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Exit manifolds for lattice differential equations

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We study the weak interaction between a pair of well-separated coherent structures in possibly non-local lattice differential equations. In particular, we prove that if a lattice differential equation in one space dimension has asymptotically stable (in the sense of a paper by Chow *et al.*) travelling-wave solutions whose profiles approach limiting equilibria exponentially fast, then the system admits solutions which are nearly the linear superposition of two such travelling waves moving in opposite directions away from one another. Moreover, such solutions are themselves asymptotically stable. This result is meant to complement analytic or numeric studies into interactions of such pulses over finite times which might result in the scenario treated here. Since the travelling waves are moving in opposite directions, these solutions are not shift-periodic and hence the framework of Chow *et al.* does not apply. We overcome this difficulty by embedding the original system in a larger one wherein the linear part can be written as a shift-periodic piece plus another piece which, although it is non-autonomous and large, has certain properties which allow us to treat it as if it were a small perturbation.

1. Introduction

1.1. The system, hypotheses and main results

This paper is concerned with weak interactions between coherent objects in lattice differential equations. These interactions include pulse–pulse interactions, the gluing of fronts and backs to make a wide pulse, front stacking and the interaction between a pulse and a front.

We study the equation

$$\dot{X} = LX + G(X) =: F(X), \quad X \in \mathcal{X} := \ell^{\infty}(\mathbb{Z}, \mathbb{R}^n), \tag{1.1}$$

where we use $\|\cdot\|$ to denote the norm for this space, $L \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ annihilates constant functions and $G\colon \mathcal{X} \to \mathcal{X}$ includes nonlinear terms which may be non-local (see (H0)).

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We say that a solution X of (1.1) is a *travelling wave* if it is of the form $X_j(t) = \phi(j - ct)$ for some continuous function ϕ which has finite limits at $\pm\infty$,

$$\phi(-\infty) = \alpha, \qquad \phi(\infty) = \omega.$$

In the case where $\alpha = \omega$, ϕ is called a *pulse*, whereas, in the case where $\alpha \neq \omega$, we call ϕ a *front*.

We are interested in proving the existence and stability of solutions which are roughly the linear superposition of two separated travelling waves which move with different speeds: in particular, when the waves are separating apart from one another as t increases, a situation we call an *exit*. Thus, we assume that (1.1) admits a pair of stable travelling-wave solutions ϕ^- and ϕ^+, each of which is either a pulse or a front. We denote their wave speeds and asymptotic values with $\pm$ subscripts. We assume that $c_- < c_+$ and that ϕ^+ is 'located' to the right of ϕ^-. Therefore, we require $\omega_- = \alpha_+$. Since L annihilates constant sequences, we can take $\alpha_+ = 0$ without loss of generality. To see this, let $\tilde{X} := X - \alpha$ and note that

$$\dot{\tilde{X}} = L\tilde{X} + \tilde{G}(\tilde{X}),$$

where $\tilde{G}(X) := G(X + \alpha)$.

Before we can state our main theorem we need to make precise the hypotheses that we impose. In what follows,

$$\mathcal{X}_b := \Big\{ X \in \ell^\infty \,\Big|\, \|X\|_b := \sup_{j \in \mathbb{Z}} |(1 + \mathrm{e}^{bj}) X_j| < \infty \Big\},$$

the space of functions which decay exponentially fast as j (or $-j$, depending on the sign of b) goes to infinity.

1.1.1. *Standing assumptions*

(H0) (Continuity of G.) $G\colon \mathcal{X} \to \mathcal{X}$ is of the form $G(X)_n = g(N_1(X)_n, \dots, N_J(X)_n)$, where $g \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^{nJ}, \mathbb{R}^n)$ with $g(0) = 0$ and $N_i \in \mathcal{L}(\mathcal{X}_\beta)$ for all $\beta \in [-b, b]$ and furthermore commutes with the shift.

(H1) (Existence of travelling waves.) There is a $b > 0$ such that the lattice differential equation (LDE) (1.1) admits travelling-wave solutions $\phi^- \in \mathcal{X}_b$ and $\phi^+ \in \mathcal{X}_{-b}$ with speeds $c_- < c_+$ and $\omega_- = \alpha_+$. We further assume that $(\phi^\pm)' \in \mathcal{X}_{\mp b}$.

(H2) (Spectral stability of travelling waves.) Let $\Phi^\pm(t, t_0)$ denote the time t map for the linear equation $\dot{Y} = (L + G'(\phi^\pm))Y$, let S denote the shift on $\mathcal{X}$, $(Sx)_j = x_{j-1}$ and let $A^\pm := S^{-1}\Phi^\pm(1/c_\pm, 0)$. Then one is a simple eigenvalue of $A^\pm$ (with eigenfunction $(\phi^\pm)'$) and $\sigma(A^\pm)\backslash\{1\}$ is contained in the open unit disc. Here the spectrum is computed regarding $A^\pm$ as an operator on $\mathcal{X}_{\mp b}$.

REMARK 1.1. Typically, the conjugated operator $A_b^\pm = (1 + \mathrm{e}^{bj})A_\pm[\cdot/(1 + \mathrm{e}^{bj})]$ is a small perturbation of $A_\pm$ so long as b is chosen sufficiently small. Thus, the spectrum of $A_\pm^b$ in $\mathcal{X}$ coincides with that of $A_\pm$ in $\mathcal{X}_b$. Hence, (H2) may be obtained in examples as a consequence of the corresponding stability criterion with $\mathcal{X}$ replacing $\mathcal{X}_b$.

Note that in [8], (H1) and (H2) are shown to be sufficient to conclude the asymptotic stability of the travelling wave. In § 1.3 we further discuss these hypotheses as they relate to a number of different systems of interest. We now state our main theorem.

THEOREM 1.2. *If* $c_- < c_+$, $\omega_- = \alpha_+$ *and* (H0)–(H2) *are satisfied, then there exists a positive constant* a *such that, for each* $\varepsilon > 0$, *there exist positive constants* C, δ_0, *and* τ^* *such that, if*

$$\|X_{\mathrm{init}} - \phi^+(\cdot - \tau_+) - \phi^-(\cdot - \tau_-)\| \leqslant \delta < \delta_0$$

with

$$\tau_+ - \tau_- \geqslant \tau^*,$$

then there are real constants γ_*^+ *and* γ_*^- *in* $(-\varepsilon, \varepsilon)$ *such that the solution* X *of (1.1) with initial condition* X_{init} *satisfies*

$$\mathrm{e}^{at}\|X(t) - \phi^+(\cdot - c_+t - \tau_+ - \gamma_*^+) - \phi^-(\cdot - c_-t - \tau_- - \gamma_*^-)\| \leqslant C(\mathrm{e}^{-a\tau_*} + \sqrt{\delta}) \quad (1.2)$$

for all $t \geqslant 0$.

We can rephrase this theorem in terms of the 'exit manifold' as follows:

$$\mathcal{M}_{\mathrm{exit}} := \{\phi^+(\cdot - \tau_+) + \phi^-(\cdot - \tau_-) : \tau_- - \tau_+ \geqslant \tau^*\}.$$

$\mathcal{M}_{\mathrm{exit}}$ is a smooth two-dimensional submanifold of ℓ^∞ (see proposition 3.4 of [8]) and consists of all linear superpositions of two well-separated travelling waves. It is not an invariant manifold for (1.1) but our main theorem implies that is a local attractor for the dynamics. We therefore have the following corollary.

COROLLARY 1.3. *If* $\mathrm{dist}_{\ell^\infty}(X_{\mathrm{init}}, \mathcal{M}_{\mathrm{exit}}) \leqslant \delta_0$, *then the solution* $X(t)$ *of (1.1) with* $X(0) = X_{\mathrm{init}}$ *satisfies*

$$\operatorname*{dist}_{\ell^\infty}(X(t), \mathcal{M}_{\mathrm{exit}}) \leqslant C\mathrm{e}^{-at}$$

REMARK 1.4. There are numerous results concerning the existence and stability of multi-pulse solutions for reaction–diffusion partial differential equations (PDEs). For instance, [1, 10, 11, 18, 27] deal with the existence and stability of multi-pulse standing solutions, [6, 12–15, 26, 31] deal with counter-propagating fronts and pulses in scalar systems using comparison principle, [9, 32], handle long distance weak interactions between standing pulses, [5, 28, 30] deal with exit or shooting solutions to systems of reaction diffusion equations; the methods used there are most similar to ours. Additionally, multi-pulse solutions in a Hamiltonian lattice have been studied by Hoffman in [19, 20].

The remainder of this paper is organized as follows. In § 1.2 we outline our approach to the proof of theorem 1.2. In § 1.3 we discuss some examples of (1.1). Section 2 decomposes the problem into stable and centre eigenspaces and we make estimates on this decomposition in § 3. Finally, § 4 contains the proof of theorem 1.2.

1.2. General strategy

We seek a solution of the form $x_j(t) = \phi^-(j - c_-t) + \phi^+(j - c_+t) + w$, where w goes to zero in ℓ^∞ as $t \to \infty$. Note that, for large values of t, the sum $\phi^-(j-c_-t) + \phi^+(j - c_+t)$ is close to zero for compact sets of spatial indices j. To that end, we embed (1.1) into the following system:

$$\left.\begin{aligned}\dot{X}^- &= LX^- + G(X^-) + H^-(t)\{G(X^- + X^+) - G(X^-) - G(X^+)\}\\ &=: F_-(X^-, X^+),\\ \dot{X}^+ &= LX^+ + G(X^+) + H^+(t)\{G(X^- + X^+) - G(X^-) - G(X^+)\}\\ &=: F_+(X^-, X^+).\end{aligned}\right\} \tag{1.3}$$

Here H^- and H^+ are localization operators defined as follows. Let $h(x) = 0$ for $x \leqslant 0$ and $h(x) = 1$ for $x > 0$ denote the usual Heaviside function, and let $\bar{c} = (c_- + c_+)/2$. Define the operator $H^+(t)$ which acts on spaces of sequences by $(H^+(t)X)_j = h(j - \bar{c}t)X_j$ and $H^-(t) = \mathrm{Id} - H^+(t)$. At time t, these operators localize sequences to the right and left half-lattices which are centred 'halfway between' ϕ^- and ϕ^+.

Note that if (X^-, X^+) solves (1.3), then $X = X^- + X^+$ solves (1.1). Thus, if a solution (X^-, X^+) solves (1.3) with $X^- = \phi^- + w^-$ and $X^+ = \phi^+ + w^+$, where $w^\pm(t)$ are decaying to zero, then $X = X^- + X^+$ is of the form that we seek with $w = w^+ + w^-$. Equation (1.3) is a perturbation of two copies of (1.1). However, the coupling terms $H^\pm(t)\{G(X^- + X^+) - G(X^-) - G(X^+)\}$ are not small, at least when viewed on $\mathcal{X}$. To wit, an application of the mean-value theorem shows (roughly speaking) that we have

$$\begin{aligned}|H^-(t)\{G(X^- + X^+) - G(X^-) - G(X^+)\}|\\ &\leqslant CH^-(t)|X^-||X^+|\\ &\leqslant CH^-(t)(|\phi^-||\phi^+| + |\phi^-||w^+| + |\phi^+||w^-| + |w^-||w^+|).\end{aligned}$$

$H^-(t)$ localizes functions to the left half-lattice, where $\phi^+ \in \mathcal{X}_{-b}$ is exponentially small. Thus, $H^-|\phi^+|$ is exponentially small and we can handle two of the four terms above. The term $|w^-||w^+|$ is quadratic, and thus can also be made small. However, $H^-(t)|\phi^-|$ is $O(1)$ and thus the term $H^-(t)|\phi^-||w^+|$ requires care. If w^+ is exponentially localized to the right half-lattice, then $H^-(t)|w^+|$ will be small just as $H^-|\phi^+|$ was. (We make this heuristic argument rigorous in proposition 3.1.)

Therefore, we will require this localization. For the remainder of the paper we regard (1.3) (after a series of non-trivial changes of coordinates) as an evolution equation in the phase space $\mathcal{Y} := \mathcal{X}_b \times \mathcal{X}_{-b}$. At first blush, this may seem to shrink the size of the space of the initial data that we allow for equation (1.1). However, for any $X(t_0) \in \mathcal{X}$, we have $H^\pm(t_0)X(t_0) \in \mathcal{X}_{\mp b}$. Therefore, we set $X^\pm(t_0) := H^\pm(t_0)X(t_0)$ so that initially X^- and X^+ are supported on the left and right half-lattices, respectively. Additionally, for any b, $\mathcal{X}_b \subset \mathcal{X}$. Thus, the study of (1.3) in $\mathcal{Y}$ contains the dynamics of (1.1) in $\mathcal{X}$. (Note that, for $t > t_0$, in general we will have $X^-(t) \neq H^-(t)X(t)$ and $X^+(t) \neq H^+(t)X(t)$.)

1.3. Examples

Note that the class of models (1.1) which satisfy (H0) is quite general. Namely, any system of lattice differential equations satisfies (H0) so long as very mild restrictions on the nonlinear piece of the non-local coupling are satisfied. The class of LDEs to which theorem 1.2 can be applied is much smaller. In many cases (H1) and (H2) are known to be false. Examples are furnished by soliton equations, conservation laws and monostable reaction diffusion equations, each of which admits travelling waves with two neutral directions, violating (H2).

The kind of equations that we have in mind are spatial discretizations of possibly non-local reaction–diffusion equations. These equations are dissipative, and thus (H2) is not immediately ruled out. However, establishing (H1) and (H2) is highly non-trivial. To demonstrate this, consider the simple scalar equation

$$\dot{u}_n = \frac{1}{h^2}(u_{n+1} + u_{n-1} - 2u_n) - f(u_n), \tag{1.4}$$

which arises as a spatial discretization of the PDE

$$u_t = u_{xx} - f(u). \tag{1.5}$$

Here f is the derivative of a double-well potential, e.g. $f(u) = u(u-1)(u-a)$ for some $a \in (0,1)$. Upon substituting the travelling-wave ansatz $u_n(t) = \phi(n - ct)$, we obtain the mixed-type equation

$$-c\phi'(\xi) = \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) - f(\phi(\xi)), \tag{1.6}$$

which is ill-posed as a dynamical system on the infinite-dimensional phase space $C([-1,1],\mathbb{R})$. Comparing it with $-c\phi' = \phi'' - f(\phi)$, which arises as a wave-profile equation for the PDE (1.5), we can see why the existence and stability theory for (1.4) has lagged behind that for (1.5). Nevertheless, both (H1) and (H2) have been established for (1.4). The existence theory (H1) can be based upon either topological fixed-point theorems [33] or comparison principles [16]. Mallet-Paret [25] developed the Fredholm theory of differential-difference operators and built a continuation argument on this theory [24] which establishes (H1) for a more general subclass of (1.1) than (1.4) under the mild assumptions of finite interaction length, spatial homogeneity and ellipticity (see [24] for details). The assumptions of finite interaction length and spatial homogeneity have been weakened [2, 7].

With regard to stability theory, it is usually the case that the essential spectrum can be easily computed, e.g. via Fourier transform. However, the eigenvalue problem is of the form (1.6) with an additional spectral parameter. When comparison principles are available, this problem is tractable. When comparison principles are not available, little is known.

We should note that comparison principles are typically available in scalar equations of reaction–diffusion type and can be used to construct a stable monotone front. Note also that if $(c, \phi(\xi))$ is a solution of equation (1.6), then $(-c, \phi(-\xi))$ is also a solution. Thus, having established the existence of one front (c_+, ϕ^+) with $c_+ > 0$, we may take $c_- = -c_+$ and $\phi^-(\xi) = \phi^+(-\xi)$. This situation, sometimes referred to as 'gluing a front and back together' is typical for the kinds of scalar equations with comparison principles for which (H1) and (H2) have been established.

We now mention some examples for which a comparison principle has recently been leveraged to obtain stability. Consider the following convolution model for phase transitions:

$$\dot{u}_n = \sum_{k\in\mathbb{Z}} J_k u_{n-k} - u_n + f(u_k)$$

with f bistable. The existence of travelling fronts was established in [2] under an ellipticity assumption on the convolution kernel J, while asymptotic stability was established in [23]. Chen *et al*. studied

$$\dot{u}_n = \sum_{|k-n|\leqslant k_0} a_{n,k} u_{n+k} + f(u_k)$$

in the case where the kernel $a_{n,k}$ is periodic in n and elliptic, and the nonlinearity f is of bistable type [7]. In both of these examples, the authors do not verify (H2) directly. However, their results imply (H2).

One example which arises in applications is the backwards reaction–diffusion equation

$$\dot{w}_n = -\frac{1}{h^2}(w_{n+1} + w_{n-1} - 2w_n) + f(w_n),$$

which Vainchtien and Van Vleck [29] derived in the study of martensitic phase transitions. Here the nonlinearity f is bistable. After making the change of variables $z_n = (-1)^n w_n$, this model falls under the general framework studied in [7]; hence (H1) and (H2) are proven for this model.

Another example which arises in applications is the discrete Fitzhugh–Nagumo (dFHN) equation

$$\dot{u}_n = d(u_{n+1} + u_{n-1} - 2u_n) + u_n(u_n - 1)(u_n - a) - v_n, \tag{1.7}$$

$$\dot{v}_n = \varepsilon(u_n - bv_n). \tag{1.8}$$

Although the dFHN equation does not admit a comparison principle, it is a singular perturbation of (1.4) for which (H1) and (H2) have been established. Geometric singular perturbation theory has recently been extended to mixed-type equations, where it has been used to establish (H1) for the dFHN equation [21].

Another situation for which results exist is front-stacking. In any of the above examples, we can replace the bistable nonlinearity f with a tristable nonlinearity, e.g. $g(u) = -u(u + 1)(u - 1)(u - a_1)(u - a_2)$ with $-1 < a_1 < 0 < a_2 < 1$. We can restrict attention to $u \in [-1, 0]$ and apply the above results for the bistable case to establish the existence and stability of a monotone front (c_-, ϕ^-) connecting -1 to 0. Similarly, we can restrict attention to $u \in [0, 1]$ to obtain a second monotone front (c_+, ϕ^+) connecting 0 to 1. In this case theorem 1.2 establishes the existence of a monotone solution connecting -1 to 1 with a long plateau at 0 which grows longer over time.

One situation to which our results do not apply is front stacking in conservation laws. This is because, in conservation laws, there is a line of equilibria at the constant solutions which generates an additional neutral eigenvalue, violating (H2). Stability for fronts in semi-discrete conservations laws was established in [4]. However, the presence of an additional neutral mode complicates the analysis both for the stability of a single wave and for the interaction. This lies beyond the scope of this paper.

We remark, finally, that stability of pulses is generally more challenging than stability of monotone fronts in PDEs. This is because monotone tools such as the Krein–Rutman theorem are not available to control the location of the discrete spectrum. Instead, the spectrum is usually controlled via Evans function methods (see, for example, [22]). The Evans function is built on top of exponential dichotomies for the spatial dynamical problem (see, for example, (1.6)) and requires finite-dimensional unstable manifolds. In the continuum case (1.6) becomes an ordinary differential equation (ODE) and this is not a problem. Exponential dichotomies have been constructed for mixed-type equations such as (1.6) (see, for example, [17]). However, the unstable manifolds are typically infinite dimensional. The development of techniques for establishing stability of travelling waves in lattice equations is an area of active research [3].

2. CMS-type decomposition to stable and centre directions

In the study of stability of travelling waves for PDEs, it is standard to change coordinates to a moving frame in which the travelling wave becomes an equilibrium. Lattices do not admit such a moving frame. Nevertheless, travelling waves on lattices are shift-periodic, that is, $\phi(n - cT_c) = \phi(n - 1)$ when $T_c = 1/c$. The stability theory for travelling waves on lattices developed in [8] is based on a Floquet theory for the time $T_c = 1/c$ map. A key step in the development of this Floquet theory is the construction of local coordinates which separate the neutral mode associated with translations of the travelling wave from the rest of the phase space. The purpose of this section is to develop a similar decomposition for the situation when two travelling waves are present.

Let

$$p^{\pm}(t) := \phi^{\pm}(\cdot - c_{\pm}t) \in \mathcal{X}_{\mp b} \quad \text{and} \quad \mathcal{V}_0^{\pm} = \{p^{\pm}(t) : t \in \mathbb{R}\} \subset \mathcal{X}_{\mp b}.$$

Lemma 4.1 of [8] shows that there exist $Z_{\pm} \in C^r(\mathbb{R}, GL(\mathcal{X}_{\mp b}))$ with the following properties, which hold for all $\theta \in \mathbb{R}$:

- $Z_{\pm}(0) = \mathrm{Id}$;
- $Z_{\pm}(\theta + 1/c_{\pm}) = SZ_{\pm}(\theta)$;
- $Z_{\pm}(\theta)\dot{p}^{\pm}(0) = \dot{p}^{\pm}(\theta)$.

Note that, in [8], the authors work in spaces l^p, which have norms that are invariant under the shift S, and thus they can conclude (by the second property) that the operator norm of $Z_{\pm}(\theta)$ is bounded independent of θ. Our spaces $\mathcal{X}_b$ are not shift-independent and thus the operator norm of $Z_{\pm}(\theta)$ may be large if θ is large.

Now fix codimension-one subspaces $E^s_{\pm} \subset \mathcal{X}_{\mp b}$ which do not contain $\dot{p}_{\pm}$ and define $\Phi^{\pm} : \mathbb{R} \times E^s_{\pm} \to l^{\infty}$ as

$$\Phi^{\pm}(\theta^{\pm}, y^{\pm}) = p^{\pm}(\theta^{\pm}) + Z_{\pm}(\theta^{\pm})y^{\pm}.$$

Proposition 4.2 of [8] ensures that $(\theta^{\pm}, y^{\pm})$ can be used as local coordinates nearby $\mathcal{V}_0^{\pm}$, where the chart is given by $\Phi^{\pm}$.

Letting $X^{\pm}(t) = \Phi^{\pm}(\theta^{\pm}(t), y^{\pm}(t))$, we now derive equations of motion for $\theta^{\pm}$ and $y^{\pm}$. We carry the details out for the minus component. Differentiating $X^{\pm}$ with respect to time and using (1.3) gives

$$Z_{-}(\theta^{-})\{[\dot{p}^{-}(0) + q^{-}(\theta^{-})y^{-}]\dot{\theta^{-}} + \dot{y}^{-}\} = F_{-}(X^{-}, X^{+}),$$

where the operator-valued function q^{-} is given by

$$q^{-}(\theta) := Z_{-}(\theta)^{-1} D Z_{-}(\theta).$$

Note that $q^{-}(\theta + 1/c_{-}) = q^{-}(\theta)$ and thus the operator norm of q^{-} is bounded uniformly in θ.

After multiplying both sides by $Z_{-}(\theta^{-})^{-1}$, apply the functional $\nu^{-} \in \mathcal{X}_{b}^{*}$, defined so as to annihilate $E_{\pm}^{s}$ (and thus $\dot{y}^{-}$) and which maps $\dot{p}^{-}(0)$ to one. This yields

$$\begin{aligned} \dot{\theta}^{-} &= \Theta_{-}(\theta^{-}, y^{-}, \theta^{+}, y^{+}) \\ &:= \frac{1}{1 + \nu^{-}(q^{-}(\theta^{-})y^{-})} \nu^{-}(Z_{-}(\theta)^{-1} F_{-}(X^{-}, X^{+})). \end{aligned} \tag{2.1}$$

For $\dot{y}^{-}$, we can solve

$$\begin{aligned} \dot{y}^{-} &= Y_{-}(\theta^{-}, y^{-}, \theta^{+}, y^{+}) \\ &:= Z_{-}(\theta^{-})^{-1} F_{-}(X^{-}, X^{+}) - [\dot{p}(0) + q^{-}(\theta^{-})y^{-}]\Theta_{-}(\theta^{-}, y^{-}, \theta^{+}, y^{+}) \\ &= \left[\mathrm{Id} - \left[\frac{\dot{p}(0) + q(\theta^{-})y^{-}}{1 + \nu_{-}(q^{-}(\theta^{-})y^{-})}\right]\nu(\cdot)\right] Z_{-}(\theta^{-})^{-1} F_{-}(X^{-}, X^{+}). \end{aligned} \tag{2.2}$$

Similarly, we choose $\nu_{+} \in \mathcal{X}_{-b}^{*}$, which annihilates $\dot{y}^{+}$ and maps $\dot{p}^{+}(0)$ to one to derive similar equations for θ^{+} and y^{+}.

Define $\gamma^{\pm}(t) := \theta^{\pm}(t) - t$ and

$$\begin{aligned} &\Gamma_{-}(\gamma^{-}, y^{-}, \gamma^{+}, y^{+}, t) \\ &\quad := \Theta_{-}(\gamma^{-} + t, y^{-}, \gamma^{+} + t, y^{+}) - 1 \\ &\quad = (1 + \nu^{-}(q(\theta^{-})y^{-}))^{-1} \\ &\qquad \times [\nu^{-} Z_{-}(\theta^{-})^{-1} F_{-}(X^{-}, X^{+}) \\ &\qquad\quad - (1 + \nu^{-}(q(\theta^{-})y^{-}))\nu^{-} Z_{-}(\theta^{-})^{-1} F_{-}(p^{-}(\theta^{-}), 0)] \qquad (2.3) \\ &\quad = [\nu^{-}(Z_{-}(\theta^{-})^{-1}[F_{-}(X^{-}, X^{+}) - F_{-}(p^{-}(\theta^{-}), 0)]) - \nu^{-}(q_{-}(\theta^{-})y^{-})] \\ &\qquad \times (1 + \nu^{-}(q(\theta^{-})y^{-}))^{-1}, \qquad (2.4) \end{aligned}$$

and similarly for Γ^{+}. In the second line we have used the fact that

$$1 = \nu^{-}(\dot{p}^{-}(0)) = \nu^{-}(Z(\theta^{-})^{-1}\dot{p}^{-}(\theta^{-})) = \nu^{-}(Z(\theta^{-})^{-1} F_{-}(p^{-}(\theta^{-}), 0)).$$

Therefore, (1.3) becomes

$$\left.\begin{aligned} \dot{y}^{-} &= Y_{-}(\theta^{-}, y^{-}, \theta^{+}, y^{+}), & \dot{y}^{+} &= Y_{+}(\theta^{-}, y^{-}, \theta^{+}, y^{+}), \\ \dot{\gamma}^{-} &= \Gamma_{-}(\gamma^{-}, y^{-}, \gamma^{+}, y^{+}, t), & \dot{\gamma}^{+} &= \Gamma_{+}(\gamma^{-}, y^{-}, \gamma^{+}, y^{+}, t). \end{aligned}\right\} \tag{2.5}$$

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Now we let

$$Y_{-0}(\gamma, y, t) = Y_{-}(t+\gamma, y, 0, 0)$$

and

$$Y_{-1}(\gamma^-, y^-, \gamma^+, y^+, t) = Y_{-}(t+\gamma^-, y^-, t+\gamma^+, y^+) - Y_{-0}(\gamma^-, y^-, t),$$

and similarly for Y_+. Let $\mathcal{A}_\pm(t) := D_y Y_{\pm 0}(t, 0)$. Then (2.5) becomes

$$\left.\begin{aligned}
\dot{y}^- &= \mathcal{A}_-(t)y^- + \{(\mathcal{A}_-(t+\gamma^-) - \mathcal{A}_-(t))y^-\} \\
&\quad + \{Y_{-0}(\gamma^-, y^-, t) - D_y Y_{-0}(\gamma^-, 0, t)y^-\} + Y_{-1}(\gamma^-, y^-, \gamma^+, y^+, t), \\
\dot{y}^+ &= \mathcal{A}_+(t)y^+ + \{(\mathcal{A}_+(t+\gamma^+) - \mathcal{A}_+(t))y^+\} \\
&\quad + \{Y_{+0}(\gamma^+, y^+, t) - D_y Y_{+0}(\gamma^+, 0, t)y^+\} + Y_{+1}(\gamma^-, y^-, \gamma^+, y^+, t), \\
\dot{\gamma}^- &= \mathit{\Gamma}_-(\gamma^-, y^-, \gamma^+, y^+, t), \\
\dot{\gamma}^+ &= \mathit{\Gamma}_+(\gamma^-, y^-, \gamma^+, y^+, t).
\end{aligned}\right\} \quad (2.6)$$

This system is equivalent to (1.3) in a neighbourhood of $\mathcal{V}_0^- \times \mathcal{V}_0^+$.

3. Estimates for the right-hand side

In this section we prove a series of useful estimates for the right-hand side of (2.6). The most important term is $Z_-(\theta^-)^{-1}[F_-(X^-, X^+) - F_-(X^-, 0)]$, which appears in both Y_{-1} and $\mathit{\Gamma}_-$.

Proposition 3.1. *We have*

$$\begin{aligned}
&\|Z_-(\theta^-)^{-1}[(F_-(X^-, X^+) - F_-(X^-, 0))]\|_{\mathcal{X}_b} \\
&\qquad \leqslant \frac{C(1+|\gamma^-|)\mathrm{e}^{|c_+\gamma^+|}}{1+\mathrm{e}^{b(c_+-c_-)t/2}}(1 + \|y^-\|_{\mathcal{X}_b} + \|y^+\|_{\mathcal{X}_{-b}} + \|y^+\|_{\mathcal{X}_{-b}}\|y^-\|_{\mathcal{X}_b}). \quad (3.1)
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
&Z_-(\theta^-)^{-1}[(F_-(X^-, X^+) - F_-(X^-, 0))] \\
&\qquad = Z_-(\theta^-)^{-1}[H^-(G(X^- + X^+) - G(X^-) - G(X^+))].
\end{aligned}$$

Note the following consequence of the mean-value theorem. Recall that $G(X)$ is of the form $G(X)_n = g((N_1X)_n, \dots, (N_JX)_n)$. Let x denote the vector

$$(N_1X^+, \dots, N_JX^+)$$

and let y denote the vector

$$(N_1X^-, \dots, N_JX^-).$$

Use the mean-value theorem to write

$$g(x+y) - g(x) = \int_0^1 Dg(x+ty)y\,\mathrm{d}t \quad \text{and} \quad g(y) = \int_0^1 Dg(ty)y\,\mathrm{d}t$$

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so that, after using the fact that Dg is locally Lipschitz, we obtain

$$\begin{aligned}|g(x+y)-g(x)-g(y)| &= \left|\int_0^1 \{Dg(x+ty)-Dg(ty)\}y\,\mathrm{d}t\right| \\ &\leqslant C|x||y| \leqslant C\sum_{i,k}|N_iX^-||N_kX^+|, \end{aligned} \tag{3.2}$$

where the constant C may be chosen uniformly on bounded sets of x and y.

Now let $\lfloor\tau\rfloor$ denote the greatest integer less than τ. Note that the second property of $Z_\pm$ implies that

$$Z_\pm(\theta^\pm) = Z_\pm\biggl(\frac{\lfloor\theta^\pm c_\pm\rfloor}{c_\pm}+\tilde{\theta}^\pm\biggr) = S^{m_\pm}Z_\pm(\tilde{\theta}^\pm) \tag{3.3}$$

and

$$Z_\pm(\theta^\pm)^{-1} = Z_\pm(\tilde{\theta}^\pm)^{-1}S^{-m_\pm},$$

where $m_\pm = \lfloor\theta^\pm c_\pm\rfloor$ and $\tilde{\theta} = c_\pm\theta - m_\pm \in [0, 1/|c_\pm|)$. Since $\tilde{\theta}$ is restricted to lie in a compact set and $\theta \mapsto Z_\pm(\theta)$ is continuous, it follows that there is a universal constant C such that the operator norm of $Z_\pm(\tilde{\theta})$ and its inverse are bounded by C.

This, together with (3.2), implies

$$\begin{aligned}
&\|Z_-(\theta^-)^{-1}[(F_-(X^-,X^+)-F_-(X^-,0))]\|_{\mathcal{X}_b} \\
&\quad\leqslant C\sum_{i,k}\|S^{-m_-}H^-\{\|N_iX^-||N_kX^+\|\}\|_{\mathcal{X}_b} \\
&\quad= C\sum_{i,k}\||S^{-m_-}H^-\{N_iX^-\}||S^{-m_-}H^-\{N_kX^+\}|\|_{\mathcal{X}_b} \\
&\quad\leqslant C\sum_{i,k}\||S^{-m_-}H^-\{N_iX^-\}|\|_{\mathcal{X}_b}\||S^{-m_-}H^-\{N_kX^+\}|\|_{\ell^\infty} \\
&\quad\leqslant C\sum_{i,k}\||S^{-m_-}H^-\{N_ip^-(\theta_-)\}|\|_{\mathcal{X}_b}\||S^{-m_-}H^-\{N_kp^+(\theta_+)\}|\|_{\ell^\infty} \\
&\qquad+ C\sum_{i,k}\||S^{-m_-}H^-\{N_ip^-(\theta_-)\}|\|_{\mathcal{X}_b}\||S^{-m_-}H^-\{N_kZ_+(\theta^+)y^+\}|\|_{\ell^\infty} \\
&\qquad+ C\sum_{i,k}\||S^{-m_-}H^-\{N_iZ_-(\theta_-)y^-\}|\|_{\mathcal{X}_b}\||S^{-m_-}H^-\{N_kp^+(\theta_+)\}|\|_{\ell^\infty} \\
&\qquad+ C\sum_{i,k}\||S^{-m_-}H^-\{N_iZ_-(\theta_-)y^-\}|\|_{\mathcal{X}_b}\||S^{-m_-}H^-\{N_kZ_+(\theta^+)y^+\}|\|_{\ell^\infty}.
\end{aligned} \tag{3.4}$$

In the second line of (3.4) we have used the fact that $H^-(xy) = (H^-x)(H^-y)$. In the third line we have used the estimate $\|UV\|_{\mathcal{X}_b} \leqslant \|U\|_{\mathcal{X}_b}\|V\|_{\ell^\infty}$. Each of the last four terms corresponds to one of the four terms on the right-hand side of the estimate in the proposition.

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We first estimate the contribution from the rightmost pulse p^+, which is small because of the cut-off function H^-:

$$\begin{aligned}
&\|S^{-m_-}H^-N_kp^+(\theta_+)\|_{\ell^\infty}\\
&\quad= \sup_{n\in\mathbb{Z}}(1-h(n-\bar{c}t+m_-))|N_k\phi^+(n-c_+\theta_++m_-)|\\
&\quad\leqslant \|N_k\|_{\mathcal{L}(\mathcal{X}_{-b})}\|\phi^+\|_{\mathcal{X}_{-b}}\sup_{n\in\mathbb{Z}}((1-h(n-\bar{c}t+m_-))(1+\mathrm{e}^{-b(n-c_+\theta_++m_-)})^{-1})\\
&\quad\leqslant C(1+\mathrm{e}^{-b(\bar{c}t-c_+\theta_+)})^{-1}\leqslant C(1+\mathrm{e}^{-b(-(c_+-c_-)/2t-c_+\gamma_+)})^{-1}\\
&\quad\leqslant C\frac{\mathrm{e}^{c_+|\gamma_+|}}{1+\mathrm{e}^{b(c_+-c_-)/2t}}.
\end{aligned}$$

We now estimate the contribution from y^+:

$$\begin{aligned}
&\|S^{-m_-}H^-N_kZ_+(\theta_+)y^+\|_{\ell^\infty}\\
&\quad= \|S^{-m_-}H^-N_kS^{m_+}Z_+(\tilde{\theta}_+)y^+\|_{\ell^\infty}\\
&\quad= \sup_{n\in\mathbb{Z}}(1-h(n-\bar{c}t+m_-))|[N_kZ_+(\tilde{\theta}_+)y^+](n+m_--m_+)|\\
&\quad\leqslant \|Z_+(\tilde{\theta}_+)\|_{\mathcal{L}(\mathcal{X}_{-b})}\|N_k\|_{\mathcal{L}(\mathcal{X}_{-b})}\|y^+\|_{\mathcal{X}_{-b}}\\
&\qquad\times\sup_{n\in\mathbb{Z}}((1-h(n-\bar{c}t+m_-))(1+\mathrm{e}^{-b(n+m_--m_+)})^{-1})\\
&\quad\leqslant C\|y^+\|_{\mathcal{X}_{-b}}(1+\mathrm{e}^{-b(\bar{c}t-m_+)})^{-1}\\
&\quad\leqslant C\|y^+\|_{\mathcal{X}_{-b}}(1+\mathrm{e}^{-b(\bar{c}t-c_+\theta_++\tilde{\theta}_+)})^{-1}\\
&\quad\leqslant C\|y^+\|_{\mathcal{X}_{-b}}\frac{\mathrm{e}^{c_+|\gamma_+|}}{1+\mathrm{e}^{b(c_+-c_-)/2t}}
\end{aligned}$$

We have used (3.3) together with the fact that S commutes with N_k.

We now estimate the contribution from the leftmost pulse p^-, which is bounded,

$$\begin{aligned}
\|S^{-m_-}H^-N_ip^-(\theta_-)\|_{\mathcal{X}_b} &\leqslant \|N_iS^{-m_-}p^-(\theta_-)\|_{\mathcal{X}_b}\\
&\leqslant \|N_i\|_{\mathcal{L}(\mathcal{X}_b)}\|\phi^-(\cdot-c_-\gamma^-+\tilde{\theta}^-)\|_{\mathcal{X}_b}\\
&\leqslant C(1+|\gamma^-|).
\end{aligned}$$

Finally, the contribution from y^- is

$$\begin{aligned}
\|S^{-m_-}H^-\{N_iZ_-(\theta^-)y^-\}\|_{\mathcal{X}_b} &\leqslant \|S^{-m_-}N_iS^{m_-}Z_-(\tilde{\theta}^-)y^-\|_{\mathcal{X}_b}\\
&\leqslant \|N_i\|_{\mathcal{L}(\mathcal{X}_b)}\|Z_-(\tilde{\theta}^-)\|_{\mathcal{L}(\mathcal{X}_b)}\|y^-\|_{\mathcal{X}_b}\\
&\leqslant C\|y^-\|_{\mathcal{X}_b}.
\end{aligned}$$

In the first line we have used the pointwise bound $|(H^-\{X\})_n|\leqslant|X_n|$ and in the second line we have used the fact that S commutes with N_i. Arranging the estimates for $p^\pm$ and $y^\pm$ completes the proof. □

Having estimated this crucial term, we are now ready to bound the right-hand sides of the evolution equation (2.6).

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PROPOSITION 3.2. *We have*

$$|\dot\gamma^-| \leqslant C\|y^-\|_{\mathcal{X}_b} + \frac{C(1+|\gamma^-|)\mathrm{e}^{|c_+\gamma^+|}}{1+\mathrm{e}^{b(c_+-c_-)t/2}} \times (1+\|y^-\|_{\mathcal{X}_b} + \|y^+\|_{\mathcal{X}_{-b}} + \|y^+\|_{\mathcal{X}_{-b}}\|y^-\|_{\mathcal{X}_b}), \tag{3.5}$$

$$\|\dot y^- - \mathcal{A}_-(t)y^-\|_{\mathcal{X}_b} \leqslant C\|y^-\|^2_{\mathcal{X}_b} + |\gamma^-|\|y^-\|_{\mathcal{X}_b} + \frac{C(1+|\gamma^-|)\mathrm{e}^{|c_+\gamma^+|}}{1+\mathrm{e}^{b(c_+-c_-)t/2}} \times (1+\|y^-\|_{\mathcal{X}_b} + \|y^+\|_{\mathcal{X}_{-b}} + \|y^+\|_{\mathcal{X}_{-b}}\|y^-\|_{\mathcal{X}_b}). \tag{3.6}$$

Proof. We first estimate

$$|\Gamma_-| \leqslant C(\|Z_-(\theta^-)^{-1}[F_-(X^+,X^-) - F_-(X^+,0)]\|_{\mathcal{X}_b} + \|Z_-(\theta^-)^{-1}[F_-(X^-,0) - F_-(p^-,0)]\|_{\mathcal{X}_b} + \|y^-\|_{\mathcal{X}_b}).$$

Here we have used (2.6) together with the fact that the terms $1/(1+\nu_-(q_-(\theta^-)y^-))$ and $\|q_-(\theta^-)\|_{\mathcal{L}(\mathcal{X}_b)}$ are bounded uniformly by a constant. We now estimate the term

$$\begin{aligned}
&\|Z_-(\theta^-)^{-1}[F_-(X^-,0) - F_-(p^-,0)]\|_{\mathcal{X}_b} \\
&\quad = \|Z_-(\tilde\theta^-)S^{-m_-}[F_-(X^-,0) - F_-(p^-,0)]\| \\
&\quad = \|Z_-(\tilde\theta^-)^{-1}[F_-(S^{-m_-}(p^- + S^{m_-}Z_-(\tilde\theta^-)y^-),0) - F_-(S^{-m_-}p^-,0)]\|_{\mathcal{X}_b} \\
&\quad \leqslant C\|y^-\|_{\mathcal{X}_b}.
\end{aligned}$$

Combining this with proposition 3.1 yields (3.5).

We compute

$$Y_{-1} = \left(1 - \frac{\dot p^-(0) + q^-(\theta^-)y^-}{1+\nu^-(q^-(\theta^-)y^-)}\nu^-(\cdot)\right)(Z_-(\theta^-)^{-1}[(F_-(X^-,X^+) - F_-(X^-,0))]).$$

Thus,

$$\begin{aligned}
\|Y_{-1}\|_{\mathcal{X}_b} &\leqslant \left\|1 - \frac{\dot p^-(0) + q^-(\theta^-)y^-}{1+\nu^-(q^-(\theta^-)y^-)}\nu^-(\cdot)\right\|_{\ell^\infty} \\
&\qquad \times \|Z_-(\theta^-)^{-1}[F_-(X^+,X^-) - F_-(X^+,0)]\|_{\mathcal{X}_b} \\
&\leqslant \frac{C(1+|\gamma^-|)\mathrm{e}^{|c_+\gamma^+|}}{1+\mathrm{e}^{b(c_+-c_-)t/2}}(1+\|y^-\|_{\mathcal{X}_b} + \|y^+\|_{\mathcal{X}_{-b}} + \|y^+\|_{\mathcal{X}_{-b}}\|y^-\|_{\mathcal{X}_b}).
\end{aligned}$$

Here we have used the fact that the operator norm of q_- is bounded uniformly in θ and the fact that $\|y^-\|$ can be made small to bound the first term, and we have used proposition 3.1 to bound the second term. Since

$$\|(\mathcal{A}_-(t+\gamma^-) - \mathcal{A}_-(t))y^-\|_{\mathcal{X}_b} \leqslant C|\gamma^-|\|y^-\|_{\mathcal{X}_b}$$

and $Y_{-0}(\gamma^-,0,t) \equiv 0$, we also have the estimate

$$\|Y_{-0}(\gamma^\pm,y^\pm,t) - D_yY_{-0}(\gamma^\pm,0,t)y^-\|_{\mathcal{X}_b} \leqslant C\|y^-\|^2_{\mathcal{X}_b}.$$

In light of (2.6), this yields (3.6) and hence completes the proof. □

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4. Proof of theorem 1.2

Proof. Let $B_\pm(t,t_0)$ denote the evolution operator associated to $\dot y = \mathcal{A}_\pm(t)y$. It follows from (H2) and statement 1 of theorem 5.3 of [8] that

$$\|B(t,t_0)\|_{\mathcal{L}(\mathcal{X}_b)} \leqslant C\mathrm{e}^{-\lambda(t-t_0)}$$

for some $C>0$, $\lambda>0$.

After applying the Duhamel formula to the equations for y and using proposition 3.2, the equations for $\dot y^-$ in (2.6) give

$$\begin{aligned}\|y^-(t)\|_{\mathcal{X}_b} &\leqslant \|B_-(t,t_0)\|_{\mathcal{L}(\mathcal{X}_b)}\|y^-(t_0)\|_{\mathcal{X}_b} \\ &\quad + \int_{t_0}^t \|B_-(t,s)\|_{\mathcal{L}(\mathcal{X}_b)}\|\dot y^-(s) - \mathcal{A}_-(s)y^-(s)\|_{\mathcal{X}_b}\,\mathrm{d}s \\ &\leqslant C\mathrm{e}^{-\lambda(t-t_0)}\|y^-(t_0)\|_{\mathcal{X}_b} + C\int_{t_0}^t \mathrm{e}^{-\lambda(t-s)} \\ &\quad \times \biggl(\|y^-(s)\|^2_{\mathcal{X}_b} + |\gamma^-(s)|\|y^-(s)\|_{\mathcal{X}_b} + \frac{(1+|\gamma^-(s)|)\mathrm{e}^{|c_+\gamma^+(s)|}}{1+\mathrm{e}^{b(c_+-c_-)/2s}} \\ &\quad \times (1+\|y^-(s)\|_{\mathcal{X}_b} + \|y^+(s)\|_{\mathcal{X}_{-b}} + \|y^+(s)\|_{\mathcal{X}_{-b}}\|y^-(s)\|_{\mathcal{X}_b})\biggr)\,\mathrm{d}s. \end{aligned} \tag{4.1}$$

Similarly, for γ^-, we have

$$\begin{aligned}|\gamma^-(t)| \leqslant C\int_{t_0}^t &(\|y^-(s)\|_{\mathcal{X}_b} + |\gamma^-(s)|\|y^-(s)\|_{\mathcal{X}_b} + \frac{(1+|\gamma^-(s)|)\mathrm{e}^{|c_+\gamma^+(s)|}}{1+\mathrm{e}^{b(c_+-c_-)/2s}} \\ &\times (1+\|y^-(s)\|_{\mathcal{X}_b} + \|y^+(s)\|_{\mathcal{X}_{-b}} + \|y^+(s)\|_{\mathcal{X}_{-b}}\|y^-(s)\|_{\mathcal{X}_b}))\,\mathrm{d}s. \end{aligned} \tag{4.2}$$

There are similar estimates for y^+ and γ^+.

Now let $\delta := \|y^-(t_0)\|_{\mathcal{X}_b} + \|y^+(t_0)\|_{\mathcal{X}_{-b}}$ and assume $\delta<1$. Let $b^* := b(c_+-c_-)/4$, let $a := \min\{\lambda/4, b^*\}$ and define

$$K_T := \sup_{t_0\leqslant t\leqslant T}[|\gamma^-(t)| + |\gamma^+(t)| + \mathrm{e}^{a(t-t_0)}(\sqrt{\delta} + \mathrm{e}^{-b^*t_0})^{-1}(\|y^-(t)\|_{\mathcal{X}_b} + \|y^+(t)\|_{\mathcal{X}_{-b}})].$$

Thus,

$$\|y^-(t)\|_{\mathcal{X}_b} + \|y^+(t)\|_{\mathcal{X}_{-b}} \leqslant \mathrm{e}^{-a(t-t_0)}(\delta + \mathrm{e}^{-b^*t_0})K_T$$

whenever $t_0 \leqslant t \leqslant T$. Note that $K_{t_0} = \delta/\sqrt{\delta} + \mathrm{e}^{-b^*t_0} < 1$ and that K_T is increasing with T. (This increase is continuous since our LDE is locally well posed.) Our theorem is proven if we can show that K_T is bounded uniformly for all $T>t_0$. We choose T so that $K_T \leqslant 1$.

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Then (4.1) gives, for $t_0 \leqslant t \leqslant T$,

$$\|y^-(t)\|_{\mathcal{X}_b} \leqslant C\delta e^{-\lambda(t-t_0)} + CK_T^2(\delta + e^{-b^* t_0}) \int_{t_0}^t e^{-\lambda(t-s)} e^{-as}\, ds + C \int_{t_0}^t e^{-\lambda(t-s)} \frac{1}{1+e^{2b^* s}}\, ds$$

for some constant C that is independent of T, δ and t_0. Here we have used the fact that one dominates $\|y^{\pm}(t)\|_{\mathcal{X}_{\mp b}}$ and that $\sqrt{\delta}+e^{-b^* t_0}$ dominates $(\sqrt{\delta}+e^{-b^* t_0})^2$. Integrating the exponentials and using the fact that, for sufficiently large s, $1/(1+e^{2b^* s})$ is well approximated by $e^{-2b^* s}$, we obtain

$$e^{a(t-t_0)}(\sqrt{\delta}+e^{b^* t_0})^{-1}\|y^-(t)\|_{\mathcal{X}_b} \leqslant C\biggl\{\frac{\delta}{\sqrt{\delta}+e^{-b^* t_0}} e^{-(\lambda-a)(t-t_0)} + K_T^2 + \frac{e^{-2b^* t_0}}{\sqrt{\delta}+e^{-b^* t_0}}(e^{-(2b^*-a)(t-t_0)} + e^{-(\lambda-a)(t-t_0)})\biggr\}.$$

We can control the right-hand side of (4.2) in much the same fashion, though we omit the details. Taking all this together, we can show there exists $C^* > 0$ (independent of T, δ and t_0) so that

$$K_T \leqslant C^*(\sqrt{\delta} + K_T^2 + e^{-b^* t_0}). \tag{4.3}$$

There exists positive constants δ_0, t_0^* and $0 < K_- < K_+ \leqslant 1$ so that if $K_- \leqslant K \leqslant K_+$, $0 < \delta < \delta_0$ and $t_0 > t_0^*$, then

$$C^*(\sqrt{\delta} + K^2 + e^{-b^* t_0}) \leqslant \tfrac{1}{2}K. \tag{4.4}$$

Let T^* be the smallest time greater than t_0 for which $K_T = K_+$, if such a T exists. Otherwise, set $T^* = +\infty$. Note that if $T^* = +\infty$, then we are done with the proof of theorem 1.2 (t_0 in this formulation corresponds to $\tau^*/(c_+ - c_-)$ in the statement of the theorem). Suppose that $T^* < +\infty$. If so, then (4.3) and (4.4) imply that

$$K_T \leqslant \tfrac{1}{2}K_T,$$

which is a contradiction. The proof is complete. □

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