}.$$

*Then* $f(\alpha)$ *is a* $U_m$*-number.*

In Theorem 1.2, we extend Theorem 1.1 to certain lacunary power series with algebraic coefficients in a finite extension of $K(x)$. We show that the values of these series at nonzero algebraic arguments in $\mathbb{K}$ are $U$-numbers. (Transcendency of lacunary power series was first treated by Mahler [5] and later improved by Braune [1].)

Theorem 1.2. *Let* $L$ *(*$L \subset \mathbb{K}$*) be a finite extension of degree* $m$ *over* $K(x)$ *and*

$$F(z) = \sum_{h=0}^{\infty} c_h z^h$$

*be a lacunary power series in* $\mathbb{K}$*, where* $c_h \in L$ *(*$h = 0, 1, 2, \ldots$*), satisfying*

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = r_n \quad (n = 1, 2, 3, \ldots), \\ c_h \neq 0, & h = s_n \quad (n = 0, 1, 2, \ldots), \end{cases}$$

*where* $\{s_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *are two infinite sequences of nonnegative rational integers with*

$$0 = s_0 \le r_1 < s_1 \le r_2 < s_2 \le r_3 < s_3 \le r_4 < s_4 \le \cdots.$$

*Suppose that the following conditions hold:*

$$\limsup_{h\to\infty} \frac{\log H(c_h)}{h} < \infty, \tag{1.1}$$

$$\lim_{n\to\infty} \frac{s_n}{r_n^2} = \infty \tag{1.2}$$

*and*

$$\limsup_{h\to\infty} \sqrt[h]{|c_h^{(j)}|} < \infty \quad (j = 1, \ldots, m),$$

*where* $c_h^{(1)}, \ldots, c_h^{(m)}$ *(*$h = 0, 1, 2, \ldots$*) denote the field conjugates of* $c_h$ *for* $L$ *(for* $h = 0, 1, 2, \ldots$*). Let* $\alpha \in \mathbb{K}$ *be algebraic over* $K(x)$ *with* $\deg(\alpha) = g$ *and*

$$0 < |\overline{\alpha}| < R, \tag{1.3}$$

*where* $|\overline{\alpha}| = \max\{|\alpha^{(1)}|, \ldots, |\alpha^{(g)}|\}$ *and*

$$R = \min_{j=1,\ldots,m}\Big\{1/\limsup_{h\to\infty} \sqrt[h]{|c_h^{(j)}|}\Big\}.$$

*Furthermore, assume that* $P_k(\alpha) \neq 0$ *and* $\deg(P_k(\alpha)) = t$ *for infinitely many* $k$*, where* $P_k(z) = \sum_{h=s_k}^{r_{k+1}} c_h z^h$ *(*$k = 0, 1, 2, \ldots$*) and* $t$ *is the degree of* $L(\alpha)$ *over* $K(x)$*. Suppose that* $p$ *does not divide* $t$*. Then* $F(\alpha)$ *is a* $U$*-number with* $\mu(F(\alpha)) \le t$*.*

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We prove Theorem 1.2 in Section 3. In the next section, we cite the lemmas we need to prove Theorem 1.2. Finally, in Section 4, we give some examples to illustrate Theorem 1.2.

## 2. Auxiliary results

We need the following three lemmas to prove Theorem 1.2.

LEMMA 2.1 (Oryan [6, Hilfssatz 4]). *Let $L$ be a finite extension of degree $m$ over $K(x)$ and let $\alpha_1, \ldots, \alpha_n$ be in $L$. Then*

$$H(\alpha_1 + \cdots + \alpha_n) \leq H(\alpha_1)^{2m^2} \cdots H(\alpha_n)^{2m^2}$$

*and*

$$H(\alpha_1 \cdots \alpha_n) \leq H(\alpha_1)^{2m^2} \cdots H(\alpha_n)^{2m^2}.$$

LEMMA 2.2 (Oryan [6, Hilfssatz 3]). *Let $\alpha_1$ and $\alpha_2$ be two distinct, conjugate elements in $\overline{K(x)}$ and denote their degree by $n$ and their height by $H$. Suppose that $p$ does not divide $n$. Then*

$$|\alpha_1 - \alpha_2| \geq H^{-n+1/2}.$$

LEMMA 2.3 (Oryan [6, page 46]). *Let $L$ be a finite extension of degree $m$ over $K(x)$ and let $\alpha$ be in $L$. Then*

$$|\alpha| \leq H(\alpha)^m.$$

## 3. Proof of Theorem 1.2

We prove Theorem 1.2 by making use of the methods of the proofs of Satz 4 and Satz 5 in Oryan [6], Satz 2 in Zeren [7, pages 120–129] and Satz 1 in Gürses [3].

We can write

$$F(\alpha) = \eta_n + \gamma_n,$$

where

$$\eta_n = \sum_{k=0}^{n-1} P_k(\alpha) = \sum_{h=s_0}^{r_n} c_h \alpha^h \quad (n = 1, 2, 3, \ldots)$$

and

$$\gamma_n = \sum_{k=n}^{\infty} P_k(\alpha) = \sum_{h=s_n}^{\infty} c_h \alpha^h \quad (n = 1, 2, 3, \ldots).$$

Since $\eta_n \in L(\alpha)$, it follows that $\deg(\eta_n) \leq t$ for $n = 1, 2, 3, \ldots$.

Note that $0 < R \leq \infty$. If $0 < R < \infty$, then, by (1.3), there exists a positive real number $\varepsilon$ such that

$$0 < |\overline{\alpha}| < R - \varepsilon$$

and so

$$|c_h^{(j)}| < (R - \varepsilon)^{-h} \quad (j = 1, \ldots, m) \tag{3.1}$$

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for sufficiently large $h$ and

$$|\gamma_n| \le \left(\frac{|\overline{\alpha}|}{R-\varepsilon}\right)^{s_n} \tag{3.2}$$

for sufficiently large $n$. If $R = \infty$, then we choose an element $\rho$ of $\mathbb{K}$ such that

$$|\overline{\alpha}| < |\rho|.$$

In this case, the series $\sum_{h=0}^{\infty} c_h^{(j)} z^h$ $(j = 1, \ldots, m)$ converge for $z = \rho$, so that

$$|c_h^{(j)}| < |\rho|^{-h} \quad (j = 1, \ldots, m) \tag{3.3}$$

for sufficiently large $h$ and it follows that

$$|\gamma_n| \le \left(\frac{|\overline{\alpha}|}{|\rho|}\right)^{s_n} \tag{3.4}$$

for sufficiently large $n$. Whether $R$ is finite or infinite, we infer from (3.1) and (3.3) that

$$|c_h^{(j)}| < r^{-h} \quad (j = 1, \ldots, m) \tag{3.5}$$

for sufficiently large $h$ and from (3.2) and (3.4) that

$$|\gamma_n| \le \left(\frac{|\overline{\alpha}|}{r}\right)^{s_n} < 1 \tag{3.6}$$

for sufficiently large $n$, where $r = \min\{R - \varepsilon, |\rho|\}$. (Note that $0 < |\overline{\alpha}|/r < 1$.)

We deduce from Lemma 2.1 that

$$H(\eta_n) \le \prod_{h=s_0}^{r_n} (H(c_h)H(\alpha)^{h2t^2})^{4t^4} \quad (n = 1, 2, 3, \ldots).$$

By (1.1), there exists a real constant $d_0 > 1$ such that

$$H(c_h) \le d_0^h \quad (h = 0, 1, 2, \ldots)$$

and it follows that

$$H(\eta_n) \le d_1^{(r_n^2)} \quad (n = 1, 2, 3, \ldots), \tag{3.7}$$

where $d_1 > 1$ is a real constant.

We now wish to prove that $\deg(\eta_n) = t$ from some $n$ onward. If $t = 1$, then $\deg(\eta_n) = 1$ $(n = 1, 2, 3, \ldots)$. If $t > 1$, then we shall show that the field conjugates of $\eta_n$ for $L(\alpha)$ are distinct from each other from some $n$ onward. Let $t > 1$, $i, j \in \{1, \ldots, t\}$ and $i \ne j$. We proceed by proving the following two claims.

*Claim 1: If $\eta_n^{(i)} \ne \eta_n^{(j)}$, then $\eta_{n+1}^{(i)} \ne \eta_{n+1}^{(j)}$ from some $n$ onward.*

Suppose that $\eta_n^{(i)} \ne \eta_n^{(j)}$. Then

$$\eta_{n+1}^{(i)} - \eta_{n+1}^{(j)} = (\eta_n^{(i)} - \eta_n^{(j)}) + \sum_{h=s_n}^{r_{n+1}} (c_h^{(i)}(\alpha^{(i)})^h - c_h^{(j)}(\alpha^{(j)})^h). \tag{3.8}$$

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From Lemma 2.2 and (3.7),

$$|\eta_n^{(i)} - \eta_n^{(j)}| \geq H(\eta_n)^{-t+1/2} \geq d_1^{-(t-1/2)r_n^2}. \tag{3.9}$$

By (3.5),

$$\left|\sum_{h=s_n}^{r_{n+1}} (c_h^{(i)}(\alpha^{(i)})^h - c_h^{(j)}(\alpha^{(j)})^h)\right| \leq \max_{h=s_n,\dots,r_{n+1}} \left\{\left(\frac{|\overline{\alpha}|}{r}\right)^h\right\} = \left(\frac{|\overline{\alpha}|}{r}\right)^{s_n} \tag{3.10}$$

for sufficiently large $n$. By (1.2),

$$\left(\frac{|\overline{\alpha}|}{r}\right)^{s_n} < d_1^{-(t-1/2)r_n^2} \tag{3.11}$$

for sufficiently large $n$. We infer from (3.8), (3.9), (3.10) and (3.11) that

$$|\eta_{n+1}^{(i)} - \eta_{n+1}^{(j)}| = |\eta_n^{(i)} - \eta_n^{(j)}|$$

for sufficiently large $n$. Hence, for sufficiently large $n$, it follows from $\eta_n^{(i)} \neq \eta_n^{(j)}$ that $\eta_{n+1}^{(i)} \neq \eta_{n+1}^{(j)}$.

*Claim 2: For any positive integer $N$, there exists an integer $n$ with $n > N$ such that $\eta_n^{(i)} \neq \eta_n^{(j)}$ or $\eta_{n+1}^{(i)} \neq \eta_{n+1}^{(j)}$.*

Let $N$ be any positive integer. By the hypothesis of the theorem, there exists an integer $n$ with $n > N$ such that $\deg(P_n(\alpha)) = t$. Assume that $\eta_n^{(i)} = \eta_n^{(j)}$ and $\eta_{n+1}^{(i)} = \eta_{n+1}^{(j)}$. Then $(P_n(\alpha))^{(i)} = (P_n(\alpha))^{(j)}$, but this contradicts $\deg(P_n(\alpha)) = t$.

Combining Claims 1 and 2, we conclude that the field conjugates of $\eta_n$ for $L(\alpha)$ are distinct from each other from some $n$ onward. Hence, $\deg(\eta_n) = t$ from some $n$ onward.

Let $B_n(y) = b_{n0} + b_{n1}y + \cdots + b_{nt}y^t$ be the minimal polynomial of $\eta_n$ over $K[x]$ for sufficiently large $n$. Then

$$B_n(F(\alpha)) = \gamma_n\beta_n,$$

where

$$\beta_n = b_{n1} + b_{n2}(2\eta_n + \gamma_n) + \cdots + b_{nt}\left(\binom{t}{1}\eta_n^{t-1} + \binom{t}{2}\eta_n^{t-2}\gamma_n + \cdots + \gamma_n^{t-1}\right).$$

It follows from Lemma 2.3, (3.6) and (3.7) that

$$|\beta_n| \leq d_2^{r_n^2}$$

for sufficiently large $n$, where $d_2$ is a real constant with $d_1 \leq d_2$. Hence, using (3.6),

$$|B_n(F(\alpha))| = |\gamma_n||\beta_n| \leq d_3^{s_n} d_2^{r_n^2}$$

for sufficiently large $n$, where $d_3 = |\overline{\alpha}|/r < 1$. Then, using (3.7),

$$|B_n(F(\alpha))| \leq H(\eta_n)^{-\theta_n} \tag{3.12}$$

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for sufficiently large $n$, where

$$\theta_n = \frac{s_n}{r_n^2}\frac{\log d_3^{-1}}{\log d_2} - 1 \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \infty.$$

The sequence $\{B_n\}$ has a subsequence $\{B_{n_i}\}$ such that $B_{n_i}(F(\alpha)) \neq 0$ $(i = 1, 2, 3, \ldots)$, $\deg(B_{n_i}) = t$ $(i = 1, 2, 3, \ldots)$ and

$$1 < H(B_{n_1}) < H(B_{n_2}) < H(B_{n_3}) < \cdots, \quad \lim_{i\to\infty} H(B_{n_i}) = \infty.$$

We infer from (3.12) and $H(\eta_{n_i}) = H(B_{n_i})$ $(i = 1, 2, 3, \ldots)$ that

$$0 < |B_{n_i}(F(\alpha))| \leq H(B_{n_i})^{-\theta_{n_i}} \quad (i = 1, 2, 3, \ldots).$$

This implies that $F(\alpha)$ is a $U$-number with $\mu(F(\alpha)) \leq t$.

## 4. Examples

We give two examples to illustrate Theorem 1.2.

Example 4.1. Let $F(z) = \sum_{h=0}^{\infty} c_h z^h$ be a lacunary power series in $\mathbb{K}$ with

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h = 1, & s_n \leq h \leq r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases}$$

where 1 denotes the identity element of $K$ and $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of nonnegative rational integers determined by

$$s_0 = 0, \quad s_n = (n+1)^{(n+1)!} \quad \text{and} \quad r_n = 2 \cdot n^{n!} \quad (n = 1, 2, 3, \ldots).$$

Then, by Theorem 1.2, $F(1/x)$ is a $U_1$-number.

Example 4.2. Let $t$ be any positive rational integer provided that $p$ does not divide $t$ and let $F(z) = \sum_{h=0}^{\infty} c_h z^h$ be a lacunary power series with

$$\begin{cases} c_h = 0, & r_n < h < s_n \quad (n = 1, 2, 3, \ldots), \\ c_h = \sqrt[t]{x}, & s_n \leq h \leq r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases}$$

where $\sqrt[t]{x}$ is defined as a root of the polynomial $y^t - x$ and $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are two infinite sequences of nonnegative rational integers determined by

$$s_0 = 0, \quad s_n = ((n+1)!)^{(n+1)!} \quad \text{and} \quad r_n = 2 \cdot (n!)^{n!} \quad (n = 1, 2, 3, \ldots).$$

Then, by Theorem 1.2, $F(1/x)$ is a $U$-number with $\mu(F(1/x)) \leq t$.

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GÜLCAN KEKEÇ, Department of Mathematics,
Faculty of Science, Istanbul University, 34134,
Vezneciler, Istanbul, Turkey
e-mail: gulkekec@istanbul.edu.tr