Ergod. Th. & Dynam. Sys. (1985), **5**, 373–377
Printed in Great Britain

Global shadowing of pseudo-Anosov homeomorphisms

MICHAEL HANDEL
Department of Mathematics, Columbia University, New York, NY 10027, *USA*

(*Received* 10 *January* 1985)

Abstract. We prove that if $f: M^2 \to M^2$ is pseudo-Anosov and if $g \simeq f$, then there is a closed subset $X \subset M^2$ and a continuous surjection $\pi: X \to M$ that is homotopic to inclusion such that $f \circ \pi = \pi \circ g|X$.

0. *Introduction*

In this paper we consider A. B. Katok's notion of K-global shadowing (defined in § 1) as it applies to a pseudo-Anosov homeomorphism $f: M^2 \to M^2$ on a closed surface. It is an equivalence relation which allows one to compare the orbits of f with the orbits of any map $g: M^2 \to M^2$ that is homotopic to f. We write $(f, x) \sim^K (g, y)$, or $(f, x) \sim (g, y)$ when K is unspecified.

Global shadowing in a pseudo-Anosov homotopy class generalizes two other well known and useful equivalence relations:

(A) When x and y are fixed points of f^n and g^n respectively $(n > 0)$, then $(f, x) \sim (g, y)$ if and only if (f^n, x) is Nielsen equivalent to (g^n, y).

(B) When $K = \varepsilon$ is sufficiently small and g is ε-close to f in the C^0-topology, then $(f, x) \sim (g, y)$ if and only if the f-orbit of x ε-shadows (in the sense of **[B]**) the pseudo-orbit of f defined by the g-orbit of y. Lewowicz [L] has considered ε-shadowing in the pseudo-Anosov context.

Thurston showed ([T]; Lemma 2.1 below) that from the point of view of Nielsen equivalence, f has the minimal number of periodic points among all maps in its homotopy class. Our first theorem shows that from the point of view of global shadowing, f has the minimal number of orbits among all maps in its homotopy class.

THEOREM 1. *Let $f: M^2 \to M^2$ be a pseudo-Anosov homeomorphism on a closed surface, and let $g: M^2 \to M^2$ be any map that is homotopic to f. Then*

(i) $(f, x_1) \sim (f, x_2) \Rightarrow x_1 = x_2$;

(ii) *for all $x \in M^2$, there exists $y \in M^2$ such that $(f, x) \sim (g, y)$; if x is f-periodic with least period n, then y can be chosen to be g-periodic with least period n.*

Our second theorem is a uniformization of theorem 1. When f is Anosov rather than pseudo-Anosov, then theorem 2 reduces to the fact (proposition 2.1 of [F]) that any map that is homotopic to an Anosov diffeomorphism is semi-conjugate (by a map which is homotopic to the identity) to that Anosov diffeomorphism.

THEOREM 2. *Let $f: M^2 \to M^2$ be a pseudo-Anosov homeomorphism of a closed surface and let $g: M^2 \to M^2$ be any map that is homotopic to f. Then there exists a closed set $Y \subset M^2$ and a surjective map $\varphi: Y \to M$ which is homotopic to the inclusion map such that $f\varphi = \varphi g | Y$.*

Theorem 1 is in response to a question of A. B. Katok. I am grateful to him for bringing it to my attention. I would also like to thank P. Boyland for several interesting conversations which provoked theorem 2.

This research was supported in part by NSF Grant No. MCS8120790.

1. *Notation and definitions*

For the remainder of the paper, $f: M^2 \to M^2$ will be a pseudo-Anosov homeomorphism of a closed surface and $g: M^2 \to M^2$ will be a map that is homotopic to f. The reader that is not familiar with pseudo-Anosov homeomorphisms should consult [T] and [**F-L-P**] as required. We will use only the following properties of $f: M^2 \to M^2$.

(1.1) The periodic points of f are dense.

(1.2) The action induced by f on the free homotopy classes of M has no periodic orbits.

(1.3) The fixed point index of a fixed point x of f^n is never 0.

(1.4) There exist $\lambda > 1$ and an equivariant metric $\tilde{D}$ on the universal cover $\tilde{M}$ of M such that $\tilde{D} = \sqrt{\tilde{D}_s^2 + \tilde{D}_u^2}$, where $\tilde{D}_s: \tilde{M} \times \tilde{M} \to [0, \infty)$ and $\tilde{D}_u: \tilde{M} \times \tilde{M} \to [0, \infty)$ are equivariant functions satisfying:

$$\tilde{D}_u(\tilde{f}\tilde{x}_1, \tilde{f}\tilde{x}_2) = \lambda \tilde{D}_u(\tilde{x}_1, \tilde{x}_2) \quad \text{and} \quad \tilde{D}_s(\tilde{f}^{-1}\tilde{x}_1, \tilde{f}^{-1}\tilde{x}_2) = \lambda \tilde{D}_s(\tilde{x}_1, \tilde{x}_2)$$

for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{M}$ and all lifts $\tilde{f}$ of f.

Remark. The stable and unstable foliations for f lift to stable and unstable foliations for $\tilde{f}$. Given $\tilde{x}_1, \tilde{x}_2 \in \tilde{M}$, $\tilde{D}_s(\tilde{x}_1, \tilde{x}_2)$ is defined to be the minimum length, with respect to the transverse measure on the stable foliation of $\tilde{f}$, of an arc connecting $\tilde{x}_1$ to $\tilde{x}_2$. The function $\tilde{D}_u$ is defined similarly with respect to the unstable foliation.

We fix once and for all a lift $\tilde{f}: \tilde{M} \to \tilde{M}$ of f to the universal cover $\tilde{M}$ of M. As g is homotopic to f there is a unique lift $\tilde{g}: \tilde{M} \to \tilde{M}$ which is equivariantly homotopic to $\tilde{f}$.

Definition 1.5 (A. B. Katok). The f-orbit of x is *K-globally shadowed* by the g-orbit of y if there are lifts $\tilde{x}$ of x and $\tilde{y}$ of y such that $\tilde{D}(\tilde{f}^k(\tilde{x}), \tilde{g}^k(\tilde{y})) \leq K$ for all $k \in \mathbb{Z}$. We write $(f, x) \sim^K (g, y)$ or $(f, x) \sim (g, y)$ if the shadowing constant K is unspecified.

Remark. The equivalence classes $(f, x) \sim (g, y)$ are not dependent on the choice of equivariant metric in the definition of global shadowing. We use $\tilde{D}$ because it is convenient for computations.

Definition 1.6. If x is a fixed point of f^n and $\tilde{x}$ is a lift of x, then $\tilde{f}^n(\tilde{x}) = s\tilde{x}$ for some covering translation s of $\tilde{M}$. Similarly, if y is a fixed point of g^n and $\tilde{y}$ is a lift of y, then $\tilde{g}^n(\tilde{y}) = t\tilde{y}$ for some covering translation t. We say that (f^n, x) and (g^n, y) are *Nielsen equivalent* if there exist $\tilde{x}$ and $\tilde{y}$ such that $s = t$.

The following lemma establishes relationship (A) stated in the introduction.

LEMMA 1.7. *If x is a fixed point of f^n and y is a fixed point of g^n, then (f^n, x) is Nielsen equivalent to (g^n, y) if and only if $(f, x) \sim (g, y)$.*

Proof. (only if) Suppose that $\tilde{f}^n(\tilde{x}) = t\tilde{x}$ and $\tilde{g}^n(\tilde{y}) = t\tilde{y}$. Then

$$\begin{aligned}\tilde{D}(\tilde{f}^k(\tilde{x}), \tilde{g}^k(\tilde{y})) &= \tilde{D}(\tilde{f}^{k-n}(t\tilde{x}), \tilde{g}^{k-n}(t\tilde{y})) \\ &= \tilde{D}(t'\tilde{f}^{k-n}(\tilde{x}), t'\tilde{g}^{k-n}(\tilde{y})) = \tilde{D}(\tilde{f}^{k-n}(\tilde{x}), \tilde{g}^{k-n}(\tilde{y})),\end{aligned}$$

since $t' = \tilde{f}^{k-n} t \tilde{f}^{-(k-n)} = \tilde{g}^{k-n} t \tilde{g}^{-(k-n)}$ is a covering translation and $\tilde{D}$ is equivariant. Thus $\tilde{D}(\tilde{f}^k(\tilde{x}), \tilde{g}^k(\tilde{y}))$ takes on only finitely many values and is bounded.

(if) Suppose that $\tilde{f}^n(\tilde{x}) = s \cdot \tilde{x}$, $\tilde{g}^n(\tilde{y}) = t \cdot y$ and that $(f, x) \sim^K (g, y)$. Then

$$\tilde{D}(\tilde{x}, (s^{-1}\tilde{g}^n)^k\tilde{y}) = \tilde{D}((s^{-1}\tilde{f}^n)^k\tilde{x}, (s^{-1}\tilde{g}^n)^k\tilde{y}) \leq K \qquad \text{all } k \in \mathbb{Z}.$$

Since any bounded subset of $\tilde{M}$ intersects only finitely many lifts of y, $(s^{-1}\tilde{g}^n)^k$ fixes both $\tilde{y}$ and $s^{-1}\tilde{g}^n(\tilde{y}) = s^{-1}t\tilde{y}$ for some $k > 0$. This implies that $(s^{-1}\tilde{g}^n)^k$ commutes with the covering translation $s^{-1}t$ and hence by property (1.2) that $s^{-1}t =$ identity. □

2. *Proofs of theorems* 1 *and* 2

We begin this section with a proof of a result of Thurston which, in conjunction with lemma 1.7, implies that f-periodic orbits of least period n are globally shadowed by g-periodic orbits of least period n. The heart of this paper (lemma 2.2) is the observation that there is a uniform bound to the shadowing constants produced by lemma 2.1.

LEMMA 2.1 [T]. (i) *If x_1 and x_2 are distinct fixed points of f^n then (f^n, x_1) and (f^n, x_2) are not Nielsen equivalent.*

(ii) *If x is f-periodic with least period n, then there exists y which is g-periodic with least period n and such that (f^n, x) is Nielsen equivalent to (g^n, y).*

Proof. Property (1.4) implies that no lift of any iterate of f can fix two distinct points. This immediately implies (i). The existence of y such that (g^n, y) is Nielsen equivalent to (f^n, x) is now a consequence of property (1.3) and [**Br**; theorem 3, p. 94]. It suffices to show that y has least period n.

There exist lifts $\tilde{x}$ and $\tilde{y}$ and a covering translation t such that $t^{-1}\tilde{f}^n(\tilde{x}) = \tilde{x}$ and $t^{-1}\tilde{g}^n(\tilde{y}) = \tilde{y}$. If y has least period $m_1 < n$, then (by the uniqueness of lift of g^n fixing $\tilde{y}$) there is a covering translation t_1 such that $t^{-1}\tilde{g}^n = (t_1\tilde{g}^{m_1})^{m_2}$ where $m_2 = n/m_1 > 1$. Since equivariantly homotopic lifts of f^n are equal,

$$t^{-1}\tilde{f}^n = (t_1\tilde{f}^{m_1})^{m_2}.$$

This implies that the entire $t_1\tilde{f}^{m_1}$ orbit of $\tilde{x}$ is fixed by $t^{-1}\tilde{f}^n$ in contradiction to our observation above that no lift of an iterate of f can fix two distinct points. □

LEMMA 2.2. *There exists* $K = K(g)$ *such that* $(f, x) \sim (g, y)$ *if and only if* $(f, x) \sim^K (g, y)$. *In particular, if* $x_n \to x$, $y_n \to y$, *and* $(f, x_n) \sim (g, y_n)$ *then* $(f, x) \sim (g, y)$.

Proof. Let $R = \max[\sup_{\tilde{x} \in \tilde{M}} \tilde{D}(\tilde{f}(\tilde{x}), \tilde{g}(\tilde{x})), \sup_{\tilde{x} \in \tilde{M}} \tilde{D}(\tilde{f}^{-1}(\tilde{x}), \tilde{g}^{-1}(\tilde{x}))]$. Note that by equivariance of $\tilde{D}$ and compactness of M, $R < \infty$. Property (1.4) implies that

$$\tilde{D}_u(\tilde{f}(\tilde{x}), \tilde{g}(\tilde{y})) \geq \lambda \tilde{D}_u(\tilde{x}, \tilde{y}) - R$$

and

$$\tilde{D}_s(\tilde{f}^{-1}(\tilde{x}), \tilde{g}^{-1}(\tilde{y})) \geq \lambda \tilde{D}_s(\tilde{x}, \tilde{y}) - R.$$

Let $K = 2(R+1)/(\lambda - 1)$. If $\tilde{D}_u(\tilde{x}, \tilde{y}) > K/2$, then $\tilde{D}_u(\tilde{f}\tilde{x}, \tilde{g}\tilde{y}) > 1 + \tilde{D}_u(\tilde{x}, \tilde{y})$. Similarly, if $\tilde{D}_s(\tilde{x}, \tilde{y}) > K/2$, then $\tilde{D}_s(\tilde{f}^{-1}(\tilde{x}), \tilde{g}^{-1}(\tilde{y})) \geq 1 + \tilde{D}_s(\tilde{x}, \tilde{y})$. It follows that if $(f, x) \sim (g, y)$ then $(f, x) \sim^K (g, y)$.

Suppose now that $\tilde{x}_n \to \tilde{x}$, $\tilde{y}_n \to \tilde{y}$ and that $\tilde{D}(\tilde{f}^k \tilde{x}_n, \tilde{g}^k \tilde{y}_n) \leq K$ for all k and n. Since $\tilde{D}(\tilde{f}^k \tilde{x}, \tilde{g}^k \tilde{y}) \leq \tilde{D}(\tilde{f}^k \tilde{x}, \tilde{f}^k \tilde{x}_n) + \tilde{D}(\tilde{f}^k \tilde{x}_n, \tilde{g}^k \tilde{y}_n) + \tilde{D}(\tilde{g}^k y_n, \tilde{g}^k y)$ for all n, and the first and third terms in this sum tend to zero as n tends to infinity,

$$\tilde{D}(\tilde{f}^k \tilde{x}, \tilde{g}^k \tilde{y}) \leq \sup_{n \in \mathbb{Z}} \tilde{D}(\tilde{f}^k \tilde{x}_n, \tilde{g}^k \tilde{y}_n) \leq K.$$

Thus $(f, x) \sim^K (g, y)$ as desired. □

Proof of theorem 1. Part (i) follows immediately from property (1.4). Part (ii) follows from lemmas 1.7, 2.1, and 2.2 and from property (1.1). □

Proof of theorem 2. Define $Y = \{y \in M^2 : \exists x \in M^2 \text{ such that } (f, x) \sim (g, y)\}$. Theorem 1 implies that for each $y_0 \in Y$ there is a unique $x_0 \in M^2$ such that $(f, x_0) \sim (g, y_0)$; define $\varphi: Y \to M$ by $\varphi(y_0) = x_0$. Lemma 2.2 implies that Y is closed and φ is continuous. By construction, $\tilde{\varphi}: \tilde{Y} \to \tilde{M}$ is a bounded distance from the inclusion. This implies that φ is homotopic to the inclusion. Finally, since $(f, x) \sim (g, y)$ implies $(f, f(x)) \sim (g, g(y))$, $f\varphi = \varphi g | Y$. □

Remark 2.3. The proof of theorem 2 carries over to the case that $M = T^2$ and f is an Anosov diffeomorphism. Note that the usual ε-shadowing arguments (e.g. [**B**]) apply to $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ with no restriction on the size of ε. Since for each $\tilde{y} \in \mathbb{R}^2$ the $\tilde{g}$-orbit of $\tilde{y}$ is an R-pseudo-orbit for $\tilde{f}$, it is always possible to find $x \in T^2$ such that $(f, x) \sim (g, y)$. Thus the set $Y \subset T^2$ constructed in the proof of theorem 2 equals all of T^2 and g is semi-conjugate to f (cf. proposition 2.1 of [**F**]).

Remark 2.4. When $\partial M \neq 0$, $\tilde{D}$ is an incomplete metric on int $\tilde{M}$. In terms of the proofs, one thinks of $\tilde{D}$ as a complete metric on $\tilde{M}^*$ which is obtained from $\tilde{M}$ by collapsing each component of $\partial \tilde{M}$ to a point. The only change required in theorem 1 is that (i) should read: $(f, x_1) \sim (f, x_2) \Rightarrow x_1, x_2$ are in the same component of ∂M. In theorem 2, change M to int M in the conclusions.

REFERENCES

[**B**] R. Bowen. On Axiom A diffeomorphisms. *Regional conference series in mathematics*; *no.* **35** (1978).

[**Br**] R. Brown. *The Lefschetz Fixed Point Theorem.* Scott Foresman, 1971.

[**F-L-P**] A. Fathi, F. Laudenbach & V. Poénaru. Travaux de Thurston sur les surfaces. *Asterisque* **66–67** 1979.

https://doi.org/10.1017/S0143385700003011 Published online by Cambridge University Press

[F] J. Franks. Anosov diffeomorphisms. *Proceedings of the Symposium in Pure Mathematics*, **14** pp. 61–94.
[L] J. Lewowicz. Persistence in expansive systems. Preprint.
[T] W. Thurston. On the geometry and dynamics of diffeomorphisms of surface. Preprint.