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BOUNDARIES OF INSTABILITY ZONES FOR SYMPLECTIC TWIST MAPS

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Abstract Very few things are known about the curves that are at the boundary of the instability zones of symplectic twist maps. It is known that in general they have an irrational rotation number and that they cannot be KAM curves. We address the following questions. Can they be very smooth? Can they be non-C^1?

Can they have a Diophantine or a Liouville rotation number? We give a partial answer for C^1 and C^2 twist maps.

In Theorem 1, we construct a C^2 symplectic twist map f of the annulus that has an essential invariant curve Γ such that

- Γ is not differentiable;
- the dynamics of $f_{|\Gamma}$ is conjugated to the one of a Denjoy counter-example;
- Γ is at the boundary of an instability zone for f.

Using the Hayashi connecting lemma, we prove in Theroem 2 that any symplectic twist map restricted to an essential invariant curve can be embedded as the dynamics along a boundary of an instability zone for some C^1 symplectic twist map.

Keywords: twist maps; instability zones; invariant curves; irregularity of invariant curves; Denjoy counter-examples; connecting lemma

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Secondary 37E10

1. Introduction

The exact symplectic twist maps of the two-dimensional annulus[1] have been studied for a long time because they represent (via a symplectic change of coordinates) the dynamics of the generic symplectic diffeomorphisms of surfaces near their elliptic periodic points (see [7]). One motivating example of such a map was introduced by Poincaré for the study of the restricted 3-body problem.

The study of such maps was initiated by Birkhoff in the 1920s (see [5]). Among other beautiful results, he proved that any essential curve that is invariant by a symplectic twist map of the annulus is the graph of a Lipschitz map (an *essential*

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[1] All the definitions are given in § 1.2.

curve is a simple loop that is not homotopic to a point and an *essential* annulus is an annulus that contains at least one essential curve). He then introduced the notion of an *instability zone*.

Definition. An *instability zone* of a symplectic twist map f of the annulus is an open subset U of the annulus $\mathbb{A}$ that is invariant by f and such that the following hold.

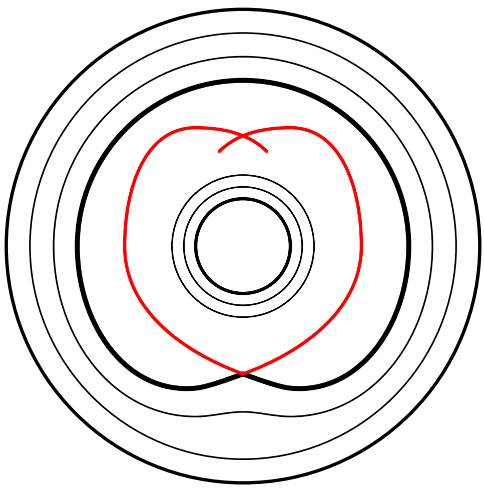
- U is homeomorphic to the (open) annulus $\mathbb{A}$ and is an essential annulus.
- The closure $\bar{U}$ of U in $\mathbb{A}$ contains no essential invariant curve that is not contained in the boundary ∂U.
- U is a maximal (for the inclusion $\subset$) subset of $\mathbb{A}$ that satisfies all these properties.

Let us notice that it is not true in general that a connected component of the boundary of a set $U \subset \mathbb{A}$ homeomorphic to $\mathbb{A}$ must be a curve. However, in the particular case where U is an instability zone of a symplectic twist map, this is true, and is proven in [19].

There are three kinds of instability zone U.

1. The whole annulus $\mathbb{A}$ can be an instability zone; this happens for example for the standard map with a large enough parameter (see [4, 18, 16]).
2. U is a connected component of the complement of an essential invariant curve.
3. U is bounded; in this case, there exist two Lipschitz functions $\psi_- < \psi_+$ whose graphs are invariant and that satisfy $U = \{(\theta, r) \in \mathbb{A};\ \psi_-(\theta) < r < \psi_+(\theta)\}$.

A lot of things are known about the existence of these instability zones. Birkhoff proved in [6] the existence of such instability zones. He even gave the first (and only) explicit example of a boundary for an instability zone. To visualize his example, imagine the time-one map T of the rigid pendulum. It is a symplectic twist map with one hyperbolic fixed point and two separatrices connecting this fixed point to itself. Perturb T to create one transverse homoclinic intersection at a point of the upper separatrix without changing the lower separatrix $\mathcal{S}$. Then $\mathcal{S}$ becomes the boundary of an instability zone that is above $\mathcal{S}$ ('above' is 'inside' in the following picture).



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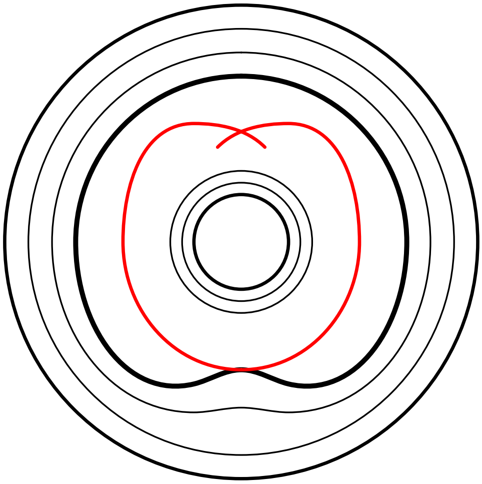
In Birkhoff's example, the boundary of the instability zone is non-smooth. Modifying the potential in such a way that it has a degenerate minimum, then we obtain a similar example for which the boundary of the instability zone is smooth:



In [12], Herman proved that, in general, the boundaries of the instability zones have an irrational rotation number. Hence Birkhoff's example is not generic.

Curiously, no other examples of explicit boundaries of instability zone are known. To be complete, let us just mention that in [17], Mather proves that the billiard map of a convex billiard whose curvature vanishes at at least one point has an instability zone bounded by the boundary of the billiard phase space. Unfortunately, in this case (vanishing curvature), the billiard map is not a twist map.

Though we do not know what the 'generic' boundaries of the instability zones are, we know some facts about what cannot be such a boundary for a sufficiently regular symplectic twist map, for example C^∞.

(1) It cannot be a curve on which the dynamics is C^∞ conjugated to a Diophantine rotation; indeed, KAM theorems (see [14, 3, 20, 23, 12], for example) imply that such a curve is accumulated from below and above by other invariant curves.

(2) It cannot be a curve on which the dynamics is C^∞ conjugated to a rational rotation; indeed, it is proved in the thesis of Douady [8] that in this case you can again apply KAM theory.

Hence a curve that is at the boundary of an instability zone either is not very regular or has a rational or Liouville rotation number. We then raise the following questions.

Question. *Can the boundary of an instability zone with an irrational rotation number be non-differentiable? Can it be smooth?*

We will give some answers to these questions in the case of low regularity (C^1 or C^2). First, we will prove the following theorem.

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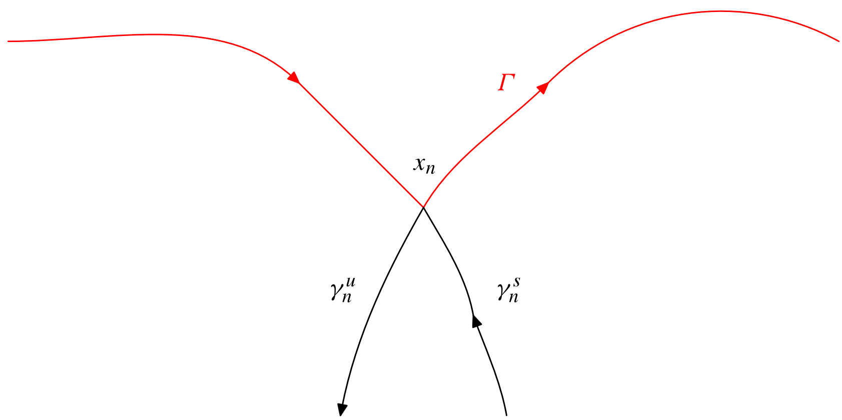
Theorem 1. *Let $\omega \in \mathbb{R}\backslash\mathbb{Q}$ be an irrational number. In any neighborhood of $(\theta, r) \to (\theta + r, r)$ in the C^2 topology, there exists a symplectic C^2 twist map f of the annulus that has an essential invariant curve Γ such that the following hold.*

- *$f_{|\Gamma}$ is C^0-conjugated to a Denjoy counter-example, and its rotation number is ω.*
- *If $\gamma : \mathbb{T} \to \mathbb{R}$ is the map whose graph is Γ, then γ is C^1 at every point except along the projection of one wandering orbit (x_n), along which γ has distinct right and left derivatives.*
- *Γ is the upper boundary of an instability zone $\mathcal{U}$ of f.*
- *There exist two families of C^2 curves $\gamma_n^s, \gamma_n^u : \mathbb{R} \to \mathbb{A}$ such that $\gamma_n^u(0) = \gamma_n^s(0) = x_n$ and*
  (a) $f \circ \gamma_n^u = \gamma_{n+1}^u$ *and* $f \circ \gamma_n^s = \gamma_{n+1}^s$;
  (b) $\forall y \in \gamma_0^s(\mathbb{R}), \lim_{n\to+\infty} d(f^n y, f^n x_0) = 0$ *and* $\forall y \in \gamma_0^u(\mathbb{R}), \lim_{n\to+\infty} d(f^{-n} y, f^{-n} x_0) = 0$;
  (c) $\gamma_n^s(]-\infty, 0]) \cup \gamma_n^u([0, +\infty[) \subset \Gamma$ *and* $\gamma_n^s(]0, +\infty[) \cup \gamma_n^u(]-\infty, 0[) \subset \mathcal{U}$.



Remark. (1) With a slight change in the construction, we can ask that Γ be the lower boundary of the instability zone $\mathcal{U}$.

(2) If we use a Denjoy counter-example with two disjoint orbits of wandering intervals, we can do the same construction along two orbits (x_n) and (y_n) and the obtain that Γ is the common boundary of *two* instability zones: the one that is above Γ and the other that is under Γ.

(3) Our counter-example is defined by $f_\varphi(\theta, r) = (\theta + r, r + \varphi(\theta + r))$ with $\int_{\mathbb{T}} \varphi = 0$. Hence $f_\varphi(\theta, r + 1) = f_\varphi(\theta, r) + (0, 1)$. If a graph Γ is invariant by f_φ, all the translated graphs $\Gamma + (0, k)$ with $k \in \mathbb{Z}$ are invariant by f_φ. This implies that the instability zones of f_φ are either the whole annulus $\mathbb{A}$ or bounded instability zones. Hence $\mathcal{U}$ is a bounded instability zone, but the theorem gives us the description of just one connected component of its boundary.

(4) In [1], we gave an example of a C^1 symplectic twist map that has a non-differentiable essential invariant curve with irrational rotation number; here, we improve the construction in the following way.

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- Using the construction of Herman that is given in [12], we manage to improve the regularity of our example and obtain a C^2 counter-example.
- Using a function $\varphi : \mathbb{T} \to \mathbb{R}$ whose restriction to a lot of intervals is linear, we manage to create a non-trivial stable set for the invariant curve; this and the fact that the rotation number of f_φ restricted to the curve is irrational imply that the invariant curve is at the boundary of an instability zone (see § 1.3 for details).

  Indeed, with the notation of this theorem, $\gamma_n^s(\mathbb{R})$ is a part of the stable set of Γ,

  $$W^s(\Gamma) = \left\{x \in \mathbb{A};\ \lim_{n\to+\infty} d(f^n x, \Gamma) = 0\right\},$$

  and $\gamma_n^u(\mathbb{R})$ is a part of the unstable set of Γ,

  $$W^u(\Gamma) = \left\{x \in \mathbb{A};\ \lim_{n\to+\infty} d(f^{-n} x, \Gamma) = 0\right\}.$$

We now explain how the dynamics restricted to any invariant curve of a symplectic twist map that has an irrational rotation number can become the dynamics at the boundary of an instability zone, as follows.

Theorem 2. *Let Γ be an essential invariant curve of a C^1 symplectic twist map $f : \mathbb{A} \to \mathbb{A}$ whose rotation number is irrational or whose rotation number is rational and the dynamics restricted to Γ is*

– *either C^0 conjugated to a rational rotation;*

– *or such that every periodic point is hyperbolic.*

Then, in any neighborhood $\mathcal{U}$ of f for the C^1 topology, there exists a C^1 symplectic twist map $g : \mathbb{A} \to \mathbb{A}$ such that

(1) *Γ is at the boundary of an instability zone of g;*

(2) $g_{|\Gamma} = f_{|\Gamma}$.

We have seen before that such a result is not valid in C^∞ topology because of the KAM theorems. The tools used to prove Theorem 2 are specific to the C^1 topology: they are the connecting lemma of Hayashi (see [11]) and more precisely some consequences of this connecting lemma that are given in [2].

Contrary to Birkhoff's counter-example or to Theorem 1, we have no idea of what is the stable/unstable set of Γ in Theorem 2. Observe too that, for our example of Theorem 1, we only know a part of the stable/unstable set. Hence we raise the following question.

Question. *Is it possible to describe (in general or for some specific examples) the stable/unstable set of the boundary of an instability zone?*

In [15], Le Calvez proves interesting facts concerning the topological structure of those sets.

We observe in § 1.3 that the existence of a non-trivial stable set for an essential invariant curve with an irrational rotation number implies that this curve is at the boundary of an instability zone. Hence a related question is the following.

Question. *Can an essential invariant curve with irrational rotation number carry a non-uniformly hyperbolic invariant measure?*

Indeed, if this happens, the union of the stable and unstable manifold of the invariant measure cannot be contained in the curve, and the curve is then at the boundary of an instability zone.

Finally, concerning the first question that we raised, we obtain an answer just in the case of low regularity. Hence the following questions remain open.

Questions. (1) *Does there exist a smooth ($C^1, C^2, \ldots$) curve with an irrational rotation number that is at the boundary of an instability zone for a C^k symplectic twist map with $k \geqslant 2$?*

(2) *Does there exist a non- C^1 curve with an irrational rotation number that is at the boundary of an instability zone for a C^k symplectic twist map with $k \geqslant 3$?*

(3) *What does a 'typical' boundary of instability zone look like (is it regular, how is its rotation number ...)?*

1.1. Structure of the article

In § 2, we will recall Herman's construction of a C^2 symplectic twist map that has an essential invariant curve on which the dynamics is Denjoy. In particular, we will give some useful estimates in § 2.3.

In § 3, we will build the counter-example that is described in Theorem 1. In § 3.1, we will construct the homeomorphism that will represent the projected dynamics along the invariant curve, and we will be more precise about the choice of the constants in § 3.2. In §§ 3.3 and 3.4, we will prove some estimates. Then, in § 3.5, we will prove that our modified example is C^2, and in § 3.6 we will determine part of the stable/unstable sets of the invariant curve.

Finally, we will prove Theorem 2 in § 4.

1.2. Notation and definitions

Notation. • $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle.

- $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is the annulus, and an element of $\mathbb{A}$ is denoted by (θ, r).
- $\mathbb{A}$ is endowed with its usual symplectic form, $\omega = d\theta \wedge dr$, and its usual Riemannian metric.
- $\pi : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ is the first projection and $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{R}$ its lift.

Definition. A C^1 diffeomorphism $f : \mathbb{A} \to \mathbb{A}$ of the annulus that is isotopic to identity is a *positive twist map* (respectively, *negative twist map*) if, for any given lift $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and for every $\tilde{\theta} \in \mathbb{R}$, the map $r \mapsto \tilde{\pi} \circ \tilde{f}(\tilde{\theta}, r)$ is an increasing (respectively, decreasing) diffeomorphism. A *twist map* may be positive or negative.

Then the maps f_φ that we defined just after Theorem 1 are positive symplectic twist maps.

Definition. Let $\gamma : \mathbb{T} \to \mathbb{R}$ be a continuous map. We say that γ is C^1 at $\theta \in \mathbb{T}$ if there exists a number $\gamma'(\theta) \in \mathbb{R}$ such that, for every sequence (θ_n^1) and (θ_n^2) of points of $\mathbb{T}$ that converge to θ such that $\theta_n^1 \neq \theta_n^2$, then

$$\lim_{n\to\infty} \frac{\gamma(\theta_n^1) - \gamma(\theta_n^2)}{\theta_n^1 - \theta_n^2} = \gamma'(\theta),$$

where we denote by $\theta_n^1 - \theta_n^2$ the unique number that represents $\theta_n^1 - \theta_n^2$ and that belongs to $]-\frac{1}{2}, \frac{1}{2}]$.

The following assertions are then equivalent.

- γ is C^1 at every point of $\mathbb{T}$.
- γ is differentiable at every point of $\mathbb{T}$, and its derivative γ' is continuous (i.e., γ is C^1 in the classical sense).

1.3. Stable set of invariant curves

A consequence of a theorem of Mather is that, if an essential curve Γ that is invariant by a symplectic twist map is at the boundary of an instability zone, then $W^s(\Gamma) \setminus \Gamma \neq \emptyset$, and this is equivalent to $W^u(\Gamma) \setminus \Gamma \neq \emptyset$, too.

More precisely, in [5], Birkhoff proved that, if U is an instability zone, and if U_1 is a neighborhood of one of its ends (i.e., eventually after compactification, a connected component of its boundary) and U_2 is a neighborhood of the other end, then there exists an orbit traveling from U_1 to U_2. This theorem was improved in [19] by Mather, who proved that, if $\mathcal{C}_1$, $\mathcal{C}_2$ are the ends of U, there exists an orbit whose α-limit set is in $\mathcal{C}_1$ and ω-limit set is in $\mathcal{C}_2$. Mather used variational arguments and, later, Le Calvez gave in [15] a purely topological proof of this result.

Conversely, let us assume that Γ is an essential invariant curve that is invariant by a symplectic twist $f : \mathbb{A} \to \mathbb{A}$, and that $W^s(\Gamma)\setminus\Gamma \neq \emptyset$. The example of the rigid pendulum proves that it can happen that Γ is not at the boundary of an instability zone.[2] Let us assume that $f_{|\Gamma}$ has an irrational rotation number or that $f_{|\Gamma}$ is C^0 conjugated to a rational rotation. Suppose that Γ is not at the boundary of an instability zone. Then there exist two sequences of essential invariant curves (Γ_n^-) and (Γ_n^+) that are different from Γ such that

1. $\forall n$, $\Gamma_n^+ \geqslant \Gamma$ and $\Gamma_n^- \leqslant \Gamma$;
2. $\lim_{n\to\infty} d(\Gamma_n^-, \Gamma) = 0$ and $\lim_{n\to\infty} d(\Gamma_n^+, \Gamma) = 0$.

Birkhoff's theorem implies that the curves Γ_n^-, Γ_n^+ are equi-Lipschitz and then relatively compact for the C^0 norm (we speak of the C^0 norm of the function whose graph is the curve of interest). Let Γ^* be any limit point of one of these two sequences. Then Γ^* in an essential invariant curve such that $\Gamma \cap \Gamma^* \neq \emptyset$. Hence $f_{|\Gamma^*}$ has the same rotation number as $f_{|\Gamma}$.

Herman proved in [12] that two curves with the same irrational rotation number are equal. Moreover, if the restriction of a symplectic twist map f restricted to an

[2] Let us recall that we asked that an instability zone be homeomorphic to the open annulus.

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essential invariant curve Γ is C^0 conjugated to a rational rotation, all the orbits are action minimizing (see, e.g., [10]) and a consequence of the results of Forni and Mather contained in [9] (see their Theorem 13.3) is that, when an essential invariant curve is filled by a periodic orbit, there exists no other minimizing orbit with the same rotation number and then no other invariant curve with the same rotation number. Hence any other invariant curve and Γ are disjoint.

Hence $\Gamma^* = \Gamma$, and the two sequences (Γ_n^-) and (Γ_n^+) converge to Γ. If $\Gamma_n^\pm$ is the graph of $\gamma_n^\pm$, this implies that the sets $\{(\theta, r); \gamma_n^-(\theta) < r < \gamma_n^+(\theta)\}$ are a base of neighborhood of Γ. Because they are invariant by f, this implies that $W^s(\gamma) = \Gamma = W^u(\Gamma)$. We now summarize this result and Mather's result as given below.

Proposition. *Let $f : \mathbb{A} \to \mathbb{A}$ be a symplectic twist map, and let Γ be an essential invariant curve. Then the following hold.*

1. *If Γ is at the boundary of an instability zone, then $W^s(\Gamma) \setminus \Gamma \neq \emptyset$ and $W^u(\Gamma) \setminus \Gamma \neq \emptyset$.*
2. *If $W^s(\Gamma) \setminus \Gamma \neq \emptyset$ or $W^u(\Gamma) \setminus \Gamma \neq \emptyset$, and if the rotation number of $f_{|\Gamma}$ is irrational, or if it is rational and if $f_{|\Gamma}$ is C^0 conjugated to a rational rotation, then Γ is at the boundary of an instability zone.*

2. An example due to Herman

In [12], Herman gives an example of a C^2 symplectic twist map $f : \mathbb{A} \to \mathbb{A}$ that has a C^1 invariant curve $\mathcal{C}$ such that $F_{|\mathcal{C}}$ is C^0-conjugated to a Denjoy counter-example. Let us recall his construction. We fix $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

2.1. Generalized standard map

The following family of symplectic twist maps was introduced by Herman in [12]. The maps are defined by

$$f_\varphi : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}; (\theta, r) \mapsto (\theta + r, r + \varphi(\theta + r)).$$

where $\varphi : \mathbb{T} \to \mathbb{R}$ is a C^1 map such that $\int_{\mathbb{T}} \varphi(\theta) d\theta = 0$.

As noticed by Herman, the main advantage of this map is the following one. Using the explicit formula of f_φ, it is easy to see that the graph of $\psi : \mathbb{T} \to \mathbb{R}$ is invariant by f_φ if and only if

$$\forall \theta \in \mathbb{T}, \quad (\theta + \psi(\theta), \psi(\theta) + \varphi(\theta + \psi(\theta))) = (\theta + \psi(\theta), \psi(\theta + \psi(\theta))).$$

If we rewrite this equality and we denote a lift of $g : \mathbb{T} \to \mathbb{T}$ by $\tilde{g} : \mathbb{R} \to \mathbb{R}$, we obtain the following criterion for the invariance of the graph of ψ. The graph of $\psi : \mathbb{T} \to \mathbb{R}$ is invariant by f_φ if and only if we have the following:

- $g = Id_{\mathbb{T}} + \psi$ is an orientation preserving homeomorphism of $\mathbb{T}$;
- $Id_{\mathbb{R}} + \frac{1}{2}\varphi = \frac{1}{2}\left(\tilde{g} + \tilde{g}^{-1}\right)$.

In this case, $g(\theta) = \pi \circ f(\theta, \psi(\theta))$, and, if we denote by p the restriction of π to the graph of ψ, we have $p \circ f \circ p^{-1} = g$. The restriction of f to the graph of ψ is then conjugated to g.

Let us give the idea of the construction of Herman: he builds a particular Denjoy counter-example $g : \mathbb{T} \to \mathbb{T}$ of rotation number ω. Because of Denjoy's theorem, such a g cannot be C^2. Using very clever estimates, Herman proves that $\varphi = \tilde{g} + \tilde{g}^{-1} - 2\mathrm{Id}$ is C^2. Hence f_φ is the wanted counter-example.

2.2. Explicit construction of a circle diffeomorphism

We use the construction that is described on p. 94 of [12], with only a slight change: we define a function η in such a way that the Denjoy counter-example is linear on some small segments.

Let us recall that we fixed $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Let us fix $\delta > 0$ and $C \gg 1$.

We introduce, for $k \in \mathbb{Z}$,

$$\ell_k = \frac{a_C}{(|k| + C)(\log(|k| + C))^{1+\delta}},$$

where a_C is chosen such that $\sum_{k\in\mathbb{Z}} \ell_k = 1$.

We use a C^∞ function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta \geqslant 0$, support$(\eta) \subset [\frac{1}{4}, \frac{3}{4}]$, $\eta_{|[\frac{3}{8}, \frac{5}{8}]} = 1$, $\eta(1 - t) = \eta(t)$, and $\int_0^1 \eta(t)dt = 1$. We define η_k by $\eta_k(t) = \eta\left(\frac{t}{\ell_k}\right)$. Then, we have $\int_0^{\ell_k} \eta_k(t)dt = \ell_k$. Moreover, there exists a constant C_0, that depends only on η, such that

$$|\eta_k| \leqslant C_0; \quad |\eta_k'| \leqslant \frac{C_0}{\ell_k}. \tag{1}$$

We assume now that $C \gg 1$ is large enough so that

$$\forall k \in \mathbb{Z}, \quad \left|\frac{\ell_{k+1}}{\ell_k} - 1\right| C_0 < 1.$$

Then the map $h_k : [0, \ell_k] \to [0, \ell_{k+1}]$ defined by $h_k(x) = \int_0^x \left(1 + \left(\frac{\ell_{k+1}}{\ell_k} - 1\right)\eta_k(t)\right) dt$ is a C^∞ diffeomorphism such that $h_k(\ell_k) = \ell_{k+1}$.

There exists a Cantor subset $K \subset \mathbb{T}$ that has zero Lebesgue measure and such that the connected components of $\mathbb{T} \setminus K$, denoted by $(I_k)_{k\in\mathbb{Z}}$, are on $\mathbb{T}$ in the same order as the sequence $(k\omega)$ and such that length$(I_k) = \ell_k$.

Let us recall an example of semi-conjugation $j : \mathbb{T} \to \mathbb{T}$ of the Denjoy counter-example to the rotation R_ω. If $x \in \{k\omega; k \in \mathbb{Z}\}$, then we define $j^{-1}(x) = \int_0^x d\mu(t)$, where μ is the probability measure $\mu = \sum_{k\in\mathbb{Z}} \ell_k \delta_{k\omega}$, $\delta_{k\omega}$ being the Dirac mass at $k\omega$. Then $j : \mathbb{T} \to \mathbb{T}$ is a continuous map with degree 1 that preserves the order on $\mathbb{T}$ and that is such that $j(I_k) = k\omega$.

Then there is a C^1 diffeomorphism $g : \mathbb{T} \to \mathbb{T}$ that fixes K, is such that K is the unique minimal subset for g, has for rotation number $\rho(g) = \omega$, and verifies $j \circ g = R_\omega \circ j$. If $k \in \mathbb{Z}$, we introduce the notation $g_{|I_k} = g_k$; then we have $g_k(I_k) = I_{k+1}$. Following [12] again, we can assume that $g_k' = g_{|I_k}' = \left(1 + \left(\frac{\ell_{k+1}}{\ell_k} - 1\right)\eta_k\right) \circ R_{-\lambda_k}$, where $R_{-\lambda_k}(I_k) = [0, \ell_k]$ and that $g_k : I_k \to I_{k+1}$ is defined by $g_k = R_{\lambda_{k+1}} \circ h_k \circ R_{-\lambda_k}$.

2.3. Some useful inequalities

We recall without proof some inequalities that are given in [12] (sometimes we give some slight improvement of these inequalities) and that are useful in proving that $\tilde{g} + \tilde{g}^{-1}$ is C^2. The constants $C_i > 0$ are independent of k and $C \gg 1$ and the limit in (6) is uniform in $C \gg 1$. Introduce the notation $K_k = \frac{\ell_{k+1}}{\ell_k} - 1$. Then there exists a bounded function $\varepsilon : \mathbb{Z} \times [1, +\infty[\to \mathbb{R}$ (i.e., $\sup\{|\varepsilon(\pm n, C)|; n \in \mathbb{N}, C \geqslant 1\} = c < +\infty$) such that, for $C \gg 1$,

$$\text{if } n \geqslant 1, \quad K_{\pm n} = \pm \frac{-1}{n+C} + \frac{\varepsilon(\pm n, C)}{(n+C)^2} \pm \frac{-(1+\delta)}{(n+C)\log(n+C)}. \tag{2}$$

For $n = 0$, consider the previous formula with a sign '+'.

We have

$$\frac{C_1}{|k|+C} \leqslant |K_k| \leqslant \frac{C_2}{|k|+C} \tag{3}$$

$$C_1 K_k^2 \leqslant |K_{k-1} - K_k| = K_k - K_{k-1} \leqslant C_2 K_k^2 \tag{4}$$

$$\frac{C_2 (\log C)^\delta}{(|k|+C)(\log(|k|+C))^{1+\delta}} \geqslant \ell_k \geqslant \frac{C_1}{(|k|+C)(\log(|k|+C))^{1+\delta}} \tag{5}$$

$$\lim_{k \to \pm\infty} \frac{K_k^2}{\ell_k} = 0. \tag{6}$$

We do not recall here how we can deduce the fact that $\tilde{g} + \tilde{g}^{-1}$ is C^2 from these inequalities, because we will give a very similar proof for the modified example in the next section.

Let us just notice the following fact that is due to our modification of the function η:

$$\forall t \in \left[\frac{3}{8}\ell_k, \frac{5}{8}\ell_k\right], \quad h_k(t) = \frac{\ell_{k+1}}{\ell_k} t.$$

Let us now give some estimates that were not given in [12]. We introduce the notation $m_k := 1 + K_k + \frac{1}{1+K_{k-1}} = \frac{\ell_{k+1}}{\ell_k} + \frac{\ell_{k+1}}{\ell_{k+2}}$. We have

$$m_{k+1} - 2 - (K_{k+1} - K_k) = \frac{K_k^2}{1+K_k};$$

hence we deduce from (4) that

$$|m_{k+1} - 2| \leqslant C_2 K_k^2 \quad \text{and} \quad |m_k - m_{k+1}| \leqslant C_2 K_k^2. \tag{7}$$

Because of (3), we deduce that

$$|m_{k+1} - 2| \leqslant \frac{C_2}{(|k|+C)^2}. \tag{8}$$

3. Modification of Herman's example

3.1. Explicit construction of a circle homeomorphism

We introduce two new functions $\gamma_-, \gamma_+ : \mathbb{R} \to \mathbb{R}$ such that

- $\text{support}(\gamma_\pm) \subset [0, 1]$;
- $\gamma_{\pm|\mathbb{R}\setminus\{\frac{1}{2}\}}$ is C^∞;

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- $\gamma_{-|[\frac{1}{2},1]} = 0$; $\gamma_{+|[0,\frac{1}{2}]} = 0$;
- $\forall t \in [\frac{3}{8}, \frac{1}{2}[$, $\gamma_-(t) = 1$ and $\forall t \in]\frac{1}{2}, \frac{5}{8}]$, $\gamma_+(t) = 1$;
- $\int_0^1 \gamma_\pm(t)dt = 0$.

Hence these two functions are C^∞ on $\mathbb{R} \setminus \{\frac{1}{2}\}$ and discontinuous at the point $\frac{1}{2}$.

We define a sequence of functions (γ_k) by

$$\gamma_k(x) = \gamma_+\left(\frac{x}{\ell_k}\right) \quad \text{if } k \geqslant 1 \quad \text{and} \quad \gamma_k(x) = \gamma_-\left(\frac{x}{\ell_k}\right) \quad \text{if } k \leqslant 0.$$

Let us fix a sequence (α_k) of real numbers such that $0 < |\alpha_k| \leqslant A \cdot |K_k|$ (where A is a constant). Then we define $\psi_k : \mathbb{R} \to \mathbb{R}$ by $\psi_k(x) = K_k\eta_k(x) + \alpha_k\gamma_k(x)$ and a new function $h_k : [0, \ell_k] \to \mathbb{R}$ by $h_k(x) = \int_0^x (1 + \psi_k(t))dt$. If C is large enough (C was the constant that is used to define (ℓ_k) and then (K_k)), then (K_k) and (α_k) are small enough (A is a fixed constant that does not depend on C) and $1 + \psi_k$ is positive. Hence h_k is a homeomorphism onto $[0, \ell_{k+1}]$.

Let us notice that h_k is differentiable everywhere except at $\frac{\ell_k}{2}$, where it has distinct left and right derivatives. More precisely, the following hold.

(1) If $k \geqslant 1$, then, $\forall x \in [\frac{3}{8}\ell_k, \frac{1}{2}\ell_k]$, $h_k(t) = \frac{\ell_{k+1}}{\ell_k}t$ and $\forall x \in [\frac{1}{2}\ell_k, \frac{5}{8}\ell_k]$, $h_k(t) = (\frac{\ell_{k+1}}{\ell_k} + \alpha_k)t - \frac{\alpha_k\ell_k}{2}$.

(2) If $k \leqslant 0$, then, $\forall x \in [\frac{3}{8}\ell_k, \frac{1}{2}\ell_k]$, $h_k(t) = (\frac{\ell_{k+1}}{\ell_k} + \alpha_k)t - \frac{\alpha_k\ell_k}{2}$ and $\forall x \in [\frac{1}{2}\ell_k, \frac{5}{8}\ell_k]$, $h_k(t) = \frac{\ell_{k+1}}{\ell_k}t$.

Then, with this new function h_k we can construct g_k and g exactly as this was done in Herman's example. The only difference is that there is a discontinuity of g' at the middle of every connected component of the wandering set, the map g being linear on a right neighborhood and on a left neighborhood of each such singularity.

Moreover, h'_k tends to 1 when k tends to $\pm\infty$. This implies (a precise proof was given in [1]) that g and the curve Γ are C^1 at all the points that are not at the middle of any connected component of the wandering set. Observe that the set of discontinuities of g' corresponds to one orbit.

3.2. Choice of a 'good' sequence (α_k).

Let us recall that we want that $\varphi = \tilde{g} + \tilde{g}^{-1} - 2\mathrm{Id}_{\mathbb{R}}$ is C^2. We need to choose the sequence (α_k) carefully to obtain that. Let us now explain how we choose (α_k), and after that we will prove that φ is C^2.

We begin by choosing two small $\alpha_1 > 0$ and $\alpha_0 < 0$ such that

$$\frac{1}{1 + K_0 + \alpha_0} + 1 + K_1 = \frac{1}{1 + K_0} + 1 + K_1 + \alpha_1.$$

We denote this quantity by m.

Then we extend the sequence $(\alpha_k)_{0\leqslant k\leqslant 1}$ by using the constants $m_k := 1 + K_k + \frac{1}{1+K_{k-1}}$:

$$\forall k \in \mathbb{Z} \setminus \{0\}, \quad 1 + K_{k+1} + \alpha_{k+1} + \frac{1}{1 + K_k + \alpha_k} = m_{k+1}. \tag{9}$$

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If we denote by Φ_k the map $\Phi_k :]0, +\infty[\to \mathbb{R}$ defined by $\Phi_k(t) = m_k - \frac{1}{t}$, each Φ_k is increasing, and we have $\Phi_{k+1}(1 + K_k) = 1 + K_{k+1}$. Because $\alpha_1 > 0$, we deduce that we can define $(\alpha_n)_{n\geqslant 1}$ by using (9), and that, $\forall n \geqslant 1, \alpha_n > 0$. In a similar way, each Φ_k^{-1} is increasing on $]-\infty, m_k[$ and $\alpha_0 < 0$; hence we can define $(\alpha_{-n})_{n\geqslant 1}$ by (9), and we then have, $\forall n \geqslant 0, \alpha_{-n} < 0$. Similar remarks were made in [1].

For this particular choice of (α_k), we can notice that, for all $k \in \mathbb{Z}$, $h_k + h_{k-1}^{-1}$ is linear in the interval $[\frac{3}{8}\ell_k, \frac{5}{8}\ell_k]$. More precisely (we use the fact that the h_k are continuous at $\frac{\ell_k}{2}$ to determine some constants), we have the following.

- If $k \geqslant 2$: if $x \in [\frac{3}{8}\ell_k, \frac{1}{2}\ell_k]$, $h_k(x) + h_{k-1}^{-1}(x) = (1 + K_k)x + \frac{1}{1+K_{k-1}}x = m_k x$, and if $x \in [\frac{1}{2}\ell_k, \frac{5}{8}\ell_k]$, then
$h_k(x) + h_{k-1}^{-1}(x) = (1 + K_k + \alpha_k)x - \frac{\alpha_k\ell_k}{2} + \frac{1}{1+K_{k-1}+\alpha_{k-1}}(x + \frac{\alpha_{k-1}\ell_{k-1}}{2}) = m_k x$.
- If $k = 1$: if $x \in [\frac{1}{2}\ell_k, \frac{5}{8}\ell_k]$,
$h_1(x) + h_0^{-1}(x) = (1 + K_1 + \alpha_1)x - \frac{\alpha_1\ell_1}{2} + \frac{1}{1+K_0}x = mx - \frac{\alpha_1\ell_1}{2} = m_1 x - \frac{\alpha_1\ell_1}{2}$, and if $x \in [\frac{3}{8}\ell_1, \frac{1}{2}\ell_1]$, then
$h_1(x) + h_0^{-1}(x) = (1 + K_1)x + \frac{1}{1+K_0+\alpha_0}(x + \frac{\alpha_0\ell_0}{2}) = m_1 x - \frac{\alpha_1\ell_1}{2}$ (notice that we change the notation for m_1 from this point).
- If $k \leqslant 0$: if $x \in [\frac{1}{2}\ell_k, \frac{5}{8}\ell_k]$, $h_k(x) + h_{k-1}^{-1}(x) = (1 + K_k)x + \frac{1}{1+K_{k-1}}x = m_k x$, and if $x \in [\frac{3}{8}\ell_k, \frac{1}{2}\ell_k]$, then
$h_k(x) + h_{k-1}^{-1}(x) = (1 + K_k + \alpha_k)x - \frac{\alpha_k\ell_k}{2} + \frac{1}{1+K_{k-1}+\alpha_{k-1}}(x + \frac{\alpha_{k-1}\ell_{k-1}}{2}) = m_k x$.

We deduce immediately that the function $\varphi = \tilde{g} + \tilde{g}^{-1} - 2\mathrm{Id}_{\mathbb{R}}$ is linear on each segment $J_k \subset I_k$ that is at the middle of I_k and has length $\frac{\ell_k}{4}$. In particular, the restriction of φ to the interior of any interval I_k is C^∞.

We denote by φ_k the C^∞ function that is equal to φ on I_k and equal to 0 everywhere else. Then, $\varphi = \sum \varphi_k$, and, to prove that φ is C^2, we just have to prove that $\lim_{k\to\pm\infty} \|D^2\varphi_k\|_{C^0} = 0$. If we want to prove that φ is close to 0 in C^2 topology, we have to prove that $\lim_{C\to+\infty} \sup\{\|D^2\varphi_k\|_{C^0}; k \in \mathbb{Z}\} = 0$.

3.3. Estimation of $(\alpha_n)_{n\geqslant 1}$

We will prove the following lemma.

Lemma 1. *There exists a constant C_2 such that, $\forall n \geqslant 1, 0 < \alpha_n \leqslant \frac{C_2}{n+C}$.*

If we want to have a control on $\|D^2\varphi_k\|_{C^0}$, we need to have a control of the sequence (α_k). We use the following notation: $\beta_k = K_k + \alpha_k$.

We have built the sequences (ℓ_k), (K_k), and (m_k) that depend on a certain constant $C \gg 1$, we have chosen $\alpha_1 > 0$ small and defined:

$$\forall n \geqslant 1, \quad 1 + \beta_{n+1} + \frac{1}{1 + \beta_n} = m_{n+1}.$$

We have considered the functions $\Phi_k :]0, +\infty[\to \mathbb{R}$ defined by $\Phi_k(t) = m_k - \frac{1}{t}$. Then, we have $1 + \beta_{k+1} = \Phi_{k+1}(1 + \beta_k)$. This function is strictly increasing and

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concave. When $m_k > 2$, Φ_k has exactly two fixed points $a_k < 1 < b_k$, and we have $b_k = \frac{1}{2}(m_k + \sqrt{m_k^2 - 4})$; hence (see (8))

$$0 < b_k - 1 < C_1\sqrt{m_k - 2} \leqslant \frac{C_2}{n + C}. \tag{10}$$

Let us now compare $1 + \beta_{n+1} = \Phi_{n+1}(1 + \beta_n)$ with $1 + \beta_n$. We fix a constant $B \gg 2$. There are three cases.

1. If $m_{n+1} \leqslant 2$ and $\beta_n \leqslant \frac{B}{n+1+C}$, then $1 + \beta_{n+1} = \Phi_{n+1}(1 + \beta_n) \leqslant \Phi_{n+1}(1 + \frac{B}{n+1+C}) \leqslant 1 + \frac{B}{n+1+C}$, because $\Phi_{n+1} \leqslant \mathrm{Id}$.
2. If $m_{n+1} > 2$ and $\beta_n \leqslant \frac{B}{n+1+C}$, then $1 + \beta_{n+1} = \Phi_{n+1}(1 + \beta_n) \leqslant \Phi_{n+1}(1 + \frac{B}{n+1+C}) \leqslant 1+\frac{B}{n+1+C}$, because $\Phi_{n+1|[b_{n+1},+\infty[} \leqslant \mathrm{Id}_{|[b_{n+1},+\infty[}$ and $1+\frac{B}{n+1+C} = 1+\frac{C_2}{n+1+C}+\frac{B-C_2}{n+1+C} \geqslant b_{n+1}$ if B is large enough (see (10)).
3. If $1 + \beta_n > 1 + \frac{B}{n+1+C}$, we introduce the notation $\delta_n = \frac{B}{2(n+1+C)}$. The function Φ_{n+1} being concave such that $D\Phi_{n+1}(1 + \delta_{n+1}) = \frac{1}{(1+\delta_n)^2}$, we have

   $$1 + \beta_{n+1} - \Phi_{n+1}(1 + \delta_n) \leqslant \frac{1}{(1 + \delta_n)^2}(1 + \beta_n - (1 + \delta_n)).$$

   If $m_{n+1} \leqslant 2$, we have $\Phi_{n+1}(1 + \delta_n) \leqslant 1 + \delta_n$ because $\Phi_{n+1} \leqslant \mathrm{Id}$; if $m_{n+1} > 2$, as $1 + \delta_n > b_{n+1}$ (see point 2), we have $\Phi_{n+1}(1 + \delta_n) \leqslant 1 + \delta_n$, and then

   $$\beta_{n+1} \leqslant \left(1 - \frac{1}{(1 + \delta_n)^2}\right)\delta_n + \frac{1}{(1 + \delta_n)^2}\beta_n.$$

   As $\delta_n \leqslant \frac{\beta_n}{2}$, we deduce that

   $$\beta_{n+1} \leqslant \left(\frac{1}{2} + \frac{1}{2}\frac{1}{(1 + \delta_n)^2}\right)\beta_n = \left(\frac{1}{2} + \frac{1}{2}\frac{1}{(1 + \frac{B}{2(n+1+C)})^2}\right)\beta_n.$$

   We deduce, for C large enough, that

   $$\beta_{n+1} \leqslant \left(1 - \frac{B}{3(n + C + 1)}\right)\beta_n.$$

We choose $B \geqslant 3$. We then have

$$\beta_{n+1} \leqslant \left(1 - \frac{1}{n + C + 1}\right)\beta_n = \frac{n + C}{n + C + 1}\beta_n. \tag{11}$$

Let us now prove some estimates for (β_n) (and then (α_n)). First, let us recall that $\beta_n > K_n$ (because we have noticed that $\alpha_n > 0$). Let us now choose $\alpha_1 > 0$ small enough such that $\beta_1 = \alpha_1 + K_1 \leqslant \frac{B}{1+C}$; this is possible because $K_1 < 0$ (see (2)). Now we prove by recurrence that, $\forall n \geqslant 1$, $\beta_n \leqslant \frac{B}{n+C}$.

The result is true for $n = 1$.

Let us assume that it is true for some $n \geqslant 1$. There are two cases.

- $\beta_n \leqslant \frac{B}{n+1+C}$. Then we have proved that $\beta_{n+1} \leqslant \frac{B}{n+1+C}$.
- $\beta_n > \frac{B}{n+1+C}$. Then, by (11), we have $\beta_{n+1} \leqslant \frac{n+C}{n+1+C}\beta_n \leqslant \frac{n+C}{n+1+C}\frac{B}{n+C} = \frac{B}{n+1+C}$.

Finally, we have proved that

$$\forall n \geqslant 1, \quad K_n \leqslant \beta_n \leqslant \frac{C_2}{n+C}.$$

Using (3), we deduce similar estimates for $\alpha_n = \beta_n - K_n$: $\forall n \geqslant 1, 0 < \alpha_n \leqslant \frac{C_2}{n+C}$.

3.4. Estimation of $(\alpha_{-n})_{n\geqslant 0}$

We will prove the following lemma.

Lemma 2. *There exists a constant C_2 such that, $\forall n \geqslant 0, -\frac{C_2}{n+C} \leqslant \alpha_{-n} < 0$.*

This time we will use the smallest fixed point $a_k = \frac{1}{2}(m_k - \sqrt{m_k^2 - 4})$ of Φ_k when $m_k > 2$. We have (because of (2), K_{-n} is positive)

$$0 > a_k - 1 > -C_1\sqrt{m_k - 2} \geqslant -\frac{C_2}{n+C}. \tag{12}$$

We have noticed that, $\forall n \geqslant 0, \beta_{-n} < K_{-n}$. Let us now compare $1 + \beta_{-n-1} = \Phi_{-n}^{-1}(1+\beta_{-n})$ with $1+\beta_{-n}$. We fix a constant $B \gg 2$. There are three cases.

1. If $m_{-n} \leqslant 2$ and $\beta_n \geqslant -\frac{B}{n+1+C}$, then $1+\beta_{-n-1} = \Phi_{-n}^{-1}(1+\beta_{-n}) \geqslant \Phi_{-n}^{-1}(1 - \frac{B}{n+1+C})) \geqslant 1 - \frac{B}{n+1+C}$, because $\Phi_{-n}^{-1} \geqslant \mathrm{Id}$.
2. If $m_{-n} > 2$ and $\beta_n \geqslant -\frac{B}{n+1+C}$, then $1+\beta_{-n-1} = \Phi_{-n}^{-1}(1+\beta_{-n}) \geqslant \Phi_{-n}^{-1}(1 - \frac{B}{n+1+C})) \geqslant 1 - \frac{B}{n+1+C}$, because $\Phi_{-n|]-\infty,a_{-n}]}^{-1} \geqslant \mathrm{Id}_{]-\infty,a_{-n}]}$ and $1 - \frac{B}{n+1+C} = 1 - \frac{C_2}{n+1+C} - \frac{B-C_2}{n+1+C} \leqslant a_{-n}$ if B is large enough (see (12)).
3. If $\beta_{-n} < -\frac{B}{n+1+C}$, we introduce the notation $\gamma_{-n} = -\frac{B}{2(n+1+C)}$. The function Φ_{-n}^{-1} being convex such that $D(\Phi_{-n}^{-1})(1+\gamma_{-n}) = \frac{1}{(m_{-n}-1-\gamma_{-n})^2}$, we have

$$\Phi_{-n}^{-1}(1+\gamma_{-n}) - (1+\beta_{-n-1}) \leqslant \frac{1}{(m_{-n}-1-\gamma_{-n})^2}((1+\gamma_{-n}) - (1+\beta_{-n})).$$

If $m_{-n} \leqslant 2$, we have $\Phi_{-n}^{-1}(1+\gamma_{-n}) \geqslant 1+\gamma_{-n}$, because $\Phi_{-n}^{-1} \geqslant \mathrm{Id}$; if $m_{-n} > 2$, as $1+\gamma_{-n} \leqslant a_{-n}$ (see point 2), we have $\Phi_{-n}^{-1}(1+\gamma_{-n}) \geqslant 1+\gamma_{-n}$, and then

$$\beta_{-n-1} \geqslant \left(1 - \frac{1}{(m_{-n}-1-\gamma_{-n})^2}\right)\gamma_{-n} + \frac{\beta_{-n}}{(m_{-n}-1-\gamma_{-n})^2}.$$

Because of (8), we have

$$\left|(m_{-n}-1-\gamma_{-n}) - \left(1 + \frac{B}{2(n+1+C)}\right)\right| \leqslant \frac{C_2}{(n+C)^2}, \tag{13}$$

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and then $1-\frac{1}{(m_{-n}-1-\gamma_{-n})^2}$ is positive if C is large enough. Because $\gamma_{-n}>\frac{\beta_{-n}}{2}$, we deduce that

$$\beta_{-n-1}\geqslant\left(\frac{1}{2}+\frac{1}{2(m_{-n}-1-\gamma_{-n})^2}\right)\beta_{-n}.$$

Then, by (13),

$$\beta_{-n-1}\geqslant\left(1-\frac{B}{3(n+1+C)}\right)\beta_{-n}.$$

If $B\geqslant 3$, we obtain

$$\beta_{-n-1}\geqslant\left(1-\frac{1}{(n+1+C)}\right)\beta_{-n}\geqslant\frac{n+C}{n+1+C}\beta_{-n}. \tag{14}$$

The end of the proof is then similar to that in the content of § 3.3, and we obtain $-\frac{C_2}{n+C}\leqslant\alpha_{-n}<0$.

3.5. Regularity of the modified example

The arguments of the proof in this subsection are very similar to those ones Herman.

Let us recall that we are interested in proving that $\lim_{k\to\pm\infty}\|D^2\varphi_k\|_{C^0}=0$ and that $\lim_{C\to+\infty}\sup\{\|D^2\varphi_k\|_{C^0};k\in\mathbb{Z}\}=0$. Because of the definition g, we have $\|D^2\varphi_k\|_{C^0}=\|D^2h_k+D^2h_{k-1}^{-1}-2\|_{C^0}$. Let us introduce the notation

$$h_k(x)=x+\Delta_k(x)=\int_0^x(1+\psi_k(t))dt.$$

Then we want to estimate the norm C^2 of

$$\zeta_k(x)=h_k(x)+h_{k-1}^{-1}(x)-2x=\Delta_k(x)-\Delta_{k-1}(h_{k-1}^{-1}x).$$

We differentiate, to obtain

$$D\zeta_k(x)=D\Delta_k(x)-D\Delta_{k-1}(h_{k-1}^{-1}x)D(h_{k-1}^{-1})(x);$$

that is,

$$D\zeta_k(x)=\psi_k(x)-\psi_{k-1}(h_{k-1}^{-1}x)D(h_{k-1}^{-1})(x).$$

We then define $f_k:[0,\ell_k]\to[0,\ell_k]$ by $f_k(x)=h_{k-1}(\frac{\ell_{k-1}}{\ell_k}x)$; then, we have $h_{k-1}^{-1}(x)=\frac{\ell_{k-1}}{\ell_k}f_k^{-1}(x)$. We have

$$D(h_{k-1}^{-1})(x)=\frac{\ell_{k-1}}{\ell_k}(Df_k^{-1})(x).$$

Let us recall that

$$\psi_k(x)=K_k\eta\left(\frac{x}{\ell_k}\right)+\alpha_k\gamma_\pm\left(\frac{x}{\ell_k}\right).$$

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Therefore,

$$\psi_{k-1}(h_{k-1}^{-1}x) = K_{k-1}\eta\left(\frac{h_{k-1}^{-1}x}{\ell_{k-1}}\right) + \alpha_{k-1}\gamma_{\pm}\left(\frac{h_{k-1}^{-1}x}{\ell_{k-1}}\right)$$
$$= K_{k-1}\eta\left(\frac{f_k^{-1}x}{\ell_k}\right) + \alpha_{k-1}\gamma_{\pm}\left(\frac{f_k^{-1}x}{\ell_k}\right).$$

Observe that

$$D(h_{k-1}^{-1})(x) = \frac{1}{Dh_{k-1}(h_{k-1}^{-1}x)} = \frac{1}{1+\psi_{k-1}(h_{k-1}^{-1}x)},$$

and then

$$D(h_{k-1}^{-1})(x) = \frac{1}{1+K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_{\pm}(\frac{f_k^{-1}x}{\ell_k})}.$$

Finally, we obtain

$$\psi_{k-1}(h_{k-1}^{-1}x)D(h_{k-1}^{-1})(x) = \frac{K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_{\pm}(\frac{f_k^{-1}x}{\ell_k})}{1+K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_{\pm}(\frac{f_k^{-1}x}{\ell_k})}.$$

Moreover, we have

$$Df_k(x) = \frac{\ell_{k-1}}{\ell_k}Dh_{k-1}\left(\frac{\ell_{k-1}}{\ell_k}x\right) = \frac{\ell_{k-1}}{\ell_k}\left(1+K_{k-1}\eta\left(\frac{x}{\ell_k}\right) + \alpha_{k-1}\gamma_{\pm}\left(\frac{x}{\ell_k}\right)\right)$$

and

$$Df_k^{-1}(x) = \frac{\ell_k}{\ell_{k-1}}Dh_{k-1}^{-1}(x) = \frac{\ell_k}{\ell_{k-1}}\frac{1}{1+K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_{\pm}(\frac{f_k^{-1}x}{\ell_k})}.$$

Let us now compute, for $x \in \mathbb{T}\setminus\{\frac{\ell_k}{2}\}$ (even if ζ_k is two times differentiable at this point, the terms in the sum are not differentiable at $\frac{\ell_k}{2}$),

$$D^2\zeta_k(x) = D\psi_k(x) - D\psi_{k-1}(h_{k-1}^{-1}x)(D(h_{k-1}^{-1})(x))^2 - \psi_{k-1}(h_{k-1}^{-1}x)D^2(h_{k-1}^{-1})(x).$$

Following [12], we define

$$II_k = \frac{K_k}{\ell_k}D\eta\left(\frac{x}{\ell_k}\right) - \frac{K_{k-1}}{\ell_k}D\eta\left(\frac{x}{\ell_k}\right) + \frac{\alpha_k}{\ell_k}D\gamma_{\pm}\left(\frac{x}{\ell_k}\right) - \frac{\alpha_{k-1}}{\ell_k}D\gamma_{\pm}\left(\frac{x}{\ell_k}\right)$$
$$III_k = -\left(\frac{K_{k-1}}{\ell_k}D\eta\left(\frac{f_k^{-1}x}{\ell_k}\right) + \frac{\alpha_{k-1}}{\ell_k}D\gamma_{\pm}\left(\frac{f_k^{-1}x}{\ell_k}\right)\right)$$
$$\times\left(\frac{Df_k^{-1}(x)}{1+K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_{\pm}(\frac{f_k^{-1}x}{\ell_k})} - 1\right)$$
$$IV_k = -\psi_{k-1}(h_{k-1}^{-1}x)D^2(h_{k-1}^{-1})(x) = \frac{\psi_{k-1}(h_{k-1}^{-1}x)D\psi_{k-1}(h_{k-1}^{-1}x)D(h_{k-1}^{-1})(x)}{(1+\psi_{k-1}(h_{k-1}^{-1}x))^2};$$

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i.e.,

$$IV_k = \frac{\ell_{k-1}}{\ell_k} \frac{\psi_{k-1}(h_{k-1}^{-1}x) D\psi_{k-1}(h_{k-1}^{-1}x) D(f_k^{-1})(x)}{(1+\psi_{k-1}(h_{k-1}^{-1}x))^2}$$

and

$$V_k = \frac{K_{k-1}}{\ell_k}\left(D\eta\left(\frac{x}{\ell_k}\right) - D\eta\left(\frac{f_k^{-1}x}{\ell_k}\right)\right) + \frac{\alpha_{k-1}}{\ell_k}\left(D\gamma_\pm\left(\frac{x}{\ell_k}\right) - D\gamma_\pm\left(\frac{f_k^{-1}x}{\ell_k}\right)\right).$$

Then,

$$D^2\zeta_k(x) = II_k + III_k + IV_k + V_k.$$

Let us now estimate each term of this sum. We need some inequalities:

$$C_1 \leqslant \|D\eta\|_{C^0}, \|D\gamma_\pm\|_{C^0}, \|D^2\eta\|_{C^0}, \|D^2\gamma_\pm\|_{C^0} \leqslant C_2; \tag{15}$$

We deduce from lemmata 1 and 2 that

$$|\alpha_k| \leqslant \frac{C_2}{|k|+C}, \tag{16}$$

and therefore we have, uniformly in $C \gg 1$ (see (5)),

$$\lim_{k\to\pm\infty} \frac{\alpha_k^2}{\ell_k} = 0. \tag{17}$$

From

$$1 + K_k + \alpha_k + \frac{1}{1+K_{k-1}+\alpha_{k-1}} = m_k,$$

we deduce that

$$|\alpha_k - \alpha_{k-1} - (m_k - 2) + K_k - K_{k-1}| \leqslant C_2(|K_{k-1}| + |\alpha_{k-1}|)^2$$

and, by (3), (4), (8), and (16),

$$|\alpha_k - \alpha_{k-1}| \leqslant \frac{C_2}{(|k|+C)^2}. \tag{18}$$

Moreover, we have

$$Df_k(x) - 1 = \frac{\ell_{k-1}}{\ell_k} - 1 + \frac{\ell_{k-1}}{\ell_k}\left(K_{k-1}\eta\left(\frac{x}{\ell_k}\right) + \alpha_{k-1}\gamma_\pm\left(\frac{x}{\ell_k}\right)\right)$$

and

$$Df_k^{-1}(x) - 1 = \frac{\ell_k}{\ell_{k-1}} \frac{1}{1 + K_{k-1}\eta(\frac{f_k^{-1}x}{\ell_k}) + \alpha_{k-1}\gamma_\pm(\frac{f_k^{-1}x}{\ell_k})} - 1;$$

and then we deduce from (3) and (16) that

$$\sup\{\|Df_k - 1\|_{C^0}, \|Df_k^{-1} - 1\|_{C^0}\} \leqslant \frac{C_2}{|k|+C}. \tag{19}$$

Let us estimate $II_k = (\frac{K_k}{\ell_k} - \frac{K_{k-1}}{\ell_k})D\eta(\frac{x}{\ell_k}) + (\frac{\alpha_k}{\ell_k} - \frac{\alpha_{k-1}}{\ell_k})D\gamma_\pm(\frac{x}{\ell_k})$; because of (4), (15) and (18), we have $|II_k| \leqslant \frac{C_2}{(|k|+C)^2\ell_k}$, and then, by (5), uniformly in $C \gg 1$, we have

$$\lim_{k\to\pm\infty} |II_k| = 0.$$

From (3), (16) and (19), we deduce that $|III_k| \leqslant \frac{C_2}{(|k|+C)^2\ell_k}$, and then, uniformly in $C \gg 1$, we have

$$\lim_{k\to\pm\infty} |III_k| = 0.$$

We have

$$IV_k = \frac{\ell_{k-1}}{\ell_k}\frac{\psi_{k-1}(h_{k-1}^{-1}x)D\psi_{k-1}(h_{k-1}^{-1}x)D(f_k^{-1})(x)}{(1+\psi_{k-1}(h_{k-1}^{-1}x))^2}.$$

We deduce from (3), (15) and (16) that $|IV_k| \leqslant \frac{C_2}{\ell_k(|k|+C)^2}$ and then that, uniformly in $C \gg 1$, we have

$$\lim_{k\to\pm\infty} |IV_k| = 0.$$

We have, for $x \in [0, \ell_k]$,

$$\left|D\eta\left(\frac{x}{\ell_k}\right) - D\eta\left(\frac{f_k^{-1}x}{\ell_k}\right)\right| \leqslant \int_0^x \frac{1}{\ell_k}\|D^2\eta\|_{C^0}\|Df_k^{-1} - 1\|_{C^0} \leqslant \|D^2\eta\|_{C^0}\|Df_k^{-1} - 1\|_{C^0};$$

then, by (15) and (19), $|D\eta(\frac{x}{\ell_k}) - D\eta(\frac{f_k^{-1}x}{\ell_k})| \leqslant \frac{C_2}{|k|+C}$.

Because, for every $x \in [0, \ell_k]$, $\frac{x}{\ell_k}$ and $\frac{f_k^{-1}x}{\ell_k}$ are in the same half interval of $[0, 1]$, $\gamma_\pm$ is smooth between $\frac{x}{\ell_k}$ and $\frac{f_k^{-1}x}{\ell_k}$ and we can do for $\gamma_\pm$ the same estimate as for η. By (3) and (16), we deduce that

$$|V_k| \leqslant \frac{C_2}{\ell_k(|k|+C)^2};$$

then, by (5), uniformly in $C \gg 1$, we have

$$\lim_{k\to\pm\infty} |V_k| = 0.$$

Finally, we have proved that φ is C^2 and even that $\|\varphi\|_{C^2}$ is small.

3.6. Stable and unstable sets of the invariant curve

We denote by Γ the invariant curve, that is, the graph of $g - \mathrm{Id}$.

We recall that the segment with length $\frac{\ell_k}{4}$ that has the same center μ_k as $I_k = [\mu_k - \frac{\ell_k}{2}, \mu_k + \frac{\ell_k}{2}]$ is denoted by $J_k = [\mu_k - \frac{\ell_k}{8}, \mu_k + \frac{\ell_k}{8}]$. Moreover, because of the definition of h_k and g_k (see § 3.1), we have the following.

1. If $k \geqslant 1$, then

$$\forall x \in \left[\mu_k - \frac{\ell_k}{8}, \mu_k\right], \quad g(x) = \mu_{k+1} + \frac{\ell_{k+1}}{\ell_k}(x - \mu_k);$$

in this case, $\frac{\ell_{k+1}}{\ell_k} < 1$.

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2. If $k \leqslant 0$, then

$$\forall x \in \left[\mu_k, \mu_k + \frac{\ell_k}{8}\right], \quad g(x) = \mu_{k+1} + \frac{\ell_{k+1}}{\ell_k}(x - \mu_k);$$

   if $k = 0$, then $\frac{\ell_{k+1}}{\ell_k} < 1$; and, if $k \leqslant -1$, then $\frac{\ell_{k+1}}{\ell_k} > 1$.

We deduce that,

1. $\forall k \geqslant 1$, $g([\mu_k - \frac{\ell_k}{8}, \mu_k]) = [\mu_{k+1} - \frac{\ell_{k+1}}{8}, \mu_{k+1}]$; then $g_{|[\mu_k - \frac{\ell_k}{8}, \mu_k]}$ is a linear contraction;
2. $\forall k \leqslant 1$, $g^{-1}([\mu_k, \mu_k + \frac{\ell_k}{8}]) = [\mu_{k-1}, \mu_{k-1} + \frac{\ell_{k-1}}{8}]$ and $(g^{-1})_{|[\mu_k, \mu_k + \frac{\ell_k}{8}]}$ is linear, a contraction if $k \leqslant 0$, and a dilatation if $k = 1$.

We introduce the family $(S_k)_{k\geqslant 1}$ and $(U_k)_{k\leqslant 0}$ of segments of $\mathbb{T} \times \mathbb{R}$ defined by

$$S_k = \left\{(x, g(x) - x); x \in \left[\mu_k - \frac{\ell_k}{8}, \mu_k\right]\right\} \quad \text{and} \quad U_k = \left\{(x, g(x) - x); x \in \left[\mu_k, \mu_k + \frac{\ell_k}{8}\right]\right\}.$$

Because the curve Γ is the graph of $g - \mathrm{Id}$, these segments are subsets of Γ. We have

$$\forall k \geqslant 1, \quad f_\varphi(S_k) = S_{k+1} \quad \text{and} \quad \forall k \leqslant 0, \quad f_\varphi^{-1}(U_k) = U_{k-1};$$

in the first case, $f_{\varphi|S_k}$ is a linear contraction with rapport $\frac{\ell_{k+1}}{\ell_k}$, and in the second case $f^{-1}_{\varphi|U_k}$ is a linear contraction with rapport $\frac{\ell_{k-1}}{\ell_k}$.

We have proved in § 3.2 some equalities for $h_k + h_{k-1}^{-1}$ that imply that

$$\forall k \in \mathbb{Z}, \quad \forall x \in J_k, \quad \varphi(x) = (m_k - 2)(x - \mu_k) + \mu_{k+1} + \mu_{k-1} - 2\mu_k.$$

Let us recall that

$$f_\varphi(\theta, r) = (\theta + r, r + \varphi(\theta + r)) \quad \text{and} \quad f_\varphi^{-1}(\theta, r) = (\theta - r + \varphi(\theta), r - \varphi(\theta));$$

therefore the restriction of f_φ^{-1} to any band $J_k \times \mathbb{R}$ is linear. If we know the expression of a linear map on a segment, we can deduce the expression of the map on the whole line supporting the segment. In particular, if we define the families of segments $(\tilde{S}_k)_{k\geqslant 1}$ and $(\tilde{U}_k)_{k\leqslant 0}$ by

$$\tilde{S}_k = \left\{\left(x, \mu_{k+1} - \mu_k + \left(\frac{\ell_{k+1}}{\ell_k} - 1\right)(x - \mu_k)\right); x \in J_k\right\} \quad \text{for } k \geqslant 1 \tag{20}$$

and

$$\tilde{U}_k = \left\{\left(x, \mu_{k+1} - \mu_k + \left(\frac{\ell_{k+1}}{\ell_k} - 1\right)(x - \mu_k)\right); x \in J_k\right\} \quad \text{for } k \leqslant 0, \tag{21}$$

then we have $U_k \subset \tilde{U}_k$, $S_k \subset \tilde{S}_k$, and

$$\forall k \geqslant 1, \quad f_\varphi(\tilde{S}_k) = \tilde{S}_{k+1} \quad \text{and} \quad \forall k \leqslant 0, \quad f_\varphi^{-1}(\tilde{U}_k) = \tilde{U}_{k-1}.$$

Moreover, the restriction of f_φ to $\tilde{S}_k$ is a linear contraction with rapport $\frac{\ell_{k+1}}{\ell_k}$, and the restriction of f_φ^{-1} to $\tilde{U}_k$ is a linear contraction with rapport $\frac{\ell_{k-1}}{\ell_k}$. We then deduce that $\tilde{S}_k$ is in the stable set of the point $(\mu_k, \mu_{k+1} - \mu_k)$ and that $\tilde{U}_k$ is in the unstable set of the point $(\mu_k, \mu_{k+1} - \mu_k)$.

We then extend these two families of segments by the following.

- If $k \leqslant 0$, then $\tilde{S}_k = f_\varphi^{k-1}(\tilde{S}_1)$.
- If $k \geqslant 0$, then $\tilde{U}_k = f_\varphi^{k}(\tilde{U}_0)$.

Let us now choose a C^∞ injective map $\gamma_1^s : \mathbb{R} \to \tilde{S}_1$ such that $\gamma_1^s(0) = x_1 = (\mu_1, \mu_2 - \mu_1)$, $\gamma_1^s(\mathbb{R})$ is $\tilde{S}_1$ without its ends and $\gamma_1^s(]-\infty, 0[)$ is S_1 without its ends.

Similarly, we choose a C^∞ injective map $\gamma_0^u : \mathbb{R} \to \tilde{U}_0$ such that $\gamma_0^u(0) = x_0 = (\mu_0, \mu_1 - \mu_0)$, $\gamma_0^u(\mathbb{R})$ is $\tilde{U}_0$ without its ends and $\gamma_0^u(]0, +\infty[)$ is U_0 without its ends.

We extend these curves to two families by $\gamma_k^s = f_\varphi^{k-1} \circ \gamma_1^s$ and $\gamma_k^u = f_\varphi^{k} \circ \gamma_0^u$. Then, we have

1. $f_\varphi \circ \gamma_k^u = \gamma_{k+1}^u$ and $f_\varphi \circ \gamma_k^s = \gamma_{k+1}^s$;
2. $\forall y \in \gamma_0^s(\mathbb{R})$, $\lim_{n\to+\infty} d(f_\varphi^n y, f_\varphi^n x_0) = 0$ and $\forall y \in \gamma_0^u(\mathbb{R})$, $\lim_{n\to+\infty} d(f_\varphi^{-n} y, f_\varphi^{-n} x_0) = 0$;
3. $\gamma_k^s(]-\infty, 0]) \cup \gamma_k^u([0, +\infty[) \subset f_\varphi^{k-1}(S_1) \cup f_\varphi^k(U_0) \subset \Gamma$.

Let us now prove that $\gamma_1^s(]0, +\infty[) \cup \gamma_0^u(]-\infty, 0[) \subset \mathbb{T} \times \mathbb{R} \setminus \Gamma$. We will deduce that $\gamma_1^s(]0, +\infty[)$ is a part of the stable set of x_1 and of Γ that does not meet Γ; hence it is in an instability zone $\mathcal{U}$, and Γ is in the boundary of $\mathcal{U}$ (see the proposition contained in § 1.3); we will even see that $\mathcal{U}$ is under Γ. Similarly, we will prove that $\gamma_0^u(]-\infty, 0[)$ is in $\mathcal{U}$. We will of course deduce that

$$\forall k \in \mathbb{Z}, \quad \gamma_k^s(]-\infty, 0[) \cup \gamma_k^u(]0, +\infty[) \subset \mathcal{U}.$$

By (20), we have an explicit expression for

$$\gamma_1^s(]0, +\infty[) = \left\{ \left(x, \mu_2 - \mu_1 + \left(\frac{\ell_2}{\ell_1} - 1 \right)(x - \mu_1) \right); x \in \left(\mu_1, \mu_1 + \frac{\ell_1}{8} \right) \right\}.$$

Moreover, because of the definition of Γ, we have (see § 3.1)

$$\forall x \in \left(\mu_1, \mu_1 + \frac{\ell_1}{8} \right), \quad g(x) - x = \left(\frac{\ell_2}{\ell_1} + \alpha_1 - 1 \right)(x - \mu_1) + \mu_2 - \mu_1.$$

As $\alpha_1 > 0$ and Γ is the graph of $g - \mathrm{Id}$, we deduce that $\gamma_1^s(]0, +\infty[)$ does not meet Γ, and even that $\gamma_1^s(]0, +\infty[)$ is under Γ. A similar argument gives the result for $\gamma_0^u(]-\infty, 0[)$.

Remark. (1) If we exchange γ_- and γ_+, we obtain an instability zone $\mathcal{U}$ that is above Γ.

(2) If we use a similar construction along two wandering intervals, we obtain a curve Γ that is at the boundary of two instability zones.

4. The case of the C^1 topology: proof of Theorem 2

If U is an open subset of $\mathbb{A}$, the set $\mathrm{Diff}^1_\omega(U)$ of C^1 symplectic diffeomorphisms of U is endowed with the strong Whitney topology (see [13, 21]). Observe that the set $\mathcal{T}$ of symplectic twist maps of $\mathbb{A}$ is open for the Whitney topology in $\mathrm{Diff}^1_\omega(\mathbb{A})$. Then, if U is any open subset of $\mathbb{A}$, the set $\mathcal{T}(U)$ of the restrictions to U of symplectic twist maps of $\mathbb{A}$ is open in $\mathrm{Diff}^1_\omega(U)$.

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The following result is Theorem 3 of [2].

Theorem ([2]). *Let (M, ω) be a non-closed manifold without boundary. There exists a dense G_δ subset $\mathcal{G}$ of $\mathrm{Diff}^1_\omega(M)$ such that, for all $f \in \mathcal{G}$, the set of points of M whose positive orbit is relatively compact in M has no interior.*

Let us now consider an essential invariant curve Γ of a symplectic twist map f of $\mathbb{A}$. The curve Γ is then the graph of a Lipschitz map $\gamma : \mathbb{T} \to \mathbb{R}$. Denoting by U one of the two connected components of $\mathbb{A} \setminus \Gamma$, we have $f(U) = U$. In order to define a neighborhood $\mathcal{U}$ of $f_{|U}$ for the C^1 strong topology, we use the function $\varepsilon : U \to \mathbb{R}_+^*$, defined by $\varepsilon(\theta, r) = (r - \gamma(\theta))^2$.

$$\mathcal{U} = \{g \in \mathcal{T}(U); \forall(\theta, r) \in U, \sup\{d(f(\theta, r), g(\theta, r)), \|Df(\theta, r) - Dg(\theta, r)\|\} \leqslant \varepsilon(\theta, r)\}.$$

The previous theorem implies that there exists $h \in \mathcal{U}$ such that the set of points of U whose positive orbit for h is relatively compact in U has no interior.

We now define $g : \mathbb{A} \to \mathbb{A}$ such that $g_{|\mathbb{A}\setminus U} = f_{|\mathbb{A}\setminus U}$ and $g_{|U} = h$. It comes from the definition of $\mathcal{U}$ and $\mathcal{T}(U)$ that $g \in \mathrm{Diff}^1_\omega(\mathbb{A})$.

Let us prove that U contains at most one essential invariant curve for g. If not, there exists a bounded invariant open region R between two such invariant curves. Then all the points of R have a positive orbit that is relatively compact in U; this is a contradiction with the choice of h.

Let us now assume that $f_{|\Gamma}$ has an irrational rotation number or that $f_{|\Gamma}$ is C^0 conjugated to a rational rotation. We have noticed in § 1.3 that, in this case, if Γ^* is another essential invariant curve of f, then $\Gamma \cap \Gamma^* = \emptyset$. Hence the closure $\bar{U}$ of U contains at most one essential invariant curve that is different from Γ, and this curve is contained in U. There are two cases.

- U contains one essential invariant curve Γ' for g. The region R between Γ and Γ' is an instability zone for g, and its boundary contains Γ.
- U contains no essential invariant curve for g. The curve Γ is at the boundary of the instability zone U of g.

Remark. Using methods contained in the (unpublished) thesis of my student Marie Girard that allow us to destroy all the invariant curves by perturbation, we can choose h such that U contains no essential invariant curve and thus is an instability zone.

The only case that we did not solve is the case of a rational rotation number for $f_{|\Gamma}$ and a hyperbolic dynamics for $f_{|\Gamma}$ (i.e., we assume that all the periodic points of $f_{|\Gamma}$ are hyperbolic).

Robinson proved the following result in [22]. Let us recall that a periodic point p of f with period τ is non-degenerate if no root of 1 is an eigenvalue of $Df^\tau(p)$.

Theorem ([22]). *Let (M, ω) be a closed manifold, and let $r \geqslant 1$. There exists a dense G_δ subset $\mathcal{G}$ of $\mathrm{Diff}^r_\omega(M)$ such that, for all $f \in \mathcal{G}$, the periodic points are non-degenerate and the stable and unstable manifolds of each pair of hyperbolic periodic orbits of f are transverse at all of their points of intersection.*

We assume now that $f \in \mathcal{T}(A)$ a symplectic twist map that has an essential invariant curve Γ such that

- its rotation number is rational;
- the periodic points of $f_{|\Gamma}$ are all hyperbolic.

Then there is a finite number of such periodic points, which we denote by $x_1, \ldots, x_n$, and Γ is the union of $\{x_1, \ldots, x_n\}$ and some branches of the stable/unstable manifolds of these periodic points. Let us notice that every $g \in \mathcal{U}$ can be extend in a unique $\tilde{g} \in \mathcal{T}(\mathbb{A})$ by $\tilde{g}_{|\mathbb{A}\backslash U} = f_{|\mathbb{A}\backslash U}$, and that, in this case, $D\tilde{g}_{|\Gamma} = Df_{|\Gamma}$. Hence $\tilde{g}$ has the same periodic points as f on Γ, and these periodic points are hyperbolic.

We can directly adapt Robinson's proof to build a dense $G_\delta \mathcal{G}$ of $\mathcal{U}$ such that, for all $g \in \mathcal{G}$, the stable and unstable branches of the stable and unstable manifolds of the x_i for $\tilde{g}$ that are contained in U are transverse at all of their points of intersection.

If now Γ^* is an essential invariant curve for $\tilde{g}$ that is contained in $\bar{U}$ and that meets Γ, then $\Gamma \cap \Gamma^*$ is a closed invariant set that contains a point of the stable manifold of a point x_i. Hence it contains this x_i. The rotation number of Γ^* is then equal to the one of Γ, and then Γ^* is the union of $\{x_1, \ldots, x_n\}$ and some branches of the stable/unstable manifolds of these periodic points. But if $\Gamma \neq \Gamma^*$, then Γ^* contains a branch in U that is a stable and an unstable branch, and this contradicts the transversality of such branches. We deduce that either $\Gamma = \Gamma^*$ or $\Gamma \cap \Gamma^* = \emptyset$, and we can conclude exactly in the same way as is the irrational case.

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