Combinatorics, Probability and Computing (2019), **28**, pp. 791–810
doi:10.1017/S0963548319000063

CAMBRIDGE UNIVERSITY PRESS

ARTICLE

Defective and clustered choosability of sparse graphs

Kevin Hendrey and David R. Wood*

School of Mathematical Sciences, Monash University, Melbourne, Australia
*Corresponding author. Email: david.wood@monash.edu

(Received 30 August 2018; revised 1 February 2019; first published online 12 April 2019)

Abstract

An (improper) graph colouring has *defect* d if each monochromatic subgraph has maximum degree at most d, and has *clustering* c if each monochromatic component has at most c vertices. This paper studies defective and clustered list-colourings for graphs with given maximum average degree. We prove that every graph with maximum average degree less than $(2d+2)/(d+2)k$ is k-choosable with defect d. This improves upon a similar result by Havet and Sereni (*J. Graph Theory*, 2006). For clustered choosability of graphs with maximum average degree m, no $(1-\varepsilon)m$ bound on the number of colours was previously known. The above result with $d=1$ solves this problem. It implies that every graph with maximum average degree m is $\lfloor \frac{3}{4}m+1 \rfloor$-choosable with clustering 2. This extends a result of Kopreski and Yu (*Discrete Math.*, 2017) to the setting of choosability. We then prove two results about clustered choosability that explore the trade-off between the number of colours and the clustering. In particular, we prove that every graph with maximum average degree m is $\lfloor \frac{7}{10}m+1 \rfloor$-choosable with clustering 9, and is $\lfloor \frac{2}{3}m+1 \rfloor$-choosable with clustering $O(m)$. As an example, the later result implies that every biplanar graph is 8-choosable with bounded clustering. This is the best known result for the clustered version of the earth–moon problem. The results extend to the setting where we only consider the maximum average degree of subgraphs with at least some number of vertices. Several applications are presented.

2010 MSC Codes: Primary 05C15

1. Introduction

This paper studies improper colourings of sparse graphs, where sparsity is measured by the following standard definition. The *maximum average degree* of a graph G, denoted by mad (G), is the maximum, taken over all subgraphs H of G, of the average degree of H. We consider improper colourings with bounded monochromatic degree or with bounded monochromatic components, for graph classes with bounded maximum average degree. We now formalize these ideas. A *colouring* of a graph G is a function that assigns a colour to each vertex. In a coloured graph G, the *monochromatic subgraph* of G is the spanning subgraph consisting of those edges whose endpoints have the same colour. A colouring has *defect* k if the monochromatic subgraph has maximum degree at most k; that is, each vertex v is adjacent to at most k vertices of the same colour as v. A connected component of the monochromatic subgraph is called a *monochromatic component*. A colouring has *clustering* k if each monochromatic component has at most k vertices. Of course, a colouring is proper if and only if it has defect 0 or clustering 1.

Our focus is on minimizing the number of colours, with small defect or small clustering as a secondary goal. This viewpoint leads to the following definitions. The *defective chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d. The *clustered chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer c, every graph in $\mathcal{G}$ is k-colourable with clustering c.

The above definitions extend in the obvious way to list-colourings and choosability. A *list-assignment* for a graph G is a function L that assigns a set $L(v)$ of colours to each vertex $v \in V(G)$. A list-assignment L is a *k-list-assignment* if $|L(v)| \geqslant k$ for each vertex $v \in V(G)$. An *L-colouring* is a colouring of G such that each vertex $v \in V(G)$ is assigned a colour in $L(v)$. Define G to be *k-choosable with defect d* if G has an L-colouring with defect d for every k-list-assignment L of G. Similarly, G is *k-choosable with clustering c* if G has an L-colouring with clustering c for every k-list-assignment L of G.

Defective and clustered (list-) colouring has been widely studied on a variety of graph classes, including bounded maximum degree [2, 29], planar [14, 16, 23], bounded genus [3, 13, 14, 15, 25, 43], excluding a minor [21, 24, 30, 36, 39, 40], excluding a topological minor [21, 40], and excluding an immersion [30]. See [42] for a survey on defective and clustered colouring. All of these classes have bounded maximum average degree. Thus our results are more widely applicable than nearly all of the previous results in the field. That said, it should be noted that some of the existing results for more specific graph classes give better bounds on the number of colours or on the defect or clustering. Generally speaking, our results give the best known bounds for graph classes that have bounded maximum average degree, unbounded maximum degree, and have no strongly sublinear separator theorem. Examples include graphs with given thickness, stack-number or queue-number.

1.1 Defective choosability

Defective choosability with respect to maximum average degree was previously studied by Havet and Sereni [27], who proved the following theorem.

Theorem 1.1. **([27]).** *For $d \geqslant 0$ and $k \geqslant 2$, every graph G with*

$$\operatorname{mad}(G) < k + \frac{kd}{k+d}$$

is k-choosable with defect d.

Our first result improves on Theorem 1.1 as follows.

Theorem 1.2. **(Section 3).** *For $d \geqslant 0$ and $k \geqslant 1$, every graph G with*

$$\operatorname{mad}(G) < \frac{2d+2}{d+2}\,k$$

is k-choosable with defect d.

Note that the two theorems are equivalent for $k = 2$. But for $k \geqslant 3$, the assumption in Theorem 1.2 is weaker than the corresponding assumption in Theorem 1.1, thus Theorem 1.2 is stronger than Theorem 1.1.

Theorem 1.1 can be restated as follows: every graph G with $\operatorname{mad}(G) = m$ is k-choosable with defect

$$\left\lfloor \frac{k(m-k)}{2k-m} \right\rfloor + 1,$$

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

whereas Theorem 1.2 says that G is k-choosable with defect

$$\left\lfloor \frac{m}{2k-m} \right\rfloor.$$

Both results require that $2k > m$, and the minimum value of k for which either theorem is applicable is $k = \lfloor m/2 \rfloor + 1$. In this case, Theorem 1.2 gives a defect bound of

$$\left\lfloor \frac{m}{2k-m} \right\rfloor,$$

which is an order of magnitude less than the defect bound of

$$(1+o(1))\left(\frac{k^2}{2k-m}\right)$$

in Theorem 1.1. Note that Havet and Sereni [27] gave a construction to show that no lower value of k is possible. That is, for $m \in \mathbb{R}^+$, the defective chromatic number of the class of graphs with maximum average degree m equals $\lfloor m/2 \rfloor + 1$; see also [42].

See [4, 5, 6, 7, 8, 9, 10, 11, 33, 34] for results about defective 2-colourings of graphs with given maximum average degree, where each of the two colour classes has a prescribed degree bound. Also note that Dorbec, Kaiser, Montassier and Raspaud [17] proved a result analogous to Theorems 1.1 and 1.2 (with weaker bounds) for defective colouring of graphs with given maximum average degree, where in addition, a given number of colour classes are stable sets.

1.2 Clustered choosability

The following theorem, due to Kopreski and Yu [35], is the only known non-trivial result for clustered colourings of graphs with given maximum average degree.[1]

Theorem 1.3. ([35]). *Every graph G is $\lfloor \frac{3}{4}\,\mathrm{mad}\,(G) + 1 \rfloor$-colourable with defect 1, and thus with clustering 2.*

There are no existing non-trivial results for clustered choosability of graphs with given maximum average degree. The closest such result, due to Dvořák and Norin [21], says that for constants $\alpha, \gamma, \varepsilon > 0$, if a graph G has at most $(k+1-\gamma)|V(G)|$ edges, and every n-vertex subgraph of G has a balanced separator of order at most $\alpha n^{1-\varepsilon}$, then G is k-choosable with clustering some function of α, γ and ε. Note that the number of colours here is roughly half the average degree of G. This result determines the clustered chromatic number of several graph classes, but for various other classes (that contain expanders) this result is not applicable because of the requirement that every subgraph has a balanced separator.

Theorem 1.2 with $d = 1$ implies the above result of Kopreski and Yu [35] and extends it to the setting of choosability.

Theorem 1.4. *Every graph G is $\lfloor \frac{3}{4}\,\mathrm{mad}\,(G) + 1 \rfloor$-choosable with defect 1, and thus with clustering 2.*

As an example of Theorem 1.4, it follows from Euler's formula that toroidal graphs have maximum average degree at most 6, implying every toroidal graph is 5-choosable with defect 1 and clustering 2, which was first proved by Dujmović and Outioua [18]. Previously, Cowen, Goddard and Jesurum [15] proved that every toroidal graph is 5-colourable with defect 1.

[1] Kopreski and Yu [35] actually proved the following stronger result: for $a \geqslant 1$ and $b \geqslant 0$, every graph G with $\mathrm{mad}\,(G) < \frac{4}{3}a + b$ is $(a+b)$-colourable, such that a colour classes have defect 1, and b colour classes are stable sets.

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

The following two theorems are our main results for clustered choosability. The first still has an absolute bound on the clustering, while the second has fewer colours but at the expense of allowing the clustering to depend on the maximum average degree.

Theorem 1.5. **(Section 6).** *Every graph G is* $\lfloor \frac{7}{10}\,\mathrm{mad}(G)+1\rfloor$*-choosable with clustering* 9.

Theorem 1.6. **(Section 7).** *Every graph G is* $\lfloor \frac{2}{3}\,\mathrm{mad}(G)+1\rfloor$*-choosable with clustering*

$$57\left\lfloor \frac{2}{3}\,\mathrm{mad}(G)\right\rfloor+6.$$

Theorem 1.6 says that the clustered chromatic number of the class of graphs with maximum average degree m is at most $\lfloor(2m)/3\rfloor+1$. This is the best known upper bound. The best known lower bound is $\lfloor m/2\rfloor+1$; see [42]. Closing this gap is an intriguing open problem.

1.3 Generalization

The above results generalize via the following definition. For a graph G and integer $n_0 \geqslant 1$, let $\mathrm{mad}(G,n_0)$ be the maximum average degree of a subgraph of G with at least n_0 vertices, unless $|V(G)|<n_0$, in which case $\mathrm{mad}(G,n_0):=0$. The next two results generalize Theorems 1.2 and 1.6 respectively, with $\mathrm{mad}(G)$ replaced by $\mathrm{mad}(G,n_0)$, where the number of colours stays the same, and the defect or clustering bound also depends on n_0.

Theorem 1.7. **(Section 3).** *For integers* $d \geqslant 0$, $n_0 \geqslant 1$ *and* $k \geqslant 1$, *every graph G with*

$$\mathrm{mad}(G,n_0)<\frac{2d+2}{d+2}\,k$$

is k-choosable with defect

$$d' := \max\left\{\left\lceil \frac{n_0-1}{k}\right\rceil -1,\ d\right\}.$$

Theorem 1.8. **(Section 7).** *For integers* $d \geqslant 0$, $n_0 \geqslant 1$ *and* $k \geqslant 1$, *every graph G with*

$$\mathrm{mad}(G,n_0)<\frac{3}{2}k$$

is k-choosable with clustering

$$c := \max\left\{\left\lceil \frac{n_0-1}{k}\right\rceil,\ 57k-51\right\}.$$

Note that Theorem 1.7 with $n_0=1$ is equivalent to Theorem 1.2, and Theorem 1.8 with $n_0=1$ and $k=\lfloor \frac{2}{3}\,\mathrm{mad}(G)\rfloor+1$ is equivalent to Theorem 1.6.

Graphs on surfaces provide motivation for this extension.[2] Graphs with Euler genus g can have average degree as high as $\Theta(\sqrt{g})$, the complete graph being one example. But such graphs necessarily have bounded size. In particular, Euler's formula implies that every n-vertex m-edge graph with Euler genus g satisfies $m<3(n+g)$. Thus, for $\varepsilon>0$, if $n \geqslant 6g/\varepsilon$ then G has average degree $2m/n<6+\varepsilon$. In particular, $\mathrm{mad}(G,6g)<7$.

[2]The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with k cross-caps is k. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds.

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Using this observation, Theorems 1.7 and 1.8 respectively imply that graphs with bounded Euler genus are 4-choosable with bounded defect and are 5-choosable with bounded clustering. Both these results are actually weaker than known results. In particular, several authors [3, 13, 15, 43] have proved that graphs with bounded Euler genus are 3-colourable or 3-choosable with bounded defect. And Dvořák and Norin [21] proved that graphs with bounded Euler genus are 4-choosable with bounded clustering. The proof of Dvořák and Norin [21] uses the fact that graphs of bounded Euler genus have strongly sublinear separators. The advantage of our approach is that it works for graph classes that do not have sublinear separator theorems. Graphs with given g-thickness are such a class [19]. We explore this direction in Theorem 8.

1.4 Clustered choosability and maximum degree

Alon, Ding, Oporowski and Vertigan [2] and Haxell, Szabó and Tardos [29] studied clustered colourings of graphs with given maximum degree. Haxell, Szabó and Tardos [29] proved that every graph with maximum degree Δ is $\lceil \frac{1}{3}(\Delta+1) \rceil$-colourable with bounded clustering. Moreover, for some Δ_0 and $\varepsilon > 0$, every graph with maximum degree $\Delta \geqslant \Delta_0$ is $\lfloor (\frac{1}{3} - \varepsilon)\Delta \rfloor$-colourable with bounded clustering. For both these results, the clustering bound is independent of Δ.

Clustered choosability of graphs with given maximum degree has not been studied in the literature (as far as we are aware). As a by-product of our work for graphs with given maximum average degree, we prove the following results for clustered choosability of graphs with given maximum degree.

Theorem 1.9. (Section 5). *Every graph G with maximum degree $\Delta \geqslant 3$ is $\lceil \frac{1}{3}(\Delta+2) \rceil$-choosable with clustering $\lceil \frac{19}{2}\Delta \rceil - 17$.*

Theorem 1.10. (Section 6). *Every graph G with maximum degree Δ is $\lceil \frac{2}{5}(\Delta+1) \rceil$-choosable with clustering 6.*

$\Delta = 5$ is the first case in which the above results for clustered choosability are weaker than the known results for clustered colouring. In particular, Haxell, Szabó and Tardos [29] proved that every graph with maximum degree 5 is 2-colourable with bounded clustering, whereas Theorems 1.9 and 1.10 only prove 3-choosability. It is open whether every graph with maximum degree 5 is 2-choosable with bounded clustering.

Finally, we remark that all our choosability results hold in the stronger setting of correspondence colouring, introduced by Dvořák and Postle [22].

2. Definitions

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $\Delta(G)$ be the maximum degree of the vertices in G. For a subset $A \subseteq V(G)$ and vertex $v \in V(G)$, let $N_A(v) := N_G(v) \cap A$ and $\deg_A(v) := |N_A(v)|$. We sometimes refer to $|V(G)|$ as $|G|$.

In a coloured graph, the *defect* of a vertex is its degree in the monochromatic subgraph. Note that a colouring with defect k also has defect $k+1$, but a vertex of defect k does not have defect $k+1$.

3. Defective choosability and maximum average degree

This section proves our result for defective choosability (Theorem 1.2). The following lemma is essentially a special case of an early result of Lovász [37].

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Lemma 3.1. *If L is a list-assignment for a graph G, such that*

$$\deg_G(v) + 1 \leqslant |L(v)|(d+1)$$

for each vertex v of G, then G is L-colourable with defect d.

Proof. Colour each vertex v in G by a colour in $L(v)$ so that the number of monochromatic edges is minimized. Suppose that some vertex v coloured α is adjacent to at least $d+1$ vertices also coloured α. Since $\deg(v) < |L(v)|(d+1)$, some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most d neighbours of v. Recolouring v by β reduces the number of monochromatic edges. This contradiction shows that no vertex v is adjacent to at least $d+1$ vertices of the same colour as v. Thus the colouring has defect d. □

Corollary 3.2. *Every graph G with* $\Delta(G) + 1 \leqslant k(d+1)$ *is k-choosable with defect d.*

The next lemma is a key idea of this paper. It provides a sufficient condition for a partial list-colouring to be extended to a list-colouring of the whole graph.

Lemma 3.3. *Let L be a k-list-assignment of a graph G. Let A, B be a partition of V(G), where G[A] is L-colourable with defect d′. If* $d \leqslant d'$ *and for every vertex* $v \in B$,

$$(d+1)\deg_A(v) + \deg_B(v) + 1 \leqslant (d+1)k,$$

then G is L-colourable with defect d′.

Proof. Let ϕ be an L-colouring of $G[A]$ with defect d'. For each vertex $v \in B$, let

$$L'(v) := L(v) \setminus \{\phi(x) : x \in N_A(v)\}.$$

Thus

$$|L'(v)| \geqslant k - \deg_A(v) \geqslant (\deg_B(v) + 1)/(d+1).$$

Lemma 3.1 implies that $G[B]$ is L-colourable with defect d. By construction, there is no monochromatic edge between A and B. Thus G is L-colourable with defect d'. □

We now prove our first main result, which is equivalent to Theorem 1.2 when $n_0 = 1$.

Theorem 1.7. *For integers* $d \geqslant 0$, $n_0 \geqslant 1$ *and* $k \geqslant 1$, *every graph G with*

$$\operatorname{mad}(G, n_0) < \frac{2d+2}{d+2}k$$

is k-choosable with defect

$$d' := \max\left\{\left\lceil \frac{n_0 - 1}{k} \right\rceil - 1,\ d\right\}.$$

Proof. We proceed by induction on $|V(G)|$. Let L be a k-list-assignment for G. For the base case, suppose that $|V(G)| \leqslant n_0 - 1$. For each vertex v of G, choose a colour in $L(v)$ so that each colour is used at most $\lceil |V(G)|/k \rceil$ times. We obtain an L-colouring with defect

$$\left\lceil \frac{n_0 - 1}{k} \right\rceil - 1.$$

Now assume that $|V(G)| \geqslant n_0$.

Let $v_1, \dots, v_p$ be a maximal sequence of distinct vertices in G, such that for all $i \in \{1, \dots, p\}$ we have

$$(d+1)\deg_{A_i}(v_i) + \deg_{B_i}(v_i) \geqslant (d+1)k,$$

where $A_i := \{v_1, \dots, v_i\}$ and $B_i := V(G) \setminus A_i$.

First suppose that $p < |V(G)|$. Let $A := \{v_1, \dots, v_p\}$ and $B := V(G) \setminus A$. By induction, $G[A]$ is L-colourable with defect d'. By the maximality of $v_1, \dots, v_p$, for every vertex $v \in B$ we have $(d+1)\deg_A(v) + \deg_B(v) + 1 \leqslant (d+1)k$. By Lemma 3.3, G is L-colourable with defect d', and we are done.

Now assume that $p = |V(G)|$. Thus

$$\begin{aligned}(d+2)|E(G)| &= \sum_{i=1}^{|V(G)|} d\deg_{A_i}(v_i) + \deg_G(v_i)\\ &= \sum_{i=1}^{|V(G)|} (d+1)\deg_{A_i}(v_i) + \deg_{B_i}(v_i)\\ &\geqslant (d+1)k|V(G)|.\end{aligned}$$

Since $|V(G)| \geqslant n_0$, we have

$$\operatorname{mad}(G, n_0) \geqslant \frac{2|E(G)|}{|V(G)|} \geqslant \frac{2d+2}{d+2}k,$$

which is a contradiction. □

4. Using independent transversals

This section introduces a useful tool called 'independent transversals', which have been previously used for clustered colouring by Alon, Ding, Oporowski and Vertigan [2] and Haxell, Szabó and Tardos [29]. Haxell [28] proved the following result.

Lemma 4.1. ([28]). *Let G be a graph with maximum degree at most Δ. Let $V_1, \dots, V_n$ be a partition of $V(G)$, with $|V_i| \geqslant 2\Delta$ for each $i \in [n]$. Then G has a stable set $\{v_1, \dots, v_n\}$ with $v_i \in V_i$ for each $i \in [n]$.*

Lemma 4.2. *Let $\Delta \geqslant 3$ and let G be a graph of maximum degree at most Δ. If H is a subgraph of G with $\Delta(H) \leqslant 2$, then G has a stable set $S \subseteq V(H)$ of vertices of degree 2 in H with the following properties.*

(1) Every subpath of H with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in H contains at least one vertex in S.
(2) Every subpath of H with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in H contains at least two vertices in S.
(3) Every connected subgraph C of H with at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S.

Proof. Consider each cycle component C of H with $|C| \geqslant 8\Delta - 12$. Say $|C| = (2\Delta - 3)a + b$, where $a \geqslant 4$ and $b \in [0, 2\Delta - 4]$. Partition C into subpaths $A_1B_1A_2B_2 \dots A_aB_a$ where $|A_i| = 2\Delta - 4$ and $|B_i| \in [1, 1 + \lceil b/a \rceil]$ for $i \in [a]$. Note that $|B_i| \leqslant 1 + \lceil b/a \rceil \leqslant \lceil \frac{1}{2}\Delta \rceil$.

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Now consider each path component P of H with $|P| \geqslant 2\Delta - 4$. Say $|P| = (2\Delta - 3)a + b - 1$, where $a \geqslant 1$ and $b \in [0, 2\Delta - 4]$. Partition P into subpaths $B_0A_1B_1 \ldots A_aB_a$ where $|A_i| = 2\Delta - 4$ for $i \in [a]$, $|B_i| = 1$ for $i \in [a-1]$, and $|B_i| \leqslant \lceil b/2 \rceil$.

Let $\mathcal{A}$ be the set of all such paths A_i taken over all the components of H. Let

$$G' := G\Big[\bigcup_{A \in \mathcal{A}} V(A)\Big] - E(H).$$

Then $\mathcal{A}$ gives a partition of $V(G')$ into parts, each of which has exactly $2\Delta - 4$ vertices, and $\Delta(G') \leqslant \Delta - 2$. By Lemma 4.1, G' has a stable set S that contains exactly one vertex in each path in $\mathcal{A}$. By construction, every vertex in S has degree 2 in H and S is a stable set in H, so S is a stable set in G.

Let P be a path in H that contains a vertex of degree 1 in H. Then H is subpath of some component path P' of H. If P contains at least $3\Delta - 6$ vertices, then $|P'| = (2\Delta - 3)a + b - 1$ where $a \geqslant 1$ and $b \in [0, 2\Delta - 4]$. Now, using our previous notation,

$$|B_0A_1| \leqslant \Delta - 2 + 2\Delta - 4 = 3\Delta - 6 \leqslant |P| \quad \text{and} \quad |A_aB_a| \leqslant \Delta - 2 + 2\Delta - 4 = 3\Delta - 6 \leqslant |P|,$$

so P is not a proper subpath of B_0A_1 or of B_aA_a. Hence P contains every vertex of A_i for some $i \in \{1, a\}$, so P contains a vertex in S.

If P contains at least $5\Delta - 9$ vertices, then $|P'| = (2\Delta - 3)a + b - 1$ where $a \geqslant 2$ and $b \in [0, 2\Delta - 4]$. Now,

$$|B_0A_1B_1A_2| \leqslant \Delta - 2 + 2(2\Delta - 4) + 1 = 5\Delta - 9 \leqslant |P| \quad \text{and} \quad |A_{a-1}B_{a-1}A_aB_a| \leqslant 5\Delta - 9 \leqslant |P|,$$

so P is not a proper subpath of $B_0A_1B_1A_2$ or of $A_{a-1}B_{a-1}A_aB_a$. Hence P contains every vertex A_i and of A_{i+1} for some $i \in \{1, a-1\}$, so P contains two vertices in S.

Suppose for contradiction there is a connected subgraph C of H on $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices with at most two vertices in S. By the definition of S, there are at most two paths $A_i \in \mathcal{A}$ with $V(A_i) \subseteq V(C)$. If C is contained in some path component of H, then C is a proper subpath of $A_jB_jA_{j+1}B_{j+1}A_{j+2}B_{j+2}A_{j+3}$ for some $j \in \{0, \ldots, a-3\}$, where we take A_0 and A_{a+1} to be the empty path for simplicity (so $|A_0B_0| = |B_0| \leqslant \Delta - 2$ and $|B_aA_{a+1}| = |B_a| \leqslant \Delta - 2$). Now

$$|A_jB_jA_{j+1}B_{j+1}A_{j+2}B_{j+2}A_{j+3}| \leqslant 4(2\Delta - 4) + 3 \leqslant \left\lceil \frac{19}{2}\Delta \right\rceil - 17.$$

If C is contained in some cycle component of H, we may assume without loss of generality that C is a subpath of the path $A_1B_1A_2B_2A_3B_3A_4$, and does not contain every vertex of A_1 and does not contain every vertex of A_4. Thus,

$$|V(C)| \leqslant |A_1B_1A_2B_2A_3B_3A_4| - 2 \leqslant 4(2\Delta - 4) + 3\left\lceil \frac{1}{2}\Delta \right\rceil - 2 \leqslant \left\lceil \frac{19}{2}\Delta \right\rceil - 17,$$

a contradiction. □

5. Clustered choosability and maximum degree

This section proves our first result about clustered choosability of graphs with given maximum degree (Theorem 1.9). The preliminary lemmas will also be used in subsequent sections.

Lemma 5.1. *If L is a list-assignment for a graph G, such that $\deg_G(v) + 2 \leqslant 3|L(v)|$ for each vertex v of G, and ϕ is an L-colouring of G that minimizes the number of monochromatic edges, then ϕ has*

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

defect 2. Moreover, for each vertex v with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$, such that at most two neighbours of v are coloured β_v under ϕ.

Proof. Suppose that some vertex v coloured α is adjacent to at least three vertices also coloured α. Since $\deg(v) < 3|L(v)|$, some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most two neighbours of v. Recolouring v by β reduces the number of monochromatic edges. This contradiction shows that every vertex has defect at most 2.

Consider a vertex v coloured α with defect 2. Suppose that v has at least three neighbours coloured β for each $\beta \in L(v) \setminus \{\alpha\}$. Thus $\deg(v) \geqslant 2 + 3(|L(v)| - 1)$, implying $\deg(v) + 1 \geqslant 3|L(v)|$, which is a contradiction. Thus some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most two neighbours of v.

Given a colouring ϕ of a graph G, let $G[\phi]$ denote the monochromatic subgraph of G given ϕ. The idea for the following lemma is by Haxell, Szabó and Tardos [29, Lemma 2.6], adapted here for the setting of list-colourings.

Lemma 5.2. *If H is a bipartite graph with bipartition (X, Y) and L is a list-assignment for H such that $|L(v)| = 2$ for all $v \in X$ and $|L(v)| = 1$ for all $v \in Y$ and every L-colouring ϕ has defect 2, then H has an L-colouring ϕ such that every connected subgraph of $H[\phi]$ at most two vertices in X.*

Proof. We begin by orienting the edges of H so that for every vertex $v \in V(H)$ and every colour $c \in L(v)$, v has at most one out-neighbour w with $c \in L(w)$ and v has at most one in-neighbour w with $c \in L(w)$. Let $L(H)$ be the union of the lists of all vertices of H. For each colour $c \in L(H)$, let H_c be the subgraph of H induced by the vertices $w \in V(H)$ with $c \in L(w)$. There is an L-colouring which assigns each vertex of H_c the colour c, so $\Delta(H_c) \leqslant 2$. Also, since every edge of H has an endpoint $y \in Y$ and $|L(y)| = 1$, every edge of H is in $E(H_c)$ for at most one $c \in L(H)$. For each $c \in L(H)$, orient the edges of H_c so that no vertex has more than one in-neighbour or out-neighbour (possible since $\Delta(H_c) \leqslant 2$). Orient all remaining edges of H arbitrarily.

We now construct an L-colouring ϕ. First, colour each vertex in Y with the unique colour in its list. Now run the following procedure, initializing $i := 1$.

1: If $i > |X|$, then exit.
2: Select $v_i \in X \setminus \{v_i : i \in [i-1]\}$ and select $\phi(v_i) \in L(v_i)$ arbitrarily. Increment i by 1 and go to **3**.
3: If there is a directed path $v_{i-1}yx$ such that $x \in X \setminus \{v_i : i \in [i-1]\}$ and $\phi(v_{i-1}) = \phi(y)$ and $\phi(v_{i-1}) \in L(x)$, let $v_i := x$, select $\phi(v_i) \in L(v_i) \setminus \{\phi(v_{i-1})\}$, increment i by 1 and go to **3**. Otherwise go to **4**.
4: If there is a directed path xyv_{i-1} such that $x \in X \setminus \{v_i : i \in [i-1]\}$ and $\phi(v_{i-1}) = \phi(y)$ and $\phi(v_{i-1}) \in L(x)$, let $v_i := x$, select $\phi(v_i) \in L(v_i) \setminus \{\phi(v_{i-1})\}$, increment i by 1 and go to **3**. Otherwise go to **1**.

Suppose for contradiction that some component C of $H[\phi]$ has at least three vertices in X. Since ϕ is an L-colouring, C has a directed subpath $x_1y_1x_2y_2x_3$ such that $\{x_1, x_2, x_3\} \subseteq X$. If x_1 was the first vertex in $\{x_1, x_2\}$ to be coloured, then x_2 was coloured next and $\phi(x_2) \neq \phi(x_1)$, a contradiction. If x_2 was the first vertex in $\{x_2, x_3\}$ to be coloured, then x_3 was coloured next and $\phi(x_3) \neq \phi(x_2)$, a contradiction. Hence, x_2 was coloured before x_1 and after x_3. But then x_1 was coloured immediately after x_2 and $\phi(x_1) \neq \phi(x_2)$, a contradiction.

We now prove our first result for clustered choosability of graphs with given maximum degree.

Theorem 1.9. *Every graph G with maximum degree $\Delta \geqslant 3$ is $\lceil \frac{1}{3}(\Delta+2) \rceil$-choosable with clustering $\lceil \frac{19}{2}\Delta \rceil - 17$.*

Proof. Let $k := \lceil (\Delta+2)/3 \rceil$. Let L be a k-list-assignment for G. Let ϕ be an L-colouring of G that minimizes the number of monochromatic edges. By Lemma 5.1, ϕ is an L-colouring with defect 2. Moreover, for each vertex v with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$ such that at most two neighbours of v are coloured β_v under ϕ. Let $L'(v) := \{\phi(v), \beta_v\}$ for each vertex v with defect 2.

Let M be the monochromatic subgraph of G. Thus $\Delta(M) \leqslant 2$. By Lemma 4.2, there is a set $S \subseteq V(M)$, such that S is stable in G, every vertex in S has defect 2 under ϕ, and the following hold.

(1) Every subpath of M with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in M contains at least one vertex in S.
(2) Every subpath of M with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in M contains at least two vertices in S.
(3) Every connected subgraph C of M on at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S.

Define a subpath of M to have *type 1* if it contains no vertex in S and at least one vertex of degree at most 1 in M. Define a subpath of M to have *type 2* if it contains at most one vertex in S and at least one vertex of degree at most 1 in M. Note that every path of type 1 is also of type 2, and every path of type 2 or 1 that does not contain a vertex of degree 1 in M contains a vertex of degree 0 in M, and hence has only one vertex. By the definition of S, every path of type 1 has at most $3\Delta - 7$ vertices and every path of type 2 has at most $5\Delta - 10$ vertices.

Let $\mathcal{T}$ be the set of connected components of $M - S$. Let H be the bipartite graph with bipartition $\{S, \mathcal{T}\}$, where $s \in S$ is adjacent to $T \in \mathcal{T}$ if and only if s is adjacent to T in G, and the colour of the vertices of T is in $L'(s)$. Define L'_H so that $L'_H(s) := L'(s)$ for every $s \in S$, and $L'_H(T)$ is the singleton containing the colour assigned to the vertices of T for every $T \in \mathcal{T}$.

Let ϕ'_H be an arbitrary L'_H-colouring of H, and let ϕ' be the corresponding L-colouring of G. Note that every vertex of $v \in S$ is assigned a colour in $L'(v)$ and every other vertex is assigned its original colour in ϕ. Since S is a stable set and by the definition of L', the number of monochromatic edges given ϕ' is at most the number of monochromatic edges given ϕ. Hence, by our choice of ϕ, no L-colouring of G yields fewer monochromatic edges than ϕ'. Hence the monochromatic subgraph M' of G given ϕ' satisfies $\Delta(M') \leqslant 2$. Let M'_H be the graph obtained from M' by contracting each $T \in \mathcal{T}$ to a single vertex. Then M'_H is isomorphic to the monochromatic subgraph of H given ϕ'_H. Since M'_H is a minor of M' and $\Delta(M') \leqslant 2$, we have $\Delta(M'_H) \leqslant 2$. Hence, every L'_H-colouring of H has defect 2.

By Lemma 5.2, H has an L'_H-colouring ϕ'_H such that no component of the monochromatic subgraph has more than two vertices in S. Let ϕ' be the corresponding L-colouring of G, and note that no component of the monochromatic subgraph M' of G given ϕ' has more than two vertices in S. In ϕ', vertices of $G - S$ keep their colour from ϕ, and vertices $v \in S$ get a colour from $L'(v)$, so ϕ' is an L-colouring that minimizes the number of monochromatic edges.

Suppose for contradiction that some vertex in $V(G-S)$ has degree 2 in M and is adjacent in M' to some vertex $s \in S$ which is not its neighbour in M (so $\phi'(s) \neq \phi(s)$). Then the L'-colouring obtained from ϕ by recolouring s with $\phi'(s)$ is not 2-defective, a contradiction.

It follows that the largest possible monochromatic component C of M' is either obtained from three disjoint paths in M of type 1 linked by two vertices in S, or is obtained from a path of type 1 and a path of type 2 linked by a vertex of S, or is a subgraph of M that contains at most two vertices in S. In each case, we have $|V(C)| \leqslant \lceil \frac{19}{2}\Delta \rceil - 17$. □

6. Clustered choosability with absolute bounded clustering

This section proves our results for clustered choosability of graphs with given maximum average degree (Theorem 1.5) or given maximum degree (Theorem 1.10), where the clustering is bounded by an absolute constant. The following lemma is the heart of the proof. With $I = \emptyset$, it immediately implies Theorem 1.10.

Lemma 6.1. *If I is a stable set of vertices in a graph G and L is a list-assignment for G such that $5|L(v)| \geqslant 2\deg(v) + 2$ for all $v \in V(G - I)$ and $5|L(v)| \geqslant 2\deg(v) + 1$ for all $v \in I$, then G has an L-colouring with clustering 9. Furthermore, if $I = \emptyset$, then G has an L-colouring with clustering 6.*

Proof. Let $\mathcal{C}$ be the class of L-colourings ϕ that minimize the number of monochromatic edges. Given $\phi \in \mathcal{C}$ and $v \in V(G)$, let $L(\phi, v)$ be the set of colours $c \in L(v)$ such the colouring ϕ' obtained from ϕ by recolouring v with c is in $\mathcal{C}$. Note that in particular $\phi(v) \in L(\phi, v)$, and that a colour $c \in L(v)$ is in $L(\phi, v)$ if and only if

$$|\{w \in N(v) : \phi(w) = c\}| = \deg_{G[\phi]}(v).$$

Claim 1. *If $\phi \in \mathcal{C}$, then $\Delta(G[\phi]) \leqslant 2$.*

Proof. Let v be a vertex of maximum degree in $G[\phi]$. If for some colour $c \in L(v)$ we have

$$|\{w \in N_G(v) : \phi(w) = c\}| < \deg_{G[\phi]}(v),$$

then the colouring ϕ' obtained from ϕ by changing the colour of v to c satisfies $|E(G[\phi'])| < |E(G[\phi])|$, contradicting the assumption that $\phi \in \mathcal{C}$. Hence, $\deg_G(v) \geqslant \deg_{G[\phi]}(v)|L(v)|$. By assumption $|L(v)| \geqslant \frac{1}{5}(2\deg_G(v) + 1)$, and the result follows. □

Claim 2. *If $\{\phi, \phi'\} \subseteq \mathcal{C}$, $v \in V(G - I)$ and $\deg_{G[\phi]}(v) = \deg_{G[\phi']}(v) = 2$, then*

$$|L(\phi, v) \cap L(\phi', v)| \geqslant 2.$$

Proof. Suppose for contradiction that $|L(\phi, v) \cap L(\phi', v)| \leqslant 1$. Note that $L(\phi, v) \cup L(\phi', v) \subseteq L(v)$. Given that

$$|L(\phi, v)| + |L(\phi', v)| = |L(\phi, v) \cup L(\phi', v)| + |L(\phi, v) \cap L(\phi', v)| \leqslant |L(v)| + 1,$$

we have $|L(\phi, v)| \leqslant (|L(v)| + 1)/2$ without loss of generality. Since $\phi \in \mathcal{C}$, for every colour $c \in L(v)$, the vertex v has at least two neighbours in G coloured c by ϕ (or else recolouring v with c would yield a colouring ϕ' with $|E(G[\phi'])| < |E(G[\phi])|$). For every colour $c \in L(v) \setminus L(\phi, v)$, the vertex v has at least three neighbours coloured c by ϕ. Hence, $\deg(v) \geqslant 3|L(v)| - (|L(v)| + 1)/2$, meaning $|L(v)| \leqslant \frac{1}{5}(2\deg(v) + 1)$, a contradiction. □

Choose $\phi_0 \in \mathcal{C}$ and $S \subseteq V(G - I)$ such that S is a stable set in $G[\phi_0]$, every vertex in S has degree 2 in $G[\phi]$, and subject to this $|S|$ is maximized. Let $S := \{s_1, s_2, \ldots, s_t\}$. For $i \in [t]$, define ϕ_i recursively so that $\phi_i(v) = \phi_{i-1}(v)$ for $v \in V(G) \setminus \{s_i\}$ and $\phi_i(s_i) \in (L(\phi_0, s_i) \cap L(\phi_{i-1}, s_i)) \setminus \{\phi_0(s_i)\}$. Such L-colourings exist by Claim 2.

Define $L'(v) := \{\phi_0(v), \phi_t(v)\}$ for all $v \in V(G)$.

Claim 3. *If ϕ is an L'-colouring of G and $s \in S$, then $|N_{G[\phi]}(s) \setminus S| = 2$.*

Proof. Note that $L'(v) = \{\phi_0(v)\}$ for $v \in V(G) \setminus S$. Hence $|N_{G[\phi]}(s) \setminus S| = |N_{G[\phi_0]}(s) \setminus S| = 2$ if $\phi(s) = \phi_0(s)$. Now suppose that $\phi(s) = \phi_t(s)$. By construction, $\phi_t(s) \in L(\phi_0, s)$, so the colouring ϕ' obtained from ϕ_0 by changing the colour of s to $\phi_t(s)$ is in $\mathcal{C}$. Now $\Delta(G[\phi']) \leqslant 2$ by Claim 1,

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

so no vertex $s' \in S$ is adjacent to s in $G[\phi']$, since s' already has two neighbours in $G[\phi_0] - S$ and hence in $G[\phi'] - S$. Since $|E(G[\phi'])| = |E(G[\phi_0])|$, we have $\deg_{G[\phi']}(s) = \deg_{G[\phi_0]}(s) = 2$. Hence

$$|N_{G[\phi]}(s) \setminus S| = |N_{G[\phi']}(s) \setminus S| = \deg_{G[\phi']}(s) = 2. \qquad \square$$

Claim 4. *If ϕ is an L'-colouring of G, then $\phi \in \mathcal{C}$.*

Proof. Suppose for contradiction that for some $\{v, w\} \subseteq S$, $vw \in E(G[\phi])$. Since S is a stable set in $G[\phi_0]$, either $\phi(v) = \phi_t(v)$ or $\phi(w) = \phi_t(w)$.

If $\phi(v) = \phi_t(v)$ and $\phi(w) = \phi_t(w)$, then v has three neighbours in $G[\phi_t]$ by Claim 3. But since $\phi_i(s_i) \in L(\phi_{i-1}, s_i)$ for $i \in [t]$, we have $\phi_t \in \mathcal{C}$, a contradiction.

Hence, without loss of generality, $\phi(v) = \phi_0(v)$ and $\phi(w) = \phi_t(w)$. Now $\phi_t(w) \in L(\phi_0, w)$, so the colouring ϕ' obtained from ϕ_0 by recolouring w with $\phi_t(w)$ is in $\mathcal{C}$. Note $vw \in E(G[\phi'])$ by assumption. By Claim 3, $|N_{G[\phi']}(v) \setminus S| = |N_{G[\phi_0]}(v) \setminus S| = 2$, so $\deg_{G[\phi']}(v) = 3$, contradicting Claim 1.

Now $|E(G[\phi])| = |E(G[\phi] - S)]| + 2|S|$ by Claim 3. But $G[\phi] - S = G[\phi_0] - S$, so $|E(G[\phi])| = |E(G[\phi_0])|$, and $\phi \in \mathcal{C}$. $\square$

Let $\mathcal{T}$ be the set of components of $G[\phi_0] - S$. Let H be the bipartite graph with bipartition $(S, \mathcal{T})$ such that $s \in S$ is adjacent to $T \in \mathcal{T}$ if s is adjacent to T in G and the colour assigned to the vertices of T by ϕ_0 is in $L'(s)$. Let L'_H be the natural restriction of L' to H. Note that an L'_H-colouring ϕ_H of H corresponds to an L'-colouring of G, and $H[\phi_H]$ is a minor of $G[\phi]$, which means $\Delta(H[\phi_H]) \leqslant 2$ by Claims 1 and 4. Hence, by Lemma 5.2, H has an L'_H-colouring ϕ_H such that no component of $H[\phi_H]$ has more than two vertices in S. Let ϕ be the corresponding L'-colouring of G. Note that each component of $G[\phi]$ has at most two vertices in S.

Suppose for contradiction that some component C of $G[\phi]$ has at least ten vertices. Now $\Delta(G[\phi]) \leqslant 2$ by Claims 1 and 4, so C is a cycle or a path. Hence C has an induced subpath $P := p_1 p_2 \dots p_8$ such that every vertex of P has degree 2 in $G[\phi]$. Since I is a stable set in G, at most one vertex in each of $\{p_1, p_2\}$, $\{p_4, p_5\}$ and $\{p_7, p_8\}$ is in I, so $C - I$ contains a stable set S_C of size 3 such that every vertex of S_C has degree 2 in $G[\phi]$. Define $S' := (S \setminus V(C)) \cup S_C$. Since $|S \cap V(C)| \leqslant 2$, we have $|S'| > |S|$. However $S' \subseteq V(G - I)$, S' is a stable set in $G[\phi]$, and every vertex of S' has degree 2 in $G[\phi]$, contradicting our choice of ϕ_0 and S.

Finally, suppose for contradiction that $I = \emptyset$ and some component C of $G[\phi]$ has at least seven vertices. As before, C is either a cycle or a path, so there is a stable set S_C in C of size 3 such that every vertex in S_C has degree 2 in $G[\phi]$. Define $S' := (S \setminus V(C)) \cup S_C$. Since $|S \cap V(C)| \leqslant 2$, we have $|S'| > |S|$. However $S' \subseteq V(G - I)$, S' is a stable set in $G[\phi]$ and every vertex of S' has degree 2 in $G[\phi]$, contradicting our choice of ϕ_0 and S. $\square$

The following lemma is analogous to Lemma 3.3.

Lemma 6.2. *Let (A, B) be a partition of the vertex set of a graph G, let $I \subseteq B$ be a stable set, and let L be a list-assignment for G. If $5|L(v)| - 5\deg_A(v) \geqslant 2\deg_B(v) + 2$ for all $v \in B \setminus I$ and $5|L(v)| - 5\deg_A(v) \geqslant 2\deg_B(v) + 1$ for all $v \in I$, then every L-colouring of $G[A]$ with clustering 9 can be extended to an L-colouring of G with clustering 9.*

Proof. Let ϕ be an L-colouring of $G[A]$ with clustering 9. For each vertex $v \in B$, let $L'(v) := L(v) \setminus \{\phi(x) : x \in N_A(v)\}$. Thus

$$|L'(v)| \geqslant |L(v)| - \deg_A(v) \geqslant \frac{2}{5}(\deg_B(v) + 1)$$

for $v \in B \setminus I$, and

$$|L'(v)| \geqslant |L(v)| - \deg_A(v) \geqslant \frac{1}{5}(2\deg_B(v) + 1)$$

for $v \in I$. Lemma 6.1 implies that $G[B]$ is L-colourable with clustering 9. By construction, there is no monochromatic edge between A and B. Thus G is L-colourable with clustering 9. □

We now prove the main result of this section.

Theorem 1.5. *Every graph G is $\lfloor \frac{7}{10}\,\mathrm{mad}(G) + 1 \rfloor$-choosable with clustering* 9.

Proof. Let $k := \lfloor \frac{7}{10}\,\mathrm{mad}(G) \rfloor + 1$. We proceed by induction on $|V(G)|$. The claim is trivial if $|V(G)| \leqslant 9$. Assume that $|V(G)| \geqslant 10$. Let L be a k-list-assignment for G.

Let p be the maximum integer for which there are pairwise disjoint sets $X_1, \dots, X_p \subseteq V(G)$, such that for each $i \in [p]$ we have $|X_i| \in \{1, 2\}$, and if $A_i := X_1 \cup \dots \cup X_{i-1}$ and $B_i := V(G) \setminus A_i$, then at least one of the following conditions holds:

- $X_i = \{v_i\}$ and $5|L(v_i)| \leqslant 5\deg_{A_i}(v_i) + 2\deg_{B_i}(v_i)$, or
- $X_i = \{v_i, w_i\}$ and $v_i w_i \in E(G)$, and

$$5|L(v_i)| \leqslant 5\deg_{A_i}(v_i) + 2\deg_{B_i}(v_i) + 1 \quad \text{and} \quad 5|L(w_i)| \leqslant 5\deg_{A_i}(w_i) + 2\deg_{B_i}(w_i) + 1.$$

First suppose that $X_1 \cup \dots \cup X_p \neq V(G)$. Let $A := X_1 \cup \dots \cup X_p$ and $B := V(G) \setminus A$. We now show that Lemma 6.2 is applicable. By the maximality of p, each vertex $v \in B$ satisfies $5|L(v)| \geqslant 5\deg_A(v) + 2\deg_B(v) + 1$. Let I be the set of vertices $v \in B$ for which $5|L(v)| = 5\deg_A(v) + 2\deg_B(v) + 1$. By the maximality of p, I is a stable set. Since $\mathrm{mad}(G[A]) \leqslant \mathrm{mad}(G)$, by induction, $G[A]$ is L-colourable with clustering 9. By Lemma 6.2, G is L-colourable with clustering 9.

Now assume that $X_1 \cup \dots \cup X_p = V(G)$. Let $R := \{i \in [p] : |X_i| = 1\}$ and $S := \{i \in [p] : |X_i| = 2\}$. Thus

$$\begin{aligned} 5k|V(G)| &\leqslant \sum_{i \in R} (3\deg_{A_i}(v_i) + 2\deg_G(v_i)) \\ &\quad + \sum_{i \in S} (3\deg_{A_i}(v_i) + 2\deg_G(v_i) + 1 + 3\deg_{A_i}(w_i) + 2\deg_G(w_i) + 1) \\ &\leqslant 3\sum_{i \in R} \deg_{A_i}(v_i) + 3\sum_{i \in S} (\deg_{A_i}(v_i) + \deg_{A_i}(w_i) + 1) + 2\sum_{v \in V(G)} \deg_G(v) \\ &= 7|E(G)|. \end{aligned}$$

Hence

$$\frac{10}{7}k \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant \mathrm{mad}(G),$$

implying $k \leqslant \frac{7}{10}\,\mathrm{mad}(G)$, which is a contradiction. □

7. Clustered choosability and maximum average degree

This section proves our final results for clustered choosability of graphs with given maximum average degree (Theorems 1.6 and 1.8).

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Lemma 7.1. *If I is a stable set in a graph G of maximum degree $\Delta \geqslant 3$, and L is a list-assignment of G, and $3|L(v)| \geqslant \deg_G(v)+1$ for each vertex $v \in I$, and $3|L(v)| \geqslant \deg_G(v)+2$ for each vertex $v \in V(G) \setminus I$, then G is L-colourable with clustering $19\Delta - 32$.*

Proof. Let ϕ be an L-colouring of G that minimizes the number of monochromatic edges. By Lemma 5.1, ϕ is an L-colouring with defect 2. Moreover, for each vertex $v \in V(G) \setminus I$ with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$, such that at most two neighbours of v are coloured β_v under ϕ. Let $L'(v) := \{\phi(v), \beta_v\}$ for each vertex $v \in V(G) \setminus I$ with defect 2.

Let M be the monochromatic subgraph of G. Thus $\Delta(M) \leqslant 2$. Each component of M is a cycle or path. Orient each cycle component of M to become a directed cycle, and orient each path component of M to become a directed path.

Let G' be obtained from G as follows: first delete all non-monochromatic edges incident to all vertices in I. Note that vertices in I now have degree at most 2. Now if vx is a directed monochromatic edge in G with $x \in I$ and x having defect 2, then contract vx into a new vertex v'. Note that $v \in V(G) \setminus I$ since I is a stable set. Note also that $\Delta(G') \leqslant \Delta(G) \leqslant \Delta$. Consider v' to be coloured by the same colour as v. Let $M_{G'}$ be the monochromatic subgraph of G'. Then $M_{G'}$ is obtained from M by the same set of contractions that formed G' from G, and $M_{G'}$ is an induced subgraph of G' with maximum degree at most 2.

By Lemma 4.2, there is a set $S' \subseteq V(M_{G'})$, such that S' is stable in G, every vertex in S' has defect 2 under ϕ, and the following hold.

(1) Every subpath of $M_{G'}$ with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in M contains at least one vertex in S'.
(2) Every subpath of $M_{G'}$ with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in M contains at least two vertices in S'.
(3) Every connected subgraph C of $M_{G'}$ with at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S'.

Let S be obtained from S' by replacing each vertex v' (arising from a contraction) by the corresponding vertex v in G. Thus $S \cap I = \emptyset$. By construction, S is a stable set in G, every vertex in S has defect 2 under ϕ, and each of the following hold.

(1) Every subpath of M with at least $6\Delta - 12$ vertices contains a vertex with degree 1 in M contains at least one vertex in S.
(2) Every subpath of M with at least $10\Delta - 18$ vertices contains a vertex with degree 1 in M contains at least two vertices in S.
(3) Every connected subgraph C of M with at least $19\Delta - 31$ vertices contains at least three vertices in S.

Define a subpath of M to have *type 1* if it contains no vertex in S and at least one vertex of degree at most 1 in M. Define a subpath of M to have *type 2* if it contains at most one vertex in S and at least one vertex of degree at most 1 in M. Note that every path of type 1 is also of type 2, and that any path of type 2 or 1 that contains no vertex of degree 1 in M contains a vertex of degree 0 in M, and hence has only one vertex. By the definition of S, every path of type 1 has at most $6\Delta - 13$ vertices and every path of type 2 has at most $10\Delta - 19$ vertices.

Let $\mathcal{T}$ be the set of connected components of $M - S$, and define a bipartite graph H with bipartition $\{S, \mathcal{T}\}$, where $s \in S$ is adjacent to $T \in \mathcal{T}$ if and only if s is adjacent to T in G, and the colour of the vertices of T is in $L'(s)$. Define L'_H so that $L'_H(s) := L'(s)$ for every $s \in S$, and $L'_H(T)$ is the singleton containing the colour assigned to the vertices of T for every $T \in \mathcal{T}$.

Let ϕ'_H be an arbitrary L'_H-colouring of H, and let ϕ' be the corresponding L-colouring of G. Note that every vertex of $v \in S$ is assigned a colour in $L'(v)$ and every other vertex is assigned its original colour in ϕ. Since S is a stable set and by the definition of L', the number of monochromatic edges given ϕ' is at most the number of monochromatic edges given ϕ. Hence by our choice of ϕ, no L-colouring of G yields fewer monochromatic edges than ϕ'. Hence the monochromatic subgraph M' of G given ϕ' satisfies $\Delta(M') \leqslant 2$. Let M'_H be the graph obtained from M' by contracting each $T \in \mathcal{T}$ to a single vertex. Then M'_H is isomorphic to the monochromatic subgraph of H given ϕ'_H. Since M'_H is a minor of M', we have $\Delta(M'_H) \leqslant 2$. Hence, every L'_H-colouring of H has defect 2.

By Lemma 5.2, H has an L'_H-colouring ϕ'_H such that no component of the monochromatic subgraph has more than two vertices in S. Let ϕ' be the corresponding L-colouring of G, and note that no component of the monochromatic subgraph M' of G given ϕ' has more than two vertices in S. In ϕ', vertices of $G - S$ keep their colour from ϕ, and vertices $v \in S$ get a colour from $L'(v)$, so ϕ' is an L-colouring which minimizes the number of monochromatic edges.

Suppose for contradiction that some vertex in $V(G - S)$ has degree 2 in M and is adjacent in M' to some vertex $s \in S$ which is not its neighbour in M (so $\phi'(s) \neq \phi(s)$). Then the L'-colouring obtained from ϕ by recolouring s with $\phi'(s)$ is not 2 defective, a contradiction.

It follows that the largest possible monochromatic component C of M' is obtained either from three disjoint paths in M of type 1 linked by two vertices in S, or is obtained from a path of type 1 and a path of type 2 linked by a vertex of S, or is a subgraph of M that contains at most two vertices in S. In each case, we have $|V(C)| \leqslant 19\Delta - 32$. □

We have the following analogue of Lemmas 3.3 and 6.2.

Lemma 7.2. *For a graph G, let A, B be a partition of $V(G)$ with $\Delta := \Delta(G[B]) \geqslant 3$, and let I be a stable set of G contained in B. Let L be a list-assignment for G and let c be an integer such that $c \geqslant 19\Delta - 32$, $G[A]$ is L-colourable with clustering c, $3|L(v)| \geqslant 3\deg_A(v) + \deg_B(v) + 1$ for each vertex $v \in I$, and $3|L(v)| \geqslant 3\deg_A(v) + \deg_B(v) + 2$ for each vertex $v \in B \setminus I$. Then G is L-colourable with clustering c.*

Proof. Let ϕ be an L-colouring of $G[A]$ with clustering c. For each vertex $v \in B$, let $L'(v) := L(v) \setminus \{\phi(x) : x \in N_A(v\}$. Thus $|L'(v)| \geqslant |L(v)| - \deg_A(v)$, implying $3|L'(v)| \geqslant \deg_B(v) + 1$ for each vertex $v \in I$, and $3|L'(v)| \geqslant \deg_B(v) + 2$ for each vertex $v \in B \setminus I$. Lemma 7.1 implies that $G[B]$ is L-colourable with clustering $19\Delta - 32$. By construction, there is no monochromatic edge between A and B. Thus G is L-colourable with clustering c. □

We now prove Theorem 1.8, which implies Theorem 1.6 when $n_0 = 1$.

Theorem 1.8. *For integers $d \geqslant 0$, $n_0 \geqslant 1$ and $k \geqslant 1$, every graph G with*

$$\operatorname{mad}(G, n_0) < \frac{3}{2}k$$

is k-choosable with clustering

$$c := \max\left\{\left\lceil \frac{n_0 - 1}{k} \right\rceil, 57k - 51\right\}.$$

Proof. We first prove the $k = 1$ case. Let G be a graph with $\operatorname{mad}(G, n_0) < 3/2$. Every component of a graph with maximum average degree less than $3/2$ has at most three vertices. Thus every component of G has at most $\max\{n_0 - 1, 3\}$ vertices. Hence, every 1-list-assignment has clustering $\max\{n_0 - 1, 3\} \leqslant c$. Now assume that $k \geqslant 2$.

We proceed by induction on $|V(G)|$. Let L be a k-list-assignment for G. If $|V(G)| \leqslant n_0 - 1$, then colour each vertex v by a colour in $L(v)$, so that each colour is used at most $\lceil (n_0 - 1)/k \rceil$ times. We obtain an L-colouring with clustering $\lceil (n_0 - 1)/k \rceil$. Now assume that $|V(G)| \geqslant n_0$.

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Let p be the maximum integer for which there are pairwise disjoint sets $X_1, \dots, X_p \subseteq V(G)$, such that for each $i \in [p]$ we have $|X_i| \in \{1, 2\}$, and if $A_i := X_1 \cup \cdots \cup X_{i-1}$ and $B_i := V(G) \setminus A_i$, then at least one of the following conditions holds:

- $X_i = \{v_i\}$ and $3|L(v_i)| \leqslant 3 \deg_{A_i}(v_i) + \deg_{B_i}(v_i)$, or
- $X_i = \{v_i, w_i\}$ and $v_i w_i \in E(G)$, and

$$3|L(v)| \leqslant 3 \deg_{A_i}(v) + \deg_{B_i}(v) + 1 \quad \text{and} \quad 3|L(w)| \leqslant 3 \deg_{A_i}(w) + \deg_{B_i}(w) + 1.$$

First suppose $X_1 \cup \cdots \cup X_p \neq V(G)$. Let $A := X_1 \cup \cdots \cup X_p$, $B := V(G) \setminus A$. Since $\operatorname{mad}(G[A], n_0) \leqslant \operatorname{mad}(G, n_0)$, by induction, $G[A]$ is L-colourable with clustering c. We now show that Lemma 7.2 is applicable. By the maximality of p, for each $v \in B$,

$$3k = 3|L(v)| \geqslant 3 \deg_A(v) + \deg_B(v) + 1 \geqslant \deg_B(v) + 1.$$

Let $\Delta := 3k - 1$. Then $\Delta(G[B]) \leqslant 3k - 1 = \Delta$. Since $k \geqslant 2$, we have $\Delta \geqslant 5$ and $19\Delta - 32 = 19(3k-1) - 32 = 57k - 51 \leqslant c$. Let I be the set of vertices $v \in B$ for which $3|L(v)| = 3 \deg_A(v) + \deg_B(v) + 1$. By the maximality of p, I is a stable set. Lemma 7.2 thus implies that G is L-colourable with clustering c.

Now assume that $X_1 \cup \cdots \cup X_p = V(G)$. Let $R := \{i \in [p] : |X_i| = 1\}$ and $S := \{i \in [p] : |X_i| = 2\}$. For $i \in R$, condition (A) holds, implying $3k \leqslant 2 \deg_{A_i}(v_i) + \deg_G(v_i)$. For $i \in S$, condition (B) holds, implying $3k \leqslant 2 \deg_{A_i}(v_i) + \deg_G(v_i) + 1$ and $3k \leqslant 2 \deg_{A_i}(w_i) + \deg_G(w_i) + 1$. Thus

$$\begin{aligned} 3k|V(G)| &\leqslant \sum_{i \in R} (2 \deg_{A_i}(v_i) + \deg_G(v_i)) \\ &\qquad + \sum_{i \in S} (2 \deg_{A_i}(v_i) + \deg_G(v_i) + 1 + 2 \deg_{A_i}(w_i) + \deg_G(w_i) + 1) \\ &= 2 \sum_{i \in R} \deg_{A_i}(v_i) + 2 \sum_{i \in S} (\deg_{A_i}(v_i) + \deg_{A_i}(w_i) + 1) + \sum_{v \in V(G)} \deg_G(v) \\ &= 4|E(G)|. \end{aligned}$$

Hence

$$\frac{3}{2} k \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant \operatorname{mad}(G),$$

and $|V(G)| \geqslant n_0$, implying $k \leqslant \frac{2}{3} \operatorname{mad}(G, n_0)$, which is a contradiction. □

8. Earth–moon colouring and thickness

The union of two planar graphs is called an *earth–moon* (or *biplanar*) graph. The famous earth–moon problem asks for the maximum chromatic number of earth–moon graphs [1, 12, 26, 31, 32, 41]. It follows from Euler's formula that every earth–moon graph has maximum average degree less than 12, and is thus 12-colourable. On the other hand, there are 9-chromatic earth–moon graphs [12, 26]. So the maximum chromatic number of earth–moon graphs is 9, 10, 11 or 12.

Defective and clustered colourings provide a way to attack the earth–moon problem. First consider defective colourings of earth–moon graphs. Since the maximum average degree of every earth–moon graph is less than 12, Theorem 1.1 by Havet and Sereni [27] implies that every earth–moon graph is k-choosable with defect d, for

$$(k, d) \in \{(7, 18), (8, 9), (9, 5), (10, 3), (11, 2)\}.$$

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

This result gives no bound with at most 6 colours. Ossona de Mendez, Oum and Wood [40] went further and showed that every earth–moon graph is k-choosable with defect d, for

$$(k, d) \in \{(5, 36), (6, 19), (7, 12), (8, 9), (9, 6), (10, 4), (11, 2)\}.$$

Examples show that 5 colours is best possible [40]. Thus the defective chromatic number of earth–moon graphs equals 5. Theorem 1.2 implies that every earth–moon graph is k-choosable with defect d for

$$(k, d) \in \{(7, 6), (8, 3), (9, 2), (11, 1)\}.$$

These results improve the best known bounds when $k \in \{7, 8, 9, 11\}$.

Now consider clustered colouring of earth–moon graphs. Wood [42] describes examples of earth–moon graphs that are not 5-colourable with bounded clustering. Thus the clustered chromatic number of earth–moon graphs is at least 6. Theorem 1.3 by Kopreski and Yu [35] proves that earth–moon graphs are 9-colourable with clustering 2. Other results for clustered colouring do not work for earth–moon graphs since they can contain expanders [19], and thus do not have sublinear separators. Since every earth–moon graph has maximum average degree strictly less than 12, Theorems 1.4 and Theorem 1.6 imply the following.

Theorem 8.1. *Every earth–moon graph is:*

- 9-*choosable with clustering* 2,
- 8-*choosable with clustering* 405.

It is open whether every earth–moon graph is 6- or 7-colourable with bounded clustering.

Earth–moon graphs are generalized as follows. The *thickness* of a graph G is the minimum integer t such that G is the union of t planar subgraphs; see [38] for a survey. It follows from Euler's formula that graphs with thickness t are $(6t-1)$-degenerate and thus $6t$-colourable. For $t \geqslant 3$, complete graphs provide a lower bound of $6t-2$. It is an open problem to improve these bounds: see [31]. Ossona de Mendez, Oum and Wood [40] studied defective colourings of graphs with given thickness, and proved the following result.

Theorem 8.2. **([40]).** *The defective chromatic number of the class of graphs with thickness t equals* $2t+1$. *In particular, every such graph is* $(2t+1)$-*choosable with defect* $2t(4t+1)$.

Now consider clustered colourings of graphs with given thickness. Obviously, the clustered chromatic number of graphs with thickness t is at most $6t$, and Wood [42] proved a lower bound of $2t+2$. Since every graph with thickness t has maximum average degree strictly less than $6t$, Theorems 1.4, 1.5 and 1.6 imply the following improved upper bounds.

Theorem 8.3. *Every graph with thickness t is:*

- $\lceil 9t/2 \rceil$-*choosable with defect 1 and clustering* 2,
- $\lceil 21t/5 \rceil$-*choosable with clustering* 9,
- $4t$-*choosable with clustering* $228t-51$.

Thickness is generalized as follows; see [32, 40, 42]. For an integer $g \geqslant 0$, the *g-thickness* of a graph G is the minimum integer t such that G is the union of t subgraphs each with Euler genus at most g. Ossona de Mendez, Oum and Wood [40] determined the defective chromatic number of this class as follows (thus generalizing Theorem 8.2).

Theorem 8.4. **([40]).** *For integers $g \geqslant 0$ and $t \geqslant 1$, the defective chromatic number of the class of graphs with g-thickness t equals $2t+1$. In particular, every such graph is $(2t+1)$-choosable with defect $2tg + 8t^2 + 2t$.*

Now consider clustered colourings of graphs with g-thickness t. Wood [42] proved that every such graph is $(6t+1)$-choosable with clustering $\max\{g, 1\}$. Euler's formula implies that every n-vertex graph with g-thickness t has less than $3t(n+g-2)$ edges (for $n \geqslant 3$), implying $\operatorname{mad}(G, 4tg - 8t + 1) < 6t + 3/2$. Hence, Theorem 1.8 implies the following improvement to this upper bound.

Theorem 8.5. *For $g \geqslant 0$ and $t \geqslant 1$, every graph with g-thickness t is $(4t+1)$-choosable with clustering*

$$\max\left\{\left\lceil \frac{4tg-8t}{4t+1} \right\rceil, 228t+6\right\}.$$

This result highlights the utility of considering $\operatorname{mad}(G, n_0)$.

9. Stack and queue layouts

This section applies our results to graphs with given stack- or queue-number. Again, previous results for clustered colouring do not work for graphs with given stack- or queue-number since they can contain expanders [19], and thus do not have sublinear separators.

A *k-stack layout* of a graph G consists of a linear ordering $v_1, \ldots, v_n$ of $V(G)$ and a partition $E_1, \ldots, E_k$ of $E(G)$ such that no two edges in E_i cross with respect to $v_1, \ldots, v_n$ for each $i \in [1, k]$. Here edges $v_a v_b$ and $v_c v_d$ *cross* if $a < c < b < d$. A graph is a *k-stack graph* if it has a k-stack layout. The *stack-number* of a graph G is the minimum integer k for which G is a k-stack graph. Stack layouts are also called *book embeddings*, and stack-number is also called *book-thickness*, *fixed outer-thickness* and *page-number*. Dujmović and Wood [20] showed that the maximum chromatic number of k-stack graphs is in $\{2k, 2k+1, 2k+2\}$.

A *k-queue layout* of a graph G consists of a linear ordering $v_1, \ldots, v_n$ of $V(G)$ and a partition $E_1, \ldots, E_k$ of $E(G)$ such that no two edges in E_i are nested with respect to $v_1, \ldots, v_n$ for each $i \in [1, k]$. Here edges $v_a v_b$ and $v_c v_d$ are *nested* if $a < c < d < b$. The *queue-number* of a graph G is the minimum integer k for which G has a k-queue layout. A graph is a *k-queue graph* if it has a k-queue layout. Dujmović and Wood [20] showed that the maximum chromatic number of k-queue graphs is in the range $[2k+1, 4k]$.

Consider clustered colourings of k-stack and k-queue graphs. Wood [42] noted the clustered chromatic number of the class of k-stack graphs is in $[k+2, 2k+2]$, and that the clustered chromatic number of the class of k-queue graphs is in $[k+1, 4k]$. The lower bounds come from standard examples, and the upper bounds hold since every k-stack graph has maximum average degree less than $2k+2$, and every k-queue graph has maximum average degree less than $4k$. Theorems 1.4, 1.5 and 1.6 thus imply the following improved upper bounds.

Theorem 9.1. *Every k-stack graph is:*

- $\lfloor (3k+4)/2 \rfloor$*-choosable with defect* 1*, and thus with clustering* 2*,*
- $\lfloor (7k+11)/5 \rfloor$*-choosable with clustering* 9*,*
- $\lfloor (4k+6)/3 \rfloor$*-choosable with clustering at most* $76k+53$*.*

https://doi.org/10.1017/S0963548319000063 Published online by Cambridge University Press

Theorem 9.2. *Every k-queue graph is:*

- *$3k$-choosable with defect 1, and thus with clustering 2,*
- *$\lfloor (14k+4)/5 \rfloor$-choosable with clustering 9,*
- *$\lfloor (8k+2)/3 \rfloor$-choosable with clustering at most $152k-13$.*

Acknowledgements. This research was initiated at the Bellairs Workshop on Graph Theory (20–27 April 2018). Many thanks to the other workshop participants for stimulating conversations and for creating a productive working environment.

References

[1] Albertson, M. O., Boutin, D. L. and Gethner, E. (2011) More results on r-inflated graphs: Arboricity, thickness, chromatic number and fractional chromatic number. *Ars Math. Contemp.* **4** 5–24.
[2] Alon, N., Ding, G., Oporowski, B. and Vertigan, D. (2003) Partitioning into graphs with only small components. *J. Combin. Theory Ser. B* **87** 231–243.
[3] Archdeacon, D. (1987) A note on defective colorings of graphs in surfaces. *J. Graph Theory* **11** 517–519.
[4] Borodin, O. V. and Ivanova, A. O. (2009) Almost proper 2-colorings of vertices of sparse graphs. *Diskretn. Anal. Issled. Oper.* **16** 16–20, 98.
[5] Borodin, O. V. and Ivanova, A. O. (2011) List strong linear 2-arboricity of sparse graphs. *J. Graph Theory* **67** 83–90.
[6] Borodin, O. V., Ivanova, A. O., Montassier, M., Ochem, P. and André Raspaud, A. (2010) Vertex decompositions of sparse graphs into an edgeless subgraph and a subgraph of maximum degree at most k. *J. Graph Theory* **65** 83–93.
[7] Borodin, O. V., Ivanova, A. O., Montassier, M. and Raspaud, A. (2011) (k, j)-coloring of sparse graphs. *Discrete Appl. Math.* **159** 1947–1953.
[8] Borodin, O. V., Ivanova, A. O., Montassier, M. and Raspaud, A. (2012) $(k, 1)$-coloring of sparse graphs. *Discrete Math.* **312** 1128–1135.
[9] Borodin, O. V. and Kostochka, A. V. (2011) Vertex decompositions of sparse graphs into an independent set and a subgraph of maximum degree at most 1. *Sibirsk. Mat. Zh.* **52** 1004–1010.
[10] Borodin, O. V. and Kostochka, A. V. (2014) Defective 2-colorings of sparse graphs. *J. Combin. Theory Ser. B* **104** 72–80.
[11] Borodin, O. V., Kostochka, A. V. and Yancey, M. (2013) On 1-improper 2-coloring of sparse graphs. *Discrete Math.* **313** 2638–2649.
[12] Boutin, D. L., Gethner, E. and Sulanke, T. (2008) Thickness-two graphs, part 1: New nine-critical graphs, permuted layer graphs, and Catlin's graphs. *J. Graph Theory* **57** 198–214.
[13] Choi, I. and Esperet, L. (2019) Improper coloring of graphs on surfaces. *J. Graph Theory* **91** 16–34.
[14] Cowen, L. J., Cowen, R. H. and Woodall, D. R. (1986) Defective colorings of graphs in surfaces: Partitions into subgraphs of bounded valency. *J. Graph Theory* **10** 187–195.
[15] Cowen, L., Goddard, W. and Jesurum, C. E. (1997) Defective coloring revisited. *J. Graph Theory* **24** 205–219.
[16] Cushing, W. and Kierstead, H. A. (2010) Planar graphs are 1-relaxed, 4-choosable. *European J. Combin.* **31** 1385–1397.
[17] Dorbec, P., Kaiser, T., Montassier, M. and Raspaud, A. (2014) Limits of near-coloring of sparse graphs. *J. Graph Theory* **75** 191–202.
[18] Dujmović, V. and Outioua, D. (2018) A note on defect-1 choosability of graphs on surfaces. arXiv:1806.06149
[19] Dujmović, V., Sidiropoulos, A. and Wood, D. R. (2016) Layouts of expander graphs. *Chicago J. Theoret. Comput. Sci.* **2016** 1.
[20] Dujmović, V. and Wood, D. R. (2004) On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.* **6** 339–358.
[21] Dvořák, Z. and Norin, S. (2017) Islands in minor-closed classes, I: Bounded treewidth and separators. arXiv:1710.02727
[22] Dvořák, Z. and Postle, L. (2018) Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory, Ser. B* **129** 38–54.
[23] Eaton, N. and Hull, T. (1999) Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl* **25** 79–87.
[24] Edwards, K., Kang, D. Y., Kim, J., Oum, S.-I. and Seymour, P. (2015) A relative of Hadwiger's conjecture. *SIAM J. Discrete Math.* **29** 2385–2388.
[25] Esperet, L. and Ochem, P. (2016) Islands in graphs on surfaces. *SIAM J. Discrete Math.* **30** 206–219.
[26] Gethner, E. and Sulanke, T. (2009) Thickness-two graphs, II: More new nine-critical graphs, independence ratio, cloned planar graphs, and singly and doubly outerplanar graphs. *Graphs Combin.* **25** 197–217.
[27] Havet, F. and Sereni, J.-S. (2006) Improper choosability of graphs and maximum average degree. *J. Graph Theory* **52** 181–199.
[28] Haxell, P. E. (2001) A note on vertex list colouring. *Combin. Probab. Comput.* **10** 345–347.

[29] Haxell, P., Szabó, T., and Tardos, G. (2003) Bounded size components: Partitions and transversals. *J. Combin. Theory Ser. B* **88** 281–297.
[30] van den Heuvel, J. and Wood, D. R. (2018) Improper colourings inspired by Hadwiger's conjecture. *J. London Math. Soc.* **98** 129–148.
[31] Hutchinson, J. P. (1993) Coloring ordinary maps, maps of empires and maps of the moon. *Math. Mag.* **66** 211–226.
[32] Jackson, B. and Ringel, G. (2000) Variations on Ringel's earth–moon problem. *Discrete Math.* **211** 233–242.
[33] Kim, J., Kostochka, A. and Zhu, X. (2014) Improper coloring of sparse graphs with a given girth, I: (0, 1)-colorings of triangle-free graphs. *European J. Combin.* **42** 26–48.
[34] Kim, J., Kostochka, A. and Zhu, X. (2016) Improper coloring of sparse graphs with a given girth, II: Constructions. *J. Graph Theory* **81** 403–413.
[35] Kopreski, M. and Yu, G. (2017) Maximum average degree and relaxed coloring. *Discrete Math.* **340** 2528–2530.
[36] Liu, C.-H. and Oum, S.-I. (2017) Partitioning H-minor free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B.*
[37] Lovász, L. (1966) On decomposition of graphs. *Studia Sci. Math. Hungar.* **1** 237–238.
[38] Mutzel, P., Odenthal, T. and Scharbrodt, M. (1998) The thickness of graphs: A survey. *Graphs Combin.* **14** 59–73.
[39] Norin, S., Scott, A., Seymour, P. and Wood, D. R. (2017) *Clustered colouring in minor-closed classes*. (S. Norin, A. Scott, P. Seymour and Wood, D. R.), *Combinatorica*, accepted in 2019. arXiv:1708.02370
[40] Ossona de Mendez, P., Oum, S.-I. and Wood, D. R. (2018) Defective colouring of graphs excluding a subgraph or minor. *Combinatorica*. doi:10.1007/s00493-018-3733-1
[41] Ringel, G. (1959) *Färbungsprobleme auf Flächen und Graphen*, Vol. 2 of Mathematische Monographien, VEB, Deutscher Verlag der Wissenschaften.
[42] Wood, D. R. (2018) Defective and clustered graph colouring. *Electron. J. Combin.* #DS23.
[43] Woodall, D. R. (2011) Defective choosability of graphs in surfaces. *Discuss. Math. Graph Theory* **31** 441–459.

Cite this article: Hendrey K and Wood DR (2019). Defective and clustered choosability of sparse graphs. *Combinatorics, Probability and Computing*. **28**, 791–810. https://doi.org/10.1017/S0963548319000063