LINEARLY STABLE POLYTOPES

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1. Introduction. Our aim in this paper is to describe a new class of convex polytopes, which will be called linearly stable. These have properties analogous to those of projectively stable polytopes (called projectively unique by Grünbaum (**1**, exercise 4.8.30)), which were first investigated early in 1966 by Grünbaum and Perles. Although many particular examples of projectively stable polytopes have been found, at present no general criteria for projective stability are known.

The main result of this paper is a theorem which enables us to classify linearly stable polytopes completely.

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2. The main theorem. The definitions of Grünbaum's recent book (**1**) will be followed throughout the paper. All d-polytopes (d-dimensional convex polytopes (**1**, § 3.1)) will be assumed to lie in d-dimensional Euclidean space E^d, unless the context demands otherwise. The polytopes which are the subject of the paper are mainly centrally symmetric (**1**, § 6.4); the centre of symmetry will always be taken to be the origin o. Throughout the paper we write c.s. for "centrally symmetric".

Two c.s. polytopes P_1 and P_2 are said to be *linearly equivalent* if there is a non-singular linear transformation A such that $P_1A = P_2$. Clearly P_1 and P_2 are combinatorially equivalent (**1**, § 3.2), that is, there is a one-to-one inclusion-preserving correspondence between the faces of P_1 and those of P_2. A c.s. polytope is called *linearly stable* if every c.s. polytope which is combinatorially equivalent to P is linearly equivalent to P.

The *regular d-cube* C^d is

$$C^d = \{x = (\xi_1, \ldots, \xi_d) \in E^d \mid |\xi_i| \leqq 1, i = 1, \ldots, d\}.$$

Clearly C^d is centrally symmetric, and any polytope linearly equivalent to C^d will be called a *d-cube*. (Strictly we should call it a *d-parallelotope*.) The symbol C^d will, however, be reserved for the regular d-cube. C^d has 2^d vertices, whose coordinates are $(\pm 1, \ldots, \pm 1)$, with all possible changes of sign.

The theorem which enables us to classify the linearly stable polytopes can now be stated.

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(1) MAIN THEOREM. *A centrally symmetric polytope is linearly stable if and only if its vertices form a (centrally symmetric) subset of the vertices of a d-cube.*

3. Examples. Before proceeding with the proof of the Main Theorem, we give a few examples of linearly stable polytopes. For this purpose, the Main Theorem and results of subsequent sections will be assumed if necessary.

Apart from the d-cube itself, the most obvious example of a linearly stable polytope is the *d-cross-polytope* X^d, defined by

$$X^d = \{x = (\xi_1, \ldots, \xi_d) \in E^d \mid |\xi_1| + \ldots + |\xi_d| \leqq 1\}.$$

X^d is the dual of C^d (**1**, § 3.4), hence the fact that it is linearly stable follows from proposition (2) below. However, since X^d has $2d$ vertices, of which any subset of d not including a pair of opposite vertices is linearly independent, it is easy to find a linearly equivalent polytope whose vertices occur among those of C^d. For example, an octahedron X^3 may be constructed by eliminating a pair of opposite vertices of the cube C^3.

Since the Cartesian product $C^d \times C^k$ of two cubes is also a cube

$$(C^d \times C^k = C^{d+k}),$$

it follows from the Main Theorem that the Cartesian product of linearly stable polytopes is linearly stable.

C^3 and X^3 are the only linearly stable 3-polytopes; in four dimensions, there are five (apart from C^4 and X^4), one of which is $X^3 \times C^1$. Using the Main Theorem, these may easily be constructed.

4. An equivalent formulation. A particular example of a dual of a c.s. d-polytope P in E^d is given by the following construction. The *polar set* P^* of P is defined by

$$P^* = \{y \in E^d \mid \langle x, y\rangle \leqq 1 \text{ for all } x \in P\}.$$

($\langle x, y\rangle$ is the usual inner product in E^d.) Then P^* is a c.s. d-polytope dual to P (**1**, Theorem 3.4.4), and $(P^*)^* = P$. For each face F of P, define $\hat{F}$ by

$$\hat{F} = \{y \in P^* \mid \langle x, y\rangle = 1 \text{ for all } x \in F\}.$$

Then $\hat{F}$ is a face of P^*, and the map $F \to \hat{F}$ yields the one-to-one inclusion-reversing correspondence between the faces of P and those of P^*. We have the following result.

(2) PROPOSITION. *Let P be a linearly stable d-polytope. Then its dual P^* is linearly stable.*

For, let P_1^* be a c.s. polytope combinatorially equivalent to P^*. Then the dual $P_1 = (P_1^*)^*$ of P_1^* is a c.s. polytope combinatorially equivalent to P, and thus, since P is linearly stable, there is a non-singular linear transformation A such that $PA = P_1$. If we let A^* be the adjoint transform of A, defined by

$$\langle xA, y\rangle = \langle x, yA^*\rangle,$$

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for all $x, y \in E^d$, then A^* is also a non-singular linear transformation, and it follows from the definition of P^* that $P_1^*A^* = P^*$. That is, P_1^* is linearly equivalent to P^*, which proves the proposition.

A set of facets ($(d - 1)$-dimensional faces) of a c.s. d-polytope P is said to be linearly independent if the corresponding outer normals to the facets are linearly independent. Since a facet of P corresponds to a vertex of the dual P^*, this implies that the corresponding subset of vert P^* (the set of vertices of P^*) is linearly independent. The following lemma yields a new formulation of the condition of the Main Theorem.

(3) LEMMA. *Let P be a c.s. d-polytope. Then there is a d-cube C such that* vert $P \subseteq$ vert C *if and only if among the facets of P there are d linearly independent facets, each of which contains half the vertices of P.*

To prove the necessity of the condition, let vert $P \subseteq$ vert C, and let D be any facet of C. Since P is centrally symmetric, D contains half the vertices of P; noticing that $P \subseteq C$, we conclude that $F = D \cap P$ is a face of P which contains half the vertices of P. If F were not a facet of P, so that dim $F < d - 1$, then since each vertex of P is in F or $-F$, we have

$$\dim P = \dim F + 1 < d,$$

contradicting the assumption that P is a d-polytope. It also follows from this that F and D have the same affine hull: aff F = aff D; hence they have the same outer normals. If $\pm D_1, \ldots, \pm D_d$ denote the $2d$ facets of C, then the facets $F_i = D_i \cap P$ $(i = 1, \ldots, d)$ satisfy the conditions of the lemma.

To show that the condition is sufficient, let $F_1, \ldots, F_d$ be the given linearly independent facets of P, each containing half the vertices of P. Then there are closed half-spaces H_i^+ $(i = 1, \ldots, d)$ of E^d, bounded by the hyperplanes H_i, such that $P \subset H_i^+$, and $F_i = H_i \cap P$. Let

$$C = \bigcap_{i=1}^{d} H_i^+ \cap \bigcap_{i=1}^{d} (-H_i^+).$$

Then C is a d-cube. For, C is bounded by d pairs of parallel hyperplanes, and thus C is a cube unless it is unbounded. However, if this were the case, C would contain a line, and then the outer normals to the facets F_i would each be perpendicular to this line, and hence linearly dependent. Since the facets F_i are linearly independent, this does not happen. Thus C is a cube, and each vertex of P is contained in d of the facets $\pm F_1, \ldots, \pm F_d$ of P, and hence in d of the facets $\pm D_1, \ldots, \pm D_d$ of C. That is, each vertex of P is a vertex of C, which completes the proof of the lemma.

Using proposition (2) and lemma (3), and bearing in mind the remark before lemma (3), we can reformulate the Main Theorem as follows.

(4) THEOREM. *Let P be a c.s. d-polytope. Then P is linearly stable if and only if among the vertices of P there are d linearly independent vertices, each of which is contained in half the facets of P.*

5. Proof of the Main Theorem. We shall now prove the Main Theorem, in the new formulation of theorem (4). The easier part of the proof is to show that the condition is sufficient. For, suppose that P is a c.s. d-polytope with some d linearly independent vertices $v_1, \ldots, v_d$, each of which is contained in half the facets of P. We shall show that the position of every other vertex of P is determined by $v_1, \ldots, v_d$. Hence, if P' is a c.s. polytope combinatorially equivalent to P, with vertices $v_1', \ldots, v_d'$ corresponding to $v_1, \ldots, v_d$, then the linear transformation A defined by

$$v_i A = v_i', \qquad i = 1, \ldots, d,$$

induces a linear equivalence between P and P'; that is, P is linearly stable.

To show this, let v be any other vertex of P. Then v belongs to some d linearly independent facets $F_1, \ldots, F_d$ of P, and is just the point of intersection of the hyperplanes aff $F_1, \ldots,$ aff F_d. For each $i = 1, \ldots, d$ and $j = 1, \ldots, d$, either F_i or $-F_i$ contains the vertex v_j; that is, F_i contains either v_j or $-v_j$. Now for $i = 1, \ldots, d$, the hyperplane aff F_i is completely determined by the linearly (and hence affinely) independent vertices $\epsilon_{ij} v_j$ ($j = 1, \ldots, d, \epsilon_{ij} = \pm 1$) which it contains. That is, v is completely determined by $v_1, \ldots, v_d$.

To prove the necessity of the condition of the theorem, we need two further results. The first of these is a stronger form of a result of Grünbaum (**1**, exercise 5.1.5).

(5) Lemma. *Let P be a c.s. d-polytope with $2n$ facets $\pm F_1, \ldots, \pm F_n$ (so that $n \geqq d$). Let C^n be the regular n-cube, with facets $\pm D_1, \ldots, \pm D_n$. Then there is a unique non-singular linear map J: $E^d \to E^n$, such that*

$$PJ = C^n \cap E^d J, \tag{1}$$

$$F_i J = D_i \cap E^d J, \qquad i = 1, \ldots, n. \tag{2}$$

In other words, a unique linear image of the given polytope P can be exhibited as a section of the n-cube C^n.

To prove this, let P^* be the polytope dual to P (that is, the polar set of P). Then P^* has $2n$ vertices, $\pm w_1, \ldots, \pm w_n$ (say), where for $i = 1, \ldots, n$, w_i is the vertex of P^* corresponding to the facet F_i of P. Then P can be written in the form

$$P = \{x \in E^d \mid |\langle x, w_i \rangle| \leqq 1, i = 1, \ldots, n\}.$$

(Compare the definition of the polar set in § 4.) Consider the map J: $E^d \to E^n$ defined by

$$xJ = (\langle x, w_1 \rangle, \ldots, \langle x, w_n \rangle);$$

that is, the ith coordinate of xJ is $\langle x, w_i \rangle$. Then J is the required map. J is clearly linear; it is non-singular since $w_1, \ldots, w_n$ span E^d, and thus $\langle x, w_i \rangle = 0$,

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for $i = 1, \ldots, n$, implies $x = o$. It satisfies properties (1) and (2), for, if $x \in P$, then $|\langle x, w_i \rangle| \leqq 1$ for each $i = 1, \ldots, n$, and hence $xJ \in C^n$. If in addition $x \in F_i$, then $\langle x, w_i \rangle = 1$, and thus $xJ \in D_i$. On the other hand, if $xJ \in C^n \cap E^dJ$, then $|\langle x, w_i \rangle| \leqq 1$ for $i = 1, \ldots, n$, therefore $x \in P$; if $xJ \in D_i \cap E^dJ$, then $\langle x, w_i \rangle = 1$ also, and hence $x \in F_i$.

Finally, J is unique. For, if I is a linear map from E^d to E^n satisfying (1) and (2), then the ith coordinate of xI yields a linear functional on E^d; that is, a linear map from E^d to the real numbers R. It is well known that a linear functional on E^d is of the form $\langle x, a_i \rangle$, for some $a_i \in E^d$. Now the facet F_i of P contains d linearly independent vertices of P, say $u_1, \ldots, u_d$. Since $F_iI \subseteq D_i$, it follows that $\langle u_j, a_i \rangle = 1$ for $j = 1, \ldots, d$. Hence the linear functionals $\langle x, w_i \rangle$ and $\langle x, a_i \rangle$ coincide on a basis of E^d, and thus are the same; that is, $a_i = w_i$. This implies that $I = J$, which completes the proof of the lemma.

We quote without proof the second result (**1**, exercise 5.1.2).

(6) Lemma. *Let L_1^d and L_2^d be two d-dimensional affine subspaces meeting an n-polytope Q. If L_1^d and L_2^d have* 0*-dimensional intersection with the same faces of Q, then the polytopes $L_1^d \cap Q$ and $L_2^d \cap Q$ are combinatorially equivalent.*

We can now show that the condition of theorem (4) is necessary. Suppose that P is a linearly stable d-polytope with $2n$ facets. Let J be the map defined in lemma (5) (after labelling the facets of P and C^n in some order), and write $L^d = E^dJ$. Then $PJ = C^n \cap L^d$ is linearly equivalent to P; since there is clearly no need to distinguish between linearly equivalent polytopes, without loss of generality we may suppose that $P = C^n \cap L^d$.

Let H be the subspace of L^d spanned by those vertices of P which are also vertices of C^n (with $H = \{o\}$ if there are no such vertices). By lemma (6), small perturbations of L^d (to L_1^d say), keeping the subspace H fixed, give rise to polytopes $P_1 = C^n \cap L_1^d$ combinatorially equivalent to P; for those vertices of P which do not lie in H are relatively interior points of faces of C^n of dimension at least 1. Clearly corresponding facets of P and P_1 lie in the same facets of C^n. Since P is linearly stable, P_1 is in fact linearly equivalent to P, which by lemma (5) implies that $P_1 = P$. We conclude that $H = L^d$, so that L^d is spanned by vertices of C^n.

In other words, P has some d linearly independent vertices, each of which is a vertex of C^n. Thus each of these vertices lies in half the facets of C^n, and so in half the facets of P itself. This completes the proof of the Main Theorem.

Reference

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