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Generation of large-scale vorticity in rotating stratified turbulence with inhomogeneous helicity: mean-field theory

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We discuss a mean-field theory of the generation of large-scale vorticity in a rotating density stratified developed turbulence with inhomogeneous kinetic helicity. We show that the large-scale non-uniform flow is produced due to either a combined action of a density stratified rotating turbulence and uniform kinetic helicity or a combined effect of a rotating incompressible turbulence and inhomogeneous kinetic helicity. These effects result in the formation of a large-scale shear, and in turn its interaction with the small-scale turbulence causes an excitation of the large-scale instability (known as a vorticity dynamo) due to a combined effect of the large-scale shear and Reynolds stress-induced generation of the mean vorticity. The latter is due to the effect of large-scale shear on the Reynolds stress. A fast rotation suppresses this large-scale instability.

Key words: astrophysical plasmas, plasma nonlinear phenomena

1. Introduction

A large-scale non-uniform flow or differential rotation in a helical small-scale turbulence can result in generation of a large-scale magnetic field by $\alpha\Omega$ or $\alpha^2\Omega$ mean-field dynamo, where α is the kinetic α effect and Ω is the angular velocity (see, e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokolov 1983; Ruzmaikin, Shukurov & Sokoloff 1988; Rüdiger, Kitchatinov & Hollerbach 2013). The kinetic α effect is related to a kinetic helicity produced, e.g. by a combined action of uniform rotation and density stratified or inhomogeneous turbulence. Formation of the non-uniform flows is caused, e.g. by a rotating anisotropic density stratified turbulence or turbulent convection. The latter effect is also related to a problem of generation of large-scale vorticity by a turbulent flow, and has various applications in geophysical and astrophysical flows (see, e.g. Lugt 1983; Pedlosky 1987; Chorin 1994).

It has been suggested by Moiseev *et al.* (1983), that the generation of the large-scale vorticity in a helical turbulence occurs due to the kinetic alpha effect. This idea is

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based on an analogy between the induction equation for the magnetic field and the vorticity equation (Batchelor 1950). The latter implies that a large-scale instability is associated with the term $\nabla \times (\alpha \overline{\boldsymbol{W}})$ in the equation for the mean vorticity, $\overline{\boldsymbol{W}}$, similarly to the mean-field equation for the magnetic field, $\overline{\boldsymbol{B}}$, where the key generation term is $\nabla \times (\alpha \overline{\boldsymbol{B}})$, see Moiseev *et al.* (1983). A mean-field equation for the vorticity has been derived by Khomenko, Moiseev & Tur (1991) using the functional technique for a compressible helical turbulence. It has been shown there that the mean vorticity grows exponentially in time due to the kinetic alpha effect.

However, the analogy between the induction equation and the vorticity equation is not complete, because the vorticity $\overline{\boldsymbol{W}} = \nabla \times \overline{\boldsymbol{U}}$, where the velocity $\overline{\boldsymbol{U}}$ is determined by the nonlinear Navier–Stokes equation. In addition, symmetry properties imply that the term $\nabla \times (\alpha \overline{\boldsymbol{W}})$ in the mean vorticity equation should originate from a Reynolds stress proportional to the mean velocity. On the one hand, the Reynolds stress enters neither the Navier–Stokes nor the vorticity equations without spatial derivatives. On the another hand, the Reynolds stress is an important turbulent characteristics, e.g. the trace of the Reynolds stress determines the turbulent kinetic energy density. To preserve the Galilean invariance, the trace of the Reynolds stress as well as the diagonal components of the Reynolds stress should be proportional to the spatial derivatives of the mean velocity, rather than to the mean velocity itself.

Frisch, She & Sulem (1987), Frisch *et al.* (1988) have investigated the effect of a non-Galilean invariant forcing that causes a large-scale instability resulting in the formation of a non-uniform flow at large scales (the so-called anisotropic kinetic alpha effect, or the AKA effect). A non-Galilean invariant forcing and generation of large-scale vorticity have been also investigated by Kitchatinov, Rüdiger & Khomenko (1994). There are various examples for turbulence driven by non-Galilean invariant forcing, e.g. supernova-driven turbulence in galaxies (Korpi *et al.* 1999) and the turbulent wakes driven by galaxies moving through the galaxy cluster (Ruzmaikin, Sokoloff & Shukurov 1989). Also, the presence of boundaries can break the Galilean invariance, see e.g. discussion in Brandenburg & Rekowski (2001), and references therein.

In a homogeneous non-helical and incompressible turbulence with an imposed mean velocity shear, the large-scale vorticity can be generated due to an excitation of a large-scale instability, referred to as a vorticity dynamo and caused by the combined effect of the large-scale shear motions and Reynolds stress-induced generation of perturbations to the mean vorticity. This effect has been studied theoretically by Elperin, Kleeorin & Rogachevskii (2003), Elperin *et al.* (2007) and detected in direct numerical simulations by Yousef *et al.* (2008), Käpylä, Mitra & Brandenburg (2009). To derive the mean-field equation for the vorticity, the spectral τ approach, which is valid for large Reynolds numbers, has been applied by Elperin *et al.* (2003). The linear stage of the large-scale instability which is saturated by nonlinear effects has been investigated by Elperin *et al.* (2003), but not a finite-time growth of large-scale vorticity as described by Chkhetiany *et al.* (1994). In particular, a first-order smoothing (a quasi-linear approach) has been used in the latter study to derive the equation for the mean vorticity in a compressible random flow with an imposed large-scale shear. The latter approach is valid only for small Reynolds numbers, and this is the reason why the large-scale instability has not been found by Chkhetiany *et al.* (1994). Importance of the vorticity dynamo has been demonstrated by Guervilly, Hughes & Jones (2015), where they suggested a mechanism for the generation of large-scale magnetic fields based on the formation of large-scale vortices in rotating turbulent convection.

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Formation of large-scale non-uniform flow by inhomogeneous helicity in a rotating incompressible turbulence has been studied theoretically (Yokoi & Yoshizawa 1993) and in direct numerical simulations (Yokoi & Brandenburg 2016). The theoretical study and numerical simulations show that a non-uniform large-scale flow is produced in the direction of angular velocity. Recent direct numerical simulations have demonstrated formation of large-scale vortices in rapidly rotating turbulent convection for both compressible (Chan 2007; Käpylä, Mantere & Hackman 2011; Mantere, Käpylä & Hackman 2011) and Boussinesq fluids (Favier, Silvers & Proctor 2014; Guervilly, Hughes & Jones 2014; Rubio *et al.* 2014). These large-scale flows consist of depth-invariant, concentrated cyclonic vortices, which form by the merger of convective thermal plumes and eventually grow to the size of the computational domain. Weaker anticyclonic circulations form in their surroundings.

In the present study we develop a mean-field theory of the generation of large-scale vorticity in a rotating density stratified turbulence with inhomogeneous helicity and large Reynolds numbers. To derive the mean-field equation for the vorticity, the spectral τ approach is applied here for a large Reynolds number turbulence. We show that a non-uniform large-scale flow is produced in a rotating fully developed turbulence due to either inhomogeneous kinetic helicity or a combined effect of a density stratified flow and uniform kinetic helicity. An interaction of the turbulence with the formed large-scale shear causes an excitation of the large-scale instability resulting in the generating of the mean vorticity (vorticity dynamo). On the other hand, a fast rotation suppresses this large-scale instability. The present study of the dynamics of large-scale vorticity in a rotating density stratified helical turbulence demonstrates that the mean-field equation for the large-scale vorticity does not contain the $\nabla \times (\alpha \overline{\boldsymbol{W}})$ term as was previously suggested by Moiseev *et al.* (1983).

2. Effect of rotation on the Reynolds stress

To study the effect of rotation on the Reynolds stress in a rotating, density stratified and inhomogeneous turbulence, we apply a mean-field approach and use Reynolds averaging. In the framework of this approach, the velocity and pressure are separated into the mean and fluctuating parts.

2.1. *Equation for velocity fluctuations*

To determine the Reynolds stress, we use equation for fluctuations of velocity $\boldsymbol{u}$, which is obtained by subtracting equation for the mean field from the corresponding equation for the instantaneous field:

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\overline{\boldsymbol{U}} \boldsymbol{\cdot} \nabla)\boldsymbol{u} - (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\overline{\boldsymbol{U}} - \frac{\nabla p}{\rho_0} + 2\boldsymbol{u} \times \boldsymbol{\Omega} + \boldsymbol{U}^N. \tag{2.1}$$

Here p are fluctuations of fluid pressure, $\overline{\boldsymbol{U}}$ is the mean fluid velocity and (2.1) is written in the reference frame rotating with the angular velocity $\boldsymbol{\Omega}$. The fluid velocity for a low Mach number fluid flow satisfies the continuity equation written in the anelastic approximation: $\operatorname{div}(\rho_0 \overline{\boldsymbol{U}}) = 0$ and $\operatorname{div}(\rho_0 \boldsymbol{u}) = 0$. The mean fluid density and pressure with the subscript '0' correspond to the hydrostatic basic reference state, given by the equation: $\nabla P_0 = \rho_0 \boldsymbol{g}$. The nonlinear term $\boldsymbol{U}^N$ which includes the molecular viscous force, $\rho_0 \boldsymbol{F}_\nu(\boldsymbol{u})$, is given by

$$U^N = \langle (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{u} \rangle - (\boldsymbol{u} \boldsymbol{\cdot} \nabla)\boldsymbol{u} + \boldsymbol{F}_\nu(\boldsymbol{u}). \tag{2.2}$$

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The derivation of the equation for the Reynolds stress includes the following steps: (i) use new variables for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0}\,\boldsymbol{u}$; (ii) derivation of the equation for the second moment of the velocity fluctuations $\langle v_i\, v_j\rangle$ in the $\boldsymbol{k}$ space; (iii) application of the spectral τ approach (see § 2.3) and solution of the derived equation for $\langle v_i\, v_j\rangle$ in the $\boldsymbol{k}$ space; (iv) returning to the physical space to obtain the formula for the Reynolds stress as a function of the rotation rate $\varOmega$. Here the angular brackets denote ensemble averaging.

2.2. *Equation for the Reynolds stress*

Applying a multi-scale approach (Roberts & Soward 1975) and using (A 2) derived in appendix A, we obtain an equation for the correlation function: $f_{ij}(\boldsymbol{k}, \boldsymbol{K}, t) = \langle v_i(\boldsymbol{k}_1, t) v_j(\boldsymbol{k}_2, t)\rangle$, where $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$, $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{K}$ correspond to the large scales, and $\boldsymbol{k}$ to the small ones. The equation for $f_{ij}(\boldsymbol{k}, \boldsymbol{R}, t) = \int f_{ij}(\boldsymbol{k}, \boldsymbol{K}, t) \exp(\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R})\, \mathrm{d}\boldsymbol{K}$ is given by

$$\frac{\partial f_{ij}(\boldsymbol{k}, \boldsymbol{R}, t)}{\partial t} = (I^U_{ijmn} + L^{\varOmega}_{ijmn}) f_{mn} + \hat{\mathcal{N}} f_{ij}, \tag{2.3}$$

where

$$\begin{aligned} I^U_{ijmn} = {}& \left[2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + \delta_{im}\delta_{jn}k_q \frac{\partial}{\partial k_p}\right] \nabla_p \overline{U}_q \\ & - \delta_{im}\delta_{jn}\,[\mathrm{div}\,\overline{\boldsymbol{U}} + \overline{\boldsymbol{U}}\boldsymbol{\cdot}\nabla], \end{aligned} \tag{2.4}$$

$$\left.\begin{aligned} L^{\varOmega}_{ijmn} &= \int \left[D^{\varOmega}_{im}(\boldsymbol{k}_1)\,\delta_{jn} + D^{\varOmega}_{jn}(\boldsymbol{k}_2)\,\delta_{im}\right] \exp(\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R})\,\mathrm{d}\boldsymbol{K}, \\ D^{\varOmega}_{ij}(\boldsymbol{k}) &= 2\varepsilon_{ijm}\varOmega_n k_{mn}. \end{aligned}\right\} \tag{2.5}$$

Here δ_{ij} is the Kronecker tensor, $k_{ij} = k_i k_j/k^2$ and ε_{ijk} is the Levi-Civita tensor. The correlation function f_{ij} is proportional to the fluid density $\rho_0(\boldsymbol{R})$ and $\hat{\mathcal{N}} f_{ij}$ are the third-order moments appearing due to the nonlinear terms:

$$\hat{\mathcal{N}} f_{ij} = \int \big(\langle P_{im}(\boldsymbol{k}_1) v^N_m(\boldsymbol{k}_1) v_j(\boldsymbol{k}_2)\rangle + \langle v_i(\boldsymbol{k}_1) P_{jm}(\boldsymbol{k}_2) v^N_m(\boldsymbol{k}_2)\rangle\big) \exp(\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R})\,\mathrm{d}\boldsymbol{K}. \tag{2.6}$$

2.3. τ *approach*

Equation (2.3) for the second-order moment $f_{ij}(\boldsymbol{k})$ contains high-order moments and a closure problem arises (see, e.g. McComb 1990; Monin & Yaglom 2013). To simplify the notation, we do not show the dependencies on t and $\boldsymbol{R}$ in the correlation function f_{ij}. We apply the spectral τ approximation, or the third-order closure procedure (see, e.g. Orszag 1970; Pouquet, Frisch & Léorat 1976; Kleeorin, Rogachevskii & Ruzmaikin 1990; Rogachevskii & Kleeorin 2004). The spectral τ approximation postulates that the deviations of the third-order moment terms, $\hat{\mathcal{N}} f_{ij}(\boldsymbol{k})$, from the contributions to these terms afforded by the background turbulence, $\hat{\mathcal{N}} f^{(0)}_{ij}(\boldsymbol{k})$, are expressed through the similar deviations of the second moments, $f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})$:

$$\hat{\mathcal{N}} f_{ij}(\boldsymbol{k}) - \hat{\mathcal{N}} f^{(0)}_{ij}(\boldsymbol{k}) = -\frac{f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})}{\tau_r(k)}, \tag{2.7}$$

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where $\tau_r(k)$ is the characteristic relaxation time of the statistical moments, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds numbers. In (2.7) the quantities with the superscript (0) correspond to the background turbulence (i.e. a turbulence with $\nabla_i \overline{\boldsymbol{U}} = 0$). We apply the τ-approximation (2.7) only to study the deviations from the background turbulence, which is assumed to be known (see below). Validation of the τ approximation for different situations has been performed in various numerical simulations and analytical studies (Brandenburg & Subramanian 2005; Rogachevskii & Kleeorin 2007; Rogachevskii *et al.* 2011, 2012; Brandenburg *et al.* 2012*a*; Käpylä *et al.* 2012).

3. Effects of rotation and kinetic helicity on the Reynolds stress

In this section we consider a combined effect of rotation and kinetic helicity on the Reynolds stress. To this end we consider a model for the background rotating helical turbulence.

3.1. *Model for the background turbulence*

We use the following model of the background rotating, density stratified and inhomogeneous turbulence with inhomogeneous kinetic helicity:

$$f_{ij}^{(0)} = \frac{E(k)[1+2k\varepsilon_u\delta(\hat{\boldsymbol{k}}\boldsymbol{\cdot}\hat{\boldsymbol{\varOmega}})]}{8\pi k^2(1+\varepsilon_u)} \left\{ \left[\delta_{ij} - k_{ij} + \frac{i}{k^2} \left(\tilde{\lambda}_i k_j - \tilde{\lambda}_j k_i \right) \right] \rho_0 \, \langle \boldsymbol{u}^2 \rangle^{(0)} - \frac{\mathrm{i}}{k^2} \left[\varepsilon_{ijp} \, k_p + \left(\varepsilon_{ipq} k_{jp} + \varepsilon_{jpq} k_{ip} \right) \tilde{\lambda}_q \right] \rho_0 \, \chi^{(0)} \right\}, \tag{3.1}$$

where $\hat{\boldsymbol{\varOmega}} = \boldsymbol{\varOmega}/\varOmega$, $\delta(x)$ is the Dirac delta function, $\chi^{(0)} = \langle \boldsymbol{u}\boldsymbol{\cdot}(\nabla\times\boldsymbol{u})\rangle^{(0)}$ is the kinetic helicity, $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda} - \nabla/2$, $\boldsymbol{\lambda} = -(\nabla\rho_0)/\rho_0$, $\tau(k) = 2\tau_\varOmega \bar{\tau}(k)$, the turbulent correlation time $\tau_\varOmega$ is given below.

We assume that the background turbulence is Kolmogorov-type turbulence with constant fluxes of energy and kinetic helicity over the spectrum, i.e. the kinetic energy spectrum $E(k) = -\mathrm{d}\bar{\tau}(k)/\mathrm{d}k$, the function $\bar{\tau}(k) = (k/k_0)^{1-q}$ with $1 < q < 3$ being the exponent of the kinetic energy spectrum ($q = 5/3$ for the Kolmogorov spectrum), $k_0 = 1/\ell_0$ and ℓ_0 is the integral scale of turbulent motions.

To derive (3.1) we use the following conditions:

(i) the anelastic approximation: $\mathrm{div}(\rho_0\,\boldsymbol{u}) = 0$, which implies that $(\mathrm{i}k_i^{(1)} - \lambda_i)f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = 0$ and $(\mathrm{i}k_j^{(2)} - \lambda_j)f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = 0$, where $\boldsymbol{k}_1 \equiv \boldsymbol{k}^{(1)} = \boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{k}_2 \equiv \boldsymbol{k}^{(2)} = -\boldsymbol{k} + \boldsymbol{K}/2$;

(ii) $\int f_{ii}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) \exp[\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R}]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{K} = \rho_0\,\langle \boldsymbol{u}^2 \rangle^{(0)}$;

(iii) $\mathrm{i}\varepsilon_{ipj} \int k_p^{(2)} f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) \exp[\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R}]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{K} = \rho_0\,\chi^{(0)}$;

(iv) $f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = f_{ji}^{*(0)}(\boldsymbol{k}, \boldsymbol{K}) = f_{ji}^{(0)}(-\boldsymbol{k}, \boldsymbol{K})$.

To introduce anisotropy of the background turbulence due to rotation, we consider an anisotropic turbulence as a combination of a three-dimensional isotropic turbulence and two-dimensional turbulence in the plane perpendicular to the rotational axis. The degree of anisotropy ε_u is defined as the ratio of the turbulent kinetic energies of two-dimensional to three-dimensional motions. In this model we neglect effects which are quadratic in $\boldsymbol{\lambda}$, $\nabla\chi^{(0)}$ and $\nabla\langle \boldsymbol{u}^2 \rangle^{(0)}$. Different contributions in equation (3.1) have been discussed by Batchelor (1953), Elperin, Kleeorin & Rogachevskii (1995), Rädler, Kleeorin & Rogachevskii (2003).

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The effect of rotation on the turbulent correlation time is described just by an heuristic argument. In particular, we assume that $\tau_\Omega^{-2} = \tau_0^{-2} + \Omega^2/C_\Omega^2$, that yields:

$$\tau_\Omega = \frac{\tau_0}{[1 + (C_\Omega^{-1} \Omega \tau_0)^2]^{1/2}}. \tag{3.2}$$

For a fast rotation, $\Omega \tau_0 \gg 1$, the parameter $\Omega \tau_\Omega$ tends to a finite value, $C_\Omega \sim 1$, where $\tau_0 = \ell_0/u_0$ and u_0 is the characteristic turbulent velocity at the integral scale ℓ_0.

3.2. *The Reynolds stress in a rotating and helical turbulence*

In this section we determine the contribution to the Reynolds stress $f_{ij}^{(\Omega,\chi)}$ caused by either rotation and stratification in helical turbulence or rotation and inhomogeneous kinetic helicity. For a slow rotation, $\Omega\,\tau_0 \ll 1$, the function $f_{ij}^{(\Omega,\chi)}$ is given by

$$f_{ij}^{(\Omega,\chi)} = \int \tau \left[\tilde{L}_{ijmn} f_{mn}^{(0,\tilde{\lambda})} + (L_{ijmn}^{\nabla} + L_{ijmn}^{\lambda})\, f_{mn}^{(0,\chi)}\right] \mathrm{d}\boldsymbol{k}, \tag{3.3}$$

where we use (A 12) derived in appendix A, the tensors $f_{ij}^{(0,\chi)} \propto \varepsilon_{ijp}\, k_p \chi^{(0)}$ and $f_{mn}^{(0,\tilde{\lambda})} \propto (\varepsilon_{ipq} k_{jp} + \varepsilon_{jpq} k_{ip}) \tilde{\lambda}_q \chi^{(0)}$ determine corresponding terms in the model (3.1) of the background turbulence, $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda} - \boldsymbol{\nabla}/2$ and all other definitions are given in appendix A. After integration in $\boldsymbol{k}$ space in (3.3), we obtain the contribution to the Reynolds stress $f_{ij}^{(\Omega,\chi)}$ caused by either rotation and stratification in helical turbulence or rotation and inhomogeneous kinetic helicity for a slow rotation, $\Omega\,\tau_0 \ll 1$:

$$f_{ij}^{(\Omega,\chi)} = \frac{(q-1)}{2q} \rho_0 \tau_0 \ell_0^2 \left[\Omega_i \lambda_j + \Omega_j \lambda_i + \frac{4}{15} \left(\Omega_i \nabla_j + \Omega_j \nabla_i\right)\right] \chi^{(0)}. \tag{3.4}$$

For a fast rotation, $\Omega\,\tau_0 \gg 1$, the contribution to the Reynolds stress $f_{ij}^{(\Omega,\chi)}$ caused by either rotation and stratification in helical turbulence or rotation and inhomogeneous kinetic helicity is given by

$$f_{ij}^{(\Omega,\chi)} = \int \tau [L_{ijmn}^{\nabla} + L_{ijmn}^{\lambda}]\, f_{mn}^{(0,\chi)}\, \mathrm{d}\boldsymbol{k}. \tag{3.5}$$

After integration in $\boldsymbol{k}$ space in (3.5), we obtain

$$f_{ij}^{(\Omega,\chi)} = C_\Omega \frac{(q-1)}{4q} \rho_0\, \ell_0^2 \{\hat{\Omega}_i \lambda_j + \hat{\Omega}_j \lambda_i + \hat{\Omega}_i \hat{\Omega}_j [\hat{\boldsymbol{\Omega}} \boldsymbol{\cdot} (\boldsymbol{\lambda} + \boldsymbol{\nabla})]\} \chi^{(0)}, \tag{3.6}$$

where $\hat{\boldsymbol{\Omega}} = \boldsymbol{\Omega}/\Omega$. In the derivation of (3.6) we take into account that the turbulent time τ_Ω for a fast rotation, $\Omega\,\tau_0 \gg 1$ is determined by (3.2). To integrate over the angles in $\boldsymbol{k}$ space for a fast rotation we use the integrals given at the end of appendix A.

3.3. *Formation of the mean velocity shear*

Let us consider the case when the angular velocity, $\boldsymbol{\Omega} = (0, \Omega, 0)$, is perpendicular to the density stratification axes, $\boldsymbol{\lambda} = (\lambda, 0, 0)$. For simplicity, also consider the case when the gradient of the kinetic helicity is parallel to $\boldsymbol{\lambda}$, i.e. $\boldsymbol{\nabla} \chi^{(0)} = (\boldsymbol{\nabla} \chi^{(0)}, 0, 0)$.

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In this case, $f_{xy}^{(\Omega,\chi)}(x)$ is only one non-zero contribution to the Reynolds stress $f_{ij}^{(\Omega,\chi)}$ caused by either rotation and stratification in helical turbulence or rotation and inhomogeneous kinetic helicity for $\Omega\,\tau_0 \ll 1$:

$$f_{xy}^{(\Omega,\chi)}(x) = f_{yx}^{(\Omega,\chi)}(x) = \frac{(q-1)}{2q}\rho_0(x)\,(\Omega\,\tau_0)\ell_0^2\left(\lambda\,\chi^{(0)} + \frac{4}{15}\nabla\chi^{(0)}\right). \tag{3.7}$$

The last term in (3.7) is in agreement with equation (30) of Yokoi & Brandenburg (2016).

The steady-state solution of the momentum equation for the y-component of the mean velocity $\overline{\boldsymbol{U}}^{(S)}$ reads:

$$\nabla_x\left[\rho_0(x)\,\nu_{_T}\left(\nabla_x\overline{U}_y^{(S)}\right) - f_{yx}^{(\Omega,\chi)}(x)\right] = 0, \tag{3.8}$$

where $\nu_{_T} = (q+3)u_0\ell_0/30$ is the turbulent viscosity (Elperin *et al.* 2002) and we take into account that the gradient of the mean pressure along $\boldsymbol{\Omega}$ vanishes. Integrating (3.8) over x, we determine the formed large-scale shear:

$$\overline{S} \equiv \nabla_x\overline{U}_y^{(S)} = \frac{f_{yx}^{(\Omega,\chi)}}{\rho_0\,\nu_{_T}} = \frac{15(q-1)}{2q(q+3)}\Omega\,\tau_0^2\left(\lambda\,\chi^{(0)} + \frac{4}{15}\nabla\chi^{(0)}\right). \tag{3.9}$$

It follows from (3.9) that the large-scale shear is produced in rotating turbulence due to either inhomogeneous kinetic helicity or a combined action of a density stratified flows and uniform kinetic helicity.

In the present study we assume that shear does not affect the background turbulence. For large values of the shear rate, the background turbulence and turbulent correlation time can be affected by the shear. In this case the quenching of the correlation time can be increased by the shear, i.e. the shear can decrease the correlation time (Zhou & Blackman 2017). The inclusion of these effects in the background turbulence is a subject of a separate study. On the other hand, the solution of (2.3) determines the deviations from the background turbulence, and the obtained solution of this equation yields (A 12), that describes the effect of shear on turbulence.

4. Generation of the large-scale vorticity

The formed large-scale shear $\overline{S}$ in a turbulent flow causes an excitation of the large-scale instability resulting in the generation of the mean vorticity due to the vorticity dynamo. The linearised equation for the small perturbations of the mean vorticity is given by

$$\frac{\partial\overline{\boldsymbol{W}}}{\partial t} = \nabla\times\left[\overline{\boldsymbol{U}}^{(S)}\times\overline{\boldsymbol{W}} + \overline{\boldsymbol{U}}\times\overline{\boldsymbol{W}}^{(S)} + 2\overline{\boldsymbol{U}}\times\boldsymbol{\Omega} + \rho_0^{-1}\left(\boldsymbol{\mathcal{F}}^{(S)} + \boldsymbol{\mathcal{F}}^{(\nu_T)}\right)\right], \tag{4.1}$$

where $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{W}}$ are perturbations of the mean velocity and mean vorticity, while $\overline{\boldsymbol{U}}^{(S)} = (0,\ \overline{S}x,\ 0)$ and $\overline{\boldsymbol{W}}^{(S)} = (0,\ 0,\ \overline{S})$ are the equilibrium mean velocity and mean vorticity related to the formed large-scale shear $\overline{S}$, given by (3.9). Here $\mathcal{F}_i^{(S)} = -\nabla_j(\rho_0\,\langle u_i\,u_j\rangle^{(S)})$ is the effective force caused the shear effect on the Reynolds stress, $\boldsymbol{\mathcal{F}}^{(\nu_T)}$ determines the turbulent viscosity, and we neglect small kinematic

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viscosity. Let us consider for simplicity small perturbations of the mean vorticity, $\overline{\boldsymbol{W}}(t, z) = (\overline{W}_x, \overline{W}_y, 0)$, so that (4.1) reads:

$$\frac{\partial \overline{W}_x}{\partial t} = \overline{S}\,\overline{W}_y + \nu_{_T} \overline{W}_x'', \tag{4.2}$$

$$\frac{\partial \overline{W}_y}{\partial t} = -\beta\,\overline{S}\,\ell_0^2\,\overline{W}_x'' - 2\varOmega\lambda \overline{U}_x + \nu_{_T} \overline{W}_y'', \tag{4.3}$$

(see appendix B), where $\overline{W}_i' = \nabla_z \overline{W}_i$ and the coefficient β has been determined in Elperin *et al.* (2003): $\beta = 4(2q^2 - 47q + 108)/315$. For a Kolmogorov energy spectrum ($q = 5/3$), the coefficient $\beta = 0.45$. In (4.2) and (4.3) we take into account the Coriolis force and the density stratification. In the presence of the density stratification due to the gravity field that is directed perpendicular to the angular velocity, we can neglect a weak centrifugal force. In (4.2) we take into account that the characteristic scale of the mean vorticity variations is much larger than the maximum scale of turbulent motions ℓ_0. Since $\overline{W}_y = \overline{U}_x'$, equation (4.3) can be rewritten as

$$\frac{\partial \overline{W}_y'}{\partial t} = -\beta\,\overline{S}\,\ell_0^2\,\overline{W}_x''' - 2\varOmega\lambda \overline{W}_y + \nu_{_T} \overline{W}_y'''. \tag{4.4}$$

We seek a solution of (4.2) and (4.4) in the form $\propto \exp[\gamma t + i(\omega + K_z z)]$, where the growth rate of the large-scale instability and the frequency of the generated waves are given by

$$\gamma = \left[\beta\,(\overline{S}\,\ell_0\,K_z)^2 - \left(\frac{\varOmega\lambda}{K_z}\right)^2\right]^{1/2} - \nu_{_T} K_z^2, \tag{4.5}$$

$$\omega = \frac{\varOmega\lambda}{K_z}. \tag{4.6}$$

Equation (4.5) implies that rotation in a density stratified turbulence decreases the growth rate of the large-scale instability. Since we consider the case when the angular velocity is perpendicular to the wave vector $\boldsymbol{K}$ of the mean vorticity perturbations, large-scale inertial waves are absent in the system. In the absence of rotation and density stratification, the expression (4.5) for the growth rate of the large-scale instability coincides with that obtained by Elperin *et al.* (2003). Equation (4.6) describes three-dimensional slow Rossby waves in rotating density stratified flows which are similar to those studied by Elperin, Kleeorin & Rogachevskii (2017), see equation (28). We remind that the system considered in this study is a three-dimensional one, where the angular velocity, $\boldsymbol{\varOmega} = (0, \varOmega, 0)$, stratification, $\boldsymbol{\lambda} = (\lambda, 0, 0)$ and the wavenumber, $\boldsymbol{K} = (0, 0, K_z)$, are perpendicular each other.

The mechanism of the large-scale instability studied here is as follows. The first term, $\overline{S}\,\overline{W}_y$, in (4.2) describes a stretching of the mean vorticity component $\overline{W}_y$ by non-uniform motions, which produces the component $\overline{W}_x$. On the other hand, the first term, $-\beta\,\overline{S}\,l_0^2\,\overline{W}_x''$, in (4.3) determines a Reynolds stress-induced generation of perturbations of the mean vorticity $\overline{W}_y$ by turbulent Reynolds stresses. In particular, this term is determined by $[\nabla \times (\rho_0^{-1}\,\boldsymbol{\mathcal{F}}^{(S)})]_y$, where $\mathcal{F}_i^{(S)}$ describes the effective force caused the shear effect on the Reynolds stress. The growth rate of the instability is caused by a combined effect of the sheared motions and the Reynolds stress-induced

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generation of perturbations of the mean vorticity (Elperin *et al.* 2003, 2007). On the other hand, the equilibrium large-scale shear is produced either rotating turbulence and inhomogeneous kinetic helicity or a combined effect of a density stratified rotating turbulence and uniform kinetic helicity (see § 3.2).

The physical explanation for why the rotation quenches the vorticity growth is the following. In the presence of the density stratified rotating turbulence, there are three effects: (i) the three-dimensional slow Rossby waves; (ii) the Reynolds stress-induced generation of perturbations of the mean vorticity $\overline{W}_y$; (iii) turbulent viscosity which decreases both the energy of the Rossby waves and the Reynolds stress-induced generation of perturbations of the mean vorticity $\overline{W}_y$. When rotation is fast, the Reynolds stress-induced generation of perturbations of the mean vorticity $\overline{W}_y$ is suppressed. A slow rotation just decreases the latter effect, so there is a competition between the generation of perturbations of the mean vorticity and the Rossby waves.

Note that additional terms in (4.2) and (4.3) caused by a combined effect of kinetic helicity and large-scale shear, are much smaller than the terms which are taken into account in these equations. The combined effects of the uniform kinetic helicity, rotation and stratification or non-uniform kinetic helicity and rotation are only important for the production of the background large-scale velocity shear.

5. Conclusions

In the present study, the following effects are investigated: (i) the effect of density stratification on the production of the large-scale vorticity by the helical rotating turbulence; (ii) the large-scale instability (vorticity dynamo) suggested by Elperin *et al.* (2003) for incompressible non-helical turbulence with a large-scale shear, has been generalised for the case of density stratified rotating and helical turbulence. In particular, we show that the large-scale flow is produced in rotating turbulence due to inhomogeneous kinetic helicity or a combined action of a density stratified flows and uniform kinetic helicity. This results in the formation of a large-scale shear determined by the balance between the turbulent viscous force and the effective force caused by the modification of the Reynolds stress by either rotation and inhomogeneous kinetic helicity or a combined action of rotation and a density stratified turbulence with a uniform kinetic helicity. This large-scale shear interacting with a turbulent flow results in an excitation of the large-scale instability generating the mean vorticity due to the vorticity dynamo, while fast rotation suppresses this instability.

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Appendix A. Derivation and solution of the Reynolds stress equation

In this appendix we derive and solve the equation for the Reynolds stress. To this end, equation (2.1) is rewritten in the new variables for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0}\,\boldsymbol{u}$:

$$\frac{1}{\sqrt{\rho_0}} \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho_0}\right) + \frac{1}{\sqrt{\rho_0}}\left[2\boldsymbol{v}\times\boldsymbol{\Omega} - (\boldsymbol{v}\cdot\nabla)\overline{\boldsymbol{U}} - G^U\,\boldsymbol{v}\right] + \boldsymbol{v}^N, \tag{A 1}$$

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where $G^U = (1/2)\,\mathrm{div}\,\overline{\boldsymbol{U}} + \overline{\boldsymbol{U}} \cdot \nabla$ and $\boldsymbol{v}^N$ are the nonlinear terms which include the molecular viscous terms. The fluid velocity fluctuations $\boldsymbol{v}$ satisfy the equation $\nabla \cdot \boldsymbol{v} = \boldsymbol{\lambda} \cdot \boldsymbol{v}/2$, where $\boldsymbol{\lambda} = -(\nabla \rho_0)/\rho_0$. To derive equation for the Reynolds stress, we rewrite the momentum equation in Fourier space:

$$\frac{\mathrm{d}v_i(\boldsymbol{k})}{\mathrm{d}t} = [D^{\Omega}_{im}(\boldsymbol{k}) + J^U_{im}(\boldsymbol{k})]v_m(\boldsymbol{k}) + v^N_i(\boldsymbol{k}), \tag{A 2}$$

where

$$J^U_{ij}(\boldsymbol{k}) = 2k_{in}\nabla_j \overline{U}_n - \nabla_j \overline{U}_i - \left[\frac{1}{2}\,\mathrm{div}\,\overline{\boldsymbol{U}} + \mathrm{i}(\overline{\boldsymbol{U}} \cdot \boldsymbol{k})\right]\delta_{ij}. \tag{A 3}$$

To derive (A 2), we multiply the momentum equation written in $\boldsymbol{k}$ space by $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_{ij}$ to exclude the pressure term from the equation of motion. Here we also use the following identities:

$$\sqrt{\rho_0}\,[\nabla \times [\nabla \times (\boldsymbol{u} \times \boldsymbol{\Omega})]] = (\boldsymbol{\Omega} \times \nabla^{(\lambda)})\,(\boldsymbol{\lambda} \cdot \boldsymbol{v}) + (\boldsymbol{\Omega} \cdot \nabla^{(\lambda)})\,(\nabla^{(\lambda)} \times \boldsymbol{v}), \tag{A 4}$$

$$\sqrt{\rho_0}\,\left[\nabla \times [\nabla \times \boldsymbol{u}]\right]_{\boldsymbol{k}} = -[\Lambda^2\,\delta_{ij} - \Lambda_i\lambda_j]v_j(\boldsymbol{k}), \tag{A 5}$$

where $\nabla^{(\lambda)} = \nabla + \boldsymbol{\lambda}/2$ and $\boldsymbol{\Lambda} = \mathrm{i}\boldsymbol{k} + \boldsymbol{\lambda}/2$.

To derive the equation for the Reynolds stress, we apply a standard multi-scale approach (Roberts & Soward 1975), i.e. the non-instantaneous two-point second-order correlation function is written as follows:

$$\begin{aligned}\langle v_i(\boldsymbol{x}, t)\, v_j(\boldsymbol{y}, t)\rangle &= \int \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 \langle v_i(\boldsymbol{k}_1, t)v_j(\boldsymbol{k}_2, t)\rangle \exp[\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{x} + \boldsymbol{k}_2 \cdot \boldsymbol{y})] \\ &= \int f_{ij}(\boldsymbol{k}, \boldsymbol{R}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})\, \mathrm{d}\boldsymbol{k},\end{aligned} \tag{A 6}$$

where we use large-scale variables: $\boldsymbol{R} = (\boldsymbol{x} + \boldsymbol{y})/2$ and $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$; and small-scale variables: $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, and $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$. Mean fields depend on the large-scale variables, while fluctuations depend on the small-scale variables, where

$$f_{ij}(\boldsymbol{k}, \boldsymbol{R}, t) = \int \langle v_i(\boldsymbol{k}_1, t)\, u_j(\boldsymbol{k}_2, t)\rangle \exp(\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{R})\, \mathrm{d}\boldsymbol{K}, \tag{A 7}$$

$\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$. Applying a multi-scale approach and using (A 2), we derive an equation for the correlation function: $f_{ij}(\boldsymbol{k}, \boldsymbol{R}, t)$, see (2.3), where $I^U_{ijmn} = \int \left(J^U_{im}(\boldsymbol{k}_1)\,\delta_{jn} + J^U_{jn}(\boldsymbol{k}_2)\,\delta_{im}\right) \exp(\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{R})\, \mathrm{d}\boldsymbol{K}$.

To solve (2.3) we extract in tensor L^{Ω}_{ijmn} the parts which depend on large-scale spatial derivatives or the density stratification effects, i.e.

$$L^{\Omega}_{ijmn} = \tilde{L}_{ijmn} + L^{\nabla}_{ijmn} + L^{\lambda}_{ijmn} + O(\lambda^2, \nabla^2), \tag{A 8}$$

where

$$\tilde{L}_{ijmn} = 2\Omega_q(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})k_{pq}, \quad L^{\nabla}_{ijmn} = -2\Omega_q(\varepsilon_{imp}\delta_{jn} - \varepsilon_{jnp}\delta_{im})k^{\nabla}_{pq}, \tag{A 9a,b}$$

$$L^{\lambda}_{ijmn} = -2\,\Omega_q\left[(\varepsilon_{imp}\delta_{jn} - \varepsilon_{jnp}\delta_{im})\,k^{\lambda}_{pq} + \frac{\mathrm{i}}{k^2}(\varepsilon_{ilq}\delta_{jn}\lambda_m - \varepsilon_{jlq}\delta_{im}\lambda_n)\,k_l\right], \tag{A 10}$$

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$$k_{ij}^{\nabla} = \frac{\mathrm{i}}{2k^2}\,[k_i \nabla_j + k_j \nabla_i - 2k_{ij}(\boldsymbol{k}\boldsymbol{\cdot}\nabla)], \quad k_{ij}^{\lambda} = \frac{\mathrm{i}}{2k^2}\,[k_i \lambda_j + k_j \lambda_i - 2k_{ij}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\lambda})]. \tag{A 11a,b}$$

Equation (2.3) in a steady state and after applying the spectral τ approximation (2.7), reads

$$f_{ij}(\boldsymbol{k}) = L^{-1}_{ijmn}\left[f^{(0)}_{mn} + \tau\,(I^{U}_{mnpq} + L^{\nabla}_{mnpq} + L^{\lambda}_{mnpq}) f_{pq}\right], \tag{A 12}$$

where we neglected terms $\sim O(\nabla^2, \lambda^2)$. Here the operator $L^{-1}_{ijmn}(\boldsymbol{\Omega})$ is the inverse of $\delta_{im}\delta_{jn} - \tau\,\tilde{L}_{ijmn}$, and it is given by

$$\begin{aligned} L^{-1}_{ijmn}(\boldsymbol{\Omega}) \;=\; & \tfrac{1}{2}\big[B_1\delta_{im}\delta_{jn} + B_2\,k_{ijmn} + B_3(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})\hat{k}_p + B_4(\delta_{im}k_{jn} + \delta_{jn}k_{im}) \\ & + B_5\varepsilon_{ipm}\varepsilon_{jqn}k_{pq} + B_6(\varepsilon_{imp}k_{jpn} + \varepsilon_{jnp}k_{ipm})\big], \end{aligned} \tag{A 13}$$

where $\hat{k}_i = k_i/k$, $B_1 = 1 + \phi(2\psi)$, $B_2 = B_1 + 2 - 4\phi(\psi)$, $B_3 = 2\psi\,\phi(2\psi)$, $B_4 = 2\phi(\psi) - B_1$, $B_5 = 2 - B_1$, $B_6 = 2\psi\,[\phi(\psi) - \phi(2\psi)]$, $\phi(\psi) = 1/(1+\psi^2)$ and $\psi = 2\tau(k)\,(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\Omega})/k$. Note that for a slow rotation, $L^{-1}_{ijmn}(\boldsymbol{\Omega}) = \delta_{im}\delta_{jn} + \tau\tilde{L}_{ijmn}$.

To integrate in (3.5) over the angles in $\boldsymbol{k}$ space for a fast rotation, we use the following integrals:

$$\int k_{ij}^{\perp}\,\mathrm{d}\varphi = \pi\delta^{(2)}_{ij}, \quad \int k_{ijmn}^{\perp}\,\mathrm{d}\varphi = \frac{\pi}{4}\Delta^{(2)}_{ijmn}, \tag{A 14a,b}$$

where $\delta^{(2)}_{ij} \equiv P_{ij}(\Omega) = \delta_{ij} - \Omega_i\Omega_j/\Omega^2$ and $\Delta^{(2)}_{ijmn} = \delta^{(2)}_{ij}\delta^{(2)}_{mn} + \delta^{(2)}_{im}\delta^{(2)}_{jn} + \delta^{(2)}_{in}\delta^{(2)}_{jm}$.

Appendix B. Effect of shear on Reynolds stress

There are two effects of shear on Reynolds stress. The first effect is related to the contribution due to the turbulent viscosity: $\langle u_i\,u_j\rangle^{(\nu_T)} = -2\nu_{_T}\,(\partial\overline{U})_{ij}$, and the second contribution determines the Reynolds stress-induced generation of perturbations of mean vorticity by the effect of large-scale shear on turbulence (Elperin *et al.* 2003):

$$\langle u_i\,u_j\rangle^{(S)} = -l_0^2[4C_1\,M_{ij} + C_2(N_{ij} + H_{ij}) + C_3\,G_{ij}], \tag{B 1}$$

where $\nu_{_T}$ is the turbulent viscosity, $(\partial\overline{U})_{ij} = (\nabla_i\overline{U}_j + \nabla_j\overline{U}_i)/2$,

$$M_{ij} = (\partial\overline{U}^{(S)})_{im}(\partial\overline{U})_{mj} + (\partial\overline{U}^{(S)})_{jm}(\partial\overline{U})_{mi}, \quad G_{ij} = \overline{W}^{(S)}_i\overline{W}_j + \overline{W}^{(S)}_j\overline{W}_i, \tag{B 2a,b}$$

$$H_{ij} = \overline{W}^{(S)}_n[\varepsilon_{nim}(\partial\overline{U})_{mj} + \varepsilon_{njm}(\partial\overline{U})_{mi}], \quad N_{ij} = \overline{W}_n[\varepsilon_{nim}(\partial\overline{U}^{(S)})_{mj} + \varepsilon_{njm}(\partial\overline{U}^{(S)})_{mi}], \tag{B 3a,b}$$

$\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{W}}$ are perturbations of the mean velocity and mean vorticity, while $\overline{\boldsymbol{U}}^{(S)} = (0,\ \overline{S}x,\ 0)$ and $\overline{\boldsymbol{W}}^{(s)} = (0,\ 0,\ \overline{S})$ are the equilibrium mean velocity and mean vorticity related to shear $\overline{S}$, the coefficients, $C_1 = 8(q^2 - 13q + 40)/315$, $C_2 = 2(6 - 7q)/45$, $C_3 = -2(q + 2)/45$, depend on the exponent of the energy spectrum. When small perturbations of the mean velocity, $\overline{\boldsymbol{U}}(t,\,z) = (\overline{U}_x,\ \overline{U}_y,\ 0)$ and

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the mean vorticity, $\overline{\boldsymbol{W}}(t, z) = (\overline{W}_x, \overline{W}_y, 0)$, depend only on z, the effective force $\rho_0^{-1}\mathcal{F}_i^{(S)} = -\nabla_j \langle u_i u_j \rangle^{(S)}$ is given by

$$\rho_0^{-1}\,\mathcal{F}_i^{(S)} = -\overline{S}\ell_0^2(\beta \overline{W}'_x,\, \beta_0\,\overline{W}'_y,\, 0), \tag{B 4}$$

where $\beta = C_1 + C_2 - C_3$ and $\beta_0 = C_2/2 - C_1 - C_3$. Here we used the following identities:

$$\nabla_j M_{ij} = -(\overline{S}/4)(\overline{W}'_x,\, -\overline{W}'_y,\, 0), \quad \nabla_j N_{ij} = -(\overline{S}/2)(\overline{W}'_x,\, 0,\, 0), \tag{B 5a,b}$$

$$\nabla_j H_{ij} = -(\overline{S}/2)(\overline{W}'_x,\, \overline{W}'_y,\, 0), \quad \nabla_j G_{ij} = \overline{S}(\overline{W}'_x,\, \overline{W}'_y,\, 0). \tag{B 6a,b}$$

Equation (B 4) yields the first term in the right-hand side of (4.3), see (4.1).

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