ON HETEROGENEITY IN THE INDIVIDUAL MODEL WITH BOTH DEPENDENT CLAIM OCCURRENCES AND SEVERITIES

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ABSTRACT

It is a common belief for actuaries that the heterogeneity of claim severities in a given insurance portfolio tends to increase its dangerousness, which results in requiring more capital for covering claims. This paper aims to investigate the effects of orderings and heterogeneity among scale parameters on the aggregate claim amount when both claim occurrence probabilities and claim severities are dependent. Under the assumption that the claim occurrence probabilities are left tail weakly stochastic arrangement increasing, the actuaries' belief is examined from two directions, i.e., claim severities are comonotonic or right tail weakly stochastic arrangement increasing. Numerical examples are provided to validate these theoretical findings. An application in assets allocation is addressed as well.

KEYWORDS

Majorization, comonotonicity, RWSAI, LWSAI, stop-loss order.

1. INTRODUCTION

Consider an insurance portfolio consisting of n policies. In the context of individual model of risk theory, let $I_i X_i$ be the claim amount from the ith insured, where $X_i \in \mathbb{R}_+$ is the random claim severity (or size), and I_i, independent of X_i, is a Bernoulli random variable with $I_i = 1$ if the claim occurs or otherwise $I_i = 0$, for $i = 1, \ldots, n$. Then, the total number of claims and aggregate claim amount can be written as $\sum_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} I_i X_i$, respectively. One of the main issues is to study the effects of heterogeneity and dependence among claim occurrence probabilities and/or claim severities on the ordering properties of aggregate claim numbers and amount. For the study of total number of claims, interested readers are referred to Karlin and Novikoff (1963), Pledger and Proschan (1971), Wang (1993), Dhaene and Goovaerts (1996, 1997), Hu and Wu (1999), Dhaene and Denuit (1999), Boland *et al.* (2002, 2004), Frostig

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(2006) and Xu and Balakrishnan (2011). In this paper, we focus on investigating the heterogeneity of claim severities on the aggregate claim amount, which would be helpful for the insurance company to prepare for the amount of capital needed for covering insured policies.

The past decade has witnessed a large amount of discussion on stochastic properties of aggregate claim amount. For the case of independent occurrence level, Ma (2000) proved that if the claim severities are exchangeable, then more dispersiveness among claim occurrence probabilities leads to smaller aggregate claim amount in the sense of the convex order. Under the assumption of X_i's arrayed with respect to the usual stochastic order, they also showed that $\sum_{i=1}^{n} I_i X_i$ stochastically increases if the vector of claim occurrence probabilities $\boldsymbol{I}$ is majorized. Frostig (2001) and Hu and Ruan (2004) established sufficient conditions to compare aggregate claim amount by means of the symmetric supermodular, and multivariate usual and symmetric stochastic orders (see Shaked and Shanthikumar, 2007). After that, many researchers paid their attention to comparing the aggregate claim amounts arising from two sets of heterogeneous insurance portfolios; see, for example, Khaledi and Ahmadi (2008), Barmalzan *et al.* (2015) and Zhang and Zhao (2015). However, these results were only developed under the independent assumption on the claim occurrence probabilities, which violates the fact that the occurrences of claims may be dependent in practice. This study will investigate the ordering properties of the aggregate claim amount when the occurrence levels are positively dependent through left tail weakly stochastic arrangement increasing (LWSAI).

Denuit and Frostig (2006) pointed out that spreading maturities by dispersing claim occurrences lessens the need for capital, while more capital is needed if the uncertainty increases in claim amounts, which validates the general belief of actuaries that the heterogeneity of the risks in a given insurance portfolio tends to increase its dangerousness. It is worthy noting that their results were established for independent claim severities. However, in actuarial practices, both occurrence levels and claim severities in insurance portfolios often exhibit positively dependency structure, though in most cases such structure is not easy to be detected. In this paper, we shall relax the independence assumed in Denuit and Frostig (2006) from two directions: (i) the dependency structure of $\boldsymbol{X} = (X_1, \dots, X_n)$ is unknown, (ii) and $\boldsymbol{X}$ is dependent through right tail weakly stochastic arrangement increasing (RWSAI). For the first case, we shall identify the worst dependency structure in terms of comonotonicity, and then study the heterogeneity of claim severities on the amount of capital needed for covering insureds. This formulation represents a conservative attitude toward the uncertainty in the dependency structure from the point of view of the insurer.

It is worthy mentioning that the dependency structure imposed on the claim occurrence probabilities and claim severities are reasonable. For instance, both the claim occurrence probabilities and claim severities are usually positively dependent for insureds in an area suffering from serious natural disasters such as drought, floods and earthquakes. Under the assumption that the claim severities belongs to the scale family (the claim severity from the ith insured is $a_i X_i$), we shall investigate how the ordering and heterogeneity of a_i's impact on the stop-loss order of $\sum_{i=1}^n I_i a_i X_i$ in the context that the $\boldsymbol{I}$ is LWSAI and independent of $\boldsymbol{I}$ the $\boldsymbol{X}$ is comonotonic or RWSAI. Here, the heterogeneity among claim severities is described by $\boldsymbol{a}$ through majorization, which characterizes the diversity of the components of vectors and plays a key role in establishing various inequalities arising from many research fields such as actuarial science, applied probability, reliability theory and operations research; see Pledger and Proschan (1971), Arnold (2007), Balakrishnan *et al.* (2018) and the excellent monograph (Marshall *et al.*, 2011).

For an insurance portfolio comprised of LWSAI occurrence probabilities and comonotonic or RWSAI claim severities, our results show that the insurance portfolio turns out to be more dangerous from the viewpoint of the insurer if the vector of scale parameters for the claim severities becomes more heterogeneous in the sense of the majorization order. As a result, more capital should be needed to cover the claims.

The remainder of this paper is rolled out as follows: Section 2 recalls some pertinent definitions and notions used in the sequel. In Section 3, we study the effects of orderings and heterogeneity among scale parameters on the aggregate claim amount when $\boldsymbol{I}$ is LWSAI and $\boldsymbol{X}$ is comonotonic. Section 4 treats the ordering properties of aggregate claim amount when $\boldsymbol{I}$ is LWSAI and $\boldsymbol{X}$ is RWSAI. Section 5 concludes the paper with an application in assets allocation and some discussions.

Throughout the paper, the occurrence level $\boldsymbol{I}$ is assumed to be independent of the claim severity $\boldsymbol{X}$. The term *increasing* is used for *monotone non-decreasing* and *decreasing* for *monotone non-increasing*. All random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and expectations exist when they appear. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of $\boldsymbol{x} = (x_1, \ldots, x_n)$. Denote $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{I}_n = \{\boldsymbol{x} : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n\}$.

2. Preliminaries

Before proceeding to the main result, we recall in this section some notions pertinent to our discussions in the sequel.

2.1. Majorization

Definition 2.1. *A real vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to*

i majorize $\boldsymbol{y} \in \mathbb{R}^n$ (written as $\boldsymbol{x} \overset{\mathrm{m}}{\succeq} \boldsymbol{y}$), if $\sum_{i=j}^n x_{(i)} \geq \sum_{i=j}^n y_{(i)}$ for $j = 2, \ldots, n$, and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$;
ii weakly submajorize $\boldsymbol{y} \in \mathbb{R}^n$ (written as $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$), if $\sum_{i=j}^n x_{(i)} \geq \sum_{i=j}^n y_{(i)}$ for $j = 1, \ldots, n$.

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For any non-negative vectors $\boldsymbol{x}$, $\boldsymbol{y}$, it is evident that $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$ implies $\boldsymbol{x} \succeq_{\text{w}} \boldsymbol{y}$, while the reverse is not true in general. Majorization is quite useful in establishing various inequalities. For more on majorization and their applications, one may refer to Marshall *et al.* (2011).

2.2. Comonotonicity

A subset $A \subseteq \mathbb{R}^n$ is said to be *comonotonic* if, for any $(x_1, \ldots, x_n) \in A$ and $(y_1, \ldots, y_n) \in A$, either $x_i \leq y_i$ for $i = 1, \ldots, n$ or $x_i \geq y_i$ for $i = 1, \ldots, n$. A random vector $\boldsymbol{X}$ is said to be *comonotonic* if there is a comonotonic subset A such that $\mathrm{P}(\boldsymbol{X} \in A) = 1$.

Since the comonotonic random vector possesses the strongest positive dependency among its coordinates, it is usually employed to model the association among risk claims in actuarial science when the real dependency structure is unknown. Let $\mathcal{R}(F_1, \ldots, F_n)$ be the Fréchet space of all the n-dimensional random vectors with univariate marginal distributions $F_1, \ldots, F_n$. It is known that the comonotonic random vector is maximal within the corresponding Fréchet space in the sense of the convex order of the sum. For comprehensive discussions on comonotonicity, readers are referred to Dhaene *et al.* (2002a,b).

2.3. Stochastic orders

Definition 2.2. *A random variable X is said to be smaller than the other one Y in the*

- *i usual stochastic order (denoted by $X \leq_{\text{st}} Y$), if $\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$ for any increasing $\phi : \mathbb{R} \mapsto \mathbb{R}$;*
- *ii hazard rate order (denoted by $X \leq_{\text{hr}} Y$), if $\mathrm{P}(Y \geq x)/\mathrm{P}(X \geq x)$ is increasing in $x \in \mathbb{R}$;*
- *iii stop-loss order (denoted by $X \leq_{\text{sl}} Y$), if $\mathrm{E}[(X - d)_+] \leq \mathrm{E}[(Y - d)_+]$ for all $d \in \mathbb{R}_+$;*
- *iv convex order (denoted by $X \leq_{\text{cx}} Y$), if $\mathrm{E}[X] = \mathrm{E}[Y]$ and $X \leq_{\text{sl}} Y$;*
- *v increasing concave order (denoted by $X \leq_{\text{icv}} Y$), if $\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$ for any increasing concave $\phi : \mathbb{R} \mapsto \mathbb{R}$.*

In the context of applied probability, the stop-loss order is usually termed as the *increasing convex order* based on the fact that $X \leq_{\text{sl}} Y \iff \mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$ for any increasing and convex $\phi : \mathbb{R} \mapsto \mathbb{R}$. It is known that the hazard rate order implies the usual stochastic order, which in turn implies the stop-loss order and the increasing concave order.

Recall that a function $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be supermodular if $\psi(\boldsymbol{x}) + \psi(\boldsymbol{y}) \leq \psi(\boldsymbol{x} \wedge \boldsymbol{y}) + \psi(\boldsymbol{x} \vee \boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where "$\wedge$" and "$\vee$" denote coordinatewise minimum and maximum, respectively.

Definition 2.3. *If two n-dimensional vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are such that*

$$\mathrm{E}[\psi(\boldsymbol{X})] \le \mathrm{E}[\psi(\boldsymbol{Y})] \quad \text{for all supermodular functions } \psi : \mathbb{R}^n \mapsto \mathbb{R},$$

then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the supermodular order (denoted by $\boldsymbol{X} \le_{\mathrm{sm}} \boldsymbol{Y}$).

The supermodular order can be used to bound some quite general random vectors through their comonotonic counterparts.

Lemma 2.4 (Shaked and Shanthikumar (2007), Theorem 9.A.21). *For a random vector $\boldsymbol{X}$ with marginal distributions $F_{X_1}, \ldots, F_{X_n}$ and a random variable U uniformly distributed on $[0, 1]$, it holds that $(X_1, X_2, \ldots, X_n) \le_{\mathrm{sm}} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U))$.*

For comprehensive discussions and applications on various stochastic orders, one may refer to Shaked and Shanthikumar (2007).

2.4. Risk measures

Let $\rho : X \mapsto \mathbb{R}_+$ be a risk measure that assigns a non-negative real number $\rho[X]$ to the risk $X \in \mathbb{R}_+$. In this paper, we adopt the following meaning of risk measure: if X is a possible loss of some financial portfolio over a time horizon, we interpret $\rho[X]$ as the amount of capital that should be added as a buffer to this portfolio so that it becomes acceptable to an internal or external risk controller. Prominent examples of risk measures include Value-at-Risk (VaR) at level α

$$\mathrm{VaR}[X; \alpha] = \inf\{t : \mathrm{P}(X \le t) \ge \alpha\}, \quad \alpha \in [0, 1],$$

Tail Value-at-Risk (TVaR) at level α

$$\mathrm{TVaR}[X; \alpha] = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}[X; a]\mathrm{d}a, \quad \alpha \in [0, 1]$$

and other distortion risk measures

$$\rho_g[X] = \int_0^{+\infty} g\left(\mathrm{P}(X > t)\right) \mathrm{d}t$$

with the distortion function g increasing with $g(0) = 0$ and $g(1) = 1$. For more discussions on these risk measures and their applications in actuarial problems, we refer interested readers to Yaari (1987) and Wang (1996, 2000, 2002).

It should be pointed out that TVaR and distortion risk measures associated with concave distortion functions agree with the stop-loss order in the sense that

$$X \le_{\mathrm{sl}} Y \Longleftrightarrow \mathrm{TVaR}[X; \alpha] \le \mathrm{TVaR}[Y; \alpha] \text{ for all } \alpha \in [0, 1];$$
$$\Longleftrightarrow \rho_g[X] \le \rho_g[Y] \text{ for all concave distortion functions } g.$$

For more detailed discussions on these relations, one may refer to Denuit *et al.* (2006).

2.5. Stochastic versions of arrangement increasing

For any (i, j) with $1 \leq i < j \leq n$, let $\tau_{ij}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ and denote

$$\mathcal{G}_{\text{sai}}^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) \geq g(\tau_{ij}(\boldsymbol{x})) \text{ for any } x_i \leq x_j\},$$
$$\mathcal{G}_{\text{wsai}}^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j\},$$
$$\mathcal{G}_{\text{lwsai}}^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is decreasing in } x_i \leq x_j\},$$
$$\mathcal{G}_{\text{rwsai}}^{i,j}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\tau_{ij}(\boldsymbol{x})) \text{ is increasing in } x_j \geq x_i\}.$$

Definition 2.5. *A random vector* $\boldsymbol{X} = (X_1, \ldots, X_n)$ *or its distribution is said to be*

i *stochastic arrangement increasing (SAI), if* $\mathrm{E}[g(\boldsymbol{X})] \geq \mathrm{E}[g(\tau_{ij}(\boldsymbol{X}))]$ *for any* $g \in \mathcal{G}_{\text{sai}}^{i,j}(n)$ *and any pair* (i, j) *such that* $1 \leq i < j \leq n$;

ii *weakly stochastic arrangement increasing (WSAI), if* $\mathrm{E}[g(\boldsymbol{X})] \geq \mathrm{E}[g(\tau_{ij}(\boldsymbol{X}))]$ *for any* $g \in \mathcal{G}_{\text{wsai}}^{i,j}(n)$ *and any pair* (i, j) *such that* $1 \leq i < j \leq n$;

iii *LWSAI, if* $\mathrm{E}[g(\boldsymbol{X})] \geq \mathrm{E}[g(\tau_{ij}(\boldsymbol{X}))]$ *for any* $g \in \mathcal{G}_{\text{lwsai}}^{i,j}(n)$ *and any pair* (i, j) *such that* $1 \leq i < j \leq n$;

iv *RWSAI, if* $\mathrm{E}[g(\boldsymbol{X})] \geq \mathrm{E}[g(\tau_{ij}(\boldsymbol{X}))]$ *for any* $g \in \mathcal{G}_{\text{rwsai}}^{i,j}(n)$ *and any pair* (i, j) *such that* $1 \leq i < j \leq n$.

It is plain that SAI implies both LWSAI and RWSAI, which in turn imply WSAI. Also, it is known that multivariate versions of Dirichlet distribution, inverted Dirichlet distribution, F distribution and Pareto distribution of type I are all SAI and hence RWSAI and LWSAI whenever the corresponding parameters are arrayed in the ascending order. In the literature, SAI is employed to model the dependence among ordered random risks in actuarial science; see for instance, Hua and Cheung (2008), You and Li (2014) and Zhang and Zhao (2015). LWSAI, RWSAI and WSAI were introduced by Cai and Wei (2014, 2015) and have been applied in the field of financial engineering and actuarial science to model dependent stochastic returns and risks, respectively; see for example, Cai and Wei (2015) and You and Li (2015, 2016).

As pointed out by Cai and Wei (2014, 2015), the notions of RWSAI, LWSAI and WSAI are multivariate generalizations of the joint hazard rate order, joint reversed hazard rate order and joint stochastic order proposed by Shanthikumar and Yao (1991). In this paper, we shall employ the two useful notions LWSAI and RWSAI to characterize the properties of multivariate distributions of claim occurrence probabilities and claim severities, respectively. It should be mentioned that LWSAI or RWSAI random vectors can be constructed through

Archimedean copulas and certain marginal distributions, which will be addressed in the next part.

2.6. Copulas

Formally, for a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ with univariate marginal distributions $F_1, \ldots, F_n$ and survival functions $\bar{F}_1, \ldots, \bar{F}_n$, there exist $C : [0, 1]^n \mapsto [0, 1]$ and $\bar{C} : [0, 1]^n \mapsto [0, 1]$ such that its distribution function F and survival function $\bar{F}$ can be represented as, for all x_i, $1 \le i \le n$,

$$F(\boldsymbol{x}) = C\big(F_1(x_1), \ldots, F_n(x_n)\big), \qquad \bar{F}(\boldsymbol{x}) = \bar{C}\big(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)\big).$$

Then, $C(\boldsymbol{u})$ and $\bar{C}(\boldsymbol{u})$ are called the *copula* and *survival copula* of $\boldsymbol{X}$, respectively.

For a decreasing and continuous function $\phi : [0, +\infty) \mapsto [0, 1]$ with $\phi(0) = 1$, $\phi(+\infty) = 0$ and the pseudo-inverse $\psi := \phi^{-1}$, the function

$$C_\phi(u_1, \ldots, u_n) = \phi\big(\psi(u_1) + \cdots + \psi(u_n)\big), \text{ for all } u_i \in [0, 1],\ i = 1, 2, \ldots, n,$$

is called an *Archimedean copula* with generator ϕ, if $(-1)^k \phi^{(k)}(x) \ge 0$ for $k = 0, \ldots, n-2$ and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex.

The past decades have witnessed the development of a large number of applications of copulas in various areas such as risk management, econometrics, actuarial science and reliability theory, see Kole *et al.* (2007), Eryilmaz (2014), Abdallah *et al.* (2015) and Zhu *et al.* (2017). According to Cai and Wei (2014), $(X_1, \ldots, X_n)$ with $X_1 \le_{\rm hr} \cdots \le_{\rm hr} X_n$ is RWSAI, if it has an Archimedean survival copula with a log-convex generator. You and Li (2016) adopted this relation to build the stop-loss order for the scalar product of absolute continuous RWSAI random vectors. For more on copulas and their properties, one may refer to Nelsen (2006).

3. Comonotonic claim severities

To begin with, let us introduce other notations to simplify our discussion. For real vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, denote the inner product $\boldsymbol{a}\cdot\boldsymbol{x} = \sum_{i=1}^n a_i x_i$, the Hadamard product $\boldsymbol{\lambda}\circ\boldsymbol{x} = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ and the sub-vector $\boldsymbol{x}_{\{i,j\}}$ of $\boldsymbol{x}$ with ith and jth entries deleted. In the sequel, we will frequently use the following compounds:

$$\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{x} = \boldsymbol{a}\cdot(\boldsymbol{\lambda}\circ\boldsymbol{x}) = \textstyle\sum_{i=1}^n a_i\lambda_i x_i, \qquad d_{i,j} = (\boldsymbol{a}\cdot\boldsymbol{x})_{\{i,j\}} = (\boldsymbol{a}_{\{i,j\}}\cdot\boldsymbol{x}_{\{i,j\}}),$$
$$c_{i,j} = (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{x})_{\{i,j\}} = \boldsymbol{a}_{\{i,j\}}\cdot\boldsymbol{\lambda}_{\{i,j\}}\circ\boldsymbol{x}_{\{i,j\}} = \textstyle\sum_{r\neq i,j}^n a_r\lambda_r x_r.$$

For ease of reference, we denote $\boldsymbol{0} = (0, \ldots, 0)$, $\boldsymbol{1} = (1, \ldots, 1)$, $p(\boldsymbol{\lambda}) = \mathrm{P}(\boldsymbol{I} = \boldsymbol{\lambda})$,

$$\Lambda_k = \{\boldsymbol{\lambda} | \lambda_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, n,\ \lambda_1 + \cdots + \lambda_n = k\}, \quad k = 0, \ldots, n,$$

and let, for $1 \leq i \neq j \leq n$ and $k = 1, \ldots, n-1$,

$$\Lambda_k^{i,j}(0,1) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 0, \lambda_j = 1\}, \quad \Lambda_k^{i,j}(0,0) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = \lambda_j = 0\},$$
$$\Lambda_k^{i,j}(1,0) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = 1, \lambda_j = 0\}, \quad \Lambda_k^{i,j}(1,1) = \{\boldsymbol{\lambda} \in \Lambda_k | \lambda_i = \lambda_j = 1\}.$$

It is plain that $\Lambda_1^{i,j}(1,1) = \Lambda_{n-1}^{i,j}(0,0) = \emptyset$ and

$$\Lambda_k = \Lambda_k^{i,j}(0,1) \cup \Lambda_k^{i,j}(0,0) \cup \Lambda_k^{i,j}(1,0) \cup \Lambda_k^{i,j}(1,1).$$

Lemma 3.1 (Cai and Wei, 2015). *A multivariate Bernoulli random vector $\boldsymbol{I}$ is LWSAI if and only if $p(\tau_{ij}(\boldsymbol{\lambda})) \leq p(\boldsymbol{\lambda})$, for all $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,1)$, $1 \leq i < j \leq n$ and $k = 1, \ldots, n-1$.*

As pointed by Cai and Wei (2015), for a multivariate Bernoulli random vector $\boldsymbol{I}$, LWSAI, RWSAI, WSAI and SAI are all equivalent. To keep consistent use of the dependence notion for $\boldsymbol{I}$ in the literature, we hereafter adopt the term LWSAI to model the occurrence levels.

Being the strongest positive dependence, comonotonicity is usually used to model the extreme dependency structure when the real dependency structure sometimes is unknown. The next result shows that the total claim amount $\sum_{i=1}^n I_i a_i X_i$ is maximized according to the convex order when the claim sizes are comonotonic.

Theorem 3.2. *Suppose $\boldsymbol{X}^c = (X_1^c, \ldots, X_n^c)$ is the comonotonic version of $\boldsymbol{X}$, and $\boldsymbol{I}$ has multivariate Bernoulli distribution. Then,*

$$\sum_{i=1}^n I_i a_i X_i \leq_{\rm cx} \sum_{i=1}^n I_i a_i X_i^c, \qquad \text{for any } \boldsymbol{a} \in \mathbb{R}_+^n.$$

Proof. For any realized I_i's and a_i's, since the vector $(I_1 a_1 X_1^c, \ldots, I_n a_n X_n^c)$ is comonotone, by using Lemma 2.4, it then holds that

$$(I_1 a_1 X_1, \ldots, I_n a_n X_n) \leq_{\rm sm} (I_1 a_1 X_1^c, \ldots, I_n a_n X_n^c),$$

which further implies $\sum_{i=1}^n I_i a_i X_i \leq_{\rm sl} \sum_{i=1}^n I_i a_i X_i^c$ by applying Theorem 3.1 of Müller (1997). Therefore, the desired results follows by noting that $\mathrm{E}[\sum_{i=1}^n I_i a_i X_i] = \mathrm{E}[\sum_{i=1}^n I_i a_i X_i^c]$. ■

From the perspective of dependence among random numbers from several collective risk models, it was proved in Corollary 3 of Denuit *et al.* (2002) that the total claim amount (or random sums) are bounded from above by means of the convex order when the claim numbers are comonotonic. Instead, Theorem 3.2 considers the individual risk model and studies the effects of dependence among claim severities on the aggregate claim amount. We show that the aggregate claim amount in the individual risk model having scaled claim severities is bounded from above by means of the convex order when the claim severities are comonotonic. In other words, Theorem 3.2 states that, for any given claim

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occurrence probabilities and scale parameters, the maximum capital needed for the insurer is attained when the vector of claim severities is comonotonic. As the comonotonicity case represents the worst scenario from the point of view of the insurer, it is thus a conservative way on detecting the effect of the heterogeneity among scale parameters on the amount of capital required for the insurer as if the comonotonicity were the real dependency structure among claim sizes.

In what follows, we study how the orderings of scale parameters impact on the total payout with comonotonic claim severities in the sense of the stop-loss order.

Theorem 3.3. *Suppose that $\boldsymbol{X}$ is comonotonic with $X_i \le_{\rm st} X_j$ for some (i, j) such that $1 \le i < j \le n$ and $\boldsymbol{I}$ is a LWSAI Bernoulli random vector. If $a_i \le a_j$, then*

$$\sum_{r=1}^{n} I_r a_r X_r \ge_{\rm sl} I_i a_j X_i + I_j a_i X_j + \sum_{r \ne i,j}^{n} I_r a_r X_r .$$

Proof. Denote $\eta(\boldsymbol{a}) = \mathrm{E}[u(\boldsymbol{a} \cdot \boldsymbol{I} \circ \boldsymbol{X})]$, where u is increasing and convex. It suffices to show $\eta(\boldsymbol{a}) \ge \eta(\tau_{ij}(\boldsymbol{a}))$. In light of

$$\begin{aligned}\eta(\boldsymbol{a}) &= \sum_{k=0}^{n} \sum_{\boldsymbol{\lambda} \in \Lambda_k} \mathrm{E}[u(\boldsymbol{a} \cdot \boldsymbol{I} \circ \boldsymbol{X}) \mid \boldsymbol{I} = \boldsymbol{\lambda}] p(\boldsymbol{\lambda}) \\ &= p(\boldsymbol{0})u(0) + p(\boldsymbol{1})\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{X}\right)\right] + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda})\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X}\right)\right] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\eta(\tau_{ij}(\boldsymbol{a})) = &\ p(\boldsymbol{0})u(0) + p(\boldsymbol{1})\mathrm{E}\left[u\left(a_j X_i + a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}\right)\right] \\ &+ \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda})\mathrm{E}\left[u\left(a_i \lambda_j X_j + a_j \lambda_i X_i + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right],\end{aligned}$$

we have

$$\begin{aligned}\eta(\boldsymbol{a}) - \eta(\tau_{ij}(\boldsymbol{a})) = &\sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \{\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X}\right)\right] \\ &- \mathrm{E}\left[u\left(a_i \lambda_j X_j + a_j \lambda_i X_i + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right]\} \\ &+ p(\boldsymbol{1}) \left\{\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{X}\right)\right] - \mathrm{E}\left[u\left(a_j X_i + a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}\right)\right]\right\}.\end{aligned} \tag{3.2}$$

Owing to the comonotonicity of $\boldsymbol{X}$, it holds that $X_i \le X_j$ almost surely. Therefore, $a_i \le a_j$ implies $\boldsymbol{a} \cdot \boldsymbol{X} \ge a_j X_i + a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}$, and hence

$$\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{X}\right)\right] \ge \mathrm{E}\left[u\left(a_j X_i + a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}\right)\right], \quad \text{for } 1 \le i < j \le n. \tag{3.3}$$

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On the other hand, for any $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0,0)$, $k = 1, 2, \ldots, n-1$, it holds that

$$\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right] = \mathrm{E}\left[u\left(a_i\lambda_j X_j + a_j\lambda_i X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]. \tag{3.4}$$

For $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(1,1)$, similar to (3.3), we have

$$\mathrm{E}[u(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})] \geq \mathrm{E}\left[u\left(a_i\lambda_j X_j + a_j\lambda_i X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]. \tag{3.5}$$

By applying (3.3), (3.4) and (3.5) to (3.2), we have

$$\begin{aligned}
\eta(\boldsymbol{a}) - \eta(\tau_{ij}(\boldsymbol{a})) \geq \sum_{k=1}^{n-1}\Bigg\{ & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right. \\
& \left. -\mathrm{E}\left[u\left(a_i\lambda_j X_j + a_j\lambda_i X_i + \boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right) + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(1,0)} p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right. \\
& \left. -\mathrm{E}\left[u\left(a_i\lambda_j X_j + a_j\lambda_i X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\right)\Bigg\} \\
= \sum_{k=1}^{n-1}\Bigg\{ & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(a_j X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\right. \\
& \left. -\mathrm{E}\left[u\left(a_i X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\right) \\
& + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\tau_{ij}(\boldsymbol{\lambda}))\left(\mathrm{E}\left[u\left(a_i X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\right. \\
& \left. -\mathrm{E}\left[u\left(a_j X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\right)\Bigg\} \\
\geq \sum_{k=1}^{n-1} & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{i,j}(0,1)} p(\tau_{ij}(\boldsymbol{\lambda}))\Big\{\mathrm{E}\left[u\left(a_j X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right] \\
& - \mathrm{E}\left[u\left(a_i X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right] \\
& + \mathrm{E}\left[u\left(a_i X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right] - \mathrm{E}\left[u\left(a_j X_i + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right]\Big\},
\end{aligned} \tag{3.6}$$

where the second inequality stems from Lemma 3.1 and the fact of

$$\mathrm{E}\left[u\left(a_j X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right] \geq \mathrm{E}\left[u\left(a_i X_j + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{i,j\}}\right)\right].$$

In what follows, we show the non-negativity of

$$\Delta_1^{i,j}(\boldsymbol{a}, \boldsymbol{\lambda}, \boldsymbol{X}) = \mathrm{E}\left[u\left(a_j X_j + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right] - \mathrm{E}\left[u\left(a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right]$$
$$+ \mathrm{E}\left[u\left(a_i X_i + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right]$$
$$- \mathrm{E}\left[u\left(a_j X_i + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{i,j\}}\right)\right]. \tag{3.7}$$

For any $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0, 1)$ and a realization $\boldsymbol{X} = \boldsymbol{x}$, it then holds that $(a_i x_i + c_{i,j}, a_j x_j + c_{i,j}) \succeq_{\mathrm{w}} (a_i x_j + c_{i,j}, a_j x_i + c_{i,j})$, which implies $(u(a_i x_i + c_{i,j}), u(a_j x_j + c_{i,j})) \succeq_{\mathrm{w}} (u(a_i x_j + c_{i,j}), u(a_j x_i + c_{i,j}))$ upon applying Theorem 5.A.1 of Marshall *et al.* (2011). So, we have

$$u(a_i x_i + c_{i,j}) + u(a_j x_j + c_{i,j}) \geq u(a_i x_j + c_{i,j}) + u(a_j x_i + c_{i,j}).$$

By applying the iterated expectation on the above inequality, we conclude $\Delta_1^{i,j}(\boldsymbol{a}, \boldsymbol{\lambda}, \boldsymbol{X}) \geq 0$, which in turn implies $\eta(\boldsymbol{a}) \geq \eta(\tau_{ij}(\boldsymbol{a}))$. Thus, the desired result is proved. ■

As a direct consequence of Theorem 3.3, Corollary 3.4 characterizes the configuration of the scale parameters that results in the maximum total claim amount for comonotonic $\boldsymbol{X}$.

Corollary 3.4. *Suppose that $\boldsymbol{X}$ is comonotonic with $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$ and $\boldsymbol{I}$ is a LWSAI Bernoulli random vector. If $\boldsymbol{a} \in \mathcal{I}_n$, then*

$$\sum_{r=1}^{n} I_r a_r X_r \geq_{\mathrm{sl}} \sum_{r=1}^{n} I_r a_{\tau(r)} X_r, \quad \textit{for any permutation } \tau \textit{ of } \{1, 2, \ldots, n\}.$$

With the help of Corollary 3.4, we shall discuss the impact of dispersiveness among scale parameters on the amount of capital needed for the insurer when $\boldsymbol{X}$ is strongest positively dependent and $\boldsymbol{I}$ is LWSAI.

Theorem 3.5. *For comonotonic $\boldsymbol{X}$ with $X_1 \leq_{\mathrm{st}} \cdots \leq_{\mathrm{st}} X_n$ and LWSAI $\boldsymbol{I}$, the $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b}$ implies*

$$\sum_{i=1}^{n} I_i a_i X_i \geq_{\mathrm{sl}} \sum_{i=1}^{n} I_i b_i X_i, \quad \textit{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathcal{I}_n.$$

Proof. By exploiting a similar proof method of Theorem 3.3, it is enough to prove

$$0 \leq \eta(\boldsymbol{a}) - \eta(\boldsymbol{b}) = \sum_{k=1}^{n-1} \sum_{\boldsymbol{\lambda} \in \Lambda_k} p(\boldsymbol{\lambda}) \left\{\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X}\right)\right] - \mathrm{E}\left[u\left(\boldsymbol{b} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X}\right)\right]\right\}$$
$$+ p(\mathbf{1}) \left\{\mathrm{E}\left[u\left(\boldsymbol{a} \cdot \boldsymbol{X}\right)\right] - \mathrm{E}\left[u\left(\boldsymbol{b} \cdot \boldsymbol{X}\right)\right]\right\}, \tag{3.8}$$

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where $\eta(\boldsymbol{a})$ is defined in (3.1) and u is increasing and convex. By the nature of majorization order, it suffices to prove the non-negativity of (3.8) under the conditions $a_1 \le a_2$, $b_1 \le b_2$, $(a_1, a_2) \overset{\text{m}}{\succeq} (b_1, b_2)$ and $a_i = b_i$ for $i = 3, \ldots, n$.

Owing to comonotonicity of (X_1, X_2) with $X_1 \le_{\text{st}} X_2$, we have $x_1 \le x_2$ given $\boldsymbol{X} = \boldsymbol{x}$. Then, the assumption $(a_1, a_2) \overset{\text{m}}{\succeq} (b_1, b_2)$ implies $(a_1x_1, a_2x_2) \succeq_{\text{w}} (b_1x_1, b_2x_2)$, and thus $a_1x_1 + a_2x_2 \ge b_1x_1 + b_2x_2$. Therefore, upon using iterated expectation formula, it follows that

$$\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{X}\right)\right] \ge \mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{X}\right)\right]. \tag{3.9}$$

Besides, for any $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0, 0)$, $k = 1, 2, \ldots, n-1$, it holds that

$$\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right] = \mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]. \tag{3.10}$$

For $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(1, 1)$, in a similar manner, we can obtain

$$\mathrm{E}\left[u\left(a_1X_1 + a_2X_2 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] \ge \mathrm{E}\left[u\left(b_1X_1 + b_2X_2 + (\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right]. \tag{3.11}$$

By using (3.9)–(3.11), we reach

$$\begin{aligned}
\eta(\boldsymbol{a}) - \eta(\boldsymbol{b}) \ge \sum_{k=1}^{n-1}\Bigg\{ & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)} p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right] - \mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right) \\
& + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)} p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right] - \mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right)\Bigg\} \\
= \sum_{k=1}^{n-1}\Bigg\{ & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)} p(\boldsymbol{\lambda})\Big(\mathrm{E}\left[u\left(a_2X_2 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] \\
& - \mathrm{E}\left[u\left(b_2X_2 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right]\Big) \\
& + \sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)} p(\tau_{12}(\boldsymbol{\lambda}))\Big(\mathrm{E}\left[u\left(a_1X_1 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] \\
& - \mathrm{E}\left[u\left(b_1X_1 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right]\Big)\Bigg\} \\
\ge \sum_{k=1}^{n-1} & \sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)} p(\tau_{12}(\boldsymbol{\lambda}))\Big\{\mathrm{E}\left[u\left(a_2X_2 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] \\
& - \mathrm{E}\left[u\left(b_2X_2 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] + \mathrm{E}\left[u\left(a_1X_1 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right] \\
& - \mathrm{E}\left[u\left(b_1X_1 + (\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X})_{\{1,2\}}\right)\right]\Big\},
\end{aligned}$$

where the second inequality is due to Lemma 3.1 and $a_2 \ge b_2$.

TABLE 1

PROBABILITY MASS FUNCTION OF (I_1, I_2, I_3).

Default	λ_1	0	0	0	0	1	1	1	1
Default	λ_2	0	0	1	1	0	0	1	1
Default	λ_3	0	1	0	1	0	1	0	1
Probability	$p(\boldsymbol{\lambda})$	0.12	0.15	0.12	0.16	0.11	0.13	0.09	0.12

Next, we need to show the non-negativity of

$$\begin{aligned}\Delta_2^{1,2}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\lambda}, \boldsymbol{X}) = \mathrm{E}\big[u\big(a_2 X_2 + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{1,2\}}\big)\big] - \mathrm{E}\big[u\big(b_2 X_2 + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{1,2\}}\big)\big] \\ +\mathrm{E}\big[u\big(a_1 X_1 + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{1,2\}}\big)\big] \\ - \mathrm{E}\big[u\big(b_1 X_1 + (\boldsymbol{a} \cdot \boldsymbol{\lambda} \circ \boldsymbol{X})_{\{1,2\}}\big)\big]. \qquad (3.12)\end{aligned}$$

Upon taking some realization $\boldsymbol{X} = \boldsymbol{x}$, for any $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0, 1)$, it holds that $(a_1 x_1 + c_{1,2}, a_2 x_2 + c_{1,2}) \succeq_{\rm w} (b_2 x_2 + c_{1,2}, b_1 x_1 + c_{1,2})$, which implies $(u(a_1 x_1 + c_{1,2}), u(a_2 x_2 + c_{1,2})) \succeq_{\rm w} (u(b_2 x_2 + c_{1,2}), u(b_1 x_1 + c_{1,2}))$ upon using Theorem 5.A.1 of Marshall *et al.* (2011). Hence, we have

$$u(a_1 x_1 + c_{1,2}) + u(a_2 x_2 + c_{1,2}) \geq u(b_2 x_2 + c_{1,2}) + u(b_1 x_1 + c_{i,j}).$$

By using iterated expectation formula, we conclude that $\Delta_2^{1,2}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\lambda}, \boldsymbol{X}) \geq 0$, and this in turn implies $\eta(\boldsymbol{a}) \geq \eta(\boldsymbol{b})$, and hence completes the proof. ■

Note that LWSAI $\boldsymbol{I}$ implies $I_1 \leq_{\rm st} \cdots \leq_{\rm st} I_n$. Theorem 3.5 suggests that increasing the heterogeneity among scale parameters makes the insurance portfolio more dangerous if the larger claim severity has larger occurrence probability.

Since the stop-loss order agrees with the TVaR risk measure and distortion risk measure with concave distortion function, the following result can be obtained from Theorem 3.5.

Corollary 3.6. *For comonotonic* $\boldsymbol{X}$ *with* $X_1 \leq_{\rm st} \cdots \leq_{\rm st} X_n$ *and LWSAI* $\boldsymbol{I}$*, if* $\boldsymbol{a} \overset{\rm m}{\succeq} \boldsymbol{b}$*, then for* $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{I}_n$*,*

i $\mathrm{TVaR}\left[\sum_{i=1}^n I_i a_i X_i, \alpha\right] \geq \mathrm{TVaR}\left[\sum_{i=1}^n I_i b_i X_i, \alpha\right]$*, for all* $\alpha \in [0, 1]$*;*

ii $\rho_g\left[\sum_{i=1}^n I_i a_i X_i\right] \geq \rho_g\left[\sum_{i=1}^n I_i b_i X_i\right]$*, for all concave distortion function* g*.*

Now, we illustrate Theorem 3.3, Corollary 3.4 and Theorem 3.5 by a numerical example.

Example 3.7. *For* $(\lambda_1, \lambda_2, \lambda_3) = (0.6, 0.4, 0.2)$ *and* U *uniformly distributed on* $(0, 1)$*, consider the realizable returns* $(X_1, X_2, X_3) = \big(-\lambda_1^{-1} \log U, -\lambda_2^{-1} \log U, -\lambda_3^{-1} \log U\big)$ *with default indicators* (I_1, I_2, I_3) *having the joint distribution given in Table 1. It is easy to check that* (I_1, I_2, I_3) *is LWSAI and* (X_1, X_2, X_3) *is comonotonic with* $X_1 \leq_{\rm st} X_2 \leq_{\rm st} X_3$*. Consider the*

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TABLE 2

STOP-LOSS PREMIUMS $\mathrm{E}[(\sum_{i=1}^{3} I_i s_i X_i - t)_+]$.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$
$\boldsymbol{s}=\boldsymbol{a}$	11.0000	10.4635	9.9532	9.4678	9.0060
$\boldsymbol{s}=\boldsymbol{b}$	9.4500	8.7436	8.1263	7.5774	7.0824
$\boldsymbol{s}=\boldsymbol{b}^*$	4.3333	3.7012	3.2029	2.7954	2.4533
$\boldsymbol{s}=\boldsymbol{c}$	7.4333	6.6444	5.9853	5.4214	4.9300
$\boldsymbol{s}=\boldsymbol{c}^*$	5.8833	5.0939	4.4378	3.8841	3.4115

following vector of scale parameters: $\boldsymbol{a}=(0,0,4)$, $\boldsymbol{b}=(0,1,3)$, $\boldsymbol{b}^*=(1,3,0)$, $\boldsymbol{c}=(1,1,2)$ *and* $\boldsymbol{c}^*=(1,2,1)$. *Due to* $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b} \overset{\mathrm{m}}{\succeq} \boldsymbol{c}$ *for* $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathcal{I}_3$, *Theorem 3.5 ensures*

$$\sum_{i=1}^{n} I_i a_i X_i \geq_{\mathrm{sl}} \sum_{i=1}^{n} I_i b_i X_i \geq_{\mathrm{sl}} \sum_{i=1}^{n} I_i c_i X_i .$$

Note that $\boldsymbol{b}^* \notin \mathcal{I}_3$ *is a permutation of* $\boldsymbol{b}$, *and* $\boldsymbol{c}^* \notin \mathcal{I}_3$ *is a permutation of* $\boldsymbol{c}$. *According to Theorem 3.3 and Corollary 3.4, the amount of capital needed for the insurer for* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *must be larger than that of* $\boldsymbol{b}^*$ *and* $\boldsymbol{c}^*$, *respectively. Table 2 collects stop-loss premiums* $\mathrm{E}[(\sum_{i=1}^{3} I_i s_i X_i - t)_+]$ *for different retentions* $t \in \mathbb{R}_+$ *under different scale parameters* $\boldsymbol{s}=\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}^*,\boldsymbol{c},\boldsymbol{c}^*$, *which verifies the above analysis. However, it should be remarked that more dispersive scale parameters may result in less dangerous portfolio if the concerned vector of scale parameters falls out of* $\mathcal{I}_n$. *For example, the stop-loss premiums for* $\boldsymbol{b}^*$ *are less than that for* $\boldsymbol{c}$ *even though* $\boldsymbol{b}^* \overset{\mathrm{m}}{\succeq} \boldsymbol{c}$.

Naturally, one may wonder whether the usual stochastic order among the claim severities could be relaxed in Theorem 3.5? The following numerical example serves as a negative answer.

Example 3.8. *For* $X_1 = -\log(1-U)$ *and* $X_2 = 2U$ *with* $U \sim \mathcal{U}(0,1)$, *one can verify that* $X_1 \leq_{\mathrm{icv}} X_2$, *however* $X_1 \leq_{\mathrm{st}} X_2$ *does not hold. Set* $p((0,0)) = 0.07$, $p((0,1)) = 0.32$, $p((1,0)) = 0.31$ *and* $p((1,1)) = 0.3$. *It is easy to check that* (I_1, I_2) *is LWSAI.*

Consider two scale vectors $\boldsymbol{a}=(2,3)$ *and* $\boldsymbol{b}=(2.5,2.5)$ *in* $\mathcal{I}_2$. *Although* $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b}$, *two stop-loss premium curves are found to cross with each other in Figure 1. Therefore, the usual stochastic order among claim severities cannot be relaxed to the increasing concave order.*

4. RWSAI CLAIM SEVERITIES

This section studies the effects of orderings and heterogeneity among scale parameters on the amount of capital needed for the insurer when the claim

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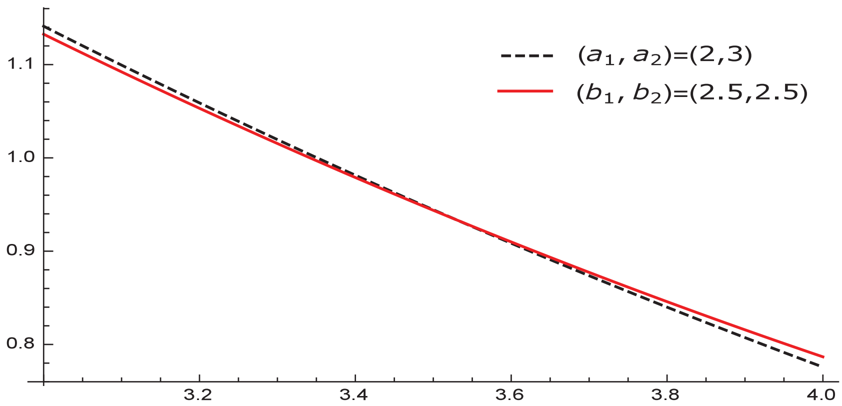

FIGURE 1: Stop-loss premiums corresponding to $\boldsymbol{a}$ and $\boldsymbol{b}$ with $t \in [3, 4]$. (Color online)

occurrence probabilities are LWSAI while the claim severities are RWSAI. To begin with, we recall one useful lemma.

Lemma 4.1 (You and Li, 2015). (X_1, X_2) *is RWSAI if and only if* $\mathrm{E}[g_2(X_1, X_2)] \geq \mathrm{E}[g_1(X_1, X_2)]$ *for all* g_1 *and* g_2 *such that*

i $g_2(x_1, x_2) - g_1(x_1, x_2)$ *is increasing in* $x_2 \geq x_1$ *for any* x_1*, and*
ii $g_2(x_1, x_2) + g_2(x_2, x_1) \geq g_1(x_1, x_2) + g_1(x_2, x_1)$ *for any* $x_2 \geq x_1$.

Theorem 4.2. *Suppose that* $\boldsymbol{X}$ *is RWSAI and* $\boldsymbol{I}$ *is a LWSAI Bernoulli random vector. If* $a_i \leq a_j$ *for* $1 \leq i < j \leq n$*, then*

$$\sum_{r=1}^{n} I_r a_r X_r \geq_{\mathrm{sl}} I_i a_j X_i + I_j a_i X_j + \sum_{r \neq i,j}^{n} I_r a_r X_r.$$

Proof. According to the proof of Theorem 3.3, it suffices to show the non-negativity of (3.2). The RWSAI $\boldsymbol{X}$ guarantees that $[(X_i, X_j)|\boldsymbol{X}_{\{i,j\}}]$ is RWSAI. Given $\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}$, denote

$$g_2(x_i, x_j) = u(a_i x_i + a_j x_j + d_{i,j}) \quad \text{and} \quad g_1(x_i, x_j) = u(a_j x_i + a_i x_j + d_{i,j}).$$

For $x_j \geq x_j' \geq 0$, $a_i \leq a_j$ and $x_i \leq x_j$, we have $a_i x_i + a_j x_j + d_{i,j} \geq a_j x_i + a_i x_j + d_{i,j}$, implying

$$(a_i x_i + a_j x_j + d_{i,j}, a_j x_i + a_i x_j' + d_{i,j}) \succeq_{\mathrm{w}} (a_j x_i + a_i x_j + d_{i,j}, a_i x_i + a_j x_j' + d_{i,j}).$$

Upon applying Theorem 5.A.1 of Marshall *et al.* (2011), we have

$$(g_2(x_i, x_j), g_1(x_i, x_j')) \succeq_{\mathrm{w}} (g_1(x_i, x_j), g_2(x_i, x_j')),$$

which further implies $g_2(x_i, x_j) - g_1(x_i, x_j) \geq g_2(x_i, x_j') - g_1(x_i, x_j')$, i.e., $g_2(x_i, x_j) - g_1(x_i, x_j)$ is increasing in $x_j \geq x_i$. On the other hand, it is plain that $g_2(x_i, x_j) + g_2(x_j, x_i) = g_1(x_i, x_j) + g_1(x_j, x_i)$ for any $x_j \geq x_i$. Therefore, from Lemma 4.1, it follows immediately that

$$\begin{aligned} &\mathrm{E}\big[u\big(a_i X_i + a_j X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}\big) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\big] \\ &\quad \geq \mathrm{E}\big[u\big(a_j X_i + a_i X_j + (\boldsymbol{a} \cdot \boldsymbol{X})_{\{i,j\}}\big) | \boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}\big]. \end{aligned} \tag{4.1}$$

By applying iterated expectation formula on inequality (4.1), we reach (3.3). Similarly, (3.4) and (3.5) can be verified for $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0, 0)$ and $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(1, 1)$, $k = 1, 2, \ldots, n-1$. As a result, we reach inequality (3.6).

For any given $\boldsymbol{\lambda} \in \Lambda_k^{i,j}(0, 1)$ and $\boldsymbol{X}_{\{i,j\}} = \boldsymbol{x}_{\{i,j\}}$, let

$$\begin{aligned} f_2(x_i, x_j) = u(a_j x_j + c_{i,j}) + u(a_i x_i + c_{i,j}) \text{ and } f_1(x_i, x_j) = u(a_i x_j + c_{i,j}) \\ + u(a_j x_i + c_{i,j}). \end{aligned}$$

It is easy to see that $f_2(x_i, x_j) - f_1(x_i, x_j)$ is increasing in $x_j \geq x_i$ by noting the increasing property of $u(a_j x_j + c_{i,j}) - u(a_i x_j + c_{i,j})$ with respect to $x_j \geq x_i$. Note that $f_2(x_i, x_j) + f_2(x_j, x_i) = f_1(x_i, x_j) + f_1(x_j, x_i)$ for any $x_j \geq x_i$. Upon using Lemma 4.1, we can conclude that $\Delta_1^{i,j}(\boldsymbol{a}, \boldsymbol{\lambda}, \boldsymbol{X}) \geq 0$ in (3.7). This invokes $\eta(\boldsymbol{a}) \geq \eta(\tau_{ij}(\boldsymbol{a}))$, yielding the desired result. ■

The following corollary is an immediate result of Theorem 4.2, which depicts the orderings of scale parameters that lead to the maximum amount of capital needed for the insurer when $\boldsymbol{X}$ is RWSAI.

Corollary 4.3. *If $\boldsymbol{X}$ is RWSAI and $\boldsymbol{I}$ is a LWSAI Bernoulli random vector If $\boldsymbol{a} \in \mathcal{I}_n$, then*

$$\sum_{r=1}^{n} I_r a_r X_r \geq_{\text{sl}} \sum_{r=1}^{n} I_r a_{\tau(r)} X_r, \quad \textit{for any permutation } \tau \textit{ of } \{1, 2, \ldots, n\}.$$

By Corollary 4.3, we shall investigate the effects of heterogeneity among scale parameters under space $\mathcal{I}_n$ on the aggregate claim amount when the claim severities are dependent through RWSAI and the claim occurrence probabilities are dependent through LWSAI.

Lemma 4.4 (You and Li (2015), Proposition 4.5). *If $\boldsymbol{X}$ is RWSAI, then $\mathcal{I}_n \ni \boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b}$ implies $\sum_{i=1}^n a_i X_i \geq_{\text{sl}} \sum_{i=1}^n b_i X_i$.*

Theorem 4.5. *If $\boldsymbol{X}$ is RWSAI and $\boldsymbol{I}$ is LWSAI, then*

$$\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b} \Longrightarrow \sum_{i=1}^{n} I_i a_i X_i \geq_{\text{sl}} \sum_{i=1}^{n} I_i b_i X_i, \quad \textit{for any } \boldsymbol{b}, \boldsymbol{a} \in \mathcal{I}_n.$$

Proof. By adopting the proof of Theorem 3.5, it is equivalent to prove (3.8) is non-negative. On the one hand, it stems from Lemma 4.4 that $\mathrm{E}[u(\boldsymbol{a} \cdot \boldsymbol{X})] \geq$

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$\mathrm{E}[u(\boldsymbol{b}\cdot\boldsymbol{X})]$. Thus, it suffices to prove the non-negativity of the first part of (3.8), i.e.,

$$\Delta_3(\boldsymbol{a},\boldsymbol{b},\boldsymbol{X})=\sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k}p(\boldsymbol{\lambda})\left\{\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]-\mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right\}\geq 0. \quad (4.2)$$

By the nature of majorization order, it suffices to prove the non-negativity of (4.2) under the conditions $a_1\leq a_2$, $b_1\leq b_2$, $(a_1,a_2)\overset{\mathrm{m}}{\succeq}(b_1,b_2)$ and $a_i=b_i$ for $i=3,\ldots,n$. For any $\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,0)$ with $k=1,2,\ldots,n-1$, it holds that

$$\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]=\mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]. \quad (4.3)$$

For any fixed $\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)$ and $\boldsymbol{X}_{\{1,2\}}=\boldsymbol{x}_{\{1,2\}}$, let

$$h_2(x_1,x_2)=u(a_1x_1+a_2x_2+c_{1,2})\quad\text{and}\quad h_1(x_1,x_2)=u(b_1x_1+b_2x_2+c_{1,2}).$$

For $x_1\leq x_2$, $a_1\leq a_2$ and $b_1\leq b_2$, by Lemma 1 of You and Li (2016), we know that $h_2(x_1,x_2)-h_1(x_1,x_2)$ is increasing in $x_2\geq x_1$. On the other hand, it is plain that

$$(a_1x_1+a_2x_2+c_{1,2},a_1x_2+a_2x_1+c_{1,2})\overset{\mathrm{m}}{\succeq}(b_1x_1+b_2x_2+c_{1,2},b_1x_2+b_2x_1+c_{1,2}).$$

By applying Theorem 5.A.1 of Marshall *et al.* (2011), we have

$$(h_2(x_1,x_2),h_2(x_2,x_1))\succeq_{\mathrm{w}}(h_1(x_1,x_2),h_1(x_2,x_1)),$$

which implies $h_2(x_1,x_2)+h_2(x_2,x_1)\geq h_1(x_1,x_2)+h_1(x_2,x_1)$. As a result, by Lemma 4.1, we have

$$\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]-\mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\geq 0.$$

So, (4.2) is non-negative for $\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,0)$ and $\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,1)$. Similarly, we also have

$$\begin{aligned}\Delta_3(\boldsymbol{a},\boldsymbol{b},\boldsymbol{X})&\geq\sum_{k=1}^{n-1}\Bigg\{\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)}p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]-\mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right)\\&\qquad+\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(1,0)}p(\boldsymbol{\lambda})\left(\mathrm{E}\left[u\left(\boldsymbol{a}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]-\mathrm{E}\left[u\left(\boldsymbol{b}\cdot\boldsymbol{\lambda}\circ\boldsymbol{X}\right)\right]\right)\Bigg\}\\&=\sum_{k=1}^{n-1}\sum_{\boldsymbol{\lambda}\in\Lambda_k^{1,2}(0,1)}p(\tau_{12}(\boldsymbol{\lambda}))\Delta_2^{1,2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\lambda},\boldsymbol{X}),\end{aligned}$$

where $\Delta_2^{1,2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\lambda},\boldsymbol{X})$ is defined in (3.12). Therefore, it suffices to show $\Delta_2^{1,2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\lambda},\boldsymbol{X})\geq 0$.

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For any fixed $\boldsymbol{\lambda} \in \Lambda_k^{1,2}(0,1)$ and $\boldsymbol{X}_{\{1,2\}} = \boldsymbol{x}_{\{1,2\}}$, let

$$l_2(x_1, x_2) = u(a_1x_1 + c_{1,2}) + u(a_2x_2 + c_{1,2}) \text{ and } l_1(x_1, x_2) = u(b_1x_1 + c_{1,2}) + u(b_2x_2 + c_{1,2}).$$

We notice that $l_2(x_1, x_2) - l_1(x_1, x_2)$ is increasing in $x_2 \geq x_1$, which can be derived from the previous observation that $u(a_2x_2 + c_{1,2}) - u(b_2x_2 + c_{1,2})$ is increasing in $x_2 \geq x_1$ for $a_2 \geq b_2$. On the other hand, for $x_2 \leq x_1$ and $a_2 \geq b_2$, we have

$$(a_2x_2 + c_{1,2}, a_2x_1 + c_{1,2}, a_1x_1 + c_{1,2}, a_1x_2 + c_{1,2}) \overset{\text{m}}{\succeq} (b_2x_2 + c_{1,2}, b_2x_1 + c_{1,2}, b_1x_1 + c_{1,2}, b_1x_2 + c_{1,2}).$$

According to Theorem 5.A.1 of Marshall *et al.* (2011), it holds that

$$(u(a_2x_2 + c_{1,2}), u(a_2x_1 + c_{1,2}), u(a_1x_1 + c_{1,2}), u(a_1x_2 + c_{1,2})) \succeq_{\text{w}} (u(b_2x_2 + c_{1,2}), u(b_2x_1 + c_{1,2}), u(b_1x_1 + c_{1,2}), u(b_1x_2 + c_{1,2})).$$

Thus, the submajorization implies $l_2(x_1, x_2) + l_2(x_2, x_1) \geq l_1(x_1, x_2) + l_1(x_2, x_1)$. In light of Lemma 4.1, we obtain $\Delta_2^{1,2}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\lambda}, \boldsymbol{X}) \geq 0$. This completes the proof. ■

The next corollary is a direct consequence of Theorem 4.5, which explains the RWSAI dependence structure among $\boldsymbol{X}$ by using an Archimedean survival copula and hazard rate order.

Corollary 4.6. *For $\boldsymbol{X}$ with $X_1 \leq_{\text{hr}} \cdots \leq_{\text{hr}} X_n$ and an Archimedean survival copula associated with a log-convex generator, if $\boldsymbol{I}$ is LWSAI, then $\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b}$ implies*

$$\sum_{i=1}^{n} I_i a_i X_i \geq_{\text{sl}} \sum_{i=1}^{n} I_i b_i X_i, \quad \textit{for any } \boldsymbol{b}, \boldsymbol{a} \in \mathcal{I}_n.$$

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed non-negative random variables, and let $I_1, I_2, \ldots, I_n$ be independent Bernoulli random variables such that $\mathrm{E}[I_i] = p_i$, which are independent of $X_1, X_2, \ldots, X_n$. Under the assumptions that $p_1 \leq \cdots \leq p_n \leq 1$, $\boldsymbol{a} \in \mathcal{I}_n$, $\boldsymbol{b} \in \mathcal{I}_n$ and $\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b}$, it was proved in Corollary 4.7 of Denuit and Frostig (2006) (also see Theorem 4.A.39 of Shaked and Shanthikumar (2007)) that $\sum_{i=1}^{n} I_i a_i X_i \geq_{\text{sl}} \sum_{i=1}^{n} I_i b_i X_i$. Obviously, the result in Corollary 4.6 substantially generalizes that of Corollary 4.7 of Denuit and Frostig (2006) to the case of heterogeneous $X_1, \ldots, X_n$ linked by an Archimedean survival copula with a log-convex generator and arrayed in the hazard rate order.

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Corollary 4.7. *For RWSAI* $\boldsymbol{X}$ *and LWSAI* $\boldsymbol{I}$*, if* $\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b}$ *for any* $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{I}_n$*, then*

i $\mathrm{TVaR}\left[\sum_{i=1}^{n} I_i a_i X_i, \alpha\right] \geq \mathrm{TVaR}\left[\sum_{i=1}^{n} I_i b_i X_i, \alpha\right]$*, for all* $\alpha \in [0, 1]$*;*
ii $\rho_g\left[\sum_{i=1}^{n} I_i a_i X_i\right] \geq \rho_g\left[\sum_{i=1}^{n} I_i b_i X_i\right]$*, for all concave distortion function* g*.*

Under the assumption that $\boldsymbol{X}$ is RWSAI and $\boldsymbol{I}$ is LWSAI and independent of $\boldsymbol{X}$, Theorem 4.5 (Corollary 4.7) asserts that an increase in the heterogeneity among scale parameters results in a set of much more dangerous insurance portfolio when the larger claim severity is accompanied with larger occurrence probability.

At the end, we employ one example to numerically illustrate Theorem 4.5.

Example 4.8. *Set* $p((0,0)) = 0.15$*,* $p((0,1)) = 0.46$*,* $p((1,0)) = 0.24$ *and* $p((1,1)) = 0.15$*, and let* $\bar{F}_1(x) = x^{-\beta_1}$ *and* $\bar{F}_2(x) = x^{-\beta_2}$ *with* $x \geq 1$ *and* $\beta_1 > \beta_2$ *be the respective survival functions of* X_1 *and* X_2*. Assume the Clayton survival copula with generator* $\phi(t) = (\theta t + 1)^{-1/\theta}$*,* $\theta > 0$*. Then, the survival function of* $\boldsymbol{X} = (X_1, X_2)$ *is*

$$\bar{F}_{\boldsymbol{X}}(x, y) = \left(x^{\theta\beta_1} + y^{\theta\beta_2} - 1\right)^{-\frac{1}{\theta}}, \quad x \geq 0, \ y \geq 0, \ \theta > 0.$$

Consider the utility function $u(x) = x^{\gamma}$ *with* $\gamma > 1$ *and the total wealth* ω*. Upon applying Corollary 1.6.12 of Denuit et al. (2006), under the assumption* $\gamma < \min\{\beta_1, \beta_2\}$ *we have*

$$\begin{aligned}
\kappa(a_1) &=: \mathrm{E}[u(a_1 I_1 X_1 + a_2 I_2 X_2)] \\
&= p((0,0))\mathrm{E}[u(0)] + p((1,0))\mathrm{E}[u(a_1 X_1)] + p((0,1))\mathrm{E}[u(a_2 X_2)] \\
&\quad + p((1,1))\mathrm{E}[u(a_1 X_1 + a_2 X_2)] \\
&= \gamma a_1^{\gamma}[p((1,0)) + p((1,1))](\beta_1 - \gamma)^{-1} + \gamma(\omega - a_1)^{\gamma}[p((0,1)) \\
&\quad + p((1,1))](\beta_2 - \gamma)^{-1} \\
&\quad + p((1,1))a_1(\omega - a_1)\gamma(\gamma - 1)\int_1^{+\infty}\int_1^{+\infty}\left(x^{\theta\beta_1} + y^{\theta\beta_2} - 1\right)^{-\frac{1}{\theta}} \\
&\quad \times [a_1 x + (\omega - a_1)y]^{\gamma - 2}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Setting $\omega = 4$*,* $\theta = 1$*,* $\beta_1 = 4$*,* $\beta_2 = 3$ *and* $\gamma = 2.5$*. It can be easily verified that* $X_1 \leq_{\mathrm{hr}} X_2$ *holds,* (I_1, I_2) *is LWSAI and* ϕ *is log-convex. Thus, the assumption of Theorem 4.5 (in particular, Corollary 4.6) is satisfied. From Figure 2,* $\kappa(a_1)$ *is maximized at* $a_1 = 0$ *for* $a_1 \in [0, 2]$*, and thus this confirms that the dispersiveness among scale parameters under space* $\mathcal{I}_2$ *increases the dangerousness of the insurance portfolio as proved in Theorem 4.5. However, as displayed in the figure, it cannot be judged whether the desired result still holds when* $a_1 \in [2, 4]$*.*

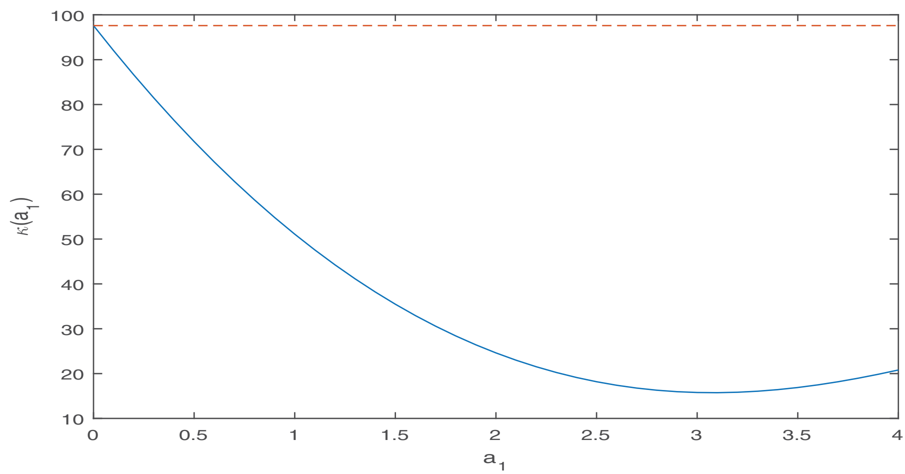

FIGURE 2: Plot of the function $\kappa(a_1)$ with respective to a_1. (Color online)

5. CONCLUDING REMARKS

For the case of LWSAI claim occurrence probabilities and comonotonic or RW-SAI claim severities, it is shown the insurance portfolio with the largest vector of scale parameters in the sense of the majorization order often turns out to be more dangerous for the insurer. This suggests that more capital is needed for covering the claims.

It is worth mentioning that the results developed here can be also applied in providing the optimal allocation policy of assets allocation for risk-seeking investors. As one eternal theme, an investor always pursues the optimal allocation of the wealth to multiple risk assets in the sense of maximizing the concerned potential risk return in financial market. In general, the investor aims to maximize the expected utility of the total potential return due to the allocation of the wealth. In modern financial markets, lots of assets such as various break-even investment financial products and some insurance policies in actuarial science bear default risks. There is a possibility that a borrower fails to pay the interest or the principal repayment obligations on a loan agreement due to either dishonesty or plain inability to do so, and the default risk plays a part in the expected total returns. For the case of risk-averse investors, interested readers may refer to Cheung and Yang (2004), Chen and Hu (2008), Cai and Wei (2015) and Li and Li (2016) for detailed treatments on optimal allocation policies.

However, there exist situations where the investors may be kind of risk-seeking; see, for example, Åstebro (2003), Post and Levy (2005) and Seiler and Seiler (2010). Consider a portfolio of n risk assets with realizable returns $\boldsymbol{X}$ and the corresponding defaults $\boldsymbol{I}$, i.e.,

$$I_i = \begin{cases} 0, & \text{the default of the } i\text{th asset occurs,} \\ 1, & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, \ldots, n.$$

Under the framework of expected utility, a risk-seeking investor with the initial wealth ω faces the optimization problem

$$\max_{\boldsymbol{a}\in\mathcal{A}_\omega} \mathrm{E}\left[u\left(\sum_{i=1}^{n} I_i a_i X_i\right)\right], \tag{5.1}$$

where $\mathcal{A}_\omega = \{\boldsymbol{a} = (a_1, \ldots, a_n) : a_1 + \cdots + a_n = \omega\}$ comprises of all admissible allocations, $\boldsymbol{X}$ is assumed to be independent of $\boldsymbol{I}$ and u is increasing and convex. According to Theorems 3.5 and 4.5, the following optimal allocation policy can be established for the risk-seeking investor, which suggests that all the wealth should be put on the asset with the largest stochastic return and smallest default probability.

Proposition 5.1. *If $\boldsymbol{I}$ is LWSAI and $\boldsymbol{X}$ is comonotonic or RWSAI, then* $(0, \ldots, 0, \omega)$ *is one solution to Problem* (5.1).

It should be remarked that the results of Theorems 3.5 and 4.5 still hold with $\boldsymbol{b} \in \mathcal{I}_n$ removed and $\boldsymbol{a} \overset{\mathrm{m}}{\succeq} \boldsymbol{b}$ replaced by $\boldsymbol{a} \succeq_{\mathrm{w}} \boldsymbol{b}$. It is of natural interest to extend our results to the case of location-scale claim severities, i.e., studying the heterogeneity among $(\boldsymbol{a}, \boldsymbol{t})$ on the aggregate payout $\sum_{i=1}^{n} I_i(a_i X_i + t_i)$.

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