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Boundary layers and domain decomposition for radiative heat transfer and diffusion equations: applications to glass manufacturing process

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In this paper domain decomposition methods for radiative transfer problems including conductive heat transfer are treated. The paper focuses on semi-transparent materials, like glass, and the associated conditions at the interface between the materials. Using asymptotic analysis we derive conditions for the coupling of the radiative transfer equations and a diffusion approximation. Several test casts are treated and a problem appearing in glass manufacturing processes is computed. The results clearly show the advantages of a domain decomposition approach. Accuracy equivalent to the solution of the global radiative transfer solution is achieved, whereas computation time is strongly reduced.

1 Introduction

Simulations of heat transfer in semitransparent materials such as, for example, glass are usually done on the basis of the radiative transfer equations or, using a diffusion approximation, on the basis of a nonlinear diffusion equation. We refer elsewhere [1–6] for a detailed description of the equations and further references.

In many applications it is not necessary to model the whole computational region by the computationally-expensive radiative transfer equation. Only in particularly sensitive regions, where the solution is far from equilibrium, as for example in boundary layers, do these equations have to be employed. In the remaining regions of the domain the diffusion approximation is valid, and will lead to sufficiently accurate results. Domain decomposition methods are thus a natural design tool in this case leading to accurate numerical codes with reasonable computation times.

The general aim of these methods is to approximate the global solution of the radiative transfer equation by the solution computed with the hybrid code. In this way a considerable amount of computing time is saved. The locations of the radiative transfer and diffusion domains are often given *a priori* by the physical situation: in the present problem the radiative transfer equations are used in a layer near the boundaries and the diffusion equation in the interior of the domain. The layer region is chosen as small as possible in order to save CPU time. The remaining problem is to obtain the correct coupling

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conditions at the interface between the two regions. This will be investigated together with the derivation of boundary conditions in the present work.

In §2 we state the equations and boundary conditions under consideration. In §3 the asymptotic procedures to derive the diffusion equation and associated boundary conditions are described. We concentrate on the description of the boundary layers. In §4 we state coupling conditions for the domain decomposition problem at the interface. These conditions are derived by an analysis of the interface layer between the two domains, using asymptotic analysis similar to boundary layer analysis to obtain half-space problems. To solve these problems approximately, methods as in Golse & Klar [7], Larsen *et al.* [1] and Pomraning [8] can be used. Simpler but less accurate approximations are given as well. In §5 we show some numerical results for one-dimensional model problems. As one observes in the simulations the developed coupling conditions lead to a very accurate approximation of the radiative transfer solution by the solution of the domain decomposition problem even for anisotropic situations. In §6 a problem in glass manufacturing processes is considered, namely the cooling down of a cylindrical piece of glass. This is an example for a problem with a very sensitive layer region near the boundaries where the process has to be computed in an accurate way using the radiative transfer equations, and an outer region, where it is sufficient to use a diffusion approximation.

2 The equations

In a domain $D \in \mathbb{R}^3$ we consider the radiative transfer equations including conductive heat transfer but without photon scattering. The space variable is denoted by $r \in D$, the direction by the unit vector $\Omega \in S$, S the unit sphere, $\nu \in \mathbb{R}^+$ denotes the frequency, $t \in \mathbb{R}^+$ the time and M the number of spectral bands. For the absorption cross-section $\tilde{\kappa} = \tilde{\kappa}(\nu)$ we assume $\tilde{\kappa}(\nu) = \kappa(k) = \text{const}, k = 1, \ldots, M$ for $\nu \in [\nu_k, \nu_{k+1})$, where $\kappa(k)$ is the absorption cross-section for band k. This assumption is justified in many cases, for example, in the case of radiative heat transfer in glass. The interval $[\nu_1, \infty)$ is called the transparent region and $[0, \nu_1)$ the opaque region of the frequency. We denote by $I = I(r, \Omega, t, k)$ the radiative intensities at r in the direction Ω in band k and by $T(r, t)$ the temperature. We consider the transport equation for the radiative intensity

$$\Omega \cdot \nabla_r I(r, \Omega, t, k) = \kappa(k)\,[B(T(r, t), k) - I(r, \Omega, t, k)], \tag{1}$$

where the spectral black body intensity for the kth band is defined by

$$B(T(r, t), k) = \int_{\nu_k}^{\nu_{k+1}} \tilde{B}(T(r, t), \nu))\, \mathrm{d}\nu$$

with the spectral black body intensity

$$\tilde{B}(T, \nu) = \frac{2h_p \nu^3}{c^2}\left(\mathrm{e}^{\frac{h_p \nu}{k_B T}} - 1\right)^{-1},$$

where h_p, c, k_B are Planck's constant, the speed of light and the Boltzmann constant, respectively. The total intensity is

$$\int_0^\infty \tilde{B}(T, \nu)\, \mathrm{d}\nu = \frac{\sigma}{\pi} T^4, \tag{2}$$

where σ is the Stefan–Boltzmann constant.

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This equation is considered together with the temperature equation

$$c_m \rho_m \partial_t T(r,t) = \nabla_r \cdot (k_h \nabla_r T(r,t)) - \sum_k \kappa(k) \int_S [B(T(r,t),k) - I(r,\Omega,t,k)]\, d\Omega. \tag{3}$$

Here c_m, ρ_m, k_h are the specific heat, the density and the thermal conductivity, respectively. Initial conditions have to be imposed on the temperature:

$$T(r,0) = T_0(r), r \in D. \tag{4}$$

Boundary conditions for I can be of absorbing, reflecting or mixed type. For example, for $\hat{r} \in \partial D$, whose inward-pointing normal is n, one can use the boundary condition

$$I(\hat{r},\Omega,t,k) = R(\hat{r},\Omega,k), \quad \text{where} \quad \Omega \cdot n > 0, \tag{5}$$

or a semitransparent boundary condition

$$I(\hat{r},\Omega,t,k) = \rho(\Omega)\, I(\hat{r},\Omega',t,k) + [1-\rho(\Omega)]\, R(\hat{r},\Omega,k), \quad \text{where} \quad \Omega \cdot n > 0; \tag{6}$$

here Ω' is the reflection of Ω in the tangent plane to ∂D:

$$\Omega' = \Omega - 2n(n \cdot \Omega),$$

and ρ is the reflectivity and R denotes the radiative intensity transmitted into the medium from the outside. The reflectivity ρ is given by the Fresnel and Snell law. This means for incident angle Θ_1 with $\cos\Theta_1 = n \cdot \Omega$ we have

$$\rho = \frac{1}{2}\left[\frac{\tan^2(\Theta_1 - \Theta_2)}{\tan^2(\Theta_1 + \Theta_2)} + \frac{\sin^2(\Theta_1 - \Theta_2)}{\sin^2(\Theta_1 + \Theta_2)}\right]$$

with

$$n_2 \sin\Theta_2 = n_1 \sin\Theta_1,$$

if $|\sin(\Theta_1)| \leqslant n_2/n_1$ and $\rho = 1$ otherwise. Here n_1 is the refractive index for the material and n_2 the coefficient for the surroundings. We assume $n_1 \geqslant n_2$.

If k_h is not equal to 0, boundary conditions for the heat transfer Eq. (3) are needed as well. One can prescribe either the temperature or the heat flux at the boundary. The heat flux is given by the total (convective plus radiative) heat input at the boundary. For example, the following conditions can be imposed at $\hat{r} \in \partial D$ (see [6]):

$$T(\hat{r},t) = h(\hat{r},t), \tag{7}$$

where h is a given function, or

$$k_h\, n \cdot \nabla_r T(\hat{r},t) = -q(T(\hat{r},t)) \tag{8}$$

with q given by

$$q(T(\hat{r},t)) = h(T_{\text{ext}}(\hat{r},t) - T(\hat{r},t)) + \alpha\pi \int_0^{\nu_1} [\tilde{B}(T_{\text{ext}}(\hat{r},t),\nu) - \tilde{B}(T(\hat{r},t),\nu)]\, \mathrm{d}\nu,$$

where T_{ext} is a fixed exterior temperature. The last equation models the heat transfer at the boundary resulting from a convective term due to the temperature difference at the boundary and a term due to the surface radiation of the body, h denoting the convective heat transfer coefficient and α the emissivity. The integration is only over the opaque frequencies, $\nu \in [0, \nu_1)$.

From the point of view of numerical simulations the full radiative transfer equations give a very accurate description of the physical situation; however, to solve them is in general

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extremely time-consuming. If certain parameters in Eqs. (1) and (3) are small (see §3.1.), the approximation that is usually used is a diffusion approximation, the Rosseland approximation [1, 2]:

$$c_m \rho_m \partial_t T = \nabla_r \cdot [(k_h + k_r(T)) \nabla_r T] \tag{9}$$

with

$$k_r(T) = \sum_k \frac{4\pi}{3} \frac{1}{\kappa(k)} \frac{\partial B}{\partial T}(T, k).$$

Eq. (9) has to be supplemented with suitable initial and boundary conditions. The equation and the associated initial and boundary conditions will be derived in the next section.

To simplify the notation we do not write in the following the explicit dependence of the functions on the number k of the frequency band. Moreover, for $f = f(k)$ the notation

$$\langle f \rangle = \sum_k f(k)$$

is used.

3 Asymptotic analysis and the boundary layer problem

In this section we describe the derivation of the diffusion approximation (the outer solution) and the boundary layer equations (the inner solution). The matching of inner and outer solution is discussed, leading to boundary conditions for the diffusion equation. For the case $k_h = 0$ we refer elsewhere [1, 8].

An important parameter in radiative transfer is the mean free path a photon can travel before being absorbed; it is proportional to the inverse absorption cross section. The general assumptions needed for the asymptotic analysis in the next sections are the following: the physical system is large compared to the mean free path, the curvature of the boundary is small and the boundary conditions for the radiative transfer equations are assumed to vary slowly with respect to $\hat{r}$ and t.

We denote by ϵ the dimensionless parameter given by the mean free path divided by a characteristic length scale and assume this parameter to be small.

To obtain the dimensionless form of the above equations we introduce the notations

$$\begin{aligned} x &= r/l_{\text{ref}}, \\ \tau &= t/t_{\text{ref}}, \\ \kappa' &= \kappa/\kappa_{\text{ref}}, \\ I' &= I/I_{\text{ref}}, \\ T' &= T/T_{\text{ref}}, \\ (B(T))' &= (B(T))/(B(T))_{\text{ref}}, \end{aligned}$$

where $l_{\text{ref}}, t_{\text{ref}}, \kappa_{\text{ref}}, I_{\text{ref}}, T_{\text{ref}}, (B(T))_{\text{ref}}$ denote the reference scales and x, τ and the primed quantities denote the nondimensional quantities.

As stated above, the quantity

$$\epsilon = \frac{1}{\kappa_{\text{ref}} l_{\text{ref}}}$$

is assumed to be small. Moreover, we have the condition $(B(T))_{\text{ref}} = I_{\text{ref}}$ and

$$t_{\text{ref}} = c_m \rho_m \kappa_{\text{ref}} l_{\text{ref}}^2 T_{\text{ref}}/I_{\text{ref}}.$$

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This is the so-called diffusion scaling [1, 9].

If k_h is not equal to 0 we need additionally

$$k_h \kappa_{\text{ref}} = I_{\text{ref}}/T_{\text{ref}}.$$

Neglecting the primed notation for the dimensionless variables one obtains from Eqs. (1) and (3) the following equations for $I(x, \Omega, \tau)$ and $T(x, \tau)$:

$$\begin{aligned} \epsilon \Omega \cdot \nabla_x I &= \kappa (B(T) - I) \\ \epsilon^2 \partial_\tau T &= \epsilon^2 \nabla_x \cdot (k_h \nabla_x T) - \langle \kappa \int_S [B(T) - I]\, d\Omega \rangle \end{aligned} \tag{10}$$

with $x \in D$, where the scaled spatial domain is again denoted by D. The parameter k_h is, in the dimensionless formulation, either $k_h = 1$ or $k_h = 0$.

Eq. (10) is considered together with the boundary conditions (5) or (6). Here we have to replace R by $R'(\hat{x}, \Omega) = R(\hat{x} l_{\text{ref}}, \Omega)$. Neglecting the primed notation again, we get the boundary condition

$$I(\hat{x}, \Omega, \tau) = R(\hat{x}, \Omega), \quad \text{where} \quad \Omega \cdot n > 0$$

and the analogous expression for (6).

In the case $k_h \neq 0$ the additional conditions (7) or (8) are used. For condition (8) one obtains, with $q'(T') = q(T' T_{\text{ref}})/q_{\text{ref}})/q_{\text{ref}}$ and $q_{\text{ref}} = k_h T_{\text{ref}}/l_{\text{ref}}$ and neglecting again the primes,

$$k_h n \cdot \nabla_x T(\hat{x}, \tau) = -q(T(\hat{x}, \tau))$$

with $\hat{x} \in \partial D$.

For the initial conditions we get with $T_0'(x) = T_0(x l_{\text{ref}})$ and, neglecting the primes, $T(x, 0) = T_0(x)$.

3.1 Outer solution (diffusion approximation)

Using in Eq. (10) an ansatz of the form

$$\begin{aligned} I &= I^0 + \epsilon I^1 + \epsilon^2 I^2 + \dots, \\ T &= T^0 + \epsilon^2 T^2 + \dots \end{aligned}$$

and

$$B(T) = B(T^0) + \epsilon^2 \frac{\partial B}{\partial T}(T^0) T^2 + O(\epsilon^3)$$

one obtains, collecting terms of the same order in ϵ:

$$O(1): \quad I^0 = B(T^0),$$

$$O(\epsilon): \quad I^1 = -\frac{1}{\kappa} \Omega \cdot \nabla_x I^0$$

$$O(\epsilon^2): \quad I^2 = -\frac{1}{\kappa} \Omega \cdot \nabla_x I^1 + \frac{\partial B}{\partial T}(T^0) T^2,$$

$$\partial_\tau T^0 = \nabla_x \cdot (k_h \nabla_x T^0) - \left\langle \kappa \int_S \left[\frac{\partial B}{\partial T}(T^0) T^2 - I^2 \right] \mathrm{d}\Omega \right\rangle.$$

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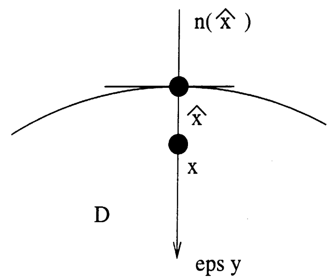

FIGURE 1. Boundary layer coordinates.

Due to the above asymptotics an approximation of the intensity accurate to $O(\epsilon)$ is given by

$$I(x, \Omega, \tau) \sim B(T^0(x,\tau)) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T^0(x,\tau)) + O(\epsilon^2). \tag{11}$$

The $O(\epsilon^2)$ terms give

$$\begin{aligned} \partial_\tau T^0 &= \nabla_x \cdot [(k_h + k_r(T^0)) \nabla_x T^0], \\ k_r(T) &= \left\langle \frac{4\pi}{3} \frac{1}{\kappa} \frac{\partial B}{\partial T}(T) \right\rangle. \end{aligned} \tag{12}$$

Eq. (12) is the nonlinear diffusion Eq. (9) for the temperature with the new time and space variables x and τ. Suitable initial and boundary conditions have to be imposed on Eq. (12). The initial conditions for the diffusion equation is given by $T^0(x, 0) = T_0(x)$. An initial layer does not exist, since we are considering a stationary transport equation. We concentrate in the following on the boundary conditions. Moreover, in this section the radiative transfer Eq. (10) is considered with fixed boundary condition (5), (7). For the treatment of reflecting boundary conditions in the linear case, see, for example, Benssousan *et al.* [10].

In the following we do not write explicitly the dependence of the functions on the time variable τ.

3.2 Inner problem (boundary layer problem)

The boundary conditions for Eq. (12) can be found by reconsidering Eqs. (10) and using an additional rescaling of the normal component of the space variable in the boundary layer. Using the standard multiple-scales approach, the new boundary layer coordinate y is given by the following (see, for example, [1, 10, 11]): We consider points $x \in D$ in a layer near the boundary. Let $\hat{x} \in \partial D$ be the point on the boundary closest to $x \in D$ and let $n(\hat{x})$ denote the inward normal at $\hat{x}$. The new coordinate y along the normal to the boundary is given by

$$y = \frac{1}{\epsilon}(x - \hat{x}) \cdot n(\hat{x}).$$

y and $\hat{x}$ give a set of coordinates in the boundary layer (see Fig. 1). For the sake of a simpler notation we use the coordinates y and x and treat them as independent variables.

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The boundary layer functions $I_b(y,x,\Omega)$ and $T_b(y,x)$ are defined in a layer of the thickness of the order of a mean free path near the boundaries.

Introducing I_b and T_b into Eqs. (10) leads up to $O(\epsilon^2)$ to

$$\begin{gathered}\mu\partial_y I_b+\epsilon\Omega\cdot\nabla_x I_b=\kappa(B(T_b)-I_b)\\ \partial_y(k_h\,\partial_y T_b)+\epsilon\partial_y(k_h[n\cdot\nabla_x T_b+\nabla_x\cdot(nT_b)])=\left\langle\kappa\int_S[B(T_b)-I_b]\,\mathrm{d}\Omega\right\rangle.\end{gathered}\tag{13}$$

Here x is a point in D near the boundary, $y\in\mathbb{R}^+$ and μ is defined by $\mu=n\cdot\Omega$. The boundary conditions for I_b and T_b at $y=0$, $x=\hat{x}$ are chosen such that they match those for I and T at $x=\hat{x}$:

$$I_b(0,\hat{x},\Omega)=R(\hat{x},\Omega),\quad \mu>0\tag{14}$$

and additionally, if $k_h\neq 0$

$$T_b(0,\hat{x})=h(\hat{x}).\tag{15}$$

Moreover, a condition at $y=\infty$ is needed:

$$\lim_{y\to\infty}\partial_y T_b(y,x)=0, x\in D.$$

3.3 Asymptotic expansion of the inner solution

We consider an $O(\epsilon^2)$ approximation of the boundary layer functions:

$$I_b(y,x,\Omega)=I_b^0(y,x,\Omega)+\epsilon I_b^1(y,x,\Omega)+O(\epsilon^2)\tag{16}$$

and

$$T_b(y,x)=T_b^0(y,x)+\epsilon T_b^1(y,x)+O(\epsilon^2)\tag{17}$$

with $x\in D$, $y\in\mathbb{R}^+$.

Lower-order boundary conditions are obtained using only an $O(\epsilon)$ approximation. This would yield sufficiently accurate results for small ϵ.

Substituting Eqs. (16) and (17) into (13) shows that $I_b^0(y,x,\Omega)$ and $T_b^0(y,x)$ solve the following half-space problem:

$$\begin{gathered}\mu\partial_y I_b^0=\kappa(B(T_b^0)-I_b^0)\\ \partial_y(k_h\,\partial_y T_b^0)=\left\langle\kappa\int_S[B(T_b^0)-I_b^0]\,\mathrm{d}\Omega\right\rangle\end{gathered}\tag{18}$$

with $y\in[0,\infty)$. The boundary conditions are given by the following: The ingoing function at $y=0$ is given by

$$I_b^0(0,\hat{x},\Omega)=R(\hat{x},\Omega),\quad \mu>0\tag{19}$$

due to Eq. (14). This is the only condition for $k_h=0$. For $k_h\neq 0$, additional conditions for the temperature T_b^0 at 0 and ∞ are required: using Eq. (15) one obtains

$$T_b^0(0,\hat{x})=h(\hat{x}).\tag{20}$$

Moreover,

$$\lim_{y\to\infty}\partial_y T_b^0(y,x)=0.$$

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Using

$$B(T_b) = B(T_b^0) + \epsilon \frac{\partial B}{\partial T}(T_b^0)\, T_b^1 + O(\epsilon^2)$$

one observes that $I_b^1(y, x, \Omega)$ and $T_b^1(y, x)$ solve

$$\begin{aligned} \mu \partial_y I_b^1 + \Omega \cdot \nabla_x I_b^0 &= \kappa \left[\frac{\partial B}{\partial T}(T_b^0)\, T_b^1 - I_b^1 \right] \\ \partial_y (k_h \partial_y T_b^1) + \partial_y (k_h [n \cdot \nabla_x T_b^0 + \nabla_x \cdot (n T_b^0)]) &= \left\langle \kappa \int_S \left[\frac{\partial B}{\partial T}(T_b^0)\, T_b^1 - I_b^1 \right] \mathrm{d}\Omega \right\rangle. \end{aligned} \tag{21}$$

The ingoing function at 0 is due to Eqs. (14) and (19) given by

$$I_b^1(0, \hat{x}, \Omega) = 0, \quad \mu > 0. \tag{22}$$

In case $k_h \neq 0$ the additional conditions for the temperature T_b^0 at 0 and ∞ are according to Eqs. (15) and (20)

$$T_b^1(0, \hat{x}) = 0.$$

Moreover, again

$$\lim_{y \to \infty} \partial_y T_b^1(y, x) = 0.$$

3.4 Solution of the inner problem

We assume that Eqs. (18) can be solved yielding $I_b^0(y, x, \Omega)$ and $T_b^0(y, x)$. In particular, it is assumed that the asymptotic values are

$$I_b^0(\infty, x, \Omega) = B(T_b^0(\infty, x)).$$

Since the boundary conditions are imposed at $x = \hat{x}$, $I_b^0(y, x, \Omega)$ and $T_b^0(y, x)$ are assumed to be uniquely determined for $x = \hat{x}$.

Eq. (21) is further treated in the following way. To obtain an equation with forcing terms vanishing at infinity one introduces (see [1]), a function $G = G(y, x, \Omega, k)$ by

$$I_b^1 = -\frac{1}{\kappa}[\Omega_T (1 - \zeta) + \mu n] \cdot \nabla_x B(T_0^b(\infty, x)) + n \cdot \nabla_x B(T_0^b(\infty, x))\, G(y, x, \Omega), \tag{23}$$

where Ω_T is the component of Ω tangential to the boundary. Here $\zeta = \zeta(y, \mu, k)$ is defined by

$$\zeta(y, \mu, k) = \exp\left(-\frac{\kappa(k)\, y}{\mu} \right)$$

for $\mu > 0$ and 0 otherwise. Using this representation of I_b^1 in Eq. (21), one obtains

$$\begin{aligned} \mu n \cdot \nabla_x B(T_0^b(\infty, x))\, \partial_y G + \Omega \cdot \nabla_x [I_b^0 - B(T_0^b(\infty, x))] & \\ = \kappa \left[\frac{\partial B}{\partial T}(T_b^0)\, T_b^1 - n \cdot \nabla_x B(T_0^b(\infty, x))\, G \right], & \quad (24) \end{aligned}$$

$$\partial_y (k_h \partial_y T_b^1) + \partial_y (k_h [n \cdot \nabla_x T_b^0 + \nabla_x \cdot (n T_b^0)]) = \left\langle \kappa \int_S \left[\frac{\partial B}{\partial T}(T_b^0)\, T_b^1 - n \cdot \nabla_x B(T_0^b(\infty, x))\, G \right] \mathrm{d}\Omega \right\rangle.$$

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The boundary conditions for G are due to Eqs. (22) and (23) given by

$$G(0, \hat{x}, \Omega) = \frac{1}{\kappa}\mu, \quad \mu > 0, \tag{25}$$

since $\zeta(0, \mu, k) = 1$. We mention that the forcing terms $\Omega \cdot \nabla_x [I_b^0 - B(T_0^b(\infty, x))]$ and $\partial_y (k_h [n \cdot \nabla_x T_b^0 + \nabla_x \cdot (n T_b^0)])$ in Eq. (24) tend to 0 as y tends to infinity. Eq. (24) is assumed to be solvable, such that G approaches a constant $\alpha = \alpha(x, \tau)$ independent of k and Ω as $y \to \infty$. Again uniqueness for $x = \hat{x}$ is assumed. We refer elsewhere [10] for a detailed investigation of kinetic half-space problems including forcing terms in the linear case.

3.5 Matching of inner and outer expansion

Matching the inner and outer asymptotic expansions one obtains the matching conditions

$$\lim_{y \to \infty} I_b(y, x, \Omega) = I(x, \Omega).$$

Due to Eq. (11), one has up to $O(\epsilon^2)$

$$I_b^0(\infty, x, \Omega) + \epsilon I_b^1(\infty, x, \Omega) = B(T^0(x)) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T^0(x)) + O(\epsilon^2).$$

Moreover

$$\lim_{y \to \infty} T_b(y, x) = T(x)$$

or to $O(\epsilon^2)$

$$T_b^0(\infty, x) + \epsilon T_b^1(\infty, x) = T^0(x) + O(\epsilon^2). \tag{26}$$

Determining the asymptotic values for I_b and T_b at $y = \infty$ and applying the matching conditions will give the correct boundary conditions for the outer solution.

3.6 Boundary conditions for the diffusion approximation

From the above inner solution one obtains

$$\begin{aligned}
\lim_{y \to \infty} & I_b(y, x, \Omega) \\
&= I_b^0(\infty, x, \Omega) + \epsilon I_b^1(\infty, x, \Omega) + O(\epsilon^2) \\
&= B(T_b^0(\infty, x)) - \frac{\epsilon}{\kappa} \Omega \cdot \nabla_x B(T_b^0(\infty, x)) + \epsilon \alpha(x)\, n \cdot \nabla_x B(T_b^0(\infty, x)) + O(\epsilon^2) \\
&= B[T_b^0(\infty, x) + \epsilon \alpha(x)\, n \cdot \nabla_x T_b^0(\infty, x)] - \frac{\epsilon}{\kappa} \Omega \cdot \nabla_x B[T_b^0(\infty, x) + \epsilon \alpha(x)\, n \cdot \nabla_x T_b^0(\infty, x)] + O(\epsilon^2).
\end{aligned}$$

Using the second equation in (21) and the conditions $\partial_y T_b^0(\infty, x) = 0 = \partial_y T_b^1(\infty, x)$ one obtains

$$\left\langle \kappa \int_S \left[\frac{\partial B}{\partial T}(T_b^0(\infty, x))\, T_b^1(\infty, x) - I_b^1(\infty, x, \Omega) \right] \mathrm{d}\Omega \right\rangle = 0.$$

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Evaluating this at $y = \infty$ and using Eq. (23) leads to

$$\left\langle \kappa \frac{\partial B}{\partial T}(T_b^0(\infty, x))\, T_b^1(\infty, x) \right\rangle = \langle \kappa n \cdot \nabla_x B(T_0^b(\infty, x))\, \alpha(x) \rangle.$$

This gives

$$T_b^1(\infty, x) = \alpha(x)\, n \cdot \nabla_x T_b^0(\infty, x).$$

Finally, we have

$$\lim_{y \to \infty} T_b(y, x) = T_b^0(\infty, x) + \epsilon T_b^1(\infty, x) + O(\epsilon^2)$$
$$= T_b^0(\infty, x) + \epsilon\alpha(x)\, n \cdot \nabla_x T_b^0(\infty, x) + O(\epsilon^2).$$

Using the matching conditions (26), one obtains

$$T^0(x) = T_b^0(\infty, x) + \epsilon\alpha(x)\, n \cdot \nabla_x T_b^0(\infty, x) + O(\epsilon^2)$$

or

$$T_b^0(\infty, x) = T^0(x) - \epsilon\alpha(x)\, n \cdot \nabla_x T^0(x) + O(\epsilon^2) \tag{27}$$

for $x \in D$. This gives a boundary condition for T^0 at $\hat{x}$:

$$T^0(\hat{x}) - \epsilon\alpha(\hat{x})\, n \cdot \nabla_x T^0(\hat{x}) = T_b^0(\infty, \hat{x}) + O(\epsilon^2). \tag{28}$$

To obtain explicit conditions from (28) one has to solve the above half-space problems in order to determine α and $T_b^0(\infty, \hat{x})$. This can be achieved using methods as in Pomraning [8], Golse & Klar [7] and Klar [12]. A simpler *ad hoc* approximation is shown in the next section.

3.7 Approximate boundary conditions near equilibrium

Here we explain a simple approach to obtain boundary conditions for the diffusion equation based on the assumption of the equality of half-range fluxes.

One uses the boundary condition given by Eq. (28). We have to approximate $T_b^0(\infty, \hat{x})$ and $\alpha(\hat{x})$: For $k_h = 0$ an approximation of $T_b^o(\infty, \hat{x})$ can be obtained by equating the half-range fluxes

$$\left\langle \int_{\mu > 0} \mu I_b^0(y, \hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle$$

at $y = 0$ and $y = \infty$:

$$\left\langle \int_{\mu > 0} \mu R(\hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle = \left\langle \int_{\mu > 0} \mu B(T_b^0(\infty, \hat{x})\, \mathrm{d}\Omega \right\rangle. \tag{29}$$

This approximation has been used in the kinetic theory of gases already by Maxwell and was later transfered to the radiative transfer case by Marshak (see Case & Zweifel [13]). It is obviously not correct in general; however, it is thought to be a good approximation near equilibrium. For $k_h \neq 0$ one can use the value for $T_b^0(0, \hat{x})$ instead of $T_b^0(\infty, \hat{x})$ obtaining

$$T_b^0(\infty, \hat{x}) = h(\hat{x}).$$

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Proceeding by a similar procedure one determines the value of α by equating the half-range fluxes

$$\left\langle \int_{\mu>0} \mu G(y, \hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle$$

at 0 and ∞:

$$\left\langle \frac{1}{\kappa} \int_{\mu>0} \mu^2\, \mathrm{d}\Omega \right\rangle = \left\langle \int_{\mu>0} \mu\alpha(\hat{x})\, \mathrm{d}\Omega \right\rangle. \tag{30}$$

Remarks. For the case $k_h = 0$, i.e. a material without conductive heat transfer a detailed asymptotic analysis has been performed elsewhere [1, 8]. In particular, an analysis of the boundary layer has been worked out there. For other approaches to obtain approximate equations, see elsewhere [14, 15].

4 Domain decomposition approaches

By solving radiative transfer and diffusion equations simultaneously in different domains, a very good approximation of the full radiative transfer solution may be obtained. Moreover, the computational complexity is in general considerably below the one needed for the full radiative transfer solution. An important point in obtaining such a coupled solution is the derivation of coupling conditions at the interface between diffusion and radiative transfer equations. This will be discussed in the following.

We assume, as mentioned in the introduction, that the computational domain is separated into a subdomain where the radiative transfer equation is solved and another subdomain where the diffusion approximation is used. That means we consider the domain D divided into two non-overlapping subdomains D_A and D_B, $D_A \cup D_B = D$ with smooth boundaries $\partial D, \partial D_A, \partial D_B$ and the interface $F = \partial D_A \cap \partial D_B$. It has to be remarked that for a fast solution of the coupling problem it is desirable to extend the diffusion domain as much as possible.

The aim is now to approximate the global radiative transfer solution by the solution of the following coupling problem: in D_A the radiative transfer Eq. (10) is solved and in D_B the diffusion Eq. (12). Providing these equations with coupling conditions at the interface F will lead to a properly-stated problem. However, the solution depends strongly on the type of coupling conditions employed. In the next sections we discuss an approach based on an interface layer analysis and its approximations. In the following I_A, T_A and I_B, T_B denote the solution of (10) in D_A and D_B, respectively. T_B^0 denotes the solution of the diffusion Eq. (12) in D_B. We concentrate on the interface region between the two domains and proceed similarly as in the above boundary layer treatment.

4.1 Interface layer problem

The interface layer near F with a thickness of the order of a mean free path is introduced in the same way as the boundary layer in §3 with D_B playing the role of D. y is now the rescaled coordinate normal to the interface F; $n = n(\hat{x})$ is the normal at $\hat{x}$ at F pointing into D_B.

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The interface layer terms $I_b(y, x, \Omega)$ and $T_b(y, x)$, $x \in D_B$ near the interface, $y \in \mathbb{R}^+$, are defined analogously to §3. Introducing I_b, T_b into Eq. (10) leads up to $O(\epsilon^2)$ to the half-space problem (13) with $\mu = n \cdot \Omega$, where n is the above normal at F. Assuming continuity of the intensity at the interface to $O(\epsilon^2)$, i.e.

$$I_A(\hat{x}, \Omega) = I_b(0, \hat{x}, \Omega) \tag{31}$$

for $\Omega \in S$ and $\hat{x} \in F$ one obtains the ingoing function at $y = 0$ for the half-space problem:

$$I_b(0, \hat{x}, \Omega) = I_A(\hat{x}, \Omega), \quad \mu > 0 \tag{32}$$

with $\mu = n \cdot \Omega$ as before.

Moreover, for $k_h \neq 0$ continuity of the temperature yields

$$T_b(0, \hat{x}) = T_A(\hat{x}). \tag{33}$$

At $y = \infty$ we use

$$\lim_{y \to \infty} \partial_y T_b(y, x) = 0.$$

4.2 Asymptotic expansion of the interface layer functions

The interface layer terms I_b, T_b are written as

$$\begin{aligned} I_b(y, x, \Omega) &= I_b^0(y, x, \Omega) + \epsilon I_b^1(y, x, \Omega) + O(\epsilon^2), \\ T_b(y, x) &= T_b^0(y, x) + \epsilon T_b^1(y, x) + O(\epsilon^2), \end{aligned} \tag{34}$$

where $x \in D_B$ near the interface.

We mention that a zeroth-order approximation neglecting the $O(\epsilon)$ term does not yield very accurate results even for small ϵ; see Fig. 2 in §5.

Substituting (34) into (13) shows that $I_b^0(y, x, \Omega)$ and $T_b^0(y, x)$ solve the half-space problem (18) as in §3. The boundary conditions for the half-space problem are given by the following: The ingoing function at $y = 0$ is

$$I_b^0(0, \hat{x}, \Omega) = I_A(\hat{x}, \Omega), \quad \mu > 0. \tag{35}$$

due to Eq. (32). This is the only condition for $k_h = 0$. To solve the half-space problem in the case $k_h \neq 0$ uniquely, additional conditions for the temperature T_b^0 at 0 and ∞ are required. Using Eq. (33) one obtains

$$T_b^0(0, \hat{x}) = T_A(\hat{x}), \lim_{y \to \infty} \partial_y T_b^0(y, x) = 0. \tag{36}$$

The first-order terms $I_b^1(y, x, \Omega)$ and $T_b^1(y, x)$ solve Eq. (21). The ingoing function at $y = 0$ is due to (32), (35) given by

$$I_b^1(0, \hat{x}, \Omega) = 0, \quad \mu > 0. \tag{37}$$

In case $k_h \neq 0$ the additional conditions for the temperature T_b^0 at 0 and ∞ are according to Eqs. (33), (36)

$$T_b^1(0, \hat{x}) = 0, \lim_{y \to \infty} \partial_y T_b^1(y, x) = 0.$$

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4.3 Solution of the interface layer problem

Again it is assumed that Eq. (18) can be solved yielding $I_b^0(y,x,\Omega)$ and $T_b^0(y,x)$. One obtains $I_b^0(\infty,x,\Omega) = B(T_b^0(\infty,x))$ and the outgoing distribution $I_b(0,\hat{x},\Omega)$, $\mu < 0$ of the half-space problem at the interface. Due to the boundary conditions $I_b^0(y,x,\Omega)$ and $T_b^0(y,x)$ are assumed to be uniquely determined for $x = \hat{x}$.

Eq. (21) is further treated in the same way as in §3: The function $G(y,x,\Omega,k)$ is introduced by

$$I_b^1 = -\frac{1}{\kappa}[\Omega_T(1-\zeta)+\mu n]\cdot\nabla_x B(T_0^b(\infty,x))+n\cdot\nabla_x B(T_0^b(\infty,x))\,G(y,x,\Omega). \tag{38}$$

It satisfies Eq. (24) with boundary conditions (25). It is assumed that G approaches a constant $\alpha(x,\tau)$ as $y\to\infty$. Since $\zeta = 0$ for $\mu < 0$, Eq. (38) yields the following outgoing function $I_b^1(0,x,\Omega)$ for $\mu < 0$:

$$I_b^1(0,x,\Omega) = -\frac{1}{\kappa}\Omega\cdot\nabla_x B(T_0^b(\infty,x))+n\cdot\nabla_x B(T_0^b(\infty,x))\,G(0,x,\Omega). \tag{39}$$

4.4 Matching of radiative transfer, interface layer and diffusion problem

Matching the interface layer and diffusion expansions one obtains

$$\lim_{y\to\infty} I_b(y,x,\Omega) = I_B(x,\Omega),$$

or

$$I_b^0(\infty,x,\Omega)+\epsilon I_b^1(\infty,x,\Omega) = B(T_B^0(x))-\epsilon\frac{1}{\kappa}\Omega\cdot\nabla_x B(T_B^0(x))+O(\epsilon^2),$$

and

$$\lim_{y\to\infty} T_b(y,x) = T_B(x),$$

or

$$T_b^0(\infty,x)+\epsilon T_b^1(\infty,x) = T_B^0(x)+O(\epsilon^2). \tag{40}$$

Determining the asymptotic values for I_b and T_b at $y = \infty$ and applying the matching conditions will give coupling conditions for the solution of the diffusion equation $T_B^0(\hat{x})$.

Matching of radiative transfer and the interface layer problem yields another coupling condition. Due to Eq. (31), the ingoing function for the radiative transfer domain D_A is

$$I_A(\hat{x},\Omega) = I_b(0,\hat{x},\Omega) \tag{41}$$

for $\mu < 0$. Up to $O(\epsilon^2)$ this is

$$I_A(\hat{x},\Omega) = I_b^0(0,\hat{x},\Omega)+\epsilon I_b^1(0,\hat{x},\Omega)+O(\epsilon^2), \quad \mu < 0. \tag{42}$$

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4.5 Coupling conditions for radiative transfer and diffusion equations

From the solution of the interface problems one obtains as in §3

$$\lim_{y\to\infty} I_b(y,x,\Omega) = B(T_b^0(\infty,x)) - \frac{\epsilon}{\kappa}\Omega\cdot\nabla_x B(T_b^0(\infty,x)) + \epsilon n\cdot\nabla_x B(T_b^0(\infty,x)\,\alpha(x) + O(\epsilon^2)$$

and

$$\lim_{y\to\infty} T_b(y,x) = T_b^0(\infty,x) + \epsilon\alpha(x)\,n\cdot\nabla_x T_b^0(\infty,x) + O(\epsilon^2).$$

Using the matching conditions (40) for the interface and diffusion problem one obtains, for $\hat{x}\in F$,

$$T_B^0(\hat{x}) - \epsilon\alpha(\hat{x})\,n\cdot\nabla_x T_B^0(\hat{x}) = T_b^0(\infty,\hat{x}) + O(\epsilon^2), \tag{43}$$

as in §3. Moreover, the matching condition (42) for the radiative transfer and interface problem yields, for $\mu < 0$, according to (39),

$$I_A(\hat{x},\Omega) = I_b^0(0,\hat{x},\Omega) - \epsilon\frac{1}{\kappa}\Omega\cdot\nabla_x B(T_b^0(\infty,\hat{x})) + \epsilon n\cdot\nabla_x B(T_b^0(\infty,\hat{x}))\,G(0,\hat{x},\Omega) + O(\epsilon^2),$$

or

$$I_A(\hat{x},\Omega) = I_b^0(0,\hat{x},\Omega) - \epsilon\frac{1}{\kappa}\Omega\cdot\nabla_x B(T_B^0(\hat{x})) + \epsilon n\cdot\nabla_x B(T_B^0(\hat{x}))\,G(0,\hat{x},\Omega) + O(\epsilon^2), \quad \text{where} \quad \mu < 0. \tag{44}$$

These conditions are sufficient, if $k_h = 0$. For $k_h \neq 0$ an additional condition is needed to determine uniquely a coupled solution:

Looking at Eqs. (10), one observes that the total flux (radiative transfer and heat flux) in the direction of the normal to the interface is given by

$$\epsilon^2 k_h\, n\cdot\nabla_x T(x) - \epsilon\left\langle \int_S n\cdot\Omega I(x,\Omega)\,\mathrm{d}\Omega \right\rangle.$$

We obtain the additional coupling condition by requiring, for $k_h \neq 0$, the continuity of the fluxes at the interface, i.e.

$$\epsilon k_h\, n\cdot\nabla_x T_A(\hat{x}) - \left\langle \int_S n\cdot\Omega I_A(\hat{x},\Omega)\,\mathrm{d}\Omega \right\rangle = \epsilon k_h\, n\cdot\nabla_x T_b(0,\hat{x}) - \left\langle \int_S n\cdot\Omega I_b(0,\hat{x},\Omega)\,\mathrm{d}\Omega \right\rangle. \tag{45}$$

Due to Eq. (31), this reduces to the continuity of the heat fluxes:

$$n\cdot\nabla_x T_A(\hat{x}) = n\cdot\nabla_x T_b(0,\hat{x}). \tag{46}$$

It is now most important to have a fast, however, accurate-enough solution procedure for the half-space problem in order to obtain explicit conditions from Eqs. (43) and (44). We note that solving the full half-space problem, for example, by a standard discretization procedure would need a lot of computing time, in particular, since it has to be solved at each point of the interface. Instead, one uses an approximation procedure [7, 8, 12], leading to easy to evaluate, however, accurate explicit coupling conditions. See §5, where the results of such a procedure are presented. One obtains in this way higher order accuracy than using the following simple conditions.

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4.6 Approximate coupling conditions

In the following simple versions of coupling conditions are stated. To get the first condition one uses (43). We approximate $T_b^0(\infty, \hat{x})$ and α. For $k_h = 0$ an approximation of $T_b^0(\infty, \hat{x})$ can be obtained by equalizing the half-range fluxes

$$\left\langle \int_{\mu>0} \mu I_b^0(y, \hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle$$

at $y = 0$ and $y = \infty$:

$$\left\langle \int_{\mu>0} \mu I_A(\hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle = \left\langle \int_{\mu>0} \mu B(T_b^0(\infty, \hat{x})\, \mathrm{d}\Omega \right\rangle. \tag{47}$$

For $k_h \neq 0$ one uses the value of $T_b^0(0, \hat{x})$ for $T_b^0(\infty, \hat{x})$ obtaining due to Eq. (33)

$$T_b^0(\infty, \hat{x}) = T_A(\hat{x}). \tag{48}$$

One proceeds as in §3 to determine α:

$$\left\langle \frac{1}{\kappa} \int_{\mu>0} \mu^2\, \mathrm{d}\Omega \right\rangle = \left\langle \int_{\mu>0} \mu\alpha(\hat{x})\, \mathrm{d}\Omega \right\rangle. \tag{49}$$

Using Eq. (46) with $T_b(\infty, \hat{x}) = T_B^0(\hat{x}) + O(\epsilon^2)$ instead of $T_b(0, \hat{x})$, one would obtain

$$n \cdot \nabla_x T_A(\hat{x}) = n \cdot \nabla_x T_B^0(\hat{x}).$$

However, to guarantee equality of the total flux, we consider Eq. (45) directly and approximate this condition by substituting $I_b(\infty, \hat{x}, \Omega) = I_B(\hat{x}, \Omega)$ instead of $I_b(0, \hat{x}, \Omega)$ and $T_b(\infty, \hat{x}) = T_B(\hat{x})$ instead of $T_b(0, \hat{x})$. Due to Eq. (11), we have

$$I_B(\hat{x}, \Omega) = B(T_B^0(x)) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T_B^0(x)) + O(\epsilon^2), \tag{50}$$

and finally, one obtains

$$\begin{aligned} &\epsilon k_h n \cdot \nabla_x T_A(\hat{x}) - \left\langle \int_S n \cdot \Omega I_A(\hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle \qquad (51) \\ &= \epsilon k_h n \cdot \nabla_x T_B^0(\hat{x}) + \epsilon \left\langle \frac{1}{\kappa} \int_S n \cdot \Omega \Omega \cdot \nabla_x B(T_B^0(\hat{x}))\, \mathrm{d}\Omega \right\rangle \\ &= \epsilon k_h n \cdot \nabla_x T_B^0(\hat{x}) + \epsilon \left\langle \frac{1}{\kappa} \frac{4\pi}{3} \frac{\partial B}{\partial T}(T_B^0(\hat{x})) \right\rangle n \cdot \nabla_x T_B^0(\hat{x}) \\ &= \epsilon [k_h + k_r(T_B^0(\hat{x}))]\, n \cdot \nabla_x T_B^0(\hat{x}). \end{aligned}$$

Eq. (51) gives a second straightforward coupling condition.

The ingoing function for the radiative transfer region $I_A(\hat{x}, \Omega)$, $\mu < 0$ is given in this approximation by substituting $I_b(\infty, \hat{x}, \Omega) = I_B(\hat{x}, \Omega)$ instead of $I_b(0, \hat{x}, \Omega)$ in Eq. (41) and using Eq. (50):

$$I_A(\hat{x}, \Omega) = I_B(\hat{x}, \Omega) = B(T_B^0(\hat{x})) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T_B^0(\hat{x})) + O(\epsilon^2), \quad \mu < 0. \tag{52}$$

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5 Numerical results for one-dimensional test cases

We consider the case of a slab with space variables (x, y, z), $x \in [0, L]$, $y, z \in \mathbb{R}$ and use model equations with the one-band approximation $M = 1$ and $k_h = 0$. The component of the angular variable Ω along the x-coordinate is denoted by ξ. We consider

$$\epsilon\xi\,\partial_x I = B(T) - I,$$

$$\epsilon^2\,\partial_\tau T = -\left(2B(T) - \int_{-1}^{1} I\,d\xi\right).$$

We use the boundary conditions (6). In the 1D case these conditions are given by

$$\rho(\xi) = \frac{1}{2}\left[\left(\frac{S-\xi}{S+\xi}\right)^2 + \left(\frac{N\xi - S}{N\xi + S}\right)^2\right],$$

if $|\xi| > \sqrt{(1-N)}$ and $\rho = 1$ otherwise. Here $S = \sqrt{(N-1+\xi^2)}$, $N = (n_2/n_1)^2$, $n_1 \geqslant n_2$. The refractive coefficients for glass and surrounding air are used: $n_1 = 1.46$ and $n_2 = 1$.

We consider a separation of $D = [0, L]$ into domains $D_A = [0, \hat{x}_1] \cup [\hat{x}_2, L]$ and $D_B = [\hat{x}_1, \hat{x}_2]$. $\hat{x}_1 < \hat{x}_2$. We concentrate on the interface point $\hat{x} = \hat{x}_1$. The boundary conditions at 0 and L are chosen in a symmetric way.

The approximate conditions, i.e. (43) combined with (47), (49) and the associated ingoing intensity (52) for the radiative transfer region are easily implemented. To implement the conditions found by the layer analysis we approximate them by the following conditions [12]. Condition (43) is given by

$$T_B^0(\hat{x}) - \epsilon\alpha(\hat{x})\,\partial_x T_B^0(\hat{x}) = T_b^0(\infty, \hat{x}) \tag{53}$$

with

$$B(T_b^0(\infty, \hat{x})) = I_b^0(\infty, \hat{x}, \xi),$$

where $I_b^0(y, \hat{x}, \xi)$ is the solution of Eqs. (18), (35) with $k_h = 0$. $I_b^0(\infty, \hat{x}, \xi)$ is approximated by

$$I_b^0(\infty, \hat{x}, \xi) \sim \frac{\int_{\xi>0} \xi I_A(\hat{x}, \xi)\,d\xi}{\int_{\xi>0} \xi\,d\xi} + \frac{3}{2}\int_{\xi'>0} (\xi')^2 \left[I_A(\hat{x}, \xi') - \frac{\int_{\xi>0} \xi I_A(\hat{x}, \xi)\,d\xi}{\int_{\xi>0} \xi\,d\xi}\right] d\xi', \tag{54}$$

as in Klar [12]. Also, α is approximated in the same way by

$$\alpha = \frac{\int_{\xi>0} \xi\xi\,d\xi}{\int_{\xi>0} \xi\,d\xi} + \frac{3}{2}\int_{\xi'>0} (\xi')^2 \left[\xi' - \frac{\int_{\xi>0} \xi\xi\,d\xi}{\int_{\xi>0} \xi\,d\xi}\right] d\xi', \tag{55}$$

since $G(0, \hat{x}, \xi) = \xi, \xi > 0$. The first terms in formulas (54), (55) would have been obtained by the approximate conditions (47), (49), respectively. The second one is a correction to this term.

The equation for the ingoing function for the radiative transfer region D_A is approximated for $\xi < 0$ due to Eq. (44) by

$$I_A(\hat{x}, \xi) = I_b^0(0, \hat{x}, \xi) - \epsilon[\xi - G(0, \hat{x}, \xi)] \cdot \partial_x B(T_B^0(\hat{x})). \tag{56}$$

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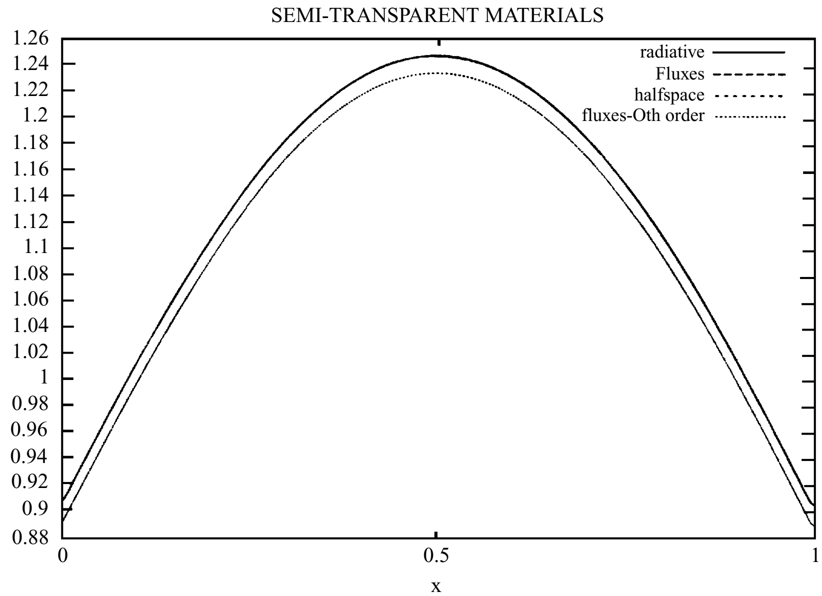

FIGURE 2. $\epsilon = 0.01$, $t = 0.4$, isotropic boundary conditions.

$I_b^0(0, \hat{x}, \xi)$, $\xi < 0$ is determined by

$$I_b^0(0,\hat{x},\xi) \sim I_b^0(\infty,\hat{x},\xi) + \frac{1}{2}\int_{\xi'>0} \frac{\xi'}{\xi'-\xi}[1-3(\xi')^2]\,[I_A(\hat{x},\xi') - I_b^0(\infty,\hat{x},\xi)]\,\mathrm{d}\xi'. \tag{57}$$

with $I_b^0(\infty, \hat{x}, \xi)$ determined by Eq. (54). See Klar [12] for a derivation of this formula. $G(0, \hat{x}, \xi)$, $\xi < 0$ is determined by

$$G(0,\hat{x},\xi) \sim \alpha + \frac{1}{2}\int_{\xi'>0} \frac{\xi'}{\xi'-\xi}[1-3(\xi')^2]\,[\xi'-\alpha]\,\mathrm{d}\xi' \tag{58}$$

with α from Eq. (55). Again, using only the first terms in these approximations one obtains exactly the approximate condition (52) in §4.6.

In the following figures results for different situations are plotted. We implement the solutions with different interface conditions. We use the conditions based on (53) combined with (54), (55) and (56) with (57), (58).

In the first three figures, the coupled problem is shown with the approximate conditions obtained by using in each of the above formulas only the first term. As mentioned these conditions are the approximate conditions developed in §4.6. They are labelled by 'fluxes'. Results obtained with the more accurate conditions derived by the full formulae presented above are shown as well. They are labelled by 'halfspace'. Moreover, we plot the global radiative transfer solution as reference solution, which is labelled by 'radiative'. We compute the radiative transfer equations in two small domains of size $0.005 \cdot L$ near the boundary. The mean free path is $\epsilon = 0.01$, $L = 1$.

For the case of isotropic boundary conditions, i.e. R in Eq. (6) not depending on ξ, we get Fig. 2. Here $R(0, \xi) = 1$ and the initial condition is $B(T_0) = 1.5$. Additionally to the

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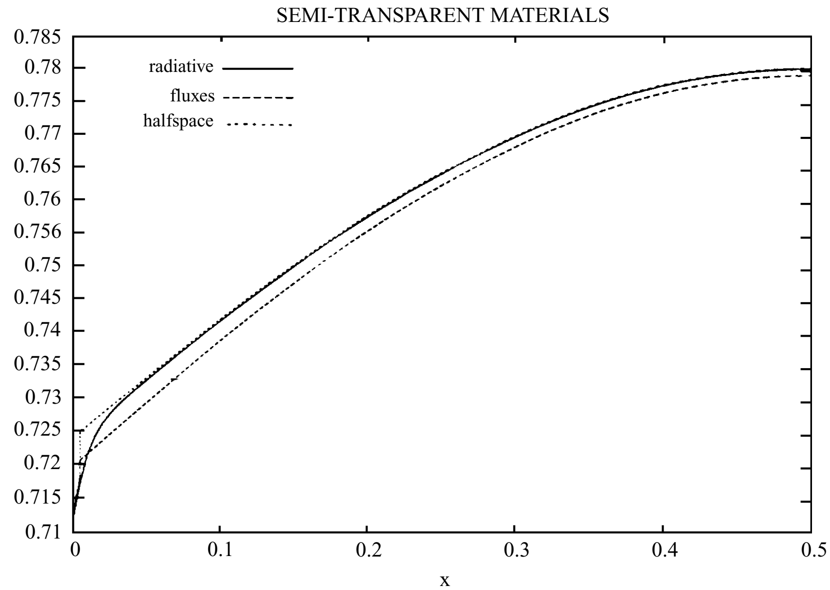

FIGURE 3. $\epsilon = 0.01$, $t = 0.4$, anisotropic boundary conditions.

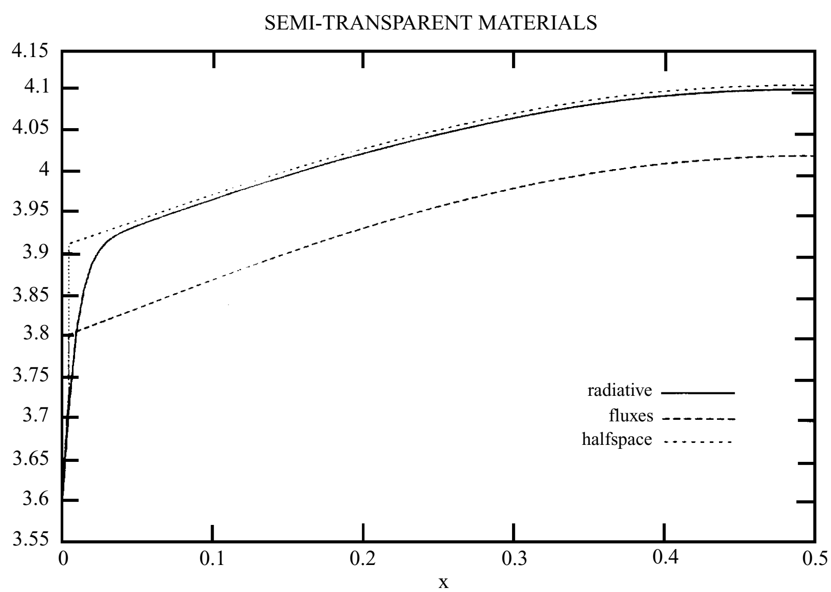

FIGURE 4. $\epsilon = 0.01$, $t = 0.4$, anisotropic boundary conditions.

conditions described above we plot another approximate condition derived from a zeroth-order approximation, i.e. an approximation where the $O(\epsilon)$ terms in Eqs. (53) and (56) are neglected. The results show that a zeroth-order approximation is not suitable. However, all other types of boundary conditions yield the same results. Using the equality of fluxes is sufficient in this situation.

Figures 3 and 4 show anisotropic situations. In Fig. 3 we use $R(0, \xi) = \xi, \xi > 0$ in Eq. (6)

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with the initial condition $B(T_0) = 0.8$. Figure 4 shows a situation with boundary conditions $R(0,\xi) = \delta(\xi - \xi_0)$, $\xi > 0$ with $0 < \xi_0 < 1$. The initial condition is $B(T_0) = 5$. We observe that for anisotropic situations a more accurate layer analysis yields a clear advantage.

Remarks. In the isotropic case the distribution function is well approximated by Eq. (11) in the whole domain. A more detailed boundary layer analysis is not necessary. The approximate conditions which are essentially based on this assumption are in this case sufficient.

6 A 3D domain decomposition problem in glass cooling processes

In this section we consider a typical example appearing in glass manufacturing processes. A cylindrical solid with radius 1 cm and height 2 cm is considered. The material is glass.

Consider Eqs. (1) and (3). Here, the number of frequency bands is $M = 18$. The coefficients are of the order $\rho_m c_m = 3 \cdot 10^6\ \mathrm{JK^{-1}\,m^{-3}}$, $k_h = 1.6\ \mathrm{Wm^{-1}\,K^{-1}}$. The absorption coefficients $\kappa(k)$ are in a range from $1\ \mathrm{m^{-1}}$ to $10^5\ \mathrm{m^{-1}}$. The boundary conditions are isotropic with outside temperature $T_{\mathrm{ext}} = 293$ K, i.e. they are given by Eq. (6) with $R = \alpha B(T_{\mathrm{ext}})$. The refractive coefficients n_1 and n_2 are chosen for glass with surrounding air, i.e. we set $n_1 = 1.46$, $n_2 = 1$ as in §5. The boundary conditions for the heat transfer equation are for $x \in \partial D$ given by Eq. (8). The initial temperature of the glass is $T_0 = 873\,K$.

In case the boundary condition (8) is used for the radiative transfer problem with $k_h \neq 0$, the boundary conditions for the diffusion approximation can be determined in the following way: We equalize fluxes at the boundary similar to the procedure in §4.5 and 4.6:

$$\epsilon k_h\, n \cdot \nabla_x T^0(\hat{x}) - \left\langle \int_S n \cdot \Omega [B(T^0(\hat{x})) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T^0(\hat{x}))]\, \mathrm{d}\Omega \right\rangle = -\epsilon q(T^0(\hat{x})) - \left\langle \int_S n \cdot \Omega I(\hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle.$$

Here Eq. (11) has been used. Using Eq. (11) again for $I(\hat{x}, \Omega)$, $\mu < 0$ and using the boundary condition for I, $\mu > 0$ we get:

$$\begin{aligned} &\epsilon[k_h + k_r(T^0(\hat{x}))]\, n \cdot \nabla_x T^0(\hat{x}) \\ &\quad = -\epsilon q(T^0(\hat{x})) - \left\langle \int_{\mu > 0} n \cdot \Omega I(\hat{x}, \Omega)\, \mathrm{d}\Omega \right\rangle \\ &\qquad - \left\langle \int_{\mu < 0} n \cdot \Omega [B(T^0(x)) - \epsilon \frac{1}{\kappa} \Omega \cdot \nabla_x B(T^0(x))]\, \mathrm{d}\Omega \right\rangle. \end{aligned} \tag{59}$$

This gives a boundary condition for T^0.

The solution is computed using the domain decomposition approach, the full radiative transfer equations and the diffusion approximation. The radiative transfer equations are solved using a method based on ray tracing and the diffusion equation by a finite element method.

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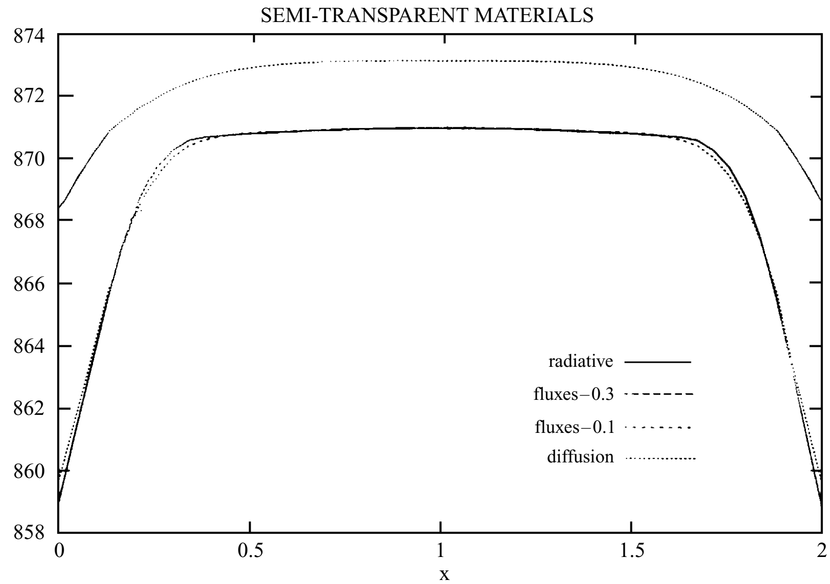

FIGURE 5. Heat transfer in glass.

The natural way to decompose the domains is to solve the diffusion equation in the interior of the cylinder and the radiative transfer equation in a domain near the boundary. We investigate solutions with larger and smaller radiative transfer regions. The interface is located 0.3 cm and 0.1 cm, respectively, away from the boundary. Here the boundary conditions are isotropic. Therefore, we use the approximate coupling conditions (43), (48) together with (51) and (52).

Figures 5 and 6 show the temperature distribution in the glass after 2 seconds. The results computed with the domain decomposition method are, as in the last section, in good agreement with the solution of the radiative transfer equation in the whole domain as in Fig. 2. In the situation with smaller radiative transfer region (0.1 cm), the CPU time for the solution of the coupled problem is about four times smaller than the time for the solution of the global radiative transfer equation. For the larger radiative transfer domain (0.3 cm) the gain in CPU time is still roughly a factor 2.

Figure 5 shows a comparison of the domain decomposition approach, the global radiative transfer solution and the diffusion approximation with boundary conditions given by Eq. (59). The temperature is plotted after two seconds considering a horizontal section in the middle of the cylinder. The radiative transfer solution and the coupled solution show a very good agreement. The solutions with radiative transfer regions of size 0.1 cm and 0.3 cm, respectively, are plotted in Fig. 5. The solution with the larger radiative transfer region is coincident with the radiative transfer solution. One observes that using the diffusion approximation with the above boundary condition does not yield very accurate results. Here a more detailed investigation of situations, where boundary conditions like (6) and (8) for the radiative transfer equations are involved, could give better results.

Figure 6 shows the full distribution of the temperature in the cylinder after two seconds.

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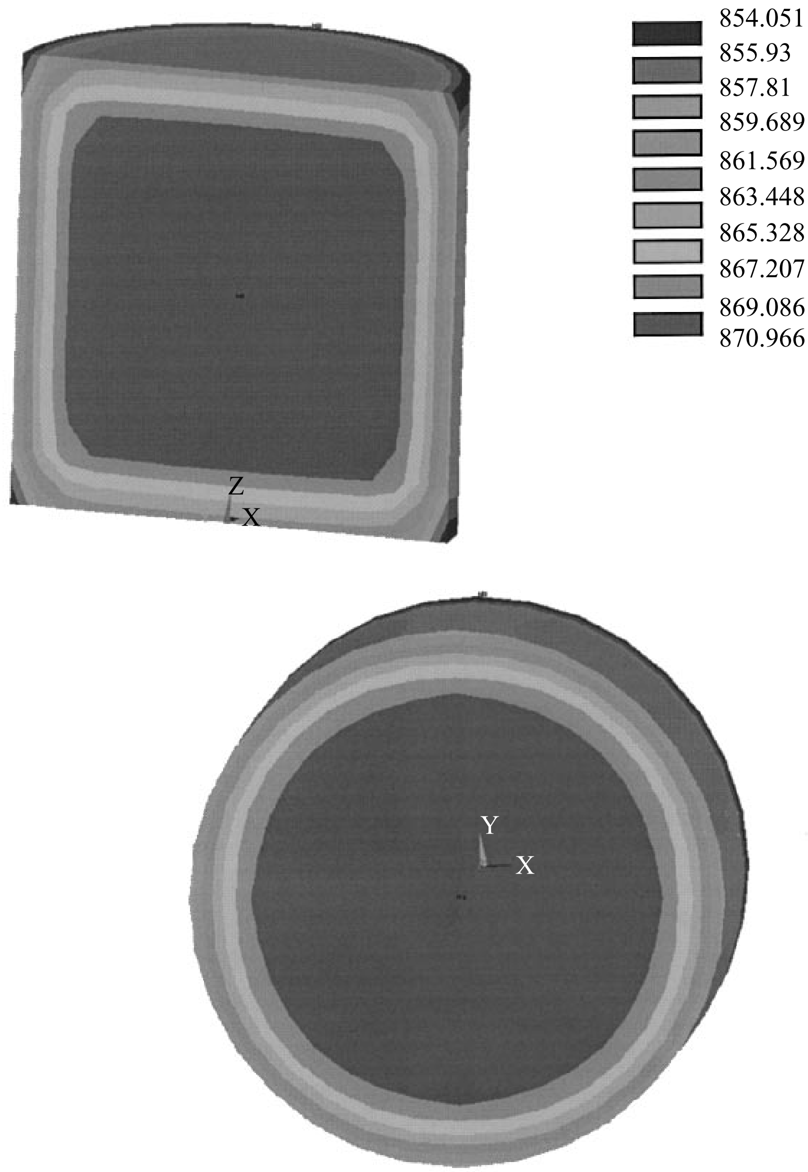

FIGURE 6. Heat transfer in glass.

7 Conclusions

- Detailed coupling conditions can be derived for radiative transfer equations with conductive heat transfer by asymptotic analysis.
- Approximations of these conditions are given. These can be used for a variety of problems with good success.
- For anisotropic situations more accurate approximations are necessary.
- Further work is required to clarify in detail the connection between the situations with $k_h = 0$ and $k_h \neq 0$.

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- Rigorous work, concerning special cases of the equations presented here, can be found elsewhere [16, 17]. However, for the full equations a rigorous analysis is still missing.

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