Canad. J. Math. Vol. 72 (5), 2020 pp. 1304–1323
http://dx.doi.org/10.4153/S0008414X19000324

Peter–Weyl Iwahori Algebras

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Abstract. The Peter–Weyl idempotent $e_{\mathcal{P}}$ of a parahoric subgroup $\mathcal{P}$ is the sum of the idempotents of irreducible representations of $\mathcal{P}$ that have a nonzero Iwahori fixed vector. The convolution algebra associated with $e_{\mathcal{P}}$ is called a *Peter–Weyl Iwahori algebra*. We show that any Peter–Weyl Iwahori algebra is Morita equivalent to the Iwahori–Hecke algebra. Both the Iwahori–Hecke algebra and a Peter–Weyl Iwahori algebra have a natural conjugate linear anti-involution $\star$, and the Morita equivalence preserves irreducible hermitian and unitary modules. Both algebras have another anti-involution, denoted by $\bullet$, and the Morita equivalence preserves irreducible and unitary modules for $\bullet$.

1 Introduction

Let k be a non-archimedean local field with ring of integers $\mathcal{R}_k$ and prime ideal $\wp_k$. Suppose $\mathcal{G} = \mathbf{G}(k)$ is the group of k-rational points of a split reductive group defined over k (for convenience, we also assume simple). After the choice of a Haar measure on $\mathcal{G}$, the vector space $C_c^\infty(\mathcal{G})$ of locally constant compactly supported functions is a convolution algebra, and any smooth representation (π, V_π) of $\mathcal{G}$ integrates to a representation of $C_c^\infty(\mathcal{G})$. The algebra $C_c^\infty(\mathcal{G})$ has a conjugate linear anti-involution $\star$ given by $f^\star(g) = \overline{f(g^{-1})}$. So while $C_c^\infty(\mathcal{G})$ is a $\star$-algebra, since it is not complete, we cannot call it a $\mathsf{C}^\star$-algebra. The algebra $C_c^\infty(\mathcal{G})$ is used to transfer problems of analysis on the group $\mathcal{G}$ to algebraic problems. In particular, we are interested in the Bernstein component of (smooth irreducible) representations with nonzero Iwahori fixed vectors. In this setting, we fix an Iwahori subgroup $\mathcal{I}$ and replace $C_c^\infty(\mathcal{G})$ by the Iwahori–Hecke algebra $\mathcal{H} := \mathcal{H}(\mathcal{G}, \mathcal{I})$ of $\mathcal{I}$–bi-invariant locally constant compactly supported functions. The Iwahori–Hecke algebra inherits a star operation from $C_c^\infty(\mathcal{G})$.

Definition 1.1 A smooth representation (π, V_π) of $\mathcal{G}$ admits an invariant hermitian form if there is a nontrivial hermitian form $\langle\cdot,\cdot\rangle$ satisfying

$$\langle \pi(x)v_1, \pi(x)v_2\rangle = \langle v_1, v_2\rangle.$$

For $f \in C_c^\infty(\mathcal{G})$, we get $\langle \pi(f)v_1, v_2\rangle = \langle v_1, \pi(f^\star)v_2\rangle$. If (π, V_π) is irreducible, the form is unique up to a nonzero scalar. The representation (π, V_π) is said to be unitarizable if V_π admits a positive definite invariant hermitian form.

Received by the editors October 4, 2018; revised May 19, 2019.
Published online on Cambridge Core June 21, 2019.
Author D.B. is partly supported by NSA grant H98230-16-1-0006. Author A.M. is partly supported by Hong Kong Research Grants Council grant CERG #16301915.
AMS subject classification: **22E50**, 22E35.
Keywords: convolution algebra, Iwahori–Hecke algebra, idempotent, Morita equivalence, parahoric subgroup, Peter–Weyl idempotent, $\star$-algebra.

The results in [BM1] and [BM2] establish that a representation V with Iwahori-fixed vectors is unitary if and only if the representation of $\mathcal{H}$ on $V^{\mathcal{I}}$ is unitary. In this setting, the fact that $\mathcal{I}$ is invariant under $\star$ is essential, but appears so obvious that it is hardly ever mentioned. The $\star$-operation on the Iwahori–Hecke algebra can be given explicitly in terms of the generators and relations of $\mathcal{H}$ (see section 5 [BM2]). So the problem of unitarity is reduced to the problem of classifying the finite dimensional unitary representations for $\star$ for $\mathcal{H}$.

Loosely, the Lefschetz principle states that results for real reductive groups should have analogues for the corresponding p-adic groups. It is natural to want to investigate the possibility of duplicating the work in [ALTV] for the Iwahori–Hecke algebra and more generally for the whole unitary dual of a p-adic group. It is extremely useful to consider the relation between signatures of several $\star$ operations at the same time. Such a comparison leads to algorithms for computing signatures of hermitian forms. In more detail, $\star$ operations correspond to real forms of a complex group. A particular real form is then compared to the compact form.

For the case of the Iwahori–Hecke algebra, an analogue of the $\star$ of the compact real form is $\bullet$ defined at the end of Section 2. It makes sense to consider hermitian and unitary modules for the $\bullet$-involution. A comparison between hermitian modules for $\bullet$ and $\star$ (with a view towards the aforementioned goal) was initiated in [BC]. Modules that are unitary for $\bullet$ have striking properties, such as being semisimple with respect to the *vector part* of the Iwahori–Hecke algebra. They are also closely related to local factors of automorphic forms.

It is natural to ask the question whether $\bullet$ comes from a conjugate linear anti-involution of the whole group (as in the real case). A natural candidate, in view of the relation between $\star$ an $\bullet$, would be to twist $\star$ by $\mathrm{Ad}(x)$ for an appropriately chosen element x. The natural candidate for x would be a pullback of the long Weyl group element. This cannot work, precisely because of the earlier observation that $w_0\mathcal{I} \neq \mathcal{I}$.

In this paper we investigate this issue from a different perspective. We look for an extension of $\bullet$ to the Bernstein block. To achieve this, we associate different algebras with the Bernstein block of unramified representations, which we call *Peter–Weyl Iwahori algebras*; $\bullet$ has a more natural connection to the group in this context. The methods of this paper are elementary and rely on the existence of full idempotents. This elementary observation has previously not been exploited much. It provides an explicit equivalence between the two categories as a tensoring process. To the extent that the arguments are formal, they apply to more general types as well.

Casselman and Borel originally established that every subquotient of an unramified principal series contains a non-zero Iwahori fixed vector, and conversely, if an irreducible representation has a non-zero Iwahori fixed vector, then it occurs as quotient of an unramified principal series. They did their work in the context of admissible representations. Later Bernstein discovered the best context to formulate results of this type is in the category of smooth representations. In particular, the unramified twists of a cuspidal representation of a Levi subgroup determine a direct summand of the category of smooth representations. Contemporaneous efforts to describe this summand representation theoretically in terms of representations of compact open subgroups were pioneered by Howe–Moy and Bushnell–Kutzko. A general notion of a special idempotent can be found in [BK], and they establish necessary conditions for a test function to be a special idempotent. These can be verified for our $e_{\mathcal{P}}$. The results in [BM1, BM2] establish the preservation of the $\star$-structure, and are generalized in [BC] and [C], so some of the results are not new. As already mentioned, the main focus of this paper is $\bullet$. We have provided a uniform treatment of both $\star$ and $\bullet$. We believe the Peter–Weyl Iwahori algebras are the more natural setting for the results in [BM1, BM2], and that the unified treatment of both $\star$ and $\bullet$ sheds new light on the role of conjugate anti-automorphisms.

In Bernstein's treatment, he gives a realization of the Iwahori–Hecke algebra as $\operatorname{End}(\mathcal{P})$, for the projective $\mathcal{P}$ associated with this block. For his presentation, it is natural to use the $\star$ that takes an operator to its hermitian dual. Results in [BC] show that, under the natural equivalence to the usual $\mathcal{H}$, this $\star$ corresponds to $\bullet$. So hermitian and unitary is not preserved under this equivalence. In our opinion, this makes the study of the Peter–Weyl algebras of additional interest.

The graded Hecke algebra possesses analogous $\star$ and $\bullet$ anti-involutions. The $\star$ involution is defined in terms of generators and relations in [BM2], while both involutions are treated from a different point of view in [Op].

We now give a more detailed version of the results in the paper. An important motivation to study the idempotents $e_{\mathcal{P}}$ for parahoric subgroups is that they are canonically attached to a facet $f(\mathcal{P})$ in the Bruhat–Tits building $B(\mathcal{G})$ of the group $\mathcal{G}$. The group $\mathcal{G}$ acts simplicially on the building $B(\mathcal{G})$, and the correspondence $f(\mathcal{P}) \mapsto e_{\mathcal{P}}$ is an equivariant process. One can then consider the Euler–Poincaré sum over the facets of the building. In [BCM], we prove that this Euler–Poincaré sum is a presentation of the Bernstein projector for a Bernstein component.

Suppose $\mathcal{P}$ is a parahoric subgroup containing our chosen Iwahori subgroup $\mathcal{I}$. Set Ξ to be the set of irreducible representations of $\mathcal{P}$ that contain the trivial representation of $\mathcal{I}$. We define the Peter–Weyl idempotent to be the idempotent

$$e_{\mathcal{P}} := \frac{1}{\operatorname{meas}(\mathcal{P})} \sum_{\sigma\in\Xi} \deg(\sigma)\Theta_{\sigma},$$

and we define the Peter–Weyl Iwahori algebra as

$$\mathcal{H}(\mathcal{G}, e_{\mathcal{P}}) := e_{\mathcal{P}} \star C_c^{\infty}(\mathcal{G}) \star e_{\mathcal{P}}.$$

When $\mathcal{P}$ equals $\mathcal{I}$, the Peter–Weyl Iwahori algebra $\mathcal{H}(\mathcal{G}, e_{\mathcal{P}})$ is the Iwahori–Hecke algebra. For any $\mathcal{P} \supset \mathcal{I}$, it is known (see Proposition 3.4) that $e_{\mathcal{P}} \star e_{\mathcal{I}} = e_{\mathcal{I}} = e_{\mathcal{I}} \star e_{\mathcal{P}}$; consequently, $\mathcal{H}(\mathcal{G}, e_{\mathcal{I}}) \subset \mathcal{H}(\mathcal{G}, e_{\mathcal{P}})$.

The Peter–Weyl Iwahori algebra $\mathcal{H}(\mathcal{G}, e_{\mathcal{P}})$ inherits a $\star$-algebra structure from $C_c^{\infty}(\mathcal{G})$. The problem we are concerned with in this paper, is to show the following:

(i) Each Peter–Weyl Iwahori algebra is Morita equivalent to the Iwahori–Hecke algebra, and furthermore, the Morita equivalence preserves $\star$-hermitian and unitary modules. The equivalence is established by showing the idempotent $e_{\mathcal{I}} \in \mathcal{H}(\mathcal{G}, e_{\mathcal{P}})$ is a full idempotent, *i.e.*, $e_{\mathcal{P}} \in \mathcal{H}(\mathcal{G}, e_{\mathcal{P}}) \star e_{\mathcal{I}} \star \mathcal{H}(\mathcal{G}, e_{\mathcal{P}})$. In fact, $e_{\mathcal{I}}$ belongs to $\mathcal{H}(\mathcal{P}, e_{\mathcal{P}})$ and is already a full idempotent, *i.e.*, $e_{\mathcal{P}} \in \mathcal{H}(\mathcal{P}, e_{\mathcal{P}}) \star e_{\mathcal{I}} \star \mathcal{H}(\mathcal{P}, e_{\mathcal{P}})$.

(ii) Each Peter–Weyl Iwahori algebra possesses an anti-involution $\bullet$ that restricts to the $\bullet$ involution on the Iwahori–Hecke algebra, and the Morita equivalence preserves $\bullet$-hermitian and unitary modules.

A recent paper, [C], gives alternatives to the techniques in [BM1,BM2]. The results on the cocenter used to prove preservation of unitarity are more complicated than ours. We rely on the explicit equivalence and Rieffel's version [R] of Morita equivalence. We only use the algebraic considerations in [R], so the completeness assumption for $C^\star$-algebras is not relevant.

2 Preliminaries on the Iwahori–Hecke Algebra

2.1 Notation

- $k \supset \mathcal{R}_k \supset \wp_k$, $\mathbf{G}$, $\mathcal{G} = \mathbf{G}(k)$ are as in the introduction. Set $\mathbb{F} = \mathcal{R}_k/\wp_k$, and let q denote the order of $\mathbb{F}$.
- For any k-subgroup $\mathbf{L} \subset \mathbf{G}$, let $\mathcal{L} = \mathbf{L}(k)$ denote the group of k-rational points. Denote by $\mathfrak{g}$ and $\mathfrak{l}$ the obvious Lie subalgebras of $(\mathrm{Lie}(\mathbf{G}))(k)$.
- Let $\mathbf{S} \subset \mathbf{G}$ denote a maximal split k-torus (which we can, in fact, assume defined over $\mathcal{R}$), and $\mathcal{S} = \mathbf{S}(k)$. So the characters $Y = \mathrm{Hom}(\mathbf{S}, \mathbf{G}_m)$ are naturally paired with the cocharacters $X = \mathrm{Hom}(\mathbf{G}_m, \mathbf{S})$. Set $\mathcal{S}^0 = \mathbf{S}(\mathcal{R})$, and denote by $\mathcal{S}^+$ the maximal pro-p-subgroup of $\mathcal{S}^0$.
- R is the set of roots of $\mathbf{G}$ with respect to $\mathbf{S}$. For a root α, we denote by $\mathbf{U}_\alpha \subset \mathbf{G}$ and $\mathcal{U}_\alpha = \mathbf{U}_\alpha(k) \subset \mathcal{G}$, the corresponding root groups. For a choice $R^+ \subset R$ of positive roots R^+, let $\mathbf{B}$ and $\mathcal{B} = \mathbf{B}(k)$ be the associated Borel subgroups and $\Pi \subset R^+$, the simple roots.

The choice of a Chevalley basis of $\mathfrak{g}$ allows us to define $\mathbf{G}$ and $\mathbf{U}_\alpha$ over the integers $\mathcal{R}_k$ and thus over $\mathbb{F}$ too (we write $\mathbf{G} \times_{\mathcal{R}_k} \mathbb{F}$ for the group over $\mathbb{F}$), so that $\mathbf{G} \times_{\mathcal{R}_k} \mathbb{F}$ is a connected reductive split $\mathbb{F}$-group and there is a canonical identification of the root systems of $\mathbf{G}$ and $\mathbf{G} \times_{\mathcal{R}_k} \mathbb{F}$.

- Let $\mathcal{B}$ denote the Bruhat–Tits building of $\mathcal{G}$. The torus $\mathbf{S}$ (defined over $\mathcal{R}_k$) yields an apartment $\mathcal{A} \subset \mathcal{B}$, and $\mathcal{B}$ is the union of all its apartments. The Chevalley basis above allows us to do the following:

(i) embed Y inside $\mathcal{A}$, so that the origin 0 becomes a hyperspecial point;
(ii) define the set of affine roots $\Psi = \{\alpha + j \mid \alpha \in R, j \in \mathbb{Z}\}$ on $\mathcal{A}$ and for each affine root ψ an affine root groups $\mathcal{U}_\psi \subset \mathcal{U}_{\mathrm{grad}(\psi)}$.

The assumption that $\mathbf{G}$ is split simple means $\mathcal{B}$ is a simplicial complex. For any facet $E \subset \mathcal{B}$, let $\mathcal{G}_E$ be the associated parahoric subgroup. When $E \subset \mathcal{A}$,

$$\begin{aligned}\mathcal{G}_E &= \text{subgroup of } \mathcal{G} \text{ generated by } \mathcal{S}^0 \text{ and } \mathcal{U}_\psi\\ &\qquad (\psi \text{ satisfying } \psi(x) \geq 0 \text{ for all } x \in E),\\ \mathcal{G}_E^+ &= \text{subgroup of } \mathcal{G} \text{ generated by } \mathcal{S}^+, \text{ and } \mathcal{U}_\psi\\ &\qquad (\psi \text{ satisfying } \psi(x) > 0 \text{ for all } x \in E).\end{aligned}$$

- Fix a Haar measure on $\mathcal{G}$ and therefore a convolution product $\star$ on $C_c^\infty(\mathcal{G})$. For any open compact subgroup $J \subset \mathcal{G}$, let 1_J denote the characteristic function,

https://doi.org/10.4153/S0008414X19000324 Published online by Cambridge University Press

and set

$$e_J := \frac{1}{\text{meas}(J)} 1_J.$$

When a facet $E \subset \mathcal{B}$ is of maximal dimension, *i.e.*, E is a chamber, the parahoric subgroup $\mathcal{J} = \mathcal{G}_E$ is an Iwahori subgroup. Set

$$\mathcal{H}(\mathcal{G}, \mathcal{J}) := e_{\mathcal{J}} \star C_c^\infty(\mathcal{G}) \star e_{\mathcal{J}} \quad \text{(Iwahori–Hecke algebra with respect to } \mathcal{J}\text{)}.$$

We recall that any two chambers of $\mathcal{B}$ belong to the same $\mathcal{G}$ orbit, so any two Iwahori subgroups are conjugate in $\mathcal{G}$.

The choices of a set of positive roots R^+ and a Chevalley basis singles out a particular Iwahori subgroup $\mathcal{I}$ that can be described as follows. For the facet $\{0\} \subset \mathcal{A}$ and its maximal parahoric subgroup $\mathcal{G}_{\{0\}}$, we consider the quotient map $\mathcal{G}_{\{0\}} \to (\mathcal{G}_{\{0\}}/\mathcal{G}^+_{\{0\}}) = (\mathbf{G} \times_{\mathcal{R}_k} \mathbb{F})(\mathbb{F})$. Then, $\mathcal{I}$ is the inverse image of the Borel subgroup of $(\mathbf{G} \times_{\mathcal{R}_k} \mathbb{F})(\mathbb{F})$ corresponding to the positive roots R^+.

We recall that the Iwahori–Hecke algebra $\mathcal{H}(\mathcal{G}, \mathcal{I})$ has a presentation in terms of the finite Iwahori–Hecke agebra $\mathcal{H}(\mathcal{G}_{\{0\}}, \mathcal{I})$ and X (which can viewed as a rational functions on the torus $Y \otimes_{\mathbb{Z}} \mathbb{C}^\times$), given as follows:

(i) Let $\mathbf{N}$ (defined over $\mathcal{R}_k$) be the normalizer of $\mathbf{S}$. For each $n \in \mathbf{N}(\mathcal{R}_k) \subset \mathcal{G}_{\{0\}}$, set

$$T_n = \frac{1}{\text{meas}(\mathcal{I})} 1_{\mathcal{I} n \mathcal{I}} \in C_c^\infty(\mathcal{G}) \qquad \text{(depends only on the coset } n\mathcal{S}^0\text{)}.$$

If $n_1, n_2 \in \mathbf{N}(\mathcal{R}_k)$ have the length property that $\ell(n_1 n_2) = \ell(n_1) + \ell(n_2)$, then $T_{n_1} \star T_{n_2} = T_{n_1 n_2}$.

For each simple root α, let $t_\alpha \in \mathbf{N}(\mathcal{R}_k)$ be an element whose action on X is the reflection s_α, and set

$$T_{s_\alpha} = \frac{1}{\text{meas}(\mathcal{I})} 1_{\mathcal{I} t_\alpha \mathcal{I}}.$$

Then $T_{s_\alpha}^2 = (q-1) T_{s_\alpha} + q\mathrm{I}$.

(ii) There is an embedding, due to Bernstein (see [Lz1, §3], [Lz2, §4]),

$$\begin{aligned} X &\longrightarrow \mathcal{H}(\mathcal{G}, \mathcal{I}) \\ x &\longrightarrow \quad \Theta_x \end{aligned}$$

satisfying

$$\Theta_x T_{s_\alpha} = T_{s_\alpha} \Theta_{s_\alpha(x)} + (q-1) \frac{\Theta_x - \Theta_{s_\alpha(x)}}{1 - \Theta_{-\alpha}}.$$

(iii) The set of elements $\{\Theta_x T_n \mid x \in X, n \in \mathbf{N}(\mathcal{R}_k)/\mathcal{S}^0\}$ is a (complex) basis of $\mathcal{H}(\mathcal{G}, \mathcal{I})$.

- The space of functions $C_c^\infty(\mathcal{G})$ admits a natural anti-involution $\star$ given by

$$f^\star(g) := \overline{f(g^{-1})}. \tag{2.1}$$

That $\mathcal{I}$ is a subgroup means the anti-involution restricts to an anti-involution of $\mathcal{H}(\mathcal{G}, \mathcal{I})$.

- The algebra $\mathcal{H}(\mathcal{G}, \mathcal{I})$ has another anti-involution $\bullet$ (see [BC]) defined in terms of the generators T_n ($n \in \mathbf{N}(\mathcal{R}_k)$), and Θ_x ($x \in X$), given by

$$T_n^\bullet = T_{n^{-1}} \text{for } n \in \mathbf{N}(\mathcal{R}_k) \text{ and } \Theta_x^\bullet = \Theta_x \text{for } x \in X.$$

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• Let $n_0 \in \mathbf{N}(\mathcal{R}_k)$ be a representative for the longest element in the Weyl group $\mathbf{N}(k)/\mathcal{S}$. There is an involution $\mathfrak{a}$ of the group $\mathcal{G}$ so that $\mathcal{S}$, $\mathcal{N} = \mathbf{N}(k)$, and $\mathcal{I}$ are $\mathfrak{a}$-invariant, and

$$\mathfrak{a}(x) = n_0 x^{-1} n_0^{-1} \qquad \forall x \in \mathcal{S},$$
$$\mathfrak{a}(n) = n_0 n n_0^{-1} \bmod \mathcal{S} \quad \forall n \in \mathcal{N}.$$

For the case of the group $\mathrm{GL}(n)$, and the standard representation realization of classical groups, the involution $\mathfrak{a}$ is

$$\mathfrak{a}(g) = n_0 (g^{-1})^{\mathrm{T}} n_0^{-1},$$

where T is transpose, and $n_0 \in \mathbf{N}(\mathcal{R}_k)$ is a monomial matrix representative of the longest Weyl element.

The involution $\mathfrak{a}$, defines an involution of the Iwahori–Hecke algebra $\mathcal{H}(\mathcal{G},\mathcal{I})$, and the relationship between the two anti-involutions $\star$ and $\bullet$ is:

$$\bullet(y) = T_{n_0}^{-1} \mathfrak{a}(y^\star) T_{n_0} \qquad \forall y \in \mathcal{H}(\mathcal{G},\mathcal{I}).$$

3 Idempotents

Let $\mathcal{J}$ be an Iwahori subgroup. The collection of pairs consisting of a minimal k-Levi subgroup of $\mathcal{G}$, *i.e.*, a maximal split torus $\mathcal{T}$, and the unramified characters of $\mathcal{T}$ is the cuspidal data to parametrize a full subcategory Ω (Bernstein component) of the smooth representations (of $\mathcal{G}$). Equivalently, Ω is the full subcategory of representations generated by their $\mathcal{J}$ fixed vectors. Furthermore, there is an essentially compact distribution P_Ω (representable by a locally L^1-function on the regular compact elements of $\mathcal{G}$), so that for any smooth representation (π, V_π), the endomorphism $\pi(P_\Omega) \in \mathrm{End}_{\mathcal{G}}(V_\pi)$ is an idempotent that projects to the Ω component of V_π.

Suppose $F \subset \mathcal{B}$ is a chamber (so $\mathcal{G}_F$ is an Iwahori subgroup), and E is a facet in F (so $\mathcal{G}_E \supset \mathcal{G}_F \supset \mathcal{G}_F^+ \supset \mathcal{G}_E^+$). The group $\mathcal{G}_F/\mathcal{G}_E^+$ is a Borel subgroup of $\mathcal{G}_E/\mathcal{G}_E^+$. Let ρ be the quotient map from $\mathcal{G}_E$ to $\mathcal{G}_E/\mathcal{G}_E^+$. We define the Peter–Weyl idempotent associated with the facet E as (see also [BCM])

$$e_E := \frac{1}{\mathrm{meas}(\mathcal{G}_E)} \Big(\sum_{\sigma \in \Xi} \deg(\sigma) \Theta_\sigma \circ \rho \Big),$$

where Ξ is the collection of irreducible representations of $\mathcal{G}_E/\mathcal{G}_E^+$ that contain a nonzero (Borel) $\mathcal{G}_F/\mathcal{G}_E^+$-fixed vector, and Θ_σ is the character of σ.

(i) If F' is another chamber containing E, then $\mathcal{G}_{F'}/\mathcal{G}_E^+$ is a Borel subgroup of $\mathcal{G}_E/\mathcal{G}_E^+$ too, and the right side of the above definition yields the same idempotent e_E.

(ii) For a chamber F, the idempotent e_F equals $e_{\mathcal{G}_F}$. In this situation, we will use both notations.

Definition 3.1 The Peter–Weyl Iwahori algebra associated with the idempotent e_E is the algebra

$$\mathcal{H}_E := e_E \star C_c^\infty(\mathcal{G}) \star e_E.$$

For convenience, we sometimes abbreviate the name to Peter–Weyl algebra. If F is a chamber, then $\mathcal{H}_F$ is the Iwahori–Hecke algebra $\mathcal{H}(\mathcal{G},\mathcal{G}_F)$.

Proposition 3.2 *Let P_Ω be the projector for the unramified Bernstein component Ω. Suppose $E \subset \mathcal{B}$ is a facet. Then*

$$e_E = P_\Omega \star \frac{1}{\operatorname{meas}(\mathcal{G}_E^+)} 1_{\mathcal{G}_E^+}.$$

To prove Proposition 3.2, we first establish the following lemma.

Lemma 3.3 *If $\phi, \psi \in C_c^\infty(\mathcal{G})$ satisfy $\pi(\phi) = \pi(\psi)$ for all (irreducible, smooth) π, then $\phi = \psi$.*

Proof Let $d\mu$ denote the Plancherel measure on $\widehat{\mathcal{G}}_{\text{temp}}$. By the Plancherel formula,

$$\forall F \in C_c^\infty(\mathcal{G}) : F(1) = \int_{\widehat{\mathcal{G}}_{\text{temp}}} \operatorname{trace}(\pi(F)) d\mu(\pi).$$

For $\phi \in C_c^\infty(\mathcal{G})$ and $x \in \mathcal{G}$, set $F_x(\phi) = \delta_x \star \phi$, so $F_x(\phi)(g) = \phi(x^{-1}g)$. If ϕ, $\psi \in C_c^\infty(\mathcal{G})$ and $\pi(\phi) = \pi(\psi)$ for all π, then for all $x \in \mathcal{G}$ we have $\pi(F_x(\phi)) = \pi(\delta_x \star \phi) = \pi(\delta_x \star \psi) = \pi(F_x(\psi))$. Thus,

$$\begin{aligned}\phi(x^{-1}) = F_x(\phi)(1) &= \int_{\widehat{\mathcal{G}}_{\text{temp}}} \operatorname{trace}(\pi(F_x(\phi)) d\mu(\pi) \\ &= \int_{\widehat{\mathcal{G}}_{\text{temp}}} \operatorname{trace}(\pi(F_x(\psi)) d\mu(\pi) = F_x(\psi)(1) = \psi(x^{-1}).\end{aligned}$$

So $\phi = \psi$. ∎

We note that we can replace $\mathcal{G}$ by a compact group J, and the analogous result holds for any two $\phi, \psi \in C_c^\infty(J)$.

Proof of Proposition 3.2 Suppose (π, V_π) is a smooth irreducible representation. The operator $\pi(P_\Omega) \in \operatorname{End}(V_\pi)$ is the scalar 1 if π has a nonzero Iwahori $\mathcal{G}_F$-fixed vector and the scalar 0 otherwise. The operator $\pi(P_\Omega \star e_{\mathcal{G}_E^+})$ is projection to the subspace $V_\pi^{\mathcal{G}_E^+}$. By [MP1, Theorem 5.2] and [MP2, Proposition 6.2], if π has a nonzero $\mathcal{G}_F$-vector, *i.e.*, an unrefined depth zero minimal K-type consisting of the trivial representation of $\mathcal{G}_F$, then any other irreducible representation of $\mathcal{G}_E$ in $V_\pi^{\mathcal{G}_E^+}$ must contain a nonzero $\mathcal{G}_F$-fixed vector. Clearly (from the representation theory of finite groups) for any irreducible representation π, the operator $\pi(e_E)$ is projection to the subspace $V_\pi^{\mathcal{G}_E^+}$. It follows that $\pi(P_\Omega \star e_{\mathcal{G}_E^+}) = \pi(e_E)$, and so, by Lemma 3.3, $P_\Omega \star e_{\mathcal{G}_E^+} = e_E$. ∎

Suppose E is a facet inside a chamber F. We recall that $\mathcal{G}_E^+ \subset \mathcal{G}_F^+ \subset \mathcal{G}_F \subset \mathcal{G}_E$ and $\mathcal{G}_E^+$ is a normal subgroup of $\mathcal{G}_E$. The idempotents $e_{\mathcal{G}_E^+}$, e_F, and e_E belong to the finite dimensional algebra

$$\mathcal{H} := e_{\mathcal{G}_E^+} \star C_c^\infty(\mathcal{G}_E) \star e_{\mathcal{G}_E^+}.$$

This algebra is equal to the canonical embedding of $C_c^\infty(\mathcal{G}_E/\mathcal{G}_E^+)$ in $C_c^\infty(\mathcal{G}_E)$ via the quotient map $\rho\colon \mathcal{G}_E \to \mathcal{G}_E/\mathcal{G}_E^+$. It is a consequence of the normality of $\mathcal{G}_E^+$ in $\mathcal{G}_E$ that $C_c^\infty(\mathcal{G}_E) \star e_F = \mathcal{H} \star e_F$ and $e_F \star C_c^\infty(\mathcal{G}_E) = e_F \star \mathcal{H}$. If κ is an irreducible representation of $\mathcal{G}_E$, with character Θ_κ, let

$$e_\kappa := \frac{1}{\operatorname{meas}(\mathcal{G}_E)} \deg(\kappa) \Theta_\kappa$$

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be the (central) idempotent in $C_c^\infty(\mathcal{G}_E)$ associated with κ. Clearly, $e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E) \star e_\kappa$ is an ideal of $C_c^\infty(\mathcal{G}_E)$ that is either a minimal ideal or zero.

Define

$$\mathcal{H}_E^{\mathrm{fin}} := e_E \star C_c^\infty(\mathcal{G}_E) \star e_E, \quad \mathcal{H}_F^{\mathrm{fin}} := e_F \star C_c^\infty(\mathcal{G}_E) \star e_F,$$
$$ {}_E\mathcal{H}_F^{\mathrm{fin}} := e_E \star C_c^\infty(\mathcal{G}_E) \star e_F, \quad {}_F\mathcal{H}_E^{\mathrm{fin}} := e_F \star C_c^\infty(\mathcal{G}_E) \star e_E.$$

Proposition 3.4 *The (finite dimensional) vector space $C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E)$ is a bi-module for $C_c^\infty(\mathcal{G}_E)$,* i.e., *an ideal of $C_c^\infty(\mathcal{G}_E)$, and*

(i) *it equals $\mathcal{H}_E^{\mathrm{fin}}$;*
(ii) *it is the span of matrix coefficients of the representations with $\mathcal{G}_F$-fixed vectors;*
(iii) *$e_E \in C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E)$,* i.e., *$e_F \in C_c^\infty(\mathcal{G}_E) \star e_{\mathcal{G}_F} \star C_c^\infty(\mathcal{G}_E)$ is a full idempotent;*
(iv) *$e_E \star e_F = e_F \star e_E = e_F$;*
(v) *$\mathcal{H}_E^{\mathrm{fin}} = {}_E\mathcal{H}_F^{\mathrm{fin}} \star {}_F\mathcal{H}_E^{\mathrm{fin}}$ and $\mathcal{H}_F^{\mathrm{fin}} = {}_F\mathcal{H}_E^{\mathrm{fin}} \star {}_E\mathcal{H}_F^{\mathrm{fin}}$.*

Proof To prove statement (i), suppose $h_1 \star e_{\mathcal{G}_F} \star h_2 \in C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E)$. For any representation κ of $\mathcal{G}_E$, we have $\kappa(h_1 \star e_F \star h_2) = \kappa(h_1)\kappa(e_F)\kappa(h_2)$. When κ is irreducible, we deduce that $e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E) \star e_\kappa$ is zero if and only if $\kappa(e_F)$ is zero.

Suppose (σ, V_σ) is an irreducible representation of $\mathcal{G}_E/\mathcal{G}_E^+$ that contains a nonzero $\mathcal{G}_F/\mathcal{G}_E^+$-fixed vector, *i.e.*, $\sigma \in \Xi$. Set $\kappa = \sigma \circ \rho$. Then $\kappa(e_F)$ is nonzero, which means that $e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E) \star e_\kappa$ is a nonzero ideal of $C_c^\infty(\mathcal{G}_E)$ that is contained in the minimal ideal $e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_\kappa$. Therefore,

$$e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E) \star e_\kappa = e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_\kappa.$$

Since $e_E = \sum_{\sigma \in \Xi} e_{\sigma \circ \rho}$, we deduce statement (i).

For the sake of completeness, we consider when (κ, V_κ) is an irreducible representation of $\mathcal{G}_E$ that does not contain a nonzero $\mathcal{G}_F$-fixed vector; *i.e.*, $\kappa(e_F)$ is zero. Then $\kappa(h_1 \star e_F \star h_2) = 0$. Consequently, $\kappa(F) = 0$ for any $F \in C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E)$. This means that $e_\kappa \star C_c^\infty(\mathcal{G}_E) \star e_F \star C_c^\infty(\mathcal{G}_E) \star e_\kappa$ is zero.

Statements (ii) and (iii) follow from statement (i).

For statement (iv), the equality $e_E \star e_F = e_F \star e_E$ follows from the fact that e_E is a central element of $C_c^\infty(\mathcal{G}_E)$. The equality $e_F \star e_E = e_F$ follows from the fact that $\kappa(e_E \star e_{\mathcal{G}_F}) = \kappa(e_{\mathcal{G}_F})$ for any irreducible representation κ of $\mathcal{G}_E$ and from Lemma 3.3.

Statement (v) is a consequence of statement (i). ∎

Suppose F is a chamber in the building, so the algebra $\mathcal{H}_F := e_F \star C_c^\infty(\mathcal{G}) \star e_F$ is an Iwahori–Hecke algebra. If E is a facet of F, we have previously named the algebra $\mathcal{H}_E := e_E \star C_c^\infty(\mathcal{G}) \star e_E$ (which contains $\mathcal{H}_F$) the Peter–Weyl Iwahori algebra (associated with E).

Proposition 3.5 *Suppose E is a facet inside a chamber F.*

(i) *The idempotent $e_F \in \mathcal{H}_E$ satisfies $\mathcal{H}_E \star e_F \star \mathcal{H}_E = \mathcal{H}_E$;* i.e., *it is a full idempotent.*
(ii) *Define ${}_E\mathcal{H}_F := e_E \star C_c^\infty(\mathcal{G}) \star e_F$ and ${}_F\mathcal{H}_E := e_F \star C_c^\infty(\mathcal{G}) \star e_E$. Then $\mathcal{H}_E = {}_E\mathcal{H}_F \star {}_F\mathcal{H}_E$ and $\mathcal{H}_F = {}_F\mathcal{H}_E \star {}_E\mathcal{H}_F$.*

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Proof The proof of statement (i) is based on the analogous fact in Proposition 3.4 for the finite dimensional algebra $\mathcal{H}_E^{\mathrm{fin}}$. We have $\mathcal{H}_E^{\mathrm{fin}} = \mathcal{H}_E^{\mathrm{fin}} \star e_F \star \mathcal{H}_E^{\mathrm{fin}}$. Since $\mathcal{H}_E^{\mathrm{fin}}$ contains the identity element e_E of $\mathcal{H}_E$, we have $\mathcal{H}_E \star \mathcal{H}_E^{\mathrm{fin}} = \mathcal{H}_E = \mathcal{H}_E^{\mathrm{fin}} \star \mathcal{H}_E$.

Therefore,

$$\mathcal{H}_E \star e_F \star \mathcal{H}_E = \mathcal{H}_E \star \mathcal{H}_E^{\mathrm{fin}} \star e_F \star \mathcal{H}_E^{\mathrm{fin}} \star \mathcal{H}_E \supset \mathcal{H}_E \star e_E \star \mathcal{H}_E = \mathcal{H}_E,$$

and so e_F is a full idempotent of $\mathcal{H}_E$, too.

Statement (ii) can be obtained from statement (i) as follows:

$$\begin{aligned}\mathcal{H}_E &= \mathcal{H}_E \star e_F \star \mathcal{H}_E = \mathcal{H}_E \star e_F \star e_F \star \mathcal{H}_E\\ &= {}_E\mathcal{H}_F \star {}_E\mathcal{H}_F \qquad (\text{since } {}_E\mathcal{H}_F = \mathcal{H}_E \star e_F \text{ and } {}_F\mathcal{H}_E = e_F \star \mathcal{H}_E).\end{aligned}$$

Similarly, $\mathcal{H}_F = e_F \star \mathcal{H}_E \star e_F = e_F \star \mathcal{H}_E \star \mathcal{H}_E \star e_F = {}_F\mathcal{H}_E \star {}_E\mathcal{H}_F$ ∎

Proposition 3.6 *Suppose F is a chamber of $\mathcal{B}$, and E is a facet contained in F.*

(i) *The left $\mathcal{H}_E$-module ${}_E\mathcal{H}_F$ is cyclic with generator e_F. Similarly, the right $\mathcal{H}_E$-module ${}_F\mathcal{H}_E$ is cyclic with generator e_F.*

(ii) *The left $\mathcal{H}_F$-module ${}_F\mathcal{H}_E$ is finitely generated. Similarly, the right $\mathcal{H}_F$-module ${}_E\mathcal{H}_F$ is finitely generated.*

Proof The first assertion of statement (i) follows from the fact that $e_E \star e_F = e_F$, while the second follows from $e_F \star e_E = e_F$.

To see statement (ii), we use the fact that for any two open compact subgroups J and J' of $\mathcal{G}$, the space $e_J \star C_c^\infty(\mathcal{G}) \star e_{J'}$ is a finitely generated $e_J \star C_c^\infty(\mathcal{G}) \star e_J$ module. We take $J = \mathcal{G}_F$ and $J' = \mathcal{G}_E^+$ to see that $e_{\mathcal{G}_F} \star C_c^\infty(\mathcal{G}) \star e_{\mathcal{G}_E^+}$ ($e_{\mathcal{G}_F}$ is e_F) is a finitely generated $\mathcal{H}_F$-module. If we convolve on the right by e_E, we deduce ${}_F\mathcal{H}_E$ is a finitely generated left $\mathcal{H}_F$-module.

Similar reasoning shows ${}_E\mathcal{H}_F$ is a finitely generated right $\mathcal{H}_F$−module. ∎

4 Morita Equivalence

Let $\mathcal{C}(\mathcal{H}_E)$, $\mathcal{C}_{\mathrm{fg}}(\mathcal{H}_E)$, and $\mathcal{C}_{\mathrm{fin}}(\mathcal{H}_E)$ denote the categories of left, left finitely generated, and left finite-dimensional $\mathcal{H}_E$-modules, respectively, and let $\mathcal{C}(\mathcal{H}_F)$, $\mathcal{C}_{\mathrm{fg}}(\mathcal{H}_F)$, and $\mathcal{C}_{\mathrm{fin}}(\mathcal{H}_F)$ be the analogous categories of (left) $\mathcal{H}_F$-modules.

Theorem 4.1 *The algebras $\mathcal{H}_E$ and $\mathcal{H}_F$ are Morita equivalent.*

(i) *The two maps*

$$\begin{array}{ccc} {}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} {}_F\mathcal{H}_E & \longrightarrow & \mathcal{H}_E \\ f \otimes_{\mathcal{H}_F} g & \longmapsto & f \star g \end{array} \quad \textit{and} \quad \begin{array}{ccc} {}_F\mathcal{H}_E \otimes_{\mathcal{H}_E} {}_E\mathcal{H}_F & \longrightarrow & \mathcal{H}_F \\ g \otimes_{\mathcal{H}_E} f & \longmapsto & g \star f \end{array}$$

are isomorphisms.

(ii) *The maps*

$$\begin{array}{ccc} \mathcal{C}(\mathcal{H}_F) & \longrightarrow & \mathcal{C}(\mathcal{H}_E) \\ X & \longmapsto & {}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} X \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{C}(\mathcal{H}_E) & \longrightarrow & \mathcal{C}(\mathcal{H}_F) \\ Y & \longmapsto & {}_F\mathcal{H}_E \otimes_{\mathcal{H}_E} Y \end{array}$$

are inverses of each other and give an equivalence between the categories of left $\mathcal{H}_E$-modules and left $\mathcal{H}_F$-modules.

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(iii) *The category equivalences of part* (i) *restrict to equivalences between* $\mathcal{C}_{\mathrm{fg}}(\mathcal{H}_E)$ *and* $\mathcal{C}_{\mathrm{fg}}(\mathcal{H}_F)$ *and between* $\mathcal{C}_{\mathrm{fin}}(\mathcal{H}_E)$ *and* $\mathcal{C}_{\mathrm{fin}}(\mathcal{H}_F)$.

Proof The statements follow from the fact that $e_F \in \mathcal{H}_E$ is a full idempotent ($\mathcal{H}_E = \mathcal{H}_E \star e_F \star \mathcal{H}_E$) (see [Lm, Chapter 18]). ∎

We remark that there is a similar Morita equivalence between $\mathcal{H}_F^{\mathrm{fin}}$ and $\mathcal{H}_E^{\mathrm{fin}}$.

The Morita equivalence of $\mathcal{H}_E$ and $\mathcal{H}_F$ means their centers are isomorphic. The center of the Peter–Weyl algebra $\mathcal{H}_E$ can be obtained from the (well known) center of the Iwahori–Hecke algebra $\mathcal{H}_F$ via the following result.

Corollary 4.2 *Express* e_E *as* $e_E = \sum_{i=1}^r a_i \star b_i$ *with* $a_i \in {}_E\mathcal{H}_F$ *and* $b_i \in {}_F\mathcal{H}_E$. *Then the isomorphism of centers in the Morita equivalence of* $\mathcal{H}_F$ *and* $\mathcal{H}_E$ *is given by*

$$z \longrightarrow \sum_{i=1}^{r} a_i \star z \star b_i.$$

Proof If z is in the center of $\mathcal{H}_F$, then for any $X \in \mathcal{C}(\mathcal{H}_F)$, the map $x \to zx$ commutes with any self-morphism of X; *i.e.*, it is in the center of the category. Under the functor $X \to {}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} X$, we have a similar self-morphism $(f \otimes_{\mathcal{H}_F} x) \to (f \otimes_{\mathcal{H}_F} zx)$ of ${}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} X$ in the category $\mathcal{C}(\mathcal{H}_E)$. We compute

$$\begin{aligned} f \otimes_{\mathcal{H}_F} zx = (e_E \star f) \otimes_{\mathcal{H}_F} zx &= \sum_{i=1}^{r} a_i \star b_i \star f \otimes_{\mathcal{H}_F} zx \\ &= \sum_{i=1}^{r} a_i \otimes_{\mathcal{H}_F} (b_i \star f) zx \qquad (\text{since } b_i \star f \in \mathcal{H}_F) \\ &= \sum_{i=1}^{r} a_i \otimes_{\mathcal{H}_F} z \star (b_i \star f) x = \sum_{i=1}^{r} a_i \star (z \star (b_i \star f)) \otimes_{\mathcal{H}_F} x \\ &= \Big(\sum_{i=1}^{r} a_i \star z \star b_i\Big)(f \otimes_{\mathcal{H}_F} x). \end{aligned}$$

So the element $\sum_{i=1}^r a_i \star z \star b_i$ is the central element of $\mathcal{H}_E$ corresponding to z. ∎

5 Matrix Coefficients

We fix a Haar measure on $\mathcal{G}$ and make $C_c^\infty(\mathcal{G})$ into a convolution algebra. The map $\star$ defined in (2.1) is an anti-involution. For any facet E, the idempotent e_E is fixed by $\star$, *i.e.*, $e_E^\star = e_E$, and therefore $\star$ is an anti-involution of the Peter–Weyl algebra $\mathcal{H}_E$. When F is a chamber, the $\star$ anti-involution on the Iwahori–Hecke algebra $\mathcal{H}_F$ is the one mentioned in the introduction.

In the Iwahori–Hecke algebra situation, using generators, [BC] defined an anti-involution $\bullet$. In this section we show that given a facet E and a chamber F containing E (so $\mathcal{H}_F \subset \mathcal{H}_E$), we can define an anti-involution that we also denote as $\bullet$. The involution depends on the chamber chosen (equivalently the Iwahori subgroup inside $\mathcal{G}_E$); but, since any two Iwahori subgroups are conjugate, the difference is up to a conjugation. To define $\bullet$, we need to exhibit a decomposition of $\mathcal{H}_E$ in terms of $\mathcal{H}_F$.

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In the next section we show the Morita equivalence between the Iwahori–Hecke algebra $\mathcal{H}_F$ and the Peter–Weyl Iwahori algebra $\mathcal{H}_E$ preserves $\star$-hermitian and $\star$-unitary representations as well as $\bullet$-hermitian and $\bullet$-unitary representations.

5.1 Preliminaries

Lemma 5.1 *Suppose (σ, V_σ) and (τ, V_τ) are irreducible representations of a compact group K, with invariant positive definite forms $\langle\cdot,\cdot\rangle_\sigma$ and $\langle\cdot,\cdot\rangle_\tau$. If the two representations are equivalent, we assume they are equal (and abbreviate the inner product to $\langle\cdot,\cdot\rangle$). Suppose $x_1, x_2 \in V_\sigma$ and $y_1, y_2 \in V_\tau$. Then*

$$\int_K \langle x_1, \sigma(h)x_2\rangle_\sigma \overline{\langle y_1, \tau(h)y_2\rangle_\tau}\, dh = \begin{cases} 0 & \text{if } \sigma \text{ is not equivalent to } \tau, \\ \frac{\operatorname{meas}(K)}{\deg(\sigma)} \langle x_1, y_1\rangle \overline{\langle x_2, y_2\rangle} & \text{if } \sigma = \tau. \end{cases}$$

Proof These are the Schur orthogonality relations. The case when the representations are inequivalent is clear. When they are equivalent, denote them both by σ. The tensor product $V_\sigma \otimes V_\sigma$ has two $(K\times K)$-invariant form given by $\langle x_1\otimes y_1, x_2\otimes y_2\rangle_{\sigma\otimes\sigma} = \langle x_1, y_1\rangle\overline{\langle x_2, y_2\rangle}$ and $\langle\langle x_1\otimes y_1, x_2\otimes y_2\rangle\rangle := \int_K \langle x_1, \sigma(h)x_2\rangle\overline{\langle y_1, \sigma(h)y_2\rangle}\, dh$, and so they must be scalar multiples of one another. Evaluation of the scalar yields

$$\langle\langle\cdot,\cdot\rangle\rangle = \frac{\operatorname{meas}(K)}{\deg(\sigma)}\langle\cdot,\cdot\rangle_{\sigma\otimes\sigma}. \qquad \blacksquare$$

When (σ, V_σ) is an irreducible representation of K, and $u, v \in V_\sigma$, we define the matrix coefficient $m^\sigma_{u,v}$ as

$$m^\sigma_{u,v}(k) := \langle u, \sigma(k)v\rangle. \tag{5.1}$$

Corollary 5.2 (i) *With the same notation as in Lemma 5.1,*

$$m^\sigma_{x_1,x_2} \star m^\tau_{y_1,y_2} = \begin{cases} 0 & \text{if } \sigma \text{ is not equivalent to } \tau, \\ \frac{\operatorname{meas}(K)}{\deg(\sigma)} \overline{\langle x_2, y_1\rangle} m^\sigma_{x_1,y_2} & \text{if } \sigma = \tau. \end{cases}$$

(ii) *If $v \in V_\sigma$ satisfies*

$$\langle v, v\rangle = \frac{\deg(\sigma)}{\operatorname{meas}(\mathcal{G}_E)}, \tag{5.2}$$

then the function $m^\sigma_{v,v}$ is a convolution idempotent.

Proof This is obvious. $\blacksquare$

Recall that $(m^\sigma_{x_1,x_2})^\star(k) = \overline{\langle x_1, \sigma(k^{-1})x_2\rangle} = \overline{\langle \sigma(k)x_1, x_2\rangle} = \langle x_2, \sigma(k)x_1\rangle = m^\sigma_{x_2,x_1}$. Thus,

$$(m^\sigma_{x_1,x_2})^\star \star m^\sigma_{y_1,y_2} = m^\sigma_{x_2,x_1} \star m^\sigma_{y_1,y_2} = \frac{\operatorname{meas}(K)}{\deg(\sigma)}\overline{\langle x_1, y_1\rangle} m^\sigma_{x_2,y_2}.$$

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We also note that if λ_h (resp. ρ_h) is the left (resp. right) translation representation, *i.e.*, $(\lambda_h(f))(k) = f(h^{-1}k)$ and $(\rho_h(f))(k) = f(kh)$, then

$$\lambda_h(m^\sigma_{u,v}) = m^\sigma_{\sigma(h)u,v} \qquad \text{and} \qquad \rho_h(m^\sigma_{u,v}) = m^\sigma_{u,\sigma(h)v}.$$

5.2 Decompositions

Suppose F is a chamber in $\mathcal{B}$, and E is a facet in F. Assume (σ, V_σ) and (τ, V_τ) are irreducible representations of $\mathcal{G}_E$ with a nonzero $\mathcal{G}_F$ fixed vector. We fix invariant positive definite forms $\langle\cdot,\cdot\rangle_\sigma$ and $\langle\cdot,\cdot\rangle_\tau$ on V_σ and V_τ respectively. For any $x, y \in V_\sigma$ and $k \in \mathcal{G}_E$, define the matrix coefficient $m^\sigma_{x,y}(k) := \langle x, \sigma(k)y\rangle$ as in (5.1). Suppose $a, b \in V_\sigma$. When $v \in V_\sigma^{\mathcal{G}_F}$, we note that the function $m^\sigma_{a,v}$ (resp. $m^\sigma_{v,b}$) is right (resp. left) $\mathcal{G}_F$-invariant. We further observe the following.

(i) If $v \in V_\sigma^{\mathcal{G}_F}$ satisfies the normalization (5.2), then $m^\sigma_{v,v}$ is both a convolution idempotent and $\mathcal{G}_F$-bi-invariant.

(ii) If $\{v_i\}$ is an orthogonal basis of V_σ with every basis vector v_i satisfying the normalization (5.2), then the idempotents $m^\sigma_{v_i,v_i}$ are mutually orthogonal and

$$e_\sigma := \sum_{i=1}^{\deg(\sigma)} m^\sigma_{v_i,v_i} \in e_E \star C_c^\infty(\mathcal{G}_E)\star e_E$$

is the central idempotent attached to σ. Set

$$\Xi := \text{collection of irreducible representations } (\sigma, V_\sigma) \text{ of } \mathcal{G}_E \text{ that have nonzero } \mathcal{G}_F\text{-fixed vectors.}$$

Then

$$e_E = \sum_{\sigma\in\Xi} e_\sigma = \sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)} m^\sigma_{v_i^\sigma,v_i^\sigma}, \tag{5.3}$$

where $\{v_i^\sigma\}$ is a orthogonal basis of V_σ satisfying (5.2).

Proposition 5.3 *Assume $F \subset \mathcal{B}$ is a chamber and $E \subset F$ is a facet. With the above notation, suppose $(\sigma, V_\sigma) \in \Xi$. Then there exist (finitely many) $a_k^\sigma \in {}_E\mathcal{H}_F$ so that*

$$\Theta_\sigma = \sum_k a_k^\sigma \star (a_k^\sigma)^\star.$$

Hence, there exist (finitely many) $b_j \in {}_E\mathcal{H}_F$ so that $e_E = \sum_j b_j \star b_j^\star$.

Proof We take $\{u_i\}$ to be an orthogonal basis for V_σ and $v \in V_\sigma^{\mathcal{G}_F}$ so that $\{m^\sigma_{u_i,u_i}\}$, and $m^\sigma_{v,v}$ are idempotents. The coefficient $a_k^\sigma = m^\sigma_{u_k,v}$ is right $\mathcal{G}_F$-invariant, and

$$a_k^\sigma \star (a_k^\sigma)^\star = m^\sigma_{u_k,v} \star m^\sigma_{v,u_k} = m^\sigma_{u_k,u_k},$$

and so $\Theta_\sigma = \sum_k m^\sigma_{u_k,u_k} = \sum_k a_k^\sigma \star (a_k^\sigma)^\star$. ■

Proposition 5.4 *For each $\sigma \in \Xi$, let $\{v_i^\sigma\}$ and $\{w_i^\sigma\}$ be two orthogonal bases of V_σ. Assume all these vectors satisfy the normalization (5.2). Then $\mathcal{H}_E$ has a direct sum*

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decomposition

(5.4) $$\mathcal{H}_E = \bigoplus_{\sigma,\tau\in\Xi} \bigoplus_{i=1}^{\deg(\sigma)} \bigoplus_{j=1}^{\deg(\tau)} m^{\sigma}_{v_i^\sigma,v_i^\sigma} \star C_c^\infty(\mathcal{G}) \star m^{\tau}_{w_j^\tau,w_j^\tau}.$$

In particular, any $f \in \mathcal{H}_E$ can be written uniquely as

(5.5) $$f = \sum_{\sigma\in\Xi} \sum_{i=1}^{\deg(\sigma)} \sum_{\tau\in\Xi} \sum_{j=1}^{\deg(\tau)} f_{\sigma,i,\tau,j},$$

where $f_{\sigma,i,\tau,j} = m^{\sigma}_{v_i^\sigma,v_i^\sigma} \star f \star m^{\tau}_{w_j^\tau,w_j^\tau}$.

Proof Suppose $f \in \mathcal{H}_E$. Since $f = e_E \star f \star e_E$, the decomposition (5.3) of e_E then yields the sum (5.5); *i.e.*, $\mathcal{H}_E$ is a sum of the indicated subspaces in (5.4). To see the sum is direct, we note that convolution on the left by $m^{\sigma}_{v_i^\sigma,v_i^\sigma}$ and on the right by $m^{\sigma}_{w_j^\tau,w_j^\tau}$ is zero on $m^{\kappa}_{v_r^\kappa,v_r^\kappa} \star C_c^\infty(\mathcal{G}) \star m^{\lambda}_{w_s^\lambda,w_s^\lambda}$ unless $(\sigma, i, \tau, j) = (\kappa, r, \lambda, s)$, and is the identity (since v_i^σ, v_j^τ are properly normalized) on $m^{\sigma}_{v_i^\sigma,v_i^\sigma} \star C_c^\infty(\mathcal{G}) \star m^{\tau}_{w_j^\tau,w_j^\tau}$. Thus, the sum is direct. ∎

6 Involutions and Forms

6.1 Extension of an Anti-involution of $\mathcal{H}_F$ to $\mathcal{H}_E$

We continue with the assumption that $F \subset \mathcal{B}$ is a chamber and $E \subset F$ a facet. Let $\star$ be the anti-involution (2.1). Suppose the Iwahori–Hecke algebra $\mathcal{H}_F$ has an anti-involution $\circ$ satisfying

(6.1) $$\forall f \in e_F \star C_c^\infty(\mathcal{G}_E) \star e_F : f^\circ = f^\star$$

We show here that it is possible to extend the anti-involution $\circ$ of $\mathcal{H}_F$ to an anti-involution of $\mathcal{H}_E$.

Lemma 6.1 *For each $\kappa \in \Xi$, choose two bases, $\{v_i^\kappa\}$ and $\{w_i^\kappa\}$, of V_κ, and choose two elements $y^\kappa, z^\kappa \in V_\kappa^{\mathcal{G}_F}$ satisfying the normalization (5.2).*

(i) *The $\mathcal{G}^F$-bi-invariant function $m^{\sigma}_{y^\sigma,v_i^\sigma} \star f \star m^{\tau}_{w_j^\tau,z^\tau}$ is convolution left invariant for $m^{\sigma}_{y^\sigma,y^\sigma}$ and convolution right invariant for $m^{\tau}_{z^\tau,z^\tau}$.*

(ii) *For all $f \in C_c^\infty(\mathcal{G})$ and $\sigma, \tau \in \Xi$,*

$$m^{\sigma}_{v_i^\sigma,v_i^\sigma} \star f \star m^{\tau}_{w_j^\tau,w_j^\tau} = m^{\sigma}_{v_i^\sigma,y^\sigma} \star F_{y^\sigma,z^\tau} \star m^{\tau}_{z^\tau,w_i^\tau},$$

where $F_{y^\sigma,z^\tau} = m^{\sigma}_{y^\sigma,v_i^\sigma} \star f \star m^{\tau}_{w_j^\tau,z^\tau}$ belongs to $\mathcal{H}_F$.

Proof This is clear. ∎

Remarks

- A consequence of statement (ii) is that

$$\mathcal{H}_E = (e_E \star C_c^\infty(\mathcal{G}_E) \star e_F) \star \mathcal{H}_F \star (e_F \star C_c^\infty(\mathcal{G}_E) \star e_E).$$

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• In statement (ii), if we replace the collection of (normalized) $\mathcal{G}_F$-invariant vectors $\{y^\kappa\}$ and $\{z^\kappa\}$ by $\{y^\kappa_\dagger\}$ and $\{z^\kappa_\dagger\}$, then the two $\mathcal{G}_F$-bi-invariant functions F_{y^σ,z^τ} and $F_{y^\sigma_\dagger,z^\tau_\dagger}$ are related by

$$F_{y^\sigma_\dagger,z^\tau_\dagger} = m^\sigma_{y^\sigma_\dagger,y^\sigma}\star F_{y^\sigma,z^\tau}\star m^\sigma_{z^\tau,z^\tau_\dagger}.$$

Assume we are in the situation of Lemma 6.1. Then any $f\in\mathcal{H}_E$ is decomposed as in (5.5); thus,

$$f=\sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)}\sum_{\tau\in\Xi}\sum_{j=1}^{\deg(\tau)} m^\sigma_{v^\sigma_i,y^\sigma}\star\left(m^\sigma_{y^\sigma,v^\sigma_i}\star f\star m^\tau_{v^\tau_j,z^\tau}\right)\star m^\tau_{z^\tau,v^\tau_j},$$

and each function $\left(m^\sigma_{y^\sigma,v^\sigma_i}\star f\star m^\tau_{v^\tau_j,z^\tau}\right)$ is $\mathcal{G}_F$-bi-invariant. Another choice $\{y^\sigma_\dagger,z^\sigma_\dagger\in V^{\mathcal{G}_F}_\sigma\mid\sigma\in\Xi\}$ yields

$$f=\sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)}\sum_{\tau\in\Xi}\sum_{j=1}^{\deg(\tau)} m^\sigma_{v^\sigma_i,y^\sigma_\dagger}\star\left(m^\sigma_{y^\sigma_\dagger,v^\sigma_i}\star f\star m^\tau_{v^\tau_j,z^\tau_\dagger}\right)\star m^\tau_{z^\tau_\dagger,v^\tau_j}.$$

We can combine these two expressions for f with the following assumptions:

(i) $\circ$ is an anti-involution of the Iwahori–Hecke algebra $\mathcal{H}_F$.

(ii) On the functions $e_F\star C(\mathcal{G}_E)\star e_F$, the maps $\circ$ and $\star$ (of (2.1)) are equal.

(iii) For any $\kappa\in\Xi$ and $a,b\in V_\kappa$, $(m^\kappa_{a,b})^\star = m^\kappa_{b,a}$.

We deduce that the linear map

$$f=\sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)}\sum_{\tau\in\Xi}\sum_{j=1}^{\deg(\tau)} m^\sigma_{v^\sigma_i,u^\sigma}\star\left(m^\sigma_{u^\sigma,v^\sigma_i}\star f\star m^\tau_{v^\tau_j,u^\tau}\right)\star m^\tau_{u^\tau,v^\tau_j}$$

$$\xrightarrow{\circ}$$

(6.2)
$$f^\circ := \sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)}\sum_{\tau\in\Xi}\sum_{j=1}^{\deg(\tau)} m^\tau_{v^\tau_j,u^\tau}\star\left(m^\sigma_{u^\sigma,v^\sigma_i}\star f\star m^\tau_{v^\tau_j,u^\tau}\right)^\circ\star m^\sigma_{u^\sigma,v^\tau_i}$$
$$=\sum_{\sigma\in\Xi}\sum_{i=1}^{\deg(\sigma)}\sum_{\sigma\in\Xi}\sum_{j=1}^{\deg(\tau)} m^\tau_{v^\tau_j,u^\tau}\star\left(m^\tau_{u^\tau,v^\tau_j}\star f^\circ\star m^\sigma_{v^\sigma_i,u^\sigma}\right)\star m^\sigma_{u^\sigma,v^\tau_i}$$

on $\mathcal{H}_E$ is well defined.

Proposition 6.2 *The linear map $\circ$ (6.2) of $\mathcal{H}_E$ is an algebra anti-involution.*

We note the following:

(i) If $\circ$ is the $\star$ anti-involution of $\mathcal{H}_F$, then the extension $\circ$ to $\mathcal{H}_F$ is the $\star$ anti-involution.

(ii) $\bullet$ satisfies (6.1), so the above computation (with $\circ$ being $\bullet$) applies to say $\bullet$ has an extension to $\mathcal{H}_E$.

Proof For each $\sigma\in\Xi$, we fix an orthogonal basis $\{v^\sigma_i\}$ of V_σ and a vector $u^\sigma\in V^{\mathcal{G}_F}_\sigma$. We assume that the vectors are normalized as in (5.2). Suppose $f,g\in\mathcal{H}_E$. Expand them as

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$$f = \sum_{\sigma,\tau} \sum_{i=1}^{\deg(\sigma)} \sum_{j=1}^{\deg(\tau)} m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j},$$

$$g = \sum_{kappa,\lambda} \sum_{r=1}^{\deg(\kappa)} \sum_{s=1}^{\deg(\lambda)} m^{\kappa}_{v^{\kappa}_r, v^{\kappa}_r} \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}.$$

By the orthogonality relations

$$\left(m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j}\right) \star \left(m^{\kappa}_{v^{\kappa}_r, v^{\kappa}_r} \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}\right) = \begin{cases} 0 & \text{unless } \tau = \kappa \text{ and } j = r, \\ \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s} & \text{when } \tau = \kappa \text{ and } j = r. \end{cases}$$

By (5.5), this must be $m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}$. From this, we use $m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} = m^{\sigma}_{v^{\sigma}_i, u^{\sigma}} m^{\sigma}_{u^{\sigma}, v^{\sigma}_i}$, $m^{\tau}_{v^{\tau}_j, v^{\tau}_j} = m^{\tau}_{v^{\tau}_j, u^{\tau}} m^{\tau}_{u^{\tau}, v^{\tau}_j}$, and $m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s} = m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} m^{\lambda}_{u^{\lambda}, v^{\lambda}_s}$ to compute

$$\begin{aligned}
&\left(m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}\right)^{\circ} \\
&\quad = \left(\sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}\right)^{\circ} \\
&\quad = \left(\sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\sigma}_{v^{\sigma}_i, u^{\sigma}} \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star m^{\lambda}_{u^{\lambda}, v^{\lambda}_s}\right)^{\circ} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \left(m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}}\right)^{\circ} \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \left(m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}} \star m^{\tau}_{u^{\tau}, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}}\right)^{\circ} \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \Big((m^{\tau}_{u^{\tau}, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}})^{\circ} \\
&\qquad\qquad \star (m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}})^{\circ} \Big) \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \Big((m^{\tau}_{u^{\tau}, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}})^{\circ} \star m^{\tau}_{u^{\tau}, u^{\tau}} \\
&\qquad\qquad \star (m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}})^{\circ} \Big) \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \Big((m^{\tau}_{u^{\tau}, v^{\tau}_j} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}})^{\circ} \star m^{\tau}_{u^{\tau}, v^{\tau}_j} \\
&\qquad\qquad \star m^{\tau}_{v^{\tau}_j, u^{\tau}} \star (m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}})^{\circ} \Big) \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \\
&\quad = \sum_{\tau} \sum_{j=1}^{\deg(\tau)} \sum_{r=1}^{\deg(\tau)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star \Big((m^{\tau}_{u^{\tau}, v^{\tau}_r} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}})^{\circ} \star m^{\tau}_{u^{\tau}, v^{\tau}_r} \\
&\qquad\qquad \star m^{\tau}_{v^{\tau}_j, u^{\tau}} \star (m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}})^{\circ} \Big) \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i}
\end{aligned}$$

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$$= \Big(\sum_{\kappa} \sum_{r=1}^{\deg(\kappa)} m^{\lambda}_{v^{\lambda}_s, u^{\lambda}} \star (m^{\kappa}_{u^{\kappa}, v^{\kappa}_r} \star g \star m^{\lambda}_{v^{\lambda}_s, u^{\lambda}})^{\circ} \star m^{\kappa}_{u^{\kappa}, v^{\kappa}_r} \Big)$$
$$\star \Big(\sum_{\tau} \sum_{j=1}^{\deg(\tau)} m^{\tau}_{v^{\tau}_j, u^{\tau}} \star (m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \star f \star m^{\tau}_{v^{\tau}_j, u^{\tau}})^{\circ} \star m^{\sigma}_{u^{\sigma}, v^{\sigma}_i} \Big)$$
$$= \big(g \star m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s} \big)^{\circ} \star \big(m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i} \star f \big)^{\circ}.$$

The above is true for any $m^{\sigma}_{v^{\sigma}_i, v^{\sigma}_i}$ and $m^{\lambda}_{v^{\lambda}_s, v^{\lambda}_s}$. We conclude $\circ$ is an algebra anti-involution. ∎

We note that the $\circ$-involution of $\mathcal{H}_E$ interchanges the two subspaces ${}_E\mathcal{H}_F$ and ${}_F\mathcal{H}_E$. We obviously have

$$\forall f \in \mathcal{H}_E, a \in {}_E\mathcal{H}_F, g \in \mathcal{H}_F : (f \star a \star g)^{\circ} = g^{\circ} \star a^{\circ} \star f^{\circ}$$

and a similar relation when $b \in {}_F\mathcal{H}_E$ instead. We have

$$\forall a \in {}_E\mathcal{H}_F, b \in {}_F\mathcal{H}_E : (a \star b)^{\circ} = b^{\circ} \star a^{\circ} \quad \text{and} \quad (b \star a)^{\circ} = a^{\circ} \star b^{\circ}.$$

We prove that the Morita equivalence of Theorem 4.1 preserves the $\circ$ hermitian and unitary modules. We continue in the context that F is a chamber and a facet $E \subset F$.

We follow the algebraic considerations in [R]. Suppose $\mathcal{A}$ is an $\mathrm{C}^{\star}$-algebra. We use $\circ$ to denote the involution of a $\mathcal{A}$. If $a \in \mathcal{A}$, we write $a \geq 0$ if there exists $x_1, \dots, x_n \in \mathcal{A}$ so that $a = \sum_{i=1}^{n} x_i^{\circ} x_i$.

In the Morita equivalence of Theorem 4.1, we assume that $\circ$ is an anti-involution of $\mathcal{H}_F$ that satisfies (6.1), so there is an extension of $\circ$ to $\mathcal{H}_E$. We want to be able to transfer the Hermitian structure of a representation of $\mathcal{H}_F$ to a representation of $\mathcal{H}_E$ and vice-versa. To effect this, ${}_E\mathcal{H}_F$ must have a $\mathcal{H}_F$-valued form $(\cdot, \cdot)_{\mathcal{H}_F} : {}_E\mathcal{H}_F \times {}_E\mathcal{H}_F \to \mathcal{H}_F$ that is sesquilinear, *i.e.*, so that

$$\forall a, b \in {}_E\mathcal{H}_F : (a, b)_{\mathcal{H}_F} = ((b, a)_{\mathcal{H}_F})^{\circ}$$
$$\forall r \in \mathcal{H}_E, a, b \in {}_E\mathcal{H}_F : (r \star a, b)_{\mathcal{H}_F} = (a, r^{\circ} \star b)_{\mathcal{H}_F}.$$

Granted the existence of the form $(\cdot, \cdot)_{\mathcal{H}_F}$, if $(\pi, V_{\pi}) \in \mathcal{C}(\mathcal{H}_F)$ has a hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{H}_F}$, then the $\mathcal{H}_E$-module ${}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} V_{\pi}$ is hermitian for the form

$$\langle f \otimes v, g \otimes w \rangle_{\mathcal{H}_E} = \langle \pi((f, g)_{\mathcal{H}_F}) v, w \rangle_{\mathcal{H}_F}.$$

This plugs into the machinery of [R], and it is formal that $\langle \cdot, \cdot \rangle_{\mathcal{H}_E}$ is a hermitian form with the appropriate invariance properties. To go in the other direction, ${}_F\mathcal{H}_E$ must have a $\mathcal{H}_E$-valued sesquilinear form $(\cdot, \cdot)_{\mathcal{H}_E} : {}_F\mathcal{H}_E \times {}_F\mathcal{H}_E \to \mathcal{H}_E$. For our situation, the two forms are

$$\forall a, b \in {}_E\mathcal{H}_F : (a, b)_{\mathcal{H}_F} := a^{\circ} \star b$$
$$\forall c, d \in {}_F\mathcal{H}_E : (c, d)_{\mathcal{H}_E} := c^{\circ} \star d$$

To show that a unitary module $(V \in \mathcal{C}(\mathcal{H}_F))$ is taken to a unitary module $(({}_E\mathcal{H}_F \otimes_{\mathcal{H}_F} V) \in \mathcal{C}(\mathcal{H}_E))$, it suffices to show $(a, a)_{\mathcal{H}_F} \geq 0$ for any $a \in {}_E\mathcal{H}_F$. Similarly, if $W \in \mathcal{C}(\mathcal{H}_E)$ is unitary, then a sufficient condition for $({}_F\mathcal{H}_E \otimes_{\mathcal{H}_E} W)$ to be unitary

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is that $(a,a)_{\mathcal{H}_E} \geq 0$ for all $a \in {}_F\mathcal{H}_E$. We write an element in ${}_F\mathcal{H}_E$ (resp. ${}_E\mathcal{H}_F$) as $a = e_E \star A \star e_F$ (resp. $a = e_F \star A \star e_E$) with $A \in \mathcal{H}_E$. Then

$$\begin{aligned}(a,a)_{\mathcal{H}_F} &= (e_E \star A \star e_F, e_E \star A \star e_F)_{\mathcal{H}_F} \\ &= (e_E \star A \star e_F)^\circ \star (e_E \star A \star e_F) = e_F \star A^\circ \star e_E \star A \star e_F.\end{aligned}$$

By Proposition 5.3, there exist $x_1, \ldots, x_r \in e_E \star C_c^\infty(\mathcal{G}_E) \star e_F$ so that $e_E = \sum_{i=1}^r x_i \star x_i^\star$. Substitution yields

$$\begin{aligned}(a,a)_{\mathcal{H}_F} &= e_F \star A^\circ \star \Big(\sum_{i=1}^r x_i \star x_i^\star\Big) \star A \star e_F \\ &= \sum_{i=1}^r (x_i^\star \star A \star e_F)^\circ \star (x_i^\star \star A \star e_F).\end{aligned}$$

So $(a,a)_{\mathcal{H}_F} \geq 0$ for all $a \in {}_E\mathcal{H}_F$. That $(b,b)_{\mathcal{H}_E} \geq 0$ for any $b \in {}_F\mathcal{H}_E$ is obvious. Hence, we have the following proposition.

Proposition 6.3 *Suppose F is a chamber and $E \subset F$ is a facet. Suppose that $\circ$ is an anti-involution of $\mathcal{H}_F$ satisfying (6.1) and $\circ$ is extended to $\mathcal{H}_E$. Then the equivalence of categories in Theorem 4.1 preserves hermitian and unitarity modules.*

7 Generalizations

7.1 Finite Field Groups

We consider a finite field $\mathbb{F}$ (with q elements) and a connected reductive group $\mathbb{G}$ defined over $\mathbb{F}$. Let G be the group of $\mathbb{F}$-rational points, and let $P = MU$ (U the radical, M a Levi factor) be the $\mathbb{F}$-rational points of a parabolic subgroup defined over $\mathbb{F}$.

Theorem 7.1 ([HC]) *Take G and $P = MU$ as above. Suppose σ and τ are irreducible cuspidal representations of M. The following are equivalent:*

(i) *There exists $n \in N_G(M)$ so that $\mathrm{Ad}(n)\sigma = \tau$.*

(ii) *Suppose (λ, V_λ) is an irreducible representation of G and $(V_\lambda)_U$ is the U-covariants (a representation of M). Then $(V_\lambda)_U$ contains σ if and only if it contains τ.*

Theorem 7.1 gives an equivalence relation on the set $\mathcal{T}$ of irreducible cuspidal representations of M. For such a representation τ, let $\Delta(\tau)$ denote the equivalence class of τ. Set

$$X_{MU} := \{\lambda \in \widehat{G} \mid (V_\lambda)_U \text{ contains a cuspidal representation of } M\}.$$

Then Theorem 7.1 also gives an equivalence relation on X_{MU} as

$$\lambda_1, \lambda_2 \in \widehat{G} : \lambda_1 \sim \lambda_2 \text{ if } (V_{\lambda_1})_U \text{ and } (V_{\lambda_2})_U \text{ share an irreducible cuspidal representation of } M.$$

Theorem 7.1 obviously provides a natural bijection between the equivalence classes of $\mathcal{T}$ and those of X_{MU}.

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We take σ to be an irreducible cuspidal representation of M. Denote by Δ its equivalence class Δ in $\mathcal{T}$ and by Ξ the corresponding equivalence class of representations of G. We define idempotent elements e_σ, e_Δ and e_Ξ in the group algebra $\mathbb{C}G$ as follows:

$$e_\sigma(g) := \begin{cases} \frac{1}{\#(MU)} \deg(\tau)\Theta_\sigma(m) & \text{if } g = mu \in MU, \\ 0 & \text{if } g \notin MU, \end{cases}$$

$$e_\Delta(g) := \begin{cases} \frac{1}{\#(MU)} \sum_{\tau\in\Delta} \deg(\tau)\Theta_\tau(m) & \text{if } g = mu \ \in MU, \\ 0 & \text{if } g \notin MU, \end{cases}$$

$$e_\Xi := \frac{1}{\#(G)} \sum_{\lambda\in\Xi} \deg(\lambda)\Theta_\lambda.$$

The element e_Ξ is the central idempotent in the group algebra $\mathbb{C}G$, and for any irreducible representation (λ, V_λ) of G,

$$\lambda(e_\Xi) = \begin{cases} \mathrm{I}_{V_\lambda} & \text{if } \lambda \in \Xi, \\ 0_{V_\lambda} & \text{if } \lambda \notin \Xi. \end{cases}$$

The idempotent e_σ is clearly the product (in any order) of the two idempotents

$$e_a(g) := \begin{cases} \frac{1}{\#(M)} \deg(\sigma)\Theta_\sigma(m) & \text{if } g = m \in M, \\ 0 & \text{if } g \notin M, \end{cases}$$

$$e_U(g) := \begin{cases} \frac{1}{\#(U)} \sum_{\tau\in\Delta} 1 & \text{if } g \in U, \\ 0 & \text{if } g \notin U, \end{cases}$$

and similarly for e_Δ. For any irreducible representation (λ, V_λ) of G, we have

$$\lambda(e_\Delta) = \lambda(e_a) \circ \lambda(e_U),$$

where $\lambda(e_U)$ projects to the U-invariants of V_λ (which we can identify with the U-covariants), and then the action of $\lambda(e_a)$ on the U-invariants is projection to the isotypical component arising from σ. Obviously, for any $\lambda \in \widehat{G}$, we have $\lambda(e_\Xi \star e_\sigma) = \lambda(e_\sigma) = \lambda(e_\sigma \star e_\Xi)$. This means that the operator Fourier transforms of the three functions $e_\Xi \star e_\sigma$, e_σ, and $e_\sigma \star e_\Xi$ are equal. This means that

$$e_\Xi \star e_\sigma = e_\sigma = e_\sigma \star e_\Xi. \tag{7.1}$$

In a completely analogous way,

$$e_\Xi \star e_\Delta = e_\Delta = e_\Delta \star e_\Xi. \tag{7.2}$$

Define

$$\mathcal{H}_\sigma := e_\sigma \star \mathbb{C}(G)\star e_\sigma, \quad \mathcal{H}_\Delta := e_\Delta \star \mathbb{C}(G)\star e_\Delta, \quad \mathcal{H}_\Xi := e_\Xi \star \mathbb{C}G\star e_\Xi,$$

and

$$_\Xi\mathcal{H}_\sigma := e_\Xi\star\mathbb{C}G \star e_\sigma, \quad _\sigma\mathcal{H}_\Xi := e_\sigma \star\mathbb{C}G\star e_\Xi$$
$$_\Xi\mathcal{H}_\Delta := e_\Xi\star\mathbb{C}G \star e_\Delta, \quad _\Delta\mathcal{H}_\Xi := e_\Delta\star\mathbb{C}G\star e_\Xi.$$

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The relations in (7.1) and (7.2) mean that $\mathcal{H}_\sigma$ and $\mathcal{H}_\Delta$ are subalgebras of $\mathcal{H}_\Xi$. The proof of Proposition 3.4 can be easily modified and combined with the referenced results on Morita equivalence to show the following proposition.

Proposition 7.2 *The ideals $\mathbb{C}G\star e_\sigma\star\mathbb{C}G$ and $\mathbb{C}G\star e_\Delta\star\mathbb{C}G$ of $\mathbb{C}G$ satisfy the following:*

(i) *Each equals $\mathcal{H}_\Xi$;* i.e., *the idempotents e_σ and e_Δ are full idempotents of $\mathcal{H}_\Xi$.*

(ii) (a) $\mathcal{H}_\Xi = {}_\Xi\mathcal{H}_\sigma \star {}_\sigma\mathcal{H}_\Xi$ *and* $\mathcal{H}_\sigma = {}_\sigma\mathcal{H}_\Xi \star {}_\Xi\mathcal{H}_\sigma$.

(b) $\mathcal{H}_\Xi = {}_\Xi\mathcal{H}_\Delta \star {}_\Delta\mathcal{H}_\Xi$ *and* $\mathcal{H}_\Delta = {}_\Delta\mathcal{H}_\Xi \star {}_\Xi\mathcal{H}_\Delta$.

(iii) *The algebras $\mathcal{H}_\sigma$ and $\mathcal{H}_\Delta$ are Morita equivalent to $\mathcal{H}_\Xi$.*

7.2 Local Field Groups

We now consider k a non-archimedean local field with notation as in the introduction. Suppose F is a facet of the building $\mathcal{B}(\mathcal{G})$, and F is a facet with a subfacet E, and $\mathcal{G}_F$ and $\mathcal{G}_E$ are the corresponding parahoric subgroups (so $\mathcal{G}_E^+ \subset \mathcal{G}_F^+ \subset \mathcal{G}_F \subset \mathcal{G}_E$). Then $G = \mathcal{G}_E/\mathcal{G}_E^+$ is the $\mathbb{F}$-rational points of a reductive group defined over $\mathbb{F}$ and $P = G_{k_F}/\mathcal{G}_E^+$ is a parabolic subgroup. Let MU be a Levi decomposition of P and let σ be an irreducible cuspidal representation of M. Define $\Delta = \Delta(\sigma)$ and Ξ as in the previous section. The inflation of the idempotent e_σ of G to $\mathcal{G}_E$ obviously has support in $\mathcal{G}_F$. For convenience of notation we continue to use the notation e_σ to denote the inflation. Denote by e_F and e_E, respectively, the inflations of e_Δ and e_Ξ to $\mathcal{G}_E$. The support of e_F is in $\mathcal{G}_F$.

Define

$$\mathcal{H}_\sigma := e_\sigma \star C_c^\infty(\mathcal{G})\star e_\sigma, \quad \mathcal{H}_F := e_F \star C_c^\infty(\mathcal{G})\star e_F, \quad \mathcal{H}_E := e_E \star C_c^\infty \star e_E,$$

and

$$\begin{aligned} {}_E\mathcal{H}_\sigma &:= e_E\star C_c^\infty(\mathcal{G})\star e_\sigma, & {}_\sigma\mathcal{H}_E &:= e_\sigma \star C_c^\infty(\mathcal{G})\star e_E \\ {}_E\mathcal{H}_\Delta &:= e_E\star C_c^\infty(\mathcal{G}) \star e_\Delta, & {}_\Delta\mathcal{H}_\Xi &:= e_\Delta\star C_c^\infty(\mathcal{G})\star e_E. \end{aligned}$$

In an entirely analogous fashion to Theorem 4.1, we have the following theorem.

Theorem 7.3 *The idempotents e_σ and e_F are full idempotents of the algebra $\mathcal{H}_E$, and so the algebras $\mathcal{H}_\sigma$ and $\mathcal{H}_F$ are Morita equivalent to $\mathcal{H}_E$.*

The $\star$-anti-involution $f^\star(g) = \overline{f(g^{-1})}$ on $C_c^\infty(\mathcal{G})$ restricts to a $\star$-anti-involution on the algebras $\mathcal{H}_\sigma$, $\mathcal{H}_F$, and $\mathcal{H}_E$. In analogy with Proposition 6.3, we have the following proposition.

Proposition 7.4 *Suppose $\circ$ is an anti-involution of $\mathcal{H}_F$ that satisfies (6.1): $\forall f \in e_F \star C_c^\infty(\mathcal{G}_E) \star e_F : f^\circ = f^\star$. Then there is an extension of $\circ$ to an anti-involution of $\mathcal{H}_E$ such that Morita equivalence of $\mathcal{H}_F$ and $\mathcal{H}_F$ preserves hermitian and unitary modules. The same holds if we replace $\mathcal{H}_F$ by $\mathcal{H}_\sigma$.*

Acknowledgments Parts of this work were done during a visit of the first author to the HKUST Mathematics Department in Spring 2014, a visit of the authors to the University of Utah Mathematics Department in Summer 2015, and visits of the second author to the Cornell University Mathematics Department in the Fall of 2017 and the University of Oxford Mathematical Institute in the Spring of 2018. The Departments and Institute are thanked for their hospitality.

We are grateful to Konstantin Ardakov, who brought to our attention the fruitfulness of establishing Morita equivalence via full idempotents.

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https://doi.org/10.4153/S0008414X19000324 Published online by Cambridge University Press