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108.41 Diophantine approximations for a class of recursive sequences

Introduction: The canonical example of a divergent sequence is $\{(-1)^n\}_{n \geqslant 1}$. It is arguably the simplest example of a sequence $\{x_n\}_{n \geqslant 1}$ for which we can explicitly compute that $\overline{\lim_{n \to \infty}}\, x_n = 1 \neq -1 = \underline{\lim_{n \to \infty}}\, x_n$, where we recall that the limit superior and limit inferior are defined, respectively, by $\overline{\lim_{n \to \infty}}\, x_n = \lim_{n \to \infty} \left(\sup_{m \geqslant n} x_m \right)$ and $\underline{\lim_{n \to \infty}}\, x_n = \lim_{n \to \infty} \left(\inf_{m \geqslant n} x_m \right)$. Two closely related divergent sequences are given by $c_n = \cos(n)$ and $s_n = \sin(n)$, $n \geqslant 1$. Similarly, we have $\overline{\lim_{n \to \infty}}\, c_n = 1 \neq -1 = \underline{\lim_{n \to \infty}}\, c_n$, but these calculations are not nearly as simple as the ones for the canonical example $\{(-1)^n\}_{n \geqslant 1}$ since they essentially rely on a deeper fact regarding the equi-distribution modulo 2π of the positive integers.

A natural way to re-write the divergence of a bounded sequence such as $\{c_n\}_{n \geqslant 1}$ is by considering a slightly modified version of it that behaves monotonically. For example, let us define recursively the sequence $\{u_n\}_{n \geqslant 1}$ by

$$u_{n+1} = \max\{u_n, c_n\}, \qquad n \geqslant 1, \tag{1}$$

with $u_1 \in \mathbb{R}$ some fixed value. Proving the convergence of the recursive sequence (1) is a straightforward exercise found in the calculus textbook [1, Exercise 106, p. 505]. Clearly, if $u_1 \geqslant 1$, the sequence is constant and equal to u_1, hence convergent to u_1. Assuming $u_1 < 1$, we see that u_n is non-decreasing and bounded above by 1, therefore convergent by the Monotone Convergence Theorem. The really interesting question however, which is not asked in [1], is finding out *precisely which value* does the sequence $\{u_n\}_{n \geqslant 1}$ converge to. On a closer inspection, we discover that computing the exact value of $\lim_{n \to \infty} u_n$ propels us into the wonderful world of Diophantine approximations, the area of mathematics concerned with the approximation of real numbers by rational ones.

Diophantine approximations

The calculation of $\lim_{n \to \infty} u_n$ makes use of Dirichlet's Approximation Theorem [3, Chapter II].

Theorem 1 (Dirichlet)

For any $\alpha \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ such that $1 \leqslant q \leqslant n$ and $|\alpha q - p| < \frac{1}{n}$.

Since the proof of Theorem 1 is a simple consequence of the Pigeonhole Principle, we briefly recall it here for the convenience of the reader. We only give the argument for $\alpha > 0$. Consider the set of $n + 1$ numbers $f_k = k\alpha - \lfloor k\alpha \rfloor$, $0 \leqslant k \leqslant n$, where the *floor* of $x \in \mathbb{R}$ is defined as $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leqslant x\}$. Since all of the numbers f_k belong to the interval

$$[0, 1) = \bigcup_{l=0}^{n-1} \left[\frac{l}{n}, \frac{l+1}{n}\right),$$

we conclude that there must exist some $l_0 \in \{0, 1, \dots, n - 1\}$ and $k_1, k_2 \in \{0, 1, \dots, n\}$ with $k_1 < k_2$ such that $f_{k_1}, f_{k_2} \in \left[\frac{l_0}{n}, \frac{l_0 + 1}{n}\right)$. In particular, $|f_{k_2} - f_{k_1}| < \frac{1}{n}$. Now letting $q = k_2 - k_1$ and $p = \lfloor k_2 \alpha \rfloor - \lfloor k_1 \alpha \rfloor$ yields $|q\alpha - p| < \frac{1}{n}$.

We are now ready to compute $\lim_{n \to \infty} u_n$. Let $\varepsilon > 0$ be given and choose $N = N(\varepsilon) \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. By Theorem 1, there exist $p, q \in \mathbb{N}$ such that $1 \leqslant q \leqslant N$ and $|2\pi q - p| < \frac{1}{N} < \varepsilon$. Recalling that the cosine function is Lipschitz, that is,

$$|\cos x - \cos y| \leqslant |x - y|, \qquad \forall x, y \in \mathbb{R},$$

we get

$$1 - \cos p = \cos 2\pi q - \cos p < \varepsilon.$$

Finally, for all $n > p$, we have

$$|u_n - 1| = 1 - u_n \leqslant 1 - u_{p+1} \leqslant 1 - \cos p < \varepsilon.$$

This proves that $\lim_{n \to \infty} u_n = 1$.

Similarly for the sequence $\{u_n\}_{n \geqslant 1}$, we could have also defined recursively the sequence $\{v_n\}_{n \geqslant 1}$ by

$$v_{n+1} = \min\{v_n, c_n\}, \qquad n \geqslant 1. \tag{2}$$

By the Monotone Convergence Theorem we see that $\{v_n\}_{n \geqslant 1}$ is convergent as well, but to what exactly? Using the substitution $v_n = -w_n$, this reduces to computing $\lim_{n \to \infty} w_n$, where $\{w_n\}_{n \geqslant 1}$ is given by

$$w_{n+1} = \max\{w_n, -c_n\} = \max\{w_n, \cos(n + \pi)\}. \tag{3}$$

The recursive sequences defined in (1) and (3) suggest that one should consider a larger class of recursive sequences that encompasses both of them.

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A class of recursive sequences

Let $\beta \in \mathbb{R}$ and $\gamma \in [0, \infty)$ be some fixed parameters, and $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ some fixed function. Consider the class of sequences $\{u_n^{\beta,\gamma}\}_{n \geqslant 1}$ given by the recursion

$$u_1^{\beta,\gamma} = u(\beta, \gamma), \; u_{n+1}^{\beta,\gamma} = \max\{u_n^{\beta,\gamma}, \cos(\gamma n + \beta)\}, \qquad n \geqslant 1. \qquad (4)$$

Without loss of generality, we can assume $\beta \leqslant 0$, since we can always replace $\cos(\gamma n + \beta)$ with $\cos(\gamma n + \beta')$, where $\beta' = \beta - 2\pi k_0$ for some $k_0 \in \mathbb{N}$ such that $\beta \leqslant 2\pi k_0$. In what follows, we break up the discussion of the convergence of the sequence $\{u_n^{\beta,\gamma}\}_{n \geqslant 1}$ into two cases.

First, let us assume that $\frac{\gamma}{2\pi} \in \mathbb{Q}$, that is, $\gamma = 2\pi t$ for some $t = \frac{r}{s} \in \mathbb{Q}_+$ with $\gcd(r, s) = 1$, $s \geqslant 1$. Note that in this case, for all $n \in \mathbb{N}$, $\cos\left(2\pi \frac{nr}{s} + \beta\right)$ takes values from the finite set

$$\mathcal{S}_{\beta,\gamma} = \left\{\cos\left(2\pi \frac{pr}{s} + \beta\right) : p \in \{0, 1, \dots, s - 1\}\right\},$$

which yields $\lim\limits_{n\to\infty} u_n^{\beta,\gamma} = \max\{u(\beta, \gamma), \max \mathcal{S}_{\beta,\gamma}\}$. In particular, if $\gamma = 2\pi r$, for some $r \in \mathbb{N}$ then $\lim\limits_{n\to\infty} u_n^{\beta,\gamma} = \max\{u(\beta, \gamma), \cos\beta\}$.

Second, consider the case where $\frac{\gamma}{2\pi} \neq \mathbb{Q}$. Our claim is the following:

$$\text{If } \beta \leqslant 0 \leqslant \gamma, \text{ and } \frac{\gamma}{2\pi} \notin \mathbb{Q}, \text{ then } \lim_{n\to\infty} u_n^{\beta,\gamma} = 1.$$

The main tool we will use to prove this claim is Kronecker's Approximation Theorem [2].

Theorem 2 (Kronecker): Any real number can be approximated by multiples of any irrational number modulo integers; that is, given $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, we have

$$\forall \varepsilon > 0, \exists q \in \mathbb{N}, \exists p \in \mathbb{Z} \text{ such that } |\alpha q - p - \beta| < \varepsilon.$$

To prove now the claim above, start by noting that if $\alpha > 0$ and $\beta \leqslant 0$, then necessarily $p \geqslant 0$ in Theorem 2. If we apply Theorem 2 to $\alpha = \frac{2\pi}{\gamma} \notin \mathbb{Q}$ and β / γ, we obtain for appropriate integers p, q that

$$\left|\frac{2\pi}{\gamma}q - p - \frac{\beta}{\gamma}\right| < \frac{\varepsilon}{\gamma};$$

that is,

$$|2\pi q - (\gamma p + \beta)| < \varepsilon.$$

From this point on, the argument concerning the computation of the limit of the sequence $\{u_n^{\beta,\gamma}\}_{n \geqslant 1}$ resembles the one given for the sequence $\{u_n\}_{n \geqslant 1}$. Letting $n > p$, we have

$$|u_n^{\beta,\gamma} - 1| = 1 - u_n^{\beta,\gamma} \leqslant 1 - u_{p+1}^{\beta,\gamma} \leqslant 1 - \cos(\gamma p + \beta)$$

$$= \cos 2\pi q - \cos(\gamma p + \beta) < \varepsilon,$$

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which proves our claim. In particular, this shows that for the sequence defined in (3), $\lim_{n\to\infty} w_n = 1$, and then for the sequence defined in (2), $\lim_{n\to\infty} v_n = -1$.

Finally, let us observe that the study of the families of recursive sequences

$$\tilde{u}_{n+1}^{\beta,\gamma} = \min\left\{\tilde{u}_n^{\beta,\gamma}, \cos\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

$$z_{n+1}^{\beta,\gamma} = \max\left\{z_n^{\beta,\gamma}, \sin\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

and

$$\tilde{z}_{n+1}^{\beta,\gamma} = \min\left\{\tilde{z}_n^{\beta,\gamma}, \sin\left(\gamma n + \beta\right)\right\}, \qquad n \geqslant 1,$$

reduces to the obvious equalities:

$$\tilde{u}_n^{\beta,\gamma} = -u_n^{\beta+\pi,\gamma},\ z_n^{\beta,\gamma} = u_n^{\beta-\frac{1}{2}\pi,\gamma} \text{ and } \tilde{z}_n^{\beta,\gamma} = -u_n^{\beta+\frac{1}{2}\pi,\gamma}.$$

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108.42 On matrices whose elements are integers with given determinant

Introduction

For matrices with large positive integer elements with a small determinant is an interesting question in a linear algebra course. In this paper, we investigate matrices of order n with large positive integer elements and having a small determinant. In [1], the author explains the method for finding an infinite family of square matrices of order 2 with large positive integer entries and small positive integer determinant. Motivated by this fact, we generalise it for the case of square matrices of any arbitrary order $n \geqslant 2$. More precisely, we prove the following result.

Theorem 1: Given positive integers d and M, there exist infinitely many matrices $A = [a_{ij}]_{1\leqslant i,j\leqslant n}$ with integer elements satisfying $a_{ij} \geqslant M$ and $\det A = d$.