THE JOURNAL OF SYMBOLIC LOGIC
Volume 83, Number 2, June 2018

ABSTRACT ω-LIMIT SETS

WILL BRIAN

Abstract. The shift map σ on ω^* is the continuous self-map of ω^* induced by the function $n \mapsto n+1$ on ω. Given a compact Hausdorff space X and a continuous function $f : X \to X$, we say that (X, f) is a quotient of (ω^*, σ) whenever there is a continuous surjection $Q : \omega^* \to X$ such that $Q \circ \sigma = \sigma \circ f$.

Our main theorem states that if the weight of X is at most $\aleph_1$, then (X, f) is a quotient of (ω^*, σ) if and only if f is weakly incompressible (which means that no nontrivial open $U \subseteq X$ has $f(\overline{U}) \subseteq U$). Under CH, this gives a complete characterization of the quotients of (ω^*, σ) and implies, for example, that (ω^*, σ^{-1}) is a quotient of (ω^*, σ).

In the language of topological dynamics, our theorem states that a dynamical system of weight $\aleph_1$ is an abstract ω-limit set if and only if it is weakly incompressible.

We complement these results by proving (1) our main theorem remains true when $\aleph_1$ is replaced by any $\kappa < \mathfrak{p}$, (2) consistently, the theorem becomes false if we replace $\aleph_1$ by $\aleph_2$, and (3) $\mathsf{OCA} + \mathsf{MA}$ implies that (ω^*, σ^{-1}) is not a quotient of (ω^*, σ).

§1. Introduction.

In [20], Parovičenko proved that every compact Hausdorff space of weight $\aleph_1$ is a continuous image of $\omega^* = \beta\omega - \omega$. In this article we prove the analogous result concerning the continuous maps on ω^* that respect the shift map.

The *shift map* $\sigma : \beta\omega \to \beta\omega$ sends an ultrafilter p to the unique ultrafilter generated by $\{A + 1 : A \in p\}$. Equivalently, σ is the unique map on $\beta\omega$ that continuously extends the map $n \mapsto n+1$ on ω. The shift map restricts to an autohomeomorphism of ω^*.

If X is a compact Hausdorff space and $f : X \to X$ is continuous, we say that (X, f) is a *quotient* of (ω^*, σ) whenever there is a continuous surjection $Q : \omega^* \to X$ such that $Q \circ \sigma = f \circ Q$. The main theorem of this article characterizes the quotients of (ω^*, σ) that have weight at most $\aleph_1$:

MAIN THEOREM. *Suppose X is a compact Hausdorff space with weight at most $\aleph_1$, and $f : X \to X$ is continuous. Then (X, f) is a quotient of (ω^*, σ) if and only if f is weakly incompressible.*

Recall that $f : X \to X$ is *weakly incompressible* if for any open $U \subseteq X$ with $\emptyset \neq U \neq X$, we have $f(\overline{U}) \not\subseteq U$. This theorem is the appropriate analogue of

Received September 12, 2016.
2010 *Mathematics Subject Classification.* Primary 54H20, Secondary, 03C98, 03E35, 37B05, 54G05.
Key words and phrases. Stone–Čech compactification, shift map, Parovičenko's theorem, abstract omega-limit sets, weak incompressibility, Continuum Hypothesis, elementary submodels, Martin's Axiom.

0022-4812/18/8302-0005
DOI:10.1017/jsl.2018.11

Parovičenko's because (ω^*, σ) is itself weakly incompressible, and this property is always preserved by taking quotients. In other words, our theorem isolates a simple property of the shift map that determines exactly when Parovičenko's topological result extends to a result of dynamics.

In order to understand the motivation for this theorem, and why we are paying such special attention to the shift map as opposed to some other continuous function $\omega^* \to \omega^*$, let us look to topological dynamics.

1.1. Connection with topological dynamics. A *dynamical system* is a pair (X, f), where X is a compact Hausdorff space and $f : X \to X$ is continuous. Such things have been studied intensively as models of time-dependent processes (we think of f as acting on X and being iterated repeatedly, and then we ask about the long-term behavior of the system). An important notion in this field of study is that of an *ω-limit set*.

Given a point $x \in X$, the ω-limit set of x is the set of all limit points of the orbit of x:

$$\omega_f(x) = \bigcap_{n \in \omega} \overline{\{f^m(x) \colon m \geq n\}}.$$

It is easy to see that $\omega_f(x)$ is closed under f, so that $(\omega_f(x), f)$ is itself a dynamical system. The structure of this system captures the topological behavior of the orbit of x.

Recall that two dynamical systems (X, f) and (Y, g) are *isomorphic* (or, for some authors, *conjugate*) if there is a homeomorphism $H : X \to Y$ with $H \circ f = g \circ H$. An *abstract ω-limit set* is a dynamical system that is isomorphic to a dynamical system of the form $(\omega_f(x), f)$.

For example, (ω^*, σ) is an abstract ω-limit set because $\omega^* = \omega_\sigma(n)$ for any $n \in \omega$ in the larger dynamical system $(\beta\omega, \sigma)$. Notice that ω^* is not an ω-limit set "internally"; that is, $\omega^* \neq \omega_\sigma(p)$ for any $p \in \omega^*$ (indeed, ω^* is not even separable). In order to realize (ω^*, σ) as an ω-limit set, it is necessary to extend it to a larger dynamical system.

A somewhat vague but very natural question is: *What do ω-limit sets look like*? Or, to put it a bit more precisely: *Is there a useful or simple characterization of abstract ω-limit sets*? Our main theorem is connected to these questions through the following result, proved in the next section:

THEOREM 2.4. *A dynamical system is an abstract ω-limit set if and only if it is a quotient of (ω^*, σ).*

In other words, (ω^*, σ) is universal (in the "mapping onto" sense) among all abstract ω-limit sets. Thus our main theorem is a characterization of abstract ω-limit sets that are not too large in weight:

MAIN THEOREM (VERSION TWO). *Suppose (X, f) is a dynamical system and the weight of X is at most $\aleph_1$. (X, f) is an abstract ω-limit set if and only if f is weakly incompressible.*

This way of stating the main theorem reveals it as an extension of the following well-known result of Bowen and Sharkovsky:

THEOREM 2.6. *A metrizable dynamical system is an abstract ω-limit set if and only if it is weakly incompressible.*

Sharkovsky proves the forward direction in [21] and Bowen proves the converse in [6]. See [2] or [17], and the references therein, for further research on the connection between weak incompressibility and ω-limit sets.

1.2. Outline of the proof. Of the various proofs of Parovičenko's theorem, ours is closest in spirit to that of Błaszczyk and Szymański in [5]. Their proof begins by writing a given compact Hausdorff space X as a length-ω_1 inverse limit of compact metrizable spaces: $X = \varprojlim \langle X_\alpha : \alpha < \omega_1 \rangle$. They then construct a coherent transfinite sequence of continuous surjections $Q_\alpha : \omega^* \to X_\alpha$, and define $Q : \omega^* \to X$ to be the inverse limit of this sequence. The Q_α are constructed recursively, using a variant of the following lifting lemma at successor stages:

Lemma 1.1. *Let Y and Z be compact metrizable spaces, and let $Q_Z : \omega^* \to Z$ and $\pi : Y \to Z$ be continuous surjections. Then there is a continuous surjection $Q_Y : \omega^* \to Y$ such that $Q_Z = \pi \circ Q_Y$.*

In our situation, the first part of Błaszczyk and Szymański's proof goes through: we prove in Corollary 3.3 below that given a dynamical system (X, f) of weight $\aleph_1$, one may always write (X, f) as a length-ω_1 inverse limit of metrizable dynamical systems. However, we run into trouble with the analogue of Lemma 1.1: the analogous lemma for dynamical systems is false (see Example 3.4).

To get around this problem, we modify Błaszczyk and Szymański's approach by using sharper tools. Rather than beginning with (X, f) and writing it as a topological inverse limit, we begin with a particular embedding of X in $[0, 1]^{\omega_1}$ and use a much stronger form of inverse limit: a continuous chain of elementary submodels of a sufficiently large fragment of the set-theoretic universe. Each model in our chain naturally gives rise to a metrizable "reflection" of (X, f), and the continuity requirement organizes these reflections into an inverse limit system with limit (X, f). Elementarity gives this system strong structural properties, and ultimately is the key that unlocks a workable analogue of Lemma 1.1.

Our use of elementarity is inspired by the work of Dow and Hart in [9], where they prove that every continuum of weight $\aleph_1$ is a continuous image of $\mathbb{H}^*$, the Stone–Čech remainder of $\mathbb{H} = [0, \infty)$. They give three proofs of this fact, each of which relies on model-theoretic notions in some essential way. The proof of our main theorem is most similar to their third proof, found in Section 3 of [9].

In Section 5, we will show that both Parovičenko's theorem about continuous images of ω^* and the Dow–Hart theorem about continuous images of $\mathbb{H}^*$ can be derived as relatively straightforward corollaries of our main theorem. In light of this, it is unsurprising that our proof uses some of the same ideas found in [5] and [9].

1.3. Extensions and limitations. Under the Continuum Hypothesis, our result gives a complete characterization of the quotients of (ω^*, σ):

Theorem 5.5. *Assuming* CH, *the following are equivalent*:

(1) *(X, f) is a quotient of (ω^*, σ).*
(2) *X has weight at most $\mathfrak{c}$ and f is weakly incompressible.*
(3) *X is a continuous image of ω^* and f is weakly incompressible.*

Every quotient of (ω^*, σ) is weakly incompressible, so (3) gives the most liberal possible characterization of quotients of (ω^*, σ): they are the weakly incompressible dynamical systems for which the topology is not an obstruction. A corollary to this is that (ω^*, σ^{-1}) is a quotient of (ω^*, σ).

In Section 5, we show that the nontrivial conclusions of Theorem 5.5 are independent of ZFC. Specifically, we show that (2) does not imply (1) or (3) in the Cohen model, and that (3) does not imply (1) under $\mathsf{OCA} + \mathsf{MA}$. In fact, we will show under $\mathsf{OCA} + \mathsf{MA}$ that (ω^*, σ^{-1}) is not a quotient of (ω^*, σ), even though σ^{-1} is weakly incompressible.

We also show in Section 5 that if $\kappa < \mathfrak{p}$ then our main theorem holds with κ in the place of $\aleph_1$:

THEOREM 5.10. *If the weight of X is less than $\mathfrak{p}$, then (X, f) is a quotient of (ω^*, σ) if and only if f is weakly incompressible.*

In the same way that our main theorem is the dynamical analogue of Parovičenko's theorem, this result is the dynamical analogue of the following result of van Douwen and Przymusiński [8]: *If X is a compact Hausdorff space with weight less than $\mathfrak{p}$, then X is a continuous image of ω^*.*

§2. First steps.

2.1. Extending maps from ω to $\beta\omega$. If X is a compact Hausdorff space and $f : \omega \to X$ is any function, then there is a unique continuous function $\beta f : \beta\omega \to X$ that extends f, the *Stone extension* of f. For a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of points in X and $p \in \beta\omega$, we will usually write $p\text{-}\lim_{n\in\omega} x_n$ for the image of p under the Stone extension of the function $n \mapsto x_n$. We will need the following facts about Stone extensions (proofs can be found in Chapter 3 of [14]):

LEMMA 2.1. *Let X be a compact Hausdorff space and $\langle x_n : n < \omega\rangle$ a sequence of points in X.*

(1) *$p\text{-}\lim_{n\in\omega} x_n = y$ if and only if for every open $U \ni y$ we have $\{n : x_n \in U\} \in p$.*
(2) *$p \mapsto p\text{-}\lim_{n\in\omega} x_n$ is a continuous function $\beta\omega \to X$.*
(3) *If $f : X \to X$ is continuous and $p \in \beta\omega$, then*

$$f(p\text{-}\lim_{n\in\omega} x_n) = p\text{-}\lim_{n\in\omega} f(x_n).$$

(4) *For each $p \in \beta\omega$, $\sigma(p)\text{-}\lim_{n\in\omega} x_n = p\text{-}\lim_{n\in\omega} x_{n+1}$.*

2.2. Extending maps from ω^* to $\beta\omega$. The following result is a fairly straightforward consequence of the Tietze Extension Theorem; a proof can be found in [10], Theorem 3.5.13.

LEMMA 2.2. *Suppose X is a compact Hausdorff space and $f : \omega^* \to X$ is continuous. Then there is a compact Hausdorff space $Y \supseteq X$, such that $Y \setminus X = \omega$, and the function $F : \beta\omega \to Y$ defined by setting $F \restriction \omega^* = f$ and $F \restriction \omega = \mathrm{id}_\omega$ is continuous.*

LEMMA 2.3. *Let (X, f) be a dynamical system, and $Q : X \to Y$ a continuous surjection such that, for all $x_1, x_2 \in X$, if $Q(x_1) = Q(x_2)$ then $Q(f(x_1)) = Q(f(x_2))$. Then there is a unique continuous $g : Y \to Y$ such that $g \circ Q = Q \circ f$.*

PROOF. The assumptions about Q imply that there is a unique function $g : Y \to Y$ such that $g \circ Q = Q \circ f$, namely $g(y) = Q(f(Q^{-1}(y)))$. We need to check that this function is continuous.

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

If K is a closed subset of Y, then $f^{-1}(Q^{-1}(K))$ is closed in X. Because X is compact, $f^{-1}(Q^{-1}(K))$ is compact, which implies $g^{-1}(K) = Q(f^{-1}(Q^{-1}(K)))$ is closed. Since K was arbitrary, g is continuous. ⊣

THEOREM 2.4. *(X, f) is an abstract ω-limit set if and only if it is a quotient of (ω^*, σ).*

PROOF. It is well known that if (X, f) is an ω-limit set then it is a quotient of (ω^*, σ). Indeed, the map $p \mapsto p\text{-}\lim_{n\in\omega} f^n(x)$ gives a quotient mapping from (ω^*, σ) to $(\omega_f(x), f)$. For details and some discussion, see Section 2 of [4]. Here we need to prove the converse.

Suppose $q : \omega^* \to X$ is a quotient mapping from (ω^*, σ) to (X, f). Using Lemma 2.2, there is a compact Hausdorff space $Y \supseteq X$ with $Y \setminus X = \omega$ such that q extends to a continuous function $Q : \beta\omega \to Y$, with $Q \restriction \omega = \mathrm{id}_\omega$. Define $g : Y \to Y$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in X, \\ n+1 & \text{if } y = n \in \omega. \end{cases}$$

Clearly $Q \circ \sigma = g \circ Q$, and g is continuous by Lemma 2.3.

To finish the proof, we will show that in (Y, g), $X = \omega_g(p)$. Notice

$$\{g^m(0) \colon m \geq n\} = \{m \colon m \geq n\}$$

for all n. Using the continuity of Q, we have

$$\overline{\{m \colon m \geq n\}}^{Y} \supseteq Q\Big(\overline{\{m \colon m \geq n\}}^{\beta\omega}\Big) \supseteq Q(\omega^*) = X.$$

Thus $\omega_g(0) \supseteq X$, and the reverse inclusion is obvious. ⊣

2.3. Chain transitivity. In this subsection we give a different but equivalent characterization of weak incompressibility. This characterization (which, for historical reasons, has a different name) is more difficult to state, but will prove more useful in what follows.

Suppose (X, f) is a dynamical system and d is a metric for X. An ε-*chain* in (X, f) is a sequence $\langle x_i \colon i \leq n\rangle$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i < n$. Roughly, an ε-chain is a piece of an orbit, but computed with a small error at each step. (X, f) is called *chain transitive* if for any $a, b \in X$ and any $\varepsilon > 0$, there is an ε-chain beginning at a and ending at b.

Using open covers in the place of ε-balls, we can reformulate this classical definition of chain transitivity so that it applies to nonmetrizable dynamical systems. Given (X, f) and an open cover $\mathcal{U}$ of X, we say that $\langle x_i \colon i \leq n\rangle$ is a $\mathcal{U}$-*chain* if, for every $i < n$, there is some $U \in \mathcal{U}$ such that $f(x_i), x_{i+1} \in U$. A dynamical system (X, f) is *chain transitive* if for any $a, b \in X$ and any open cover $\mathcal{U}$ of X, there is a $\mathcal{U}$-chain beginning at a and ending at b.

LEMMA 2.5.

(1) *A dynamical system is chain transitive if and only if it is weakly incompressible.*
(2) *Every quotient of (ω^*, σ) is weakly incompressible.*

The proof of (1) is essentially the same as the proof for metrizable dynamical systems (see, e.g., Theorem 4.12 in [1]). Proofs of both (1) and (2) can be found in Section 5 of [7].

2.4. The Bowen–Sharkovsky theorem. In the interest of keeping this article self-contained, we will now sketch a proof of the theorem of Bowen and Sharkovsky mentioned in the introduction.

THEOREM 2.6 (Bowen–Sharkovsky). *A metrizable dynamical system is an abstract ω-limit set if and only if it is weakly incompressible.*

PROOF SKETCH. The forward direction is a consequence of Theorem 2.4 and Lemma 2.5. To prove the reverse direction, we will use chain transitivity instead of weak incompressibility.

Let (X, f) be a metrizable, chain transitive dynamical system. Fix a sequence $\langle d_n : n \in \omega \rangle$ of points in X such that every tail of the sequence is dense in X. For each $n \in \omega$, we may use chain transitivity to connect d_n to d_{n+1} via a $\frac{1}{n+1}$-chain, resulting in an infinite sequence

$$\langle d_0, x_1^0, \dots, x_{\ell_0}^0, d_1, x_1^1, \dots, x_{\ell_1}^1, d_2, x_1^2, \dots, x_{\ell_2}^2, d_3, \dots \rangle.$$

Re-indexing this sequence as $\langle y_n : n \in \omega \rangle$ and defining $Q : \omega^* \to X$ to be the Stone extension of the map $n \mapsto y_n$, one may check that Q is a quotient mapping from (ω^*, σ) to (X, f). By Theorem 2.4, (X, f) is an abstract ω-limit set. ⊣

§3. A few lemmas.

In this section we begin the proof of our main theorem in the form of several lemmas (the main part of the proof is in the next section). The purpose of these lemmas is to give a detailed description of which functions on ω induce quotient mappings on ω^*.

Given an ordinal δ, the *standard basis* for $[0,1]^\delta$ is the basis generated by sets of the form $\pi_\alpha^{-1}(p, q)$, where $p, q \in [0,1] \cap \mathbb{Q}$ and π_α is the projection mapping a point of $[0,1]^\delta$ to its α^{th} coordinate. Whenever we mention basic open subsets of $[0,1]^\delta$, this is the basis we mean. Notice that every basic open subset of $[0,1]^\delta$ can be defined using finitely many ordinals less than δ and finitely many rational numbers.

Suppose X is a closed subset of $[0,1]^\delta$. By an *open cover* of X, we will mean a set $\mathcal{U}$ of open subsets of $[0,1]^\delta$ with $X \subseteq \bigcup \mathcal{U}$. A *nice open cover* of X is a finite open cover $\mathcal{U}$ of X consisting of finitely many basic open subsets of $[0,1]^\delta$, such that $U \cap X \neq \emptyset$ for all $U \in \mathcal{U}$.

If $\mathcal{U}$ is a collection of subsets of $[0,1]^\delta$ and $A \subseteq [0,1]^\delta$,

$$\mathcal{U}_\star(A) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

For convenience, if $A = \{a\}$ we write $\mathcal{U}_\star(a)$ instead of $\mathcal{U}_\star(\{a\})$.

If $\mathcal{U}$ and $\mathcal{V}$ are collections of open sets, recall that $\mathcal{U}$ *refines* $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$. $\mathcal{U}$ is a *star refinement* of $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ such that $\mathcal{U}_\star(U) \subseteq V$. It is known (see, e.g., Theorem 5.1.12 in [10]) that every open cover of a compact Hausdorff space has a star refinement.

LEMMA 3.1. *Let X be a closed subset of $[0,1]^\delta$. A function $f : X \to X$ is continuous if and only if for every open cover $\mathcal{U}$ of X there is a nice open cover $\mathcal{V}$ of X such that*

$$\{\mathcal{V}_\star(f(\mathcal{V}_\star(x) \cap X)) : x \in X\}$$

is an open cover of X that refines $\mathcal{U}$.

The proof of this lemma is an elementary exercise in general topology, and we omit it.

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

Given a countable ordinal δ, define $\Pi_\delta : [0,1]^{\omega_1} \to [0,1]^\delta$ to be the natural projection onto the first δ coordinates, namely $\Pi_\delta = \Delta_{\alpha<\delta}\pi_\alpha$. A form of the following lemma was proved by Noble and Ulmer in [19], and later rediscovered by Shchepin in [22].

Lemma 3.2. *Let X be a closed subset of $[0,1]^{\omega_1}$ and let $f : X \to X$ be continuous. There is a closed unbounded $C \subseteq \omega_1$ such that for every $\delta \in C$ and $x, y \in X$, if $\Pi_\delta(x) = \Pi_\delta(y)$ then $\Pi_\delta(f(x)) = \Pi_\delta(f(y))$.*

Corollary 3.3. *Every dynamical system of weight $\aleph_1$ can be written as an inverse limit of metrizable dynamical systems.*

Proof. Let (X, f) be a dynamical system of weight $\aleph_1$. Embed X in $[0,1]^{\omega_1}$, and let C be the closed unbounded set of ordinals described in the previous lemma. For each $\delta \in C$, let $X_\delta = \Pi_\delta(X)$ and define $f_\delta : X_\delta \to X_\delta$ by $f_\delta(\Pi_\delta(x)) = \Pi_\delta(f(x))$, which is continuous by Lemma 2.3. Then $\langle(\Pi_\delta(X), f_\delta) \colon \delta \in C\rangle$ is an inverse limit system, having the natural projections as bonding maps, and the limit of this system is (X, f). $\dashv$

Before moving on to our next lemma, we take a moment to justify the use of elementary submodels in the next section. Naïvely, one might wonder why we cannot simply prove our main theorem in the style of Błaszczyk and Szymański, using Corollary 3.3 and the appropriate analogue of Lemma 1.1:

($*$) *Let (Y, g) and (Z, h) be metrizable dynamical systems, and suppose $Q_Z : \omega^* \to Z$ and $\pi : Y \to Z$ are quotient mappings. Then there is a quotient mapping $Q_Y : \omega^* \to Y$ such that $Q_Z = \pi \circ Q_Y$.*

The following example shows that ($*$) is not true, so that we need more than a simple topological inverse limit structure in order to make Błaszczyk and Szymański's proof go through. We will simply sketch the example and leave detailed proofs to the reader.

Example 3.4. $([0,1], \mathrm{id})$ is a weakly incompressible dynamical system, and for our example it will play the role of both (Y, g) and (Z, h) in ($*$). Define $\pi : [0,1] \to [0,1]$ by setting $\pi(0) = 0$, $\pi(\frac{2}{3}) = 1$, and $\pi(1) = \frac{1}{2}$, and then extending π linearly to the rest of $[0,1]$. We will define a quotient mapping π_Z from (ω^*, σ) to $([0,1], \mathrm{id})$ that does not lift through π.

Define $p_Z : \omega \to [0,1]$ so that $p_Z(n)$ is the distance from $s(n) = \sum_{m \leq n} \frac{1}{m}$ to the nearest even integer. Letting $\pi_Z : \omega^* \to [0,1]$ be the map induced by p_Z, it is easy to check (using Lemma 3.5 below) that π_Z is a quotient mapping from (ω^*, σ) to $([0,1], \mathrm{id})$.

Suppose $\pi_Y : \omega^* \to [0,1]$ is another quotient mapping from (ω^*, σ) to $([0,1], \mathrm{id})$. By the Tietze Extension Theorem, π_Y is induced by a map $p_Y : \omega \to [0,1]$. If $\pi_Z = \pi \circ \pi_Y$, then $\lim_{n\to\infty} |p_Z(n) - \pi(p_Y(n))| = 0$. Since $\pi_Y \circ \sigma = \pi_Y$, $\lim_{n\to\infty} |p_Y(n) - p_Y(n+1)| = 0$ also. Putting these facts together, one may show that, for large enough n, $p_Y(n) \in [0, \frac{2}{3} + \varepsilon)$ for any prescribed $\varepsilon > 0$, contradicting the surjectivity of π_Y.

Suppose $X \subseteq [0,1]^\delta$ and $f : X \to X$ is continuous. If $\mathcal{U}$ is a nice open cover of X, we say that a sequence $\langle x_n \colon n < \omega\rangle$ is *eventually compliant with $\mathcal{U}$* if there exists some $m \in \omega$ such that

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

(1) $\{x_n : n \geq m\} \subseteq \bigcup \mathcal{U}$,
(2) $\{x_n : n \geq m\} \cap U$ is infinite for all $U \in \mathcal{U}$, and
(3) for all $n \geq m$, we have $x_{n+1} \in \mathcal{U}_\star(f(\mathcal{U}_\star(x_n) \cap X))$.

Roughly, the idea behind this definition is that if our vision is blurred (with the amount of blurriness prescribed by $\mathcal{U}$), then (1) it appears that every x_n could be in X, (2) it appears that $\{x_n : n \geq m\}$ could be dense in X, and (3) for each n, not only does it seem that x_n could be in X, but also that $x_{n+1} = f(x_n)$. A finite sequence $\langle x_n : n \leq k\rangle$ is said to be compliant with $\mathcal{U}$ if condition (3) holds for every $n < k$.

Lemma 3.5. *Let X be a closed subset of $[0,1]^\delta$ and let $f : X \to X$ be continuous. If $\langle x_n : n < \omega\rangle$ is a sequence of points in $[0,1]^\delta$ that is eventually compliant with every nice open cover of X, then the map $p \mapsto p\text{-}\lim_{n\in\omega} x_n$ is a quotient mapping from (ω^*, σ) to (X, f).*

Conversely, if Q is a quotient mapping from (ω^*, σ) to (X, f), then there is a sequence $\langle x_n : n < \omega\rangle$ in $[0,1]^\delta$ such that $Q(p) = p\text{-}\lim_{n\in\omega} x_n$ for all $p \in \omega^*$, and this sequence is eventually compliant with every nice open cover of X.

Proof. Fix $X \subseteq [0,1]^\delta$ and $f : X \to X$, and suppose $\langle x_n : n < \omega\rangle$ is a sequence of points in $[0,1]^\delta$ that is eventually compliant with every nice open cover of X. Define $Q : \omega^* \to [0,1]^\delta$ by $Q(p) = p\text{-}\lim_{n\in\omega} x_n$. From the definitions, we know that Q is a continuous function with domain ω^*. We need to check that $Q(\omega^*) = X$ and that $Q \circ \sigma = f \circ Q$.

First we show that $Q(\omega^*) \subseteq X$. Let U be any open subset of $[0,1]^\delta$ containing X. There is some nice open cover $\mathcal{U}$ of X such that $\bigcup \mathcal{U} \subseteq U$. By part (1) of our definition of eventual compliance, $p\text{-}\lim_{n\in\omega} x_n \in \overline{U}$ for every $p \in \omega^*$. Since U was arbitrary, $F(\omega^*) \subseteq X$.

Next we show that $X \subseteq Q(\omega^*)$. Let U be any basic open subset of $[0,1]^\delta$ with $U \cap X \neq \emptyset$. We may find a nice open cover $\mathcal{U}$ of X such that $U \in \mathcal{U}$. By part (2) of the definition of eventual compliance, $Q(\omega^*) \cap U \neq \emptyset$. Because $Q(\omega^*)$ is the continuous image of a compact space, and therefore closed, this shows $X \subseteq Q(\omega^*)$.

Lastly, we show that $Q \circ \sigma = f \circ Q$. Fix $p \in \omega^*$, and let U be an open neighborhood of $f(Q(p))$. We may find an open cover $\mathcal{U}$ of X such that $U \in \mathcal{U}$ and U is the only member of $\mathcal{U}$ containing $f(Q(p))$. Applying Lemma 3.1, we obtain a nice open cover $\mathcal{V}$ of X such that $\mathcal{V}_\star(f(\mathcal{V}_\star(Q(p)) \cap X)) \subseteq U$.

Let m be large enough to witness the fact that $\langle x_n : n < \omega\rangle$ is eventually compliant with $\mathcal{V}$. Because p is nonprincipal,

$$A = \{n \geq m : x_n \in \mathcal{V}_\star(Q(p))\} \in p.$$

Using part (3) of the definition of eventual compliance, $x_{n+1} \in U$ for every $n \in A$. Thus

$$Q(\sigma(p)) = \sigma(p)\text{-}\lim_{n\in\omega} x_n = p\text{-}\lim_{n\in\omega} x_{n+1} \in \overline{U}.$$

Because U was an arbitrary open neighborhood of $f(Q(p))$, this shows $Q(\sigma(p)) = f(Q(p))$. Since p was arbitrary, $Q \circ \sigma = f \circ Q$ as desired. This finishes the proof of the first assertion of the lemma.

For the converse direction, suppose Q is a quotient mapping from (ω^*, σ) to (X, f). By the Tietze Extension Theorem, Q extends to a continuous function on $\beta\omega$. In other words, there is a sequence $\langle x_n : n < \omega\rangle$ of points in $[0,1]^\delta$ such that $Q(p) = p\text{-}\lim_{n\in\omega} x_n$ for every $p \in \omega^*$. We want to show that this sequence is eventually compliant with every nice open cover of X. Using the fact that $Q(\omega^*) = X$, it is easy to check parts (1) and (2) of the definition of eventual compliance.

To verify (3), let $\mathcal{U}$ be a nice open cover of X and suppose $\langle x_n : n < \omega \rangle$ is not eventually compliant with $\mathcal{U}$. Then there is an infinite $A \subseteq \omega$ such that, for every $a \in A$, $x_{a+1} \notin \mathcal{U}_\star(f(\mathcal{U}_\star(x_a) \cap X))$. Let $p \in A^*$, let $x = Q(p)$, and fix $U \in \mathcal{U}$ with $x \in U$. By definition, $x = p\text{-}\lim_{n\in\omega} x_n \in U$ implies that for some infinite $B \in p$, $\{x_n : n \in B\} \subseteq U$. Replacing B with $B \cap A$ if necessary, we may assume $B \subseteq A$. $B+1 \in \sigma(p)$, and for all $b \in B$ we have $x_{b+1} \notin \mathcal{U}_\star(f(\mathcal{U}_\star(x_b) \cap X)) \supseteq \mathcal{U}_\star(f(U \cap X))$. Thus

$$Q(\sigma(p)) = \sigma(p)\text{-}\lim_{n\in\omega} x_n = p\text{-}\lim_{n\in\omega} x_{n+1} \notin \mathcal{U}_\star(f(U \cap X)) \ni f(Q(p)).$$

Thus $Q \circ \sigma(p) \neq f \circ Q(p)$, contradicting the assumption that Q is a quotient mapping. $\dashv$

§**4. The main theorem.** We are now in a position to prove the main theorem. As mentioned in the introduction, our proof technique parallels that in Section 3 of Dow and Hart's article [9]. In order to make things easier for the reader (especially the reader already familiar with [9]), we have tried to match our notation to that of [9] wherever possible.

THEOREM 4.1 (Main theorem). *Suppose (X, f) is a dynamical system with weight $\aleph_1$. Then (X, f) is a quotient of (ω^*, σ) if and only if f is weakly incompressible.*

PROOF. Every quotient of (ω^*, σ) is weakly incompressible by Lemma 2.5. We must prove that a weakly incompressible dynamical system with weight $\aleph_1$ is a quotient of (ω^*, σ).

Let (X, f) be a weakly incompressible dynamical system with weight $\aleph_1$. Using transfinite recursion, we will construct maps $q_\xi : \omega \to [0,1]$ for all $\xi < \omega_1$. In the end, the diagonal mapping $Q = \Delta_{\xi<\omega_1} q_\xi$ will define a sequence $\langle Q(n) : n < \omega \rangle$ in $[0,1]^{\omega_1}$ that is eventually compliant with every nice open cover of X. By Lemma 3.5, this suffices to prove the theorem.

The recursion will be guided by a sequence of elementary submodels of a large fragment of the set-theoretic universe. Fix a large enough regular cardinal κ (to be specific, "large enough" means $\kappa \geq (2^{\aleph_1})^+$) and let H denote the set of all sets hereditarily smaller than κ. A large enough choice of κ guarantees $X, f \in H$. The regularity of κ guarantees that H is a model of ZFC—the power set axiom, and even the power set axiom fails only for sets X with $|\mathcal{P}(X)| \geq \kappa$. Thus H is a good substitute for the universe of all sets. The argument below, which does not mention the power set of any very large sets, can take place "inside" H. Using the Löwenheim–Skolem Theorem (see Chapter 3 of [15] for a reference), fix a sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of H such that

(1) $X, f \in M_0$.
(2) $M_\beta \subseteq M_\alpha$ whenever $\beta < \alpha$.
(3) for limit α, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.
(4) for each α, $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$.

For each $\alpha < \omega_1$, define $\delta_\alpha = \omega_1 \cap M_\alpha$. Recall that δ_α is a countable ordinal, the supremum of all countable ordinals in M_α.

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

For each $\alpha < \omega_1$, let $X_\alpha = \Pi_{\delta_\alpha}(X)$. In Shchepin's proof of Lemma 3.2, it is clear that the closed unbounded set C mentioned in the statement of the lemma is a subset of $\{\delta_\alpha : \alpha < \omega_1\}$. (Moreover, even if this were not already true we could simply observe that $\{\delta_\alpha : \alpha < \omega_1\}$ is closed unbounded, and replace it with $C \cap \{\delta_\alpha : \alpha < \omega_1\}$, discarding all those M_α with $\alpha \notin C$.) Thus we may define $f_\alpha : X_\alpha \to X_\alpha$ to be the unique self-map of X_α satisfying $\Pi_{\delta_\alpha} \circ f = f_\alpha \circ \Pi_{\delta_\alpha}$, and have f_α continuous by Lemma 2.3.

(X_α, f_α) is a dynamical system, and Π_{δ_α} provides a natural quotient mapping from (X, f) to (X_α, f_α). X_α is metrizable because it is a subset of $[0,1]^{\delta_\alpha}$, and f_α is weakly incompressible by Lemma 2.5 (alternatively, weak incompressibility can be proved directly by an elementarity argument). We may think of the (X_α, f_α) as metrizable "reflections" of (X, f).

If $\langle x_n : n < \omega \rangle$ is a sequence of points in $[0,1]^{\delta_\alpha}$ for some α, let us say that a sequence $\langle y_n : n < \omega \rangle$ of points in $[0,1]^{\omega_1}$ is a *lifting* of $\langle x_n : n < \omega \rangle$ if $\Pi_{\delta_\alpha}(y_n) = x_n$ for all n.

Let $\mathcal{U}$ be a nice open cover of X with $\mathcal{U} \in M_\alpha$. Only ordinals less than δ_α can be used in the definition of $\mathcal{U}$, so $\mathcal{U}$ naturally projects to a nice open cover of X_α in $[0,1]^{\delta_\alpha}$, namely $\Pi_{\delta_\alpha}(\mathcal{U}) = \{\Pi_{\delta_\alpha}(U) : U \in \mathcal{U}\}$. Conversely, every nice open cover $\mathcal{U}$ of X_α in $[0,1]^{\delta_\alpha}$ lifts to a nice open cover of X, namely $\Pi_{\delta_\alpha}^{-1}(\mathcal{U}) = \left\{\Pi_{\delta_\alpha}^{-1}(U) : U \in \mathcal{U}\right\}$. Using this correspondence, one may easily verify the following:

Observation: A sequence of points in $[0,1]^{\delta_\alpha}$ is eventually compliant with every nice open cover of X_α (with respect to the map f_α) if and only if any lifting of that sequence to $[0,1]^{\omega_1}$ is eventually compliant with every nice open cover of X that is defined using ordinals $< \delta_\alpha$.

We are now in a position to begin our recursive construction of the maps q_ξ. Step α of the recursion will be used to construct simultaneously all the maps q_ξ with $\xi \in \delta_\alpha \setminus \bigcup_{\beta<\alpha} \delta_\beta$.

By Theorems 2.4 and 2.6, (X_0, f_0) is a quotient of (ω^*, σ). By Lemma 3.5, there is a sequence $\langle x_n : n < \omega \rangle$ of points in $[0,1]^{\delta_0}$ eventually compliant with every nice open cover of X_0 in $[0,1]^{\delta_0}$. For $\xi < \delta_0$, define $q_\xi(n) = \pi_\beta(x_n)$ (in other words, we define the q_β so that $\Delta_{\beta<\delta_0} q_\beta$ maps ω to the sequence just constructed). This completes the base step of the recursion.

For later stages of the recursion, we assume two inductive hypotheses:

(H1) the sequence $\langle q_\xi : \xi < \delta_\alpha \rangle$ is in $M_{\alpha+1}$.
(H2) the sequence $\langle \Delta_{\xi<\delta_\alpha}(n) : n \in \omega \rangle$ is eventually compliant with every nice open cover of X_α (with respect to the map f_α).

The first hypothesis is preserved at every stage simply because we are assuming that $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$ for each α. This means that any construction involving only models M_β with $\beta < \alpha$ can be carried out "inside" of $M_{\alpha+1}$. (At least, this is true provided that any arbitrary choices made in the construction are made in some canonical fashion; let us assume this from now on.) We have listed the first hypothesis only because it is necessary for performing the construction of the q_ξ with $\xi \in [\delta_\alpha, \delta_{\alpha+1})$ at stage α of the recursion. The second hypothesis is not automatically

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

preserved at every stage like the first; indeed, we will need to choose the maps q_ξ very carefully in order to preserve (H2).

At limit stages there is nothing to construct: due to our choice of the M_α, we have $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ for limit α, so that all the q_ξ with $\xi < \delta_\alpha$ have already been defined by stage α. We need only check that (H2) is preserved at limit stages. For this, suppose $\mathcal{U}$ is any nice open cover of X_α. $\mathcal{U}$ is defined using only finitely many ordinals, so because $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$, there is some $\beta < \alpha$ such that $\mathcal{U}$ is defined using only ordinals $< \delta_\beta$. Because (H2) holds at β, $\langle \Delta_{\xi<\delta_\alpha}(n) : n \in \omega \rangle$ is eventually compliant with $\mathcal{U}$. As $\mathcal{U}$ was arbitrary, (H2) holds at α.

For the successor stage, let $\alpha < \omega_1$ and suppose the functions q_ξ have already been constructed for every $\xi < \delta_\alpha$. Let $Q_\alpha = \Delta_{\xi<\delta_\alpha} q_\xi$.

Because $M_{\alpha+1}$ is countable, there are only countably many nice open covers of X in $M_{\alpha+1}$, namely those that are definable from ordinals $< \delta_{\alpha+1}$. Also, any two nice open covers of X in $M_{\alpha+1}$ have a common refinement that is also a nice open cover of X in $M_{\alpha+1}$. Thus we may find a countable sequence $\langle \mathcal{U}_m : m < \omega \rangle$ of nice open covers of X such that

(1) $\mathcal{U}_m \in M_{\alpha+1}$ for every m,
(2) $\mathcal{U}_n$ refines $\mathcal{U}_m$ whenever $m \leq n$, and
(3) if $\mathcal{U}$ is any nice open cover of X in $M_{\alpha+1}$, then some $\mathcal{U}_m$ refines $\mathcal{U}$.

Note that this part of the construction occurs "outside" $M_{\alpha+1}$, because we are using the fact that $M_{\alpha+1}$ is countable.

Fix $m \in \omega$ and consider $\mathcal{U}_m$. The finite set of ordinals used in the definition of $\mathcal{U}_m$ may be split into two parts: those ordinals below δ_α, which we call F^0_m, and those in the interval $[\delta_\alpha, \delta_{\alpha+1})$, which we call F^1_m. The ordinals F^1_m are not in M_α, but we may use elementarity to find a set of ordinals G_m in M_α that "reflects" the set F^1_m.

More formally, suppose that we write down in the language of first-order logic a (very long) formula φ^m that does all of the following:

(1) φ^m defines $\mathcal{U}_m$ in terms of $F^0_m \cup F^1_m$,
(2) φ^m asserts that $\mathcal{U}_m$ is a nice open cover of X,
(3) φ^m records information about how $\mathcal{U}_m$ interacts with X and f:
    (a) for all $\mathcal{J} \subseteq \mathcal{U}_m$, φ^m asserts either that $\bigcap \mathcal{J} \cap X = \emptyset$ or that $\bigcap \mathcal{J} \cap X \neq \emptyset$,
    (b) if $\mathcal{J} \subseteq \mathcal{U}_m$, $\bigcap \mathcal{J} \cap X \neq \emptyset$, and $U \in \mathcal{U}_m$, then φ^m asserts either that $f(\bigcup \mathcal{J} \cap X) \cap U = \emptyset$ or that $f(\bigcup \mathcal{J} \cap X) \cap U \neq \emptyset$.

Given a finite sequence of points, the information contained in (1) is enough to determine precisely which elements of $\mathcal{U}_m$ contain each member of the sequence. Once that is known, the information in (3) is enough to determine whether the sequence is compliant with $\mathcal{U}_m$.

By elementarity, there is a finite set G_m of ordinals in M_α such that φ^m remains true when the members of F^1_m are replaced with the members of G_m. For each $\xi \in F^1_m$, let ξ_m denote the corresponding member of G_m.

Let $\mathcal{V}_m$ be the nice open cover of X that is defined via φ^m, but substituting the members of G_m in place of the corresponding members of F^1_m. We think of $\mathcal{V}_m$ as the reflection of $\mathcal{U}_m$ in M_α. Let $k(m) \in \omega$ be the least natural number with the property that for all $k \geq k(m)$, $Q_\alpha(k) \in \bigcup \mathcal{V}^m$ and $\langle Q_\alpha(k), Q_\alpha(k+1) \rangle$ is $\mathcal{V}_m$-compliant. This $k(m)$ exists by the inductive hypothesis (H2). If $m < m'$, then $\mathcal{V}^{m'}$ refines $\mathcal{V}_m$, which implies $k(m) \leq k(n)$. Thus the function $m \mapsto k(m)$ is nondecreasing.

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

We are now in a position to define the maps q_ξ for $\delta_\alpha \leq \xi < \delta_{\alpha+1}$:

$$q_\xi(n) = \begin{cases} 0 & \text{if } k(m) \leq n < k(m+1) \text{ and } \xi \notin F^1_m, \\ q_{\xi_m}(n) & \text{if } k(m) \leq n < k(m+1) \text{ and } \xi \in F^1_m. \end{cases}$$

Roughly, this says that q_ξ assumes the behavior of its mirror image q_{ξ_m} on the interval between $k(m)$ and $k(m+1)$, provided some suitable mirror image has already been found. As m increases, the ξ_m become better and better reflections of ξ, because the formulas φ^m include more and more information about X and f.

With the q_ξ thus defined, we need to check that the inductive hypothesis (H2) remains true at the next stage of the recursion. Let $Q_{\alpha+1} = \Delta_{\xi<\delta_{\alpha+1}} q_\xi$.

Let $\mathcal{U}$ be a nice open cover of X with $\mathcal{U} \in M_{\alpha+1}$ and fix m large enough so that $\mathcal{U}_m$ refines $\mathcal{U}$. By the definition of $k(m)$, if $k(m) \leq k < k(m+1)$ then $Q_\alpha(k) \in \bigcup \mathcal{V}_m$ and $\langle Q_\alpha(k), Q_\alpha(k+1)\rangle$ is $\mathcal{V}_m$-compliant. Of course, only the coordinates in $F^0_m \cup G_m$ are relevant to determining these facts. More importantly, all of this relevant information is captured by the formula φ^m. By elementarity and our choice of the set G_m, if $k(m) < k \leq k(m+1)$ then $Q_{\alpha+1}(k) \in \bigcup \mathcal{U}_m$ and $\langle Q_{\alpha+1}(k), Q_{\alpha+1}(k+1)\rangle$ is $\mathcal{U}_m$-compliant.

Given any $k > k(m)$, the same argument shows that $Q_\alpha(k) \in \bigcup \mathcal{U}_{m'}$ and $\langle Q_{\alpha+1}(k), Q_{\alpha+1}(k+1)\rangle$ is $\mathcal{U}_{m'}$-compliant, for some $m' \geq m$. Because $\mathcal{U}_{m'}$ refines $\mathcal{U}_m$, this shows that the sequence $\langle Q_{\alpha+1}(n) \colon n \in \omega\rangle$ satisfies parts (1) and (3) of the definition of $\mathcal{U}_m$-compliance.

For part (2), it suffices to recall that statements of the form "$Q_\alpha(n)$ is a member of the ℓ^{th} member of $\mathcal{V}_m$" are completely determined by the data recorded in φ_m. Thus, by elementarity, $Q_\alpha(n)$ is in the ℓ^{th} member of $\mathcal{V}_m$ if and only if $Q_{\alpha+1}(n)$ is in the ℓ^{th} member of $\mathcal{U}_m$. From this, and the fact $\langle Q_\alpha(n) \colon n \in \omega\rangle$ satisfies part (2) of the definition of $\mathcal{V}_m$-compliance, it follows easily that $\langle Q_{\alpha+1}(n) \colon n < \omega\rangle$ satisfies part (2) of the definition of $\mathcal{U}_m$-compliance. This finishes the proof that (H2) is preserved at successor stages, and this in turn completes the successor step of our recursion.

We claim that the map $Q = \Delta_{\alpha<\omega_1} q_\alpha$ is as required; i.e., that the sequence $\langle Q(n) \colon n < \omega\rangle$ is eventually compliant with every nice open cover of X. Indeed, if $\mathcal{U}$ is a nice open cover of X, then $\mathcal{U}$ is defined by finitely many ordinals, so it was considered at some stage α of our recursion. At stage α, we guaranteed that any lifting of $\langle Q_\alpha(n) \colon n < \omega\rangle$ is eventually compliant with $\mathcal{U}$. $\langle Q(n) \colon n < \omega\rangle$ is such a lifting, so it is eventually compliant with $\mathcal{U}$. As $\mathcal{U}$ was arbitrary, Q is as required. $\dashv$

§5. Related results.

5.1. A few corollaries. Consider the following two theorems, both discussed in the introduction:

- (Parovičenko, [20]). *Every compact Hausdorff space of weight $\aleph_1$ is a continuous image of ω^*.*
- (Dow-Hart, [9]). *Every connected compact Hausdorff space of weight $\aleph_1$ is a continuous image of $\mathbb{H}^*$, where $\mathbb{H} = [0, \infty)$.*

We begin this section by showing that both of these theorems can be derived as fairly straightforward consequences of Theorem 4.1.

LEMMA 5.1. *Let Y be a compact Hausdorff space of weight κ. There is a weakly incompressible dynamical system (X, f) such that X has weight $\aleph_0 \cdot \kappa$ and Y is clopen in X.*

PROOF. Let Y be a compact Hausdorff space of weight κ. Let X be the one-point compactification of $\mathbb{Z} \times Y$, where $\mathbb{Z}$ is given the discrete topology. Let $*$ denote the unique point of $X - \mathbb{Z} \times Y$, and define $f : X \to X$ so that $f(*) = *$, and $f(n, y) = (n+1, y)$. Clearly, f is continuous, X has weight $\aleph_0 \cdot \kappa$, and Y is (homeomorphic to) a clopen subset of X.

It remains to show that (X, f) is chain transitive. Let $\mathcal{U}$ be any open cover of X and $a, b \in X$. To find a $\mathcal{U}$-chain from a to b, fix $U \in \mathcal{U}$ with $* \in U$. If $a = *$ and $b = (n, y)$, we may choose m small enough that $m < n$ and $(m, y) \in U$. Then

$$\langle *, (m, y), (m+1, y), \ldots, (n, y) \rangle$$

is a $\mathcal{U}$-chain from a to b. Similarly if $a = (m, y)$ and $b = *$, choose n large enough that $n > m$ and $(n, y) \in U$. Then

$$\langle (m, y), (m+1, y), \ldots, (n, y), * \rangle$$

is a $\mathcal{U}$-chain from a to b. If $a \neq * \neq b$, then we may get a $\mathcal{U}$-chain from a to b by concatenating a $\mathcal{U}$-chain from a to $*$ with a $\mathcal{U}$-chain from $*$ to b. Thus (X, f) is chain transitive. $\dashv$

Parovičenko's theorem follows immediately from Theorem 4.1 and the next result:

PROPOSITION 5.2. *Suppose every weakly incompressible dynamical system of weight κ is a quotient of (ω^*, σ). Then every compact Hausdorff space of weight κ is a continuous image of ω^*.*

PROOF. Suppose every weakly incompressible dynamical system of weight κ is a quotient of (ω^*, σ), and let Y be a compact Hausdorff space of weight κ. Let (X, f) be the dynamical system guaranteed by Lemma 5.1. (X, f) is a quotient of (ω^*, σ), so in particular there is a continuous surjection $Q : \omega^* \to X$. $Q^{-1}(Y)$ is clopen in ω^*, and therefore homeomorphic to ω^*. Thus $Q \restriction Q^{-1}(Y)$ is a continuous surjection from (a copy of) ω^* to Y. $\dashv$

Observe that a compact Hausdorff space X is connected if and only if the dynamical system (X, id) is weakly incompressible. Thus Theorem 4.1 and the following proposition immediately imply the Dow–Hart theorem:

PROPOSITION 5.3. *If (X, id) is a quotient of (ω^*, σ) then X is a continuous image of $\mathbb{H}^*$.*

PROOF. Suppose (X, id) is a quotient of (ω^*, σ), and assume that $X \subseteq [0,1]^\delta$ for some δ. By Theorem 4.1 and the second part of Lemma 3.5, there is a sequence $\langle x_n : n < \omega \rangle$ of points in $[0,1]^\delta$ that is eventually compliant with every nice open cover of X.

Define a map $q : \mathbb{H} \to [0,1]^\delta$ by sending n to x_n for $n \in \omega$, and then extending q linearly to the rest of $\mathbb{H}$. This function on $\mathbb{H}$ induces a map $Q : \mathbb{H}^* \to [0,1]^\delta$, and we claim Q is a continuous surjection from $\mathbb{H}^*$ to X.

Q is continuous by definition. We see that $Q(\mathbb{H}^*) \supseteq X$ by considering those elements of $\mathbb{H}^*$ that are supported on the integers. It remains to show $Q(\mathbb{H}^*) \subseteq X$. Let W be an open set containing X and let $\mathcal{U}$ be a nice open cover with $\bigcup \mathcal{U} \subseteq W$. Let

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

$\mathcal{V}$ be a star refinement of a star refinement of $\mathcal{U}$. Because $\langle x_n \colon n < \omega \rangle$ is eventually compliant with $\mathcal{V}$, there is some m such that for all $n \geq m$, $x_{n+1} \in \mathcal{V}_\star(\mathcal{V}_\star(x_n))$. By our choice of $\mathcal{V}$, there is some $U \in \mathcal{U}$ with $x_n, x_{n+1} \in U$. As every basic open subset of $[0,1]^\delta$ is convex, $q(r) \in U$ for all $r \in [x_n, x_{n+1}]$. Thus $q(r) \in W$ for every $r \in [m, \infty)$, which implies $Q(\mathbb{H}^*) \subseteq W$. Since W was an arbitrary open set containing X, $Q(\mathbb{H}^*) \subseteq X$. ⊣

We end this subsection with a third corollary of Theorem 4.1, articulating a seemingly new universal property of (ω^*, σ). Roughly, it states that any small enough dynamical system can be obtained from (ω^*, σ) by first taking a subsystem and then taking a quotient.

Proposition 5.4. *Let (X, f) be any dynamical system with the weight of X at most $\aleph_1$. There is a closed, G_δ, shift-invariant subset K of ω^* such that (X, f) is a quotient of (K, σ).*

Proof. We begin with a slightly stronger version of Lemma 5.1:

Claim. *Let (X, f) be a dynamical system, with X of weight κ. There is a weakly incompressible dynamical system (Y, g) such that $X \subseteq Y$ and $f = g \restriction X$. Moreover, X is G_δ in Y and Y has weight $\aleph_0 \cdot \kappa$.*

Proof of claim. Let (X, f) be a dynamical system. Let Y be the one-point compactification of $X \times (\mathbb{Z} \cup \{\infty\})$, where $\mathbb{Z} \cup \{\infty\}$ is given the usual topology, with the positive integers converging to ∞. Let $*$ denote the unique point of $Y - X \times (\mathbb{Z} \cup \{\infty\})$. Define $g : Y \to Y$ so that $g(*) = *$ and otherwise

$$g(x,z) = \begin{cases} (f(x), z+2) & \text{if } z \in \mathbb{Z} \text{ and } z \text{ is even,} \\ (f(x), z-2) & \text{if } z \in \mathbb{Z} \text{ and } z \text{ is odd,} \\ (f(x), \infty) & \text{if } z = \infty. \end{cases}$$

We omit the proof that this dynamical system is chain transitive: it is just as in the proof of Lemma 5.1, but with a few extra cases to check. Identifying X with $X \times \{\infty\}$, it is clear that X is G_δ in Y and that $f = g \restriction X$. ⊣

Returning to the proof of the proposition, let (X, f) be a dynamical system where the weight of X is at most $\aleph_1$. Let (Y, g) be as in the claim. By Theorem 4.1, there is a quotient mapping Q from (ω^*, σ) to (Y, g). Clearly $K = Q^{-1}(X)$ is a closed, G_δ, shift-invariant subset of ω^*, and $Q \restriction K$ provides a quotient mapping from (K, σ) to (X, f). ⊣

5.2. The first and fourth heads of $\beta\omega$. If we assume the Continuum Hypothesis, then Theorem 4.1 gives a complete internal characterization of the quotients of (ω^*, σ):

Theorem 5.5. *Assuming* CH, *the following are equivalent*:

(1) *(X, f) is a quotient of (ω^*, σ).*
(2) *X has weight at most $\mathfrak{c}$ and f is weakly incompressible.*
(3) *X is a continuous image of ω^* and f is weakly incompressible.*

Proof. (1) $\Leftrightarrow$ (2) is a straightforward consequence of Theorem 4.1 and CH. (2) $\Leftrightarrow$ (3) is a straightforward consequence of Parovičenko's characterization of the continuous images of ω^* under CH. ⊣

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

Of the six implications this theorem entails, three are provable from ZFC: (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), and (3) $\Rightarrow$ (2). We will now consider the other three, and show that each of them is independent of ZFC.

Lemma 5.1 shows that (2) $\Rightarrow$ (3) if and only if every compact Hausdorff space of weight $\leq \mathfrak{c}$ is a continuous image of ω^*. This is a purely topological question about ω^* that is considered elsewhere, e.g., in [18]. It is known to be independent: for example, a result of Kunen states that $\omega_2 + 1$ is not a continuous image of ω^* in the Cohen model.

Because (1) $\Rightarrow$ (3) is a theorem of ZFC, the previous paragraph also shows that (2) $\Rightarrow$ (1) is independent.

The independence of (3) $\Rightarrow$ (1) requires a different argument. Consider the following corollary to Theorem 5.5:

Corollary 5.6. *Assuming* CH, (ω^*, σ^{-1}) *is a quotient of* (ω^*, σ).

Proof. The proof is immediate from Theorem 5.5 and the following observation: *If X is a compact Hausdorff space and $f : X \to X$ is a homeomorphism, then f is weakly incompressible if and only if f^{-1} is.*

This is easy to see using chain transitivity: (X, f) has a $\mathcal{U}$-chain from a to b if and only if (X, f^{-1}) has a $\mathcal{U}$-chain from b to a. ⊣

To show that (3) $\Rightarrow$ (1) is independent, it is enough to prove that the conclusion of Corollary 5.6 is independent.

Theorem 5.7. *Assuming* OCA + MA, (ω^*, σ^{-1}) *is not a quotient of* (ω^*, σ).

Recall that a continuous function $F : \omega^* \to \omega^*$ is *trivial* if there is a function $f : \omega \to \beta\omega$ such that $F = \beta f \restriction \omega^*$. Similarly, given $A \subseteq \omega$, $F : A^* \to \omega^*$ is trivial if it is induced by a function $A \to \beta\omega$. To prove Theorem 5.7, we use a deep theorem greatly restricting the self-maps of ω^* under OCA + MA. A strong version of the result is proved by Farah in [11], but we need only a special case, already implicit in the work of Velickovic [25], with precursors in the work of Shelah-Steprāns [24] and Shelah [23].

Theorem 5.8 (Farah, et al.). *Assuming* OCA + MA, *for any continuous* $F : \omega^* \to \omega^*$ *there is some* $A \subseteq \omega$ *such that* $F \restriction A^*$ *is trivial and* $F(\omega^* - A^*)$ *is nowhere dense.*

Proof of Theorem 5.7. Suppose Q is a quotient mapping from (ω^*, σ) to (ω^*, σ^{-1}). Using Theorem 5.8, fix $A \subseteq \omega$ such that $Q \restriction A^*$ is trivial and $Q(\omega^* - A^*)$ is nowhere dense. Also, fix $q : A \to \beta\omega$ such that $Q \restriction A^* = \beta q \restriction A^*$.

Because Q is surjective, A must be infinite.

Let $X = \{a \in A : q(a) \in \omega\}$. Observe that $Q \restriction X$ remains trivial and that $Q(\omega^* - X^*)$ remains nowhere dense. Thus, replacing A with X if necessary, we may (and do) assume that $q(a) \in \omega$ for all $a \in A$.

If q is not finite-to-one on A, there is an infinite set $Y \subseteq A$ and some $n \in \omega$ with $q(Y) = n$, but then $Q(p) = n$ for any $p \in Y^*$, a contradiction. Thus q is finite-to-one on A.

Suppose A is not co-finite. Then $B = \{a \in A : a + 1 \notin A\}$ is infinite. Observe that $\sigma^{-1} \circ Q(B^*) = \sigma^{-1}(q(B)^*) = (q(B) - 1)^*$. This set is clopen and, in particular, has nonempty interior. Thus there is some $p \in B^*$ such that $\sigma^{-1} \circ Q(p) \notin Q(\omega^* - A^*)$, since the latter is nowhere dense. However,

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

$$Q \circ \sigma(p) \in Q \circ \sigma(B^*) = Q((B+1)^*) \subseteq Q(\omega^* - A^*),$$

so that $\sigma^{-1} \circ Q(p) \neq Q \circ \sigma(p)$, a contradiction. Thus A is co-finite.

To summarize: $Q = \beta q \restriction \omega^*$ for some finite-to-one function q defined on a co-finite subset of ω. Changing q on a finite set does not change $Q = \beta q \restriction \omega^*$, so we may assume Q is induced by a finite-to-one function $q : \omega \to \omega$.

We now construct an infinite sequence of natural numbers as follows. Pick $b_0 \in \omega$ arbitrarily. Assuming $b_0, b_1, \ldots, b_n$ are given, there are co-finitely many $b \in \omega$ satisfying

(1) $b \neq b_0, b_1, \ldots, b_n$,
(2) $q(b) - 1 \neq q(b_0 + 1), q(b_1 + 1), \ldots, q(b_n + 1)$, and
(3) $q(b + 1) \neq q(b_0) - 1, q(b_1) - 1, \ldots, q(b_n) - 1$.

This follows from the fact that q is finite-to-one. Also, there are infinitely many $b \in \omega$ satisfying

(4) $q(b) - 1 \neq q(b + 1)$

since otherwise q would be an order-reversing map on ω, which is absurd. Thus, given $b_0, b_1, \ldots, b_n$, we may choose some $b_{n+1} \in \omega$ satisfying $(1) - (4)$.

Let $B = \{b_n : n < \omega\}$. B is infinite by (1), so $B^* \neq \emptyset$. By (2) - (4), we have $q(B + 1) \cap (q(B) - 1) = \emptyset$. However, observe that

$$Q \circ \sigma(B^*) = Q((B+1)^*) = q(B+1)^*,$$

$$\sigma^{-1} \circ Q(B^*) = \sigma^{-1}(q(B)^*) = (q(B) - 1)^*.$$

Hence $Q \circ \sigma(B^*) \cap \sigma^{-1} \circ Q(B^*) = \emptyset$. This contradicts our assumption that Q is a quotient mapping from (ω^*, σ) to (ω^*, σ^{-1}), which would imply that these two sets should be equal instead of disjoint. ⊣

We do not know whether Corollary 5.6 can be improved from a quotient mapping to an isomorphism:

QUESTION 5.9. *Is it consistent that there is a homeomorphism $H : \omega^* \to \omega^*$ with $H \circ \sigma = \sigma^{-1} \circ H$?*

We point out that if the answer to this question is yes, then it seems likely that CH will imply the existence of such an isomorphism already (see Section 5.1 of [12]). See [13] for some partial results.

5.3. An extension using Martin's Axiom. Our final theorem extends Theorem 4.1 to cardinals $\kappa < \mathfrak{p}$.

THEOREM 5.10. *Let (X, f) be a dynamical system with the weight of X less than $\mathfrak{p}$. Then (X, f) is a quotient of (ω^*, σ) if and only if f is weakly incompressible.*

PROOF. Let (X, f) be a weakly incompressible dynamical system, and let κ be the weight of X. Suppose $\kappa < \mathfrak{p}$. By a theorem of M. Bell in [3], this is equivalent to the assumption $\mathsf{MA}_\kappa(\sigma\text{-centered})$, Martin's Axiom at κ for σ-centered posets. We may (and do) assume that $X \subseteq [0, 1]^\kappa$.

We will use $\mathsf{MA}_\kappa(\sigma\text{-centered})$ to construct a sequence of points in $[0, 1]^\kappa$ that is eventually compliant with every nice open cover of X.

Recall that $[0, 1]^\kappa$ is separable, and fix a countable dense $D \subseteq [0, 1]^\kappa$.

Fix $x \in X$, and without loss of generality suppose $x, f(x) \in D$. Let $\mathbb{P}$ be the set of all pairs $\langle s, \mathcal{U} \rangle$, such that s is a sequence of distinct points in D with final

https://doi.org/10.1017/jsl.2018.11 Published online by Cambridge University Press

element x, and $\mathcal{U}$ is a nice open cover of X. Order $\mathbb{P}$ by defining $\langle t, \mathcal{V}\rangle \leq \langle s, \mathcal{U}\rangle$ if and only if

- $\mathcal{V}$ refines $\mathcal{U}$.
- s is an initial segment of t.
- if $t \neq s$, then $t - s$ is a $\mathcal{U}$-compliant sequence of points in $D \cap \bigcup \mathcal{U}$, beginning at $f(x)$, ending at x, and meeting every $U \in \mathcal{U}$.

Ultimately, we will use $\mathsf{MA}_\kappa(\sigma\text{-centered})$ to obtain a suitably generic $G \subseteq \mathbb{P}$, and then $\gamma = \bigcup\{s : \langle s, \mathcal{U}\rangle \in G\}$ will be the desired sequence of points. Roughly, a condition $\langle s, \mathcal{U}\rangle$ is a promise that s is an initial segment of γ, and that the part of γ after s is $\mathcal{U}$-compliant.

Because D is countable, there are only countably many possibilities for the first coordinate of a condition in $\mathbb{P}$. Thus to show that $\mathbb{P}$ is σ-centered, it suffices to show that any two conditions $\langle s, \mathcal{U}\rangle$, $\langle s, \mathcal{V}\rangle$ with the same first coordinate s have a common extension. But this is obvious: taking $\mathcal{W}$ to be any nice open cover refining both $\mathcal{U}$ and $\mathcal{V}$, $\langle s, \mathcal{W}\rangle$ is as desired.

For each nice open cover $\mathcal{U}$ of X, define $D_{\mathcal{U}} = \{\langle s, \mathcal{V}\rangle \in \mathbb{P} : \mathcal{V} \text{ refines } \mathcal{U}\}$. We claim that $D_{\mathcal{U}}$ is dense in $\mathbb{P}$. To see this, fix a nice open cover $\mathcal{U}$ of X and let $\langle s, \mathcal{V}\rangle \in \mathbb{P}$. By the previous paragraph, $\langle s, \mathcal{V}\rangle$ and $\langle s, \mathcal{U}\rangle$ have a common extension. This common extension is in $D_{\mathcal{U}}$ and below $\langle s, \mathcal{V}\rangle$, as desired. So $D_{\mathcal{U}}$ is dense in $\mathbb{P}$.

For each $n \in \mathbb{N}$, define $E_n = \{\langle s, \mathcal{V}\rangle \in \mathbb{P} : |s| \geq n\}$. We claim that each E_n is dense in $\mathbb{P}$. To see this, let $\langle s, \mathcal{U}\rangle \in \mathbb{P}$. Using the chain transitivity of X, we may find a $\mathcal{U}$-compliant sequence t of points in $D \cap \bigcup \mathcal{U}$, beginning at $f(x)$, ending at x, and meeting every $U \in \mathcal{U}$. Then $\langle s^\frown t, \mathcal{U}\rangle$ extends $\langle s, \mathcal{U}\rangle$ and has a longer first coordinate. Repeating this process if needed, we may find extensions of $\langle s, \mathcal{U}\rangle$ with arbitrarily long first coordinates. Thus each E_n is dense in $\mathbb{P}$.

By $\mathsf{MA}_\kappa(\sigma\text{-centered})$, there is a filter G on $\mathbb{P}$ meeting all the $D_{\mathcal{U}}$ and all the E_n. Then $\gamma = \bigcup\{s : \langle s, \mathcal{U}\rangle \in G\}$ is an infinite sequence (infinite because $G \cap E_n \neq \emptyset$), and for any nice open cover $\mathcal{U}$ of X, γ is eventually compliant with $\mathcal{U}$ (because $G \cap D_{\mathcal{U}} \neq \emptyset$). Applying Lemma 3.5 completes the proof. ⊣

We end with a question linking weakly incompressible dynamical systems with the Katowice problem:

QUESTION 5.11. *Is it consistent to have a weakly incompressible autohomeomorphism of* ω_1^*?

If F were such a map, then F could not be trivial on any set A^* with A co-countable. It is consistent that no such map exists, but it is not currently known whether the opposite is also consistent. See [16] for some discussion of this problem and related results. We leave it as an exercise to show that there is no weakly incompressible dynamical system on κ^* for any $\kappa \geq \omega_2$.

§6. **Acknowledgment.** The author would like to thank Brian Raines and Alan Dow for helpful conversations, and Brian especially for patiently listening to several preliminary (and wrong) ideas for proving the main theorem of this article.

REFERENCES

[1] E. Akin, ***The General Topology of Dynamical Systems***, Graduate Studies in Mathematics, reprint edition, American Mathematical Society, Providence, RI, 2010.

[2] A. D. Barwell, C. Good, P. Oprocha, and B. E. Raines, *Characterizations of ω-limit sets of topologically hyperbolic spaces*. ***Discrete and Continuous Dynamical Systems***, vol. 33 (2013), no. 5, pp. 1819–1833.

[3] M. Bell, *On the combinatorial principle $P(c)$*. ***Fundamenta Mathematicae***, vol. 114 (1981), no. 2, pp. 149–157.

[4] A. Blass, *Ultrafilters: Where topological dynamics = algebra = combinatorics*. ***Topology Proceedings***, vol. 18 (1993), pp. 33–56.

[5] A. Błaszczyk and A. Szymański, *Concerning Parovičenko's theorem*. ***Bulletin de L'Academie Polonaise des Sciences Série des sciences mathématiques***, vol. 28 (1980), no. 7-8, pp. 311–314.

[6] R. Bowen, *ω-limit sets for axiom A diffeomorphisms*. ***Journal of Differential Equations***, vol. 18 (1975), no. 2, pp. 333–339.

[7] W. R. Brian, *P-sets and minimal right ideals in $\mathbb{N}^*$*. ***Fundamenta Mathematicae***, vol. 229 (2015), no. 3, pp. 277–293.

[8] E. K. van Douwen and T. C. Przymusiński, *Separable extensions of first-countable spaces*. ***Fundamenta Mathematicae***, vol. 111 (1980), pp. 147–158.

[9] A. Dow and K. P. Hart *A universal continuum of weight $\aleph$*. ***Transactions of the American Mathematical Society***, vol. 353 (2000), no. 5, pp. 1819–1838.

[10] R. Engelking, ***General Topology***, revised ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.

[11] I. Farah, *Analytic quotients: Theory of lifting for quotients over analytic ideals on the integers*. ***Memoirs of the American Mathematical Society***, vol. 148 (2000), no. 702, pp. 1–177.

[12] ———, *The fourth head of $\beta\mathbb{N}$*, ***Open Problems in Topology II*** (E. Pearl, editor), Elsevier, Amsterdam, 2007, pp. 145–150.

[13] S. Geschke, *The shift on $\mathcal{P}(\omega)/\mathrm{fin}$*, unpublished manuscript. Available at `www.math.uni-hamburg.de/home/geschke/publikationen.html.en`.

[14] N. Hindman and D. Strauss, ***Algebra in the Stone-Čech Compactification***, second ed., De Gruyter, Berlin, 2012.

[15] W. Hodges, ***Model Theory***, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[16] P. Larson and P. McKenney, *Automorphisms of $\mathcal{P}(\lambda)/\mathcal{I}_\kappa$*. ***Fundamenta Mathematicae***, vol. 233 (2016), no. 3, pp. 271–291.

[17] J. Meddaugh and B. E. Raines, *Shadowing and internal chain transitivity*. ***Fundamenta Mathematicae***, vol. 222 (2013), no. 3, pp. 279–287.

[18] J. van Mill, *An introduction to $\beta\omega$*, ***Handbook of Set-Theoretic Topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 503–560.

[19] N. Noble and M. Ulmer, *Factoring functions on Cartesian products*. ***Transactions of the American Mathematical Society***, vol. 163 (1972), pp. 329–339.

[20] I. I. Parovičenko, *A universal bicompact of weight $\aleph_1$*. ***Soviet Mathematics Doklady***, vol. 4 (1963), pp. 592–595.

[21] O. M. Sharkovsky, *On attracting and attracted sets*. ***Doklady Akademii Nauk SSSR***, vol. 160 (1965), pp. 1036–1038.

[22] E. V. Shchepin, *Real functions and canonical sets in Tikhonov products and topological groups*. ***Russian Mathematical Surveys***, vol. 31 (1976), no. 6, pp. 17–27.

[23] S. Shelah, ***Proper Forcing***, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin, 1982.

[24] S. Shelah and J. Steprāns, PFA *implies all automorphisms are trivial*. ***Proceedings of the American Mathematical Society***, vol. 104 (1988), no. 4, pp. 1220–1225.

[25] B. Velickovic, OCA *and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$*. ***Topology and its Applications***, vol. 49 (1992), no. 1, pp. 1–12.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE
9201 UNIVERSITY CITY BLVD.
CHARLOTTE, NC 28223-0001, USA
E-mail: wbrian.math@gmail.com
URL: wrbrian.wordpress.com