J. Inst. Math. Jussieu (2013) **12**(2), 335–389
doi:10.1017/S1474748012000746

THE LOW MACH NUMBER LIMIT FOR THE ISENTROPIC EULER SYSTEM WITH AXISYMMETRIC INITIAL DATA

TAOUFIK HMIDI
IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes CEDEX, France (thmidi@univ-rennes1.fr)

(Received 25 September 2011; accepted 29 February 2012; first published online 24 May 2012)

Abstract This paper is devoted to the study of the low Mach number limit for the isentropic Euler system with axisymmetric initial data without swirl. In the first part of the paper we analyze the problem corresponding to the subcritical regularities, that is H^s with $s > \frac{5}{2}$. Taking advantage of the Strichartz estimates and using the special structure of the vorticity we show that the lifespan T_ε of the solutions is bounded below by $\log\log\log\frac{1}{\varepsilon}$, where ε denotes the Mach number. Moreover, we prove that the incompressible parts converge to the solution of the incompressible Euler system when the parameter ε goes to zero. In the second part of the paper we address the same problem but for the Besov critical regularity $B^{\frac{5}{2}}_{2,1}$. This case turns out to be more subtle because of at least two features. The first one is related to the Beale–Kato–Majda criterion which is not known to be valid for rough regularities. The second one concerns the critical aspect of the Strichartz estimate $L^1_T L^\infty$ for the acoustic parts $(\nabla\Delta^{-1}\mathrm{div}\, v_\varepsilon, c_\varepsilon)$: it scales in the space variables like the space of the initial data.

Keywords: incompressible limit; axisymmetrix flows; critical Besov spaces

AMS 2000 *Mathematics subject classification:* Primary 35Q35
Secondary 76B03; 76N10

1. Introduction

The object of this paper is to study the incompressible limit problem for classical solutions of the compressible isentropic Euler equations. The fluid is assumed to evolve in the whole space $\mathbb{R}^3$ and possesses a special geometric structure: the vector fields are invariant under the group of rotations around the vertical axis (Oz). Recall that the state of the fluid is described by the velocity field v_ε and the sound speed c_ε, through a penalized quasilinear hyperbolic system,

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon\cdot\nabla v_\varepsilon + \bar{\gamma} c_\varepsilon \nabla c_\varepsilon + \dfrac{1}{\varepsilon}\nabla c_\varepsilon = 0 \\ \partial_t c_\varepsilon + v_\varepsilon\cdot\nabla c_\varepsilon + \bar{\gamma} c_\varepsilon \mathrm{div}\, v_\varepsilon + \dfrac{1}{\varepsilon}\mathrm{div}\, v_\varepsilon = 0 \\ (v_\varepsilon, c_\varepsilon)|_{t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}), \end{cases} \tag{1.1}$$

with $\bar{\gamma}$ a strictly positive number and ε a small parameter called the Mach number. We point out that the derivation of this model can be carried out from the compressible isentropic equations after rescaling the time and changes of variables; see for instance [15, 20, 26]. This model has been widely considered through the last few decades and special attention is focused on the construction of a family of solutions with a nondegenerate time existence. Nevertheless, the most relevant task is to study rigorously the convergence towards the incompressible Euler equations when the Mach number goes to zero. We recall that the incompressible Euler system is given by

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0. \end{cases} \tag{1.2}$$

The answer to these problems depends on several factors: the domain where the fluid is assumed to evolve: the full space $\mathbb{R}^d$, the torus $\mathbb{T}^d$, bounded or unbounded domains. The second factor is the state of the initial data: whether they are well-prepared or not. In the well-prepared case [20, 21], we assume that the initial data are slightly compressible, which means that $\operatorname{div} v_{0,\varepsilon} = O(\varepsilon)$ and $\nabla c_{0,\varepsilon} = O(\varepsilon)$ as $\epsilon \to 0$. However, in the ill-prepared case [33], we only assume that the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_\varepsilon$ is bounded in some Sobolev spaces H^s with $s > \frac{d}{2} + 1$ and that the incompressible parts of $(v_{0,\varepsilon})_\varepsilon$ tend to some vector field v_0. Remark that in the well-prepared case we have a uniform bound of $(\partial_t v_\varepsilon)_\varepsilon$, and this allows us to pass to the limit by using the Aubin–Lions compactness lemma. Nevertheless, the ill-prepared case is more subtle because the time derivative $\partial_t v_\varepsilon$, is of size $O(\frac{1}{\varepsilon})$. To overcome this difficulty, Ukai [33] used the dispersive effects generated by the acoustic waves in order to prove that the compressible part of the velocity and the acoustic term vanish when ε goes to zero. In [15], we deal with a more degenerate case: we allow the initial data to be so ill-prepared that corresponding solutions can tend to a vortex patch or even to a Yudovich solution. Since these solutions do not belong to the Sobolev space H^s for any $s > 2$, we allow initial data that are not uniformly bounded in these spaces. We point out that this problem has already been studied in numerous papers; see for instance [5, 12–14, 20, 21, 23–26, 33].

It is well-known that in space dimension 2 and contrary to the incompressible Euler system case, equations (1.1) develop singularities in finite time even for smooth initial data; see [27]. This result remains true for higher dimensions; see [30]. On the other hand it is known from the work [21] that in the case of the full space $\mathbb{R}^d$ or the torus domain $\mathbb{T}^d$, the lifespan $T^\star_\varepsilon$ of the system (1.1) converges to the lifespan of the limit system (1.2). This result was established for the well-prepared case and with initial data lying in H^s, with $s > \frac{d}{2} + 2$. We point out that this approach cannot work for lower regularities $\frac{d}{2} + 2 > s > \frac{d}{2} + 1$. This is due to the fact that we use a perturbation argument, through the estimate of the difference between the solutions in H^{s-1}, and we require the embedding $H^{s-1} \hookrightarrow W^{1,\infty}$ to achieve this program. As an immediate application we get in dimension 2 that $\lim_{\varepsilon \to 0} T^\star_\varepsilon = +\infty$. This last result can be improved and the fact that the solutions of (1.2) have a double-exponential growth in Sobolev space induces an explicit lower bound for the lifespan $T^\star_\varepsilon \geqslant C \log \log \frac{1}{\varepsilon}$. It seems that we can get better information about the lifespan of the solutions when the initial data have some special structures. In [4], Alinhac proved that in space dimension 2 and for axisymmetric initial data the lifespan of the solution is equivalent to $\frac{1}{\varepsilon}$. For dimension 3 and for irrotational velocity, Sideris [30] established that the solutions are almost global in time, that is, their lifespans are bounded below by $e^{\frac{1}{\varepsilon}}$. Finally, we mention the results of [18, 29] dealing with the global existence under suitable conditions on the initial data: the initial density must be small and has a compact support and the spectrum of ∇u_0 must be far away from the set of the negative real numbers.

In this paper, we try to accomplish the same program in dimension 3 for axisymmetric initial data. This is motivated by the works of [28, 32] where it is proven that the incompressible Euler system is globally well-posed when the initial data are axisymmetric and belong to $H^s, s > \frac{5}{2}$. For the definition of the axisymmetry, see Definition 2. The proof relies on the special structure of the vorticity Ω which leads to a global bound of $\|\Omega(t)\|_{L^\infty}$ and then we use the Beale–Kato–Majda criterion [7]. Recently, we established in [1] the global well-posedness for (1.2) with initial data lying in borderline Besov spaces $v_0 \in B_{p,1}^{\frac{3}{p}+1}, 1 \leqslant p \leqslant \infty$. It is important to mention that in this context the Beale–Kato–Majda criterion is not known to be valid and the geometry is crucially used in different steps of the proof, and it is combined with a dynamical interpolation method.

Our main goal here is to study the incompressible limit problem for both subcritical and critical cases with ill-prepared axisymmetric initial data. Concerning the subcritical regularities we obtain the following result.

Theorem 1. *Let $s > \frac{5}{2}$ and $\{(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon\leqslant 1}\}$ be a H^s-bounded family of axisymmetric initial data, that is*

$$\sup_{0<\varepsilon\leqslant 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^s} < +\infty.$$

Then the system (1.1) has a unique solution $(v_\varepsilon, c_\varepsilon) \in C([0, T_\varepsilon[; H^s)$, with

$$T_\varepsilon \geqslant C \log\log\log\left(\frac{1}{\varepsilon}\right) := \tilde{T}_\varepsilon.$$

The constant C does not depend on ε. Moreover, there exists $\sigma > 0$ such that

$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_{\tilde{T}_\varepsilon} L^\infty} \leqslant C_0 \varepsilon^\sigma \quad \text{and} \quad \|(v_\varepsilon, c_\varepsilon)(t)\|_{H^s} \leqslant C_0 e^{\exp C_0 t}, \quad \forall t \in [0, \tilde{T}_\varepsilon].$$

Assume in addition that the incompressible parts $(\mathcal{P}v_{0,\varepsilon})$ converge in L^2 to some v_0. Then the incompressible parts of the solutions tend to the Kato's solution v of the system (1.2). More precisely, for every $T > 0$ and $\forall\, \eta < s$,

$$\lim_{\varepsilon\to 0} \|\mathcal{P}v_\varepsilon - v\|_{L^\infty_T H^\eta} = 0.$$

Remark 1. To study the lifespan of the solutions we do not use the approach of [21] based on the stability of the incompressible Euler system. More precisely, it seems that there is no need to use the limit system: we can only work with (1.1) and use the special

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

structure of the vorticity dynamics. On the other hand, because we are working with the vorticity, and by taking advantage of its special structure, we can improve the regularity required for the stability: we can work in the framework of Sobolev spaces H^s with $s > \frac{5}{2}$. We emphasize that in [21] the Sobolev regularity must be larger than $\frac{7}{2}$.

The proof of Theorem 1 relies on the use of Strichartz estimates for the compressible parts ($\mathcal{Q}v_\varepsilon$) and the acoustic ones (c_ε); see Corollary 2. Thus interpolating this result with the energy estimates (see Proposition 1), we obtain the following result described in Proposition 2: there exists $\sigma > 0$ such that

$$\|\operatorname{div} v_\varepsilon\|_{L^1_T L^\infty} + \|\nabla c_\varepsilon\|_{L^1_T L^\infty} \leqslant C_0 \varepsilon^\sigma (1+T^2) e^{CV_\varepsilon(T)}, \quad V_\varepsilon(T) = \|(\nabla v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T L^\infty}. \tag{1.3}$$

It is worth pointing out that working with the subcritical regularities is very valuable for getting the preceding inequality. However, this argument fails in the critical spaces as we shall see next. The second ingredient of the proof is the use of the special structure of the vorticity in the axisymmetric case combined with the Beale–Kato–Majda criterion. In what follows, we will briefly discuss the main feature of the axisymmetric flows; for the complete computations, see the following sections. By the definition, the velocity takes the form $v(x) = v^r(r,z)e_r + v^z(r,z)e_z$ in the cylindrical basis, and consequently the vorticity $\Omega := \operatorname{curl} v$ is given by $\Omega = (\partial_z v^r - \partial_r v^z)e_\theta := \Omega^\theta e_\theta$. Therefore, the vorticity dynamics is described by

$$\partial_t \Omega_\varepsilon + v_\varepsilon \cdot \nabla \Omega_\varepsilon + \Omega_\varepsilon \operatorname{div} v_\varepsilon = \frac{v_\varepsilon^r}{r} \Omega_\varepsilon. \tag{1.4}$$

It follows that the quantity $\frac{\Omega_\varepsilon}{r}$ satisfies the transport equation:

$$\big(\partial_t + v_\varepsilon \cdot \nabla + \operatorname{div} v_\varepsilon\big) \frac{\Omega_\varepsilon}{r} = 0.$$

We observe that this is analogous to the vorticity structure for the compressible Euler equations in space dimension 2. Performing energy estimates we obtain a law of almost conservation:

$$\Big\|\frac{\Omega_\varepsilon}{r}(t)\Big\|_{L^p} \leqslant \Big\|\frac{\Omega_{0,\varepsilon}}{r}\Big\|_{L^p} e^{(1-\frac{1}{p})\,\|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}, \quad \forall p \in [1,\infty].$$

When the fluid is incompressible, we obtain exact conservation laws which are sufficient for leading to the global well-posedness. As regards the proof of the incompressible limit in the case of ill-prepared initial data, it is done in a straightforward manner by using the Strichartz estimates. In the second part of this paper we will focus on the low Mach number limit for initial data with critical regularities. In our context a function space $\mathcal{X}$ is called critical if it is embedded in a Lipschitz class and both have the same scaling; for example we cite the Besov space $B^{\frac{5}{2}}_{2,1}$ or more generally $B^{\frac{3}{p}+1}_{p,1}$, with $p \in [1,\infty]$. These spaces have been until now the largest ones in the Besov space hierarchy for which we can establish the local well-posedness of the incompressible Euler equations or more generally for the quasilinear hyperbolic systems of order 1. In [1], it is proven that the system (1.2) is globally well-posed for axisymmetric initial data with critical Besov regularity. Therefore it is legitimate to try to accomplish the same program for the system (1.1) as in the subcritical case and in particular to quantify a lower bound for the lifespan of the solutions. Nevertheless, to get uniform bounds with respect to the parameter ε and remove the penalization term we need to work with critical spaces which are constructed over the Hilbert space L^2 like $B^{\frac{5}{2}}_{2,1}$ or other modified spaces, as we will see next. Although one can prove the local well-posedness for the system (1.1) with a uniform time existence, the extension of the results of Theorem 1 to the critical case seems to be much more relevant. We distinguish at least two principal difficulties. The first one has a connection with Strichartz estimates (1.3): in the critical framework the quantities $\|\mathrm{div}\, v_\varepsilon\|_{L^\infty}$ and $\|v_\varepsilon\|_{B^{\frac{5}{2}}_{2,1}}$ have the same scaling and the interpolation argument used in the proof of Theorem 1 cannot work without doing refined improvement. Indeed, we have no sufficient information about the decay of the remainder series $\sum_{q\geqslant N} 2^{\frac{5}{2}q}\|(\Delta_q v_\varepsilon, \Delta_q c_\varepsilon)(t)\|_{L^2}$ and there is no explicit dependence of this decay with respect to the parameters, N, t and ε : it seems that in general the number N can implicitly depend on the variable time and on the parameter ε and this makes the task very hard. To overcome this difficulty, we start with the important observation that any function $f \in B^{\frac{5}{2}}_{2,1}$ belongs to some heterogeneous Besov spaces $B^{\frac{5}{2},\Psi}_{2,1}$, where $\Psi : [-1, +\infty[\to \mathbb{R}^*_+$ is a nondecreasing function depending on the profile of f and satisfying $\lim_{q\to+\infty} \Psi(q) = +\infty$. These latter spaces are defined by the norm

$$\|u\|_{B^{\frac{5}{2},\Psi}_{2,1}} = \sum_{q\geqslant -1} \Psi(q) 2^{\frac{5}{2}q}\|\Delta_q u\|_{L^2}\,.$$

Further details and more discussion of such spaces will be found in the next section. We point out that the function Ψ measures the decay of the remainder series in the space of the initial data and we will see that the same decay will occur for the solution uniformly with respect to ε. Therefore one can set up the interpolation argument in the critical framework like for the subcritical case but without any explicit dependence of the lifespan with respect to the initial data.

Concerning the second difficulty, it is related to the Beale–Kato–Majda criterion which is not applicable in the context of critical regularities. In this case the estimate of $\|\Omega_\varepsilon(t)\|_{L^\infty}$ is not sufficient for propagating the initial regularities and one should estimate $\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}}$ instead. We will see next that many geometric properties of the axisymmetric flows are used and play a central role in the critical framework. Our result reads as follows.

Theorem 2. *Let* $\{(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon\leqslant 1}\}$ *be a family of axisymmetric initial data such that*

$$\sum_{q\geqslant -1} 2^{\frac{5}{2}q} \sup_{0<\varepsilon\leqslant 1} \|(\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})\|_{L^2} < +\infty. \tag{1.5}$$

Then the system (1.1) has a unique solution $(v_\varepsilon, c_\varepsilon) \in C([0, T_\varepsilon]; B^{\frac{5}{2}}_{2,1})$, *with*

$$\lim_{\varepsilon\to 0} T_\varepsilon = +\infty.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Moreover the acoustic parts of the solutions go to zero:

$$\lim_{\varepsilon\to 0}\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_{T_\varepsilon}L^\infty} = 0.$$

Assume in addition that the incompressible parts $(\mathcal{P}v_{0,\varepsilon})$ *converge in* L^2 *to some* v_0. *Then the incompressible parts of the solutions tend to the Kato's solution* v *of the system (1.2). More precisely, for any* $\eta < \frac{5}{2}$,

$$\lim_{\varepsilon\to 0}\|\mathcal{P}v_\varepsilon - v\|_{L^\infty_t H^\eta} = 0 \quad \textit{and} \quad \|(v_\epsilon, c_\epsilon)(t)\|_{B^{\frac{5}{2}}_{2,1}} \leqslant C_0 e^{e^{\exp C_0 t}}, \quad \forall t \in [0, T_\varepsilon].$$

We shall make some useful remarks.

Remarks 1. (1) In the preceding theorem we need the assumption (1.5) which is much stronger than $\sup_{0<\varepsilon\leqslant 1}\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{\frac{5}{2}}_{2,1}} < +\infty$. This is crucial for proving a uniform decay for higher frequencies; see Corollary 1.

(2) This theorem is a special case of a general result that will be discussed in Theorem 3. More precisely, we can extend these results for initial data lying in the heterogeneous Besov spaces $B^{\frac{5}{2},\Psi}_{2,1}$, with $\Psi \in \mathcal{U}_\infty$; see Definition 1. When Ψ has a slow growth at infinity then the lifespan T_ε of the solutions is bounded below as follows:

$$T_\varepsilon \geqslant C_0 \log\log\big|\log\Psi\big(\log\varepsilon^{-1}\big)\big|.$$

This covers the results of Theorem 1: it suffices to choose $\Psi(x) = 2^{(s-\frac{5}{2})x}$, $s > \frac{5}{2}$.

We have already mentioned that in the critical case the Beale–Kato–Majda criterion is inapplicable, and yet the significant quantity that one should estimate is $\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}}$. This does not seem an easy task due to the nonlinearities in the vorticity equation and to the lack of the incompressibility of the velocity vector field. It is worth pointing out that the incompressible case corresponding to the constraint $\operatorname{div} v_\varepsilon = 0$ was studied a few years ago in [1], where the following linear growth was established:

$$\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}} \leqslant C\,\|\Omega_\varepsilon(0)\|_{B^0_{\infty,1}}\, e^{\|v^r_\varepsilon/r\|_{L^1_t L^\infty}}\left(1 + \int_0^t \|\nabla v_\varepsilon(\tau)\|_{L^\infty}\,d\tau\right).$$

One of the main technical parts of this paper is extending this result to the compressible model (1.4). What we are able to prove is the following:

$$\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}} \leqslant C\,\|\Omega_\varepsilon(0)\|_{B^0_{\infty,1}}\, e^{\|v^r_\varepsilon/r\|_{L^1_t L^\infty}}\left(1 + e^{C\|v_\varepsilon\|_{L^1_t \mathrm{Lip}}}\|\operatorname{div} v_\varepsilon\|^2_{L^1_t B^{\frac{3}{p}}_{p,1}}\right)$$
$$\times\left(1 + \int_0^t \|v_\varepsilon(\tau)\|_{\mathrm{Lip}}\,d\tau\right).$$

where $p \in [1, \infty[$ and $\|\cdot\|_{\mathrm{Lip}}$ stands for Lipschitz norm. We observe that when we take $\operatorname{div} v_\varepsilon := 0$ then we get the previous linear estimate. The proof of this result uses, but with important modifications, the approach developed in [1] for the incompressible case. This method is based on a suitable splitting of the vorticity and the use of the dynamical interpolation techniques. The geometry of axisymmetric flows plays a crucial role in the proof, and we use also some tools of paradifferential calculus and harmonic analysis.

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

The rest of the paper is organized as follows. In § 2, we recall some functional spaces and some of their basic properties. Section 3 is devoted to the establishing of some energy estimates in the heterogeneous Besov spaces $B^{s,\Psi}_{2,1}$. In § 4, we prove some useful Strichartz estimates for the compressible and acoustic parts of the fluid. In § 5, we discuss some basic notions of axisymmetric geometry and we study some important geometric properties of the vorticity and prove some *a priori* estimates. In § 6, we prove Theorem 1. Section 7 is deserved to the proof of Theorem 2 and in particular to the proof of the logarithmic estimate described previously. Finally, in Appendix we establish some elementary lemmas.

2. The functional toolbox

In this section we review some of the basic tools of the paradifferential calculus and recall some elementary properties of Besov and Lorentz spaces. Before going further into the details we will give some notation that we will use intensively in this work.

Notation:

- Throughout this paper, C stands for some real positive constant which may be different in each occurrence and C_0 a constant which depends on the initial data.
- We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leqslant CY$ where C is a constant independent of X and Y.
- For any pair of operators P and Q acting in the same Banach space $\mathcal{X}$, the commutator $[P, Q]$ is given by $PQ - QP$.

2.1. Littlewood–Paley theory

Hereafter, the space dimension is fixed as $d = 3$, but all of the results of this section are valid for any dimension if the necessary modifications are made. To define Littlewood–Paley operators we need to recall the dyadic partition of unity—for a proof see for instance [10]: there exist two positive radial functions $\chi \in \mathcal{D}(\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3 \setminus \{0\})$ such that:

(i) $\chi(\xi) + \sum_{q\geqslant 0} \varphi(2^{-q}\xi) = 1 \ \forall \xi \in \mathbb{R}^3$,

(ii) $\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1$ if $\xi \neq 0$,

(iii) $\text{supp}\varphi(2^{-p}\cdot) \cap \text{supp}\varphi(2^{-q}\cdot) = \varnothing$, if $|p - q| \geqslant 2$,

(iv) $q \geqslant 1 \Rightarrow \text{supp}\chi \cap \text{supp}\varphi(2^{-q}) = \varnothing$.

For every $u \in \mathcal{S}'(\mathbb{R}^3)$ we define the dyadic blocks by

$$\Delta_{-1}u = \chi(\mathrm{D})u; \quad \forall q \in \mathbb{N}, \quad \Delta_q u = \varphi(2^{-q}\mathrm{D})u \quad \text{and} \quad S_q u = \sum_{-1\leqslant j\leqslant q-1} \Delta_j u.$$

One can easily prove that for every tempered distribution u, the following identity holds true in the weak sense:

$$u = \sum_{q\geqslant -1} \Delta_q u. \tag{2.1}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

In the same way we define the homogeneous operators

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q u = \sum_{j \leqslant q-1} \dot{\Delta}_j u.$$

We notice that these operators are of convolution type. For example for $q \in \mathbb{Z}$, we have

$$\dot{\Delta}_q u = 2^{3q} h(2^q \cdot) \star u, \quad \text{with } h \in \mathcal{S}, \quad \widehat{h}(\xi) = \varphi(\xi).$$

For the homogeneous decomposition, the identity (2.1) is not true due to the polynomials but we can write

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials.

We will make continuous use of Bernstein inequalities (see for example [10]).

Lemma 1. *There exists a constant C such that for $k \in \mathbb{N}, q \geqslant -1,\ 1 \leqslant a \leqslant b$ and for $u \in L^a(\mathbb{R}^3)$,*

$$\sup_{|\alpha| \leqslant k} \|\partial^\alpha S_q u\|_{L^b} \leqslant C^k\, 2^{q(k+3(\frac{1}{a}-\frac{1}{b}))}\, \|S_q u\|_{L^a},$$

$$C^{-k} 2^{qk}\, \|\dot{\Delta}_q u\|_{L^a} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_q u\|_{L^a} \leqslant C^k 2^{qk}\, \|\dot{\Delta}_q u\|_{L^a}.$$

Let us now introduce Bony's decomposition [9] which is the basic tool of paradifferential calculus. In the product uv of two distributions, which is not always well-defined, we distinguish three parts:

$$uv = T_u v + T_v u + \mathcal{R}(u, v),$$

where $T_u v$ is the paraproduct of v by u and $\mathcal{R}(u, v)$ the remainder term. They are defined as follows:

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \mathcal{R}(u, v) = \sum_q \Delta_q u \widetilde{\Delta}_q v, \quad \widetilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

2.2. Usual and heterogeneous Besov spaces

Now we will define the nonhomogeneous and homogeneous Besov spaces by using Littlewood–Paley operators. Let $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$; then the nonhomogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \left(2^{qs}\, \|\Delta_q u\|_{L^p}\right)_{\ell^r} < +\infty.$$

We remark that the usual Sobolev space H^s coincides with $B^s_{2,2}$ for $s \in \mathbb{R}$ and the Hölder space C^s coincides with $B^s_{\infty,\infty}$ when s is not an integer. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left(2^{qs}\, \|\dot{\Delta}_q u\|_{L^p}\right)_{\ell^r(\mathbb{Z})} < +\infty.$$

The following embeddings, valid for both homogeneous and nonhomogeneous cases, are an easy consequence of Bernstein inequalities (see for instance [10]):

$$B^s_{p_1,r_1} \hookrightarrow B^{s+3(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}, \quad p_1 \leqslant p_2 \quad \text{and} \quad r_1 \leqslant r_2.$$

Let $T > 0$ and $\rho \geqslant 1$; we denote by $L^\rho_T B^s_{p,r}$ the space of distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \left\| \left(2^{qs} \|\Delta_q u\|_{L^p} \right)_{\ell^r} \right\|_{L^\rho_T} < +\infty.$$

Now we will introduce the heterogeneous Besov spaces which are an extension of the classical Besov spaces.

Definition 1. Let $\Psi : [-1, +\infty[\to \mathbb{R}^*_+$ be a given function.

(i) We say that Ψ belongs to the class $\mathcal{U}$ if the following conditions are satisfied:
  (1) Ψ is a nondecreasing function.
  (2) There exists $C > 0$ such that

$$\sup_{q \in \mathbb{N} \cup \{-1\}} \frac{\Psi(q+1)}{\Psi(q)} \leqslant C.$$

(ii) We define the class $\mathcal{U}_\infty$ in terms of the set of functions $\Psi \in \mathcal{U}$ satisfying $\lim_{x \to +\infty} \Psi(x) = +\infty$.

(iii) Let $s \in \mathbb{R}$, $p, r \in [1, +\infty]$ and $\Psi \in \mathcal{U}$. We define the heterogeneous Besov space $B^{s,\Psi}_{p,r}$ as follows:

$$u \in B^{s,\Psi}_{p,r} \iff \|u\|_{B^{s,\Psi}_{p,r}} = \left(\Psi(q) 2^{qs} \|\Delta_q u\|_{L^p} \right)_{\ell^r} < +\infty.$$

Remark 2. (1) We observe that when we take $\Psi(x) = 2^{\alpha x}$ with $\alpha \in \mathbb{R}_+$, the space $B^{s,\Psi}_{p,r}$ reduces to the classical Besov space $B^{s+\alpha}_{p,r}$.

(2) The condition (2) seems to be necessary for the definition of $B^{s,\Psi}_{p,r}$: it allows us to obtain a definition which is independent of the choice of the dyadic partition.

The following lemma is important for the proof of Theorem 2. Roughly speaking, we will prove that any element of a given Besov space is always more regular than the prescribed regularity.

Lemma 2. *Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $r \in [1, +\infty[$ and $f \in B^s_{p,r}$. Then there exists a function Ψ belonging to $\mathcal{U}_\infty$ such that $f \in B^{s,\Psi}_{p,r}$.*

Proof. We observe that the proof reduces to the following statement: assume that a strictly positive sequence $(c_q)_{q \geqslant -1}$ satisfies $\sum_q c_q < +\infty$; then there exists a nondecreasing sequence $(a_q)_{q \geqslant -1}$ satisfying

$$\lim_{q \to +\infty} a_q = +\infty, \qquad \sup_{q \in \mathbb{N} \cup \{-1\}} \frac{a_{q+1}}{a_q} \leqslant C \tag{2.2}$$

and such that

$$\sum_{q \geqslant -1} a_q c_q < +\infty. \tag{2.3}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Let $b_q = \left(\sum_{n\geqslant q} c_n\right)^{-\frac{1}{2}}$; then $(b_q)_{q\geqslant -1}$ is a nondecreasing sequence going to infinity. Moreover,

$$\sum_{q\geqslant -1} b_q c_q \leqslant 2\left(\sum_{q\geqslant -1} c_q\right)^{\frac{1}{2}}. \tag{2.4}$$

Indeed, we introduce the piecewise function $f : [-1, +\infty[\to \mathbb{R}_+$ defined by

$$f(x) = c_q, \quad \text{for } x \in [q, q+1[\text{ and } q \in \mathbb{N} \cup \{-1\}.$$

Then we get, by obvious computations,

$$\begin{aligned}
\sum_{q\geqslant -1} b_q c_q &= \sum_{q\geqslant -1} \frac{\displaystyle\int_q^{q+1} f(x)dx}{\left(\displaystyle\int_q^{+\infty} f(x)dx\right)^{\frac{1}{2}}} \\
&\leqslant \sum_{q\geqslant -1} \int_q^{q+1} \frac{f(x)}{\left(\int_x^{+\infty} f(y)dy\right)^{\frac{1}{2}}} dx \\
&\leqslant \int_{-1}^{+\infty} \frac{f(x)}{\left(\int_x^{+\infty} f(y)dy\right)^{\frac{1}{2}}} dx \\
&\leqslant 2\left(\int_{-1}^{+\infty} f(x)dx\right)^{\frac{1}{2}} = 2\left(\sum_{q\geqslant -1} c_q\right)^{\frac{1}{2}}.
\end{aligned}$$

Now we will construct by a recursive procedure a sequence $(a_q)_{q\geqslant -1}$ satisfying (2.2) and (2.3), and such that

$$\sum_{q\geqslant -1} a_q c_q \leqslant 2\left(\sum_{q\geqslant -1} c_q\right)^{\frac{1}{2}}.$$

Let $(a_q)_{q\geqslant -1}$ be the sequence defined by the following recursive formula:

$$\begin{cases} a_{q+1} = \frac{1}{2}\left(a_q + \min(b_{q+1}, 2a_q)\right) \\ a_{-1} = b_{-1}. \end{cases} \tag{2.5}$$

We will show first that $a_q \leqslant b_q$. This is true for $q = -1$ and since $(b_q)_{q\geqslant -1}$ is nondecreasing, then

$$a_{q+1} - b_{q+1} \leqslant \frac{1}{2}(a_q - b_{q+1}) \leqslant \frac{1}{2}(a_q - b_q).$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Thus we find by the principle of recurrence that $a_q \leqslant b_q, \forall q \geqslant -1$. From this property and (2.4) we get the convergence of the series $\sum_{q\geqslant -1} a_q c_q$ and more precisely

$$\sum_{q\geqslant -1} a_q c_q \leqslant 2\Big(\sum_{q\geqslant -1} c_q\Big)^{\frac{1}{2}}.$$

Let us now prove that the sequence $(a_q)_{q\geqslant -1}$ is nondecreasing. Indeed, by easy computations and from the nondecreasing property of $(b_q)_{q\geqslant -1}$ and the fact that $a_q \leqslant b_q$ we get

$$\begin{aligned} a_{q+1} - a_q &= \frac{1}{2}\big(\min(b_{q+1}, 2a_q) - a_q\big) \\ &= \frac{1}{2}\min(b_{q+1} - a_q, a_q) \\ &\geqslant \frac{1}{2}\min(b_q - a_q, a_q) \geqslant 0. \end{aligned}$$

Now we will prove that $(a_q)_{q\geqslant -1}$ converges to $+\infty$; otherwise it will converge to a finite real number $\ell > 0$. Using the relation (2.5) combined with the fact that $\lim_{q\to+\infty} b_q = +\infty$ yields necessarily $\ell = \frac{3}{2}\ell$, which contradicts the fact that $\ell \in]0, +\infty[$. On the other hand, we have

$$1 \leqslant \frac{a_{q+1}}{a_q} \leqslant \frac{3}{2}.$$

This ends the proof of all the properties of the sequence $(a_q)_{q\geqslant -1}$. □

Remark 3. From the proof of Lemma 2 we may easily check that we can replace the value of b_q by any expression $(\sum_{n\geqslant q} c_n)^{-\alpha}$, with $\alpha < 1$. The case $\alpha = 1$ is not true, at least for any convergent geometric series.

As a consequence we get the following result.

Corollary 1. *Let $s \in \mathbb{R}, p \in [1, \infty], r \in [1, \infty[$ and $(f_\varepsilon)_{0<\varepsilon\leqslant 1}$ be a family of smooth functions satisfying*

$$\Big(2^{qs} \sup_{0<\varepsilon\leqslant 1} \|\Delta_q f_\varepsilon\|_{L^p}\Big)_{\ell^r} < +\infty$$

Then there exists $\Psi \in \mathcal{U}_\infty$ such that

$$f_\varepsilon \in B^{s,\Psi}_{p,r}, \quad \forall\, \varepsilon \in]0, 1].$$

Proof. We set $c_q := 2^{qsr} \sup_{0<\varepsilon\leqslant 1} \|\Delta_q f_\varepsilon\|^r_{L^p}$; then $\sum_q c_q < +\infty$ and thus we can use Lemma 2. Therefore there exists $\Psi \in \mathcal{U}_\infty$ such that

$$\sum_{q\geqslant -1} \Psi^r(q) 2^{qsr} \sup_{0<\varepsilon\leqslant 1} \|\Delta_q f_\varepsilon\|^r_{L^p} < +\infty.$$

This achieves the proof. □

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

2.3. Lorentz spaces

Let us now introduce the Lorentz spaces that will be used later and especially to analyze the critical regularities. There are two ways to define these spaces: by a rearrangement procedure or by using real interpolation theory. We will briefly give both descriptions. For any measurable function f we define its nonincreasing rearrangement by

$$f^*(t) := \inf\Big\{s,\ \mu\big(\{x,\ |f(x)| > s\}\big) \leqslant t\Big\},$$

where μ denotes the usual Lebesgue measure. For $(p,q) \in [1,+\infty]^2$, the Lorentz space $L^{p,q}$ is the set of functions f such that $\|f\|_{L^{p,q}} < \infty$, with

$$\|f\|_{L^{p,q}} := \begin{cases} \Big(\displaystyle\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\Big)^{\frac{1}{q}}, & \text{for } 1 \leqslant q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & \text{for } q = \infty. \end{cases}$$

The second definition of Lorentz spaces which is equivalent to the first one is given by real interpolation theory:

$$(L^{p_0}, L^{p_1})_{(\theta,q)} = L^{p,q},$$

where $1 \leqslant p_0 < p < p_1 \leqslant \infty$, and θ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \leqslant q \leqslant \infty$. These spaces inherit from the Lebesgue spaces L^p the stability property of multiplication by a bounded function:

$$\|uv\|_{L^{p,q}} \leqslant C\,\|u\|_{L^\infty}\,\|v\|_{L^{p,q}}\,. \tag{2.6}$$

On the other hand we have the following embeddings:

$$L^{p,q} \hookrightarrow L^{p,q'}, \quad \forall\, 1 \leqslant p \leqslant \infty;\ 1 \leqslant q \leqslant q' \leqslant \infty \quad \text{and} \quad L^{p,p} = L^p. \tag{2.7}$$

3. Energy estimates

This section is devoted to the establishing of some energy estimates for the system (1.1) in the framework of heterogeneous Besov spaces $B^{s,\Psi}_{2,r}$. This is an extension of the classical estimates known for the usual Besov spaces $B^s_{2,r}$ and whose proof can be found for example in [15].

Proposition 1. *Let $(v_\varepsilon, c_\varepsilon)$ be a smooth solution of (1.1) and $\Psi \in \mathcal{U}$; see Definition 1. Then:*

(1) *the L^2-estimate: there exists $C > 0$ such that $\forall t \geqslant 0$,*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{L^2} \leqslant C\,\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{L^2}\, e^{C\|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}.$$

(2) *Besov estimates: for $s > 0$, $s \geqslant 0$, $r \in [1,+\infty]$, there exists $C > 0$ such that*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}} \leqslant C\,\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{s,\Psi}_{2,r}}\, e^{CV_\varepsilon(t)}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

with

$$V_\varepsilon(t) := \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

The above estimate holds true for homogeneous Besov spaces $\dot{B}^s_{2,r}$.

Proof. (1) Taking the L^2 inner product of the first equation of (1.1) with v_ε and integrating by parts,

$$\frac{1}{2}\frac{d}{dt}\|v_\varepsilon(t)\|^2_{L^2} - \frac{1}{2}\int_{\mathbb{R}^3} \operatorname{div} v_\varepsilon(|v_\varepsilon|^2 + \bar{\gamma} c_\varepsilon^2)dx - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} c_\varepsilon \operatorname{div} v_\varepsilon dx = 0.$$

Multiplying the second equation of (1.1) by c_ε and integrating by parts,

$$\frac{1}{2}\frac{d}{dt}\|c_\varepsilon(t)\|^2_{L^2} + \left(\bar{\gamma} - \frac{1}{2}\right)\int_{\mathbb{R}^3} (\operatorname{div} v_\varepsilon) c_\varepsilon^2 dx + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} c_\varepsilon \operatorname{div} v_\varepsilon dx = 0.$$

Thus summing these identities yields

$$\begin{aligned}\frac{d}{dt}\big(\|v_\varepsilon(t)\|^2_{L^2} + \|v_\varepsilon(t)\|^2_{L^2}\big) &= \int_{\mathbb{R}^3} \operatorname{div} v_\varepsilon \left(|v_\varepsilon|^2 + (1-\bar{\gamma})c_\varepsilon^2\right)dx \\ &\lesssim \|\operatorname{div} v_\varepsilon(t)\|_{L^\infty}\big(\|v_\varepsilon(t)\|^2_{L^2} + \|c_\varepsilon(t)\|^2_{L^2}\big).\end{aligned}$$

The result follows easily from Gronwall's lemma.

(2) We will use localize the equations in frequency and use some results on the commutators. Let $q \geqslant -1$ and set $f_q := \Delta_q f$; then

$$\begin{cases} \partial_t v_{\varepsilon,q} + v_\varepsilon \cdot \nabla v_{\varepsilon,q} + \bar{\gamma} c_\varepsilon \nabla c_{\varepsilon,q} + \dfrac{1}{\varepsilon}\nabla c_{\varepsilon,q} = -[\Delta_q, v_\varepsilon\cdot\nabla]v_\varepsilon - \bar{\gamma}[\Delta_q, c_\varepsilon]\nabla c_\varepsilon := T^1_{\varepsilon,q} \\ \partial_t c_{\varepsilon,q} + v_\varepsilon \cdot \nabla c_{\varepsilon,q} + \bar{\gamma} c_\varepsilon \operatorname{div} v_{\varepsilon,q} + \dfrac{1}{\varepsilon}\operatorname{div} v_{\varepsilon,q} \\ \quad = -[\Delta_q, v_\varepsilon\cdot\nabla]c_\varepsilon - \bar{\gamma}[\Delta_q, c_\varepsilon]\operatorname{div} v_\varepsilon := T^2_{\varepsilon,q}. \end{cases} \tag{3.1}$$

Define $w_{\varepsilon,q}(t) := |c_{\varepsilon,q}(t)|^2 + |v_{\varepsilon,q}(t)|^2$; then taking the L^2 inner product like in the first part (1), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w_{\varepsilon,q}(t)\|_{L^1} &= \frac{1}{2}\int_{\mathbb{R}^3} \operatorname{div} v_\varepsilon\, w_{\varepsilon,q}dx - \bar{\gamma}\int_{\mathbb{R}^3} c_\varepsilon\big(\nabla c_{\varepsilon,q}\cdot v_{\varepsilon,q} + c_{\varepsilon,q}\operatorname{div} v_{\varepsilon,q}\big)dx \\ &\quad + \int_{\mathbb{R}^3}(T^1_{\varepsilon,q}v_{\varepsilon,q} + T^2_{\varepsilon,q}c_{\varepsilon,q})dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \operatorname{div} v_\varepsilon\, w_{\varepsilon,q}dx + \bar{\gamma}\int_{\mathbb{R}^3} \nabla c_\varepsilon\cdot\big(c_{\varepsilon,q}v_{\varepsilon,q}\big)dx \\ &\quad + \int_{\mathbb{R}^3}(T^1_{\varepsilon,q}v_{\varepsilon,q} + T^2_{\varepsilon,q}c_{\varepsilon,q})dx \\ &\lesssim \Big(\|\operatorname{div} v_\varepsilon(t)\|_{L^\infty} + \|\nabla c_\varepsilon(t)\|_{L^\infty}\Big)\|w_{\varepsilon,q}(t)\|_{L^1} \\ &\quad + \|(T^1_{\varepsilon,q}, T^2_{\varepsilon,q})\|_{L^2}\|w_{\varepsilon,q}(t)\|^{\frac{1}{2}}_{L^1}.\end{aligned}$$

We have used in the last line the Young inequality $|ab| \leqslant \frac{1}{2}(a^2+b^2)$. It follows that

$$\frac{d}{dt}\|(v_{\varepsilon,q}, c_{\varepsilon,q})(t)\|_{L^2} \lesssim \Big(\|\operatorname{div} v_\varepsilon(t)\|_{L^\infty} + \|\nabla c_\varepsilon(t)\|_{L^\infty}\Big)\|(v_{\varepsilon,q}, c_{\varepsilon,q})(t)\|_{L^2} + \|(T^1_{\varepsilon,q}, T^2_{\varepsilon,q})\|_{L^2}.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Multiplying by $2^{qs}\Psi(q)$ and summing over q we get

$$\frac{d}{dt}\|(v_\varepsilon,c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}} \leqslant \big(\|\mathrm{div}\, v_\varepsilon(t)\|_{L^\infty}+\|\nabla c_\varepsilon(t)\|_{L^\infty}\big)\,\|(v_\varepsilon,c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}} + \Big(2^{qs}\Psi(q)\,\|(T^1_{\varepsilon,q},T^2_{\varepsilon,q})\|_{L^2}\Big)_{\ell^r}.$$

Now according to Lemma 6 in Appendix we have: for $s>0$, $r\in[1,\infty]$ and $\Psi\in\mathcal{U}$,

$$\big(2^{qs}\Psi(q)\,\|[\Delta_q,v\cdot\nabla]u\|_{L^2}\big)_{\ell^r}\lesssim \|\nabla v\|_{L^\infty}\,\|u\|_{B^{s,\Psi}_{2,r}}+\|\nabla u\|_{L^\infty}\,\|v\|_{B^{s,\Psi}_{2,r}}.$$

This yields

$$\frac{d}{dt}\|(v_\varepsilon,c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}}\lesssim\big(\|\nabla v_\varepsilon(t)\|_{L^\infty}+\|\nabla c_\varepsilon(t)\|_{L^\infty}\big)\,\|(v_\varepsilon,c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}}.$$

It suffices to use Gronwall's inequality to get the desired estimate. □

4. Strichartz estimates

The main goal of this section is to establish some Strichartz estimates for the compressible and acoustic parts which are governed by coupled nonlinear wave equations. As was shown in the pioneering work of Ukai [33], the use of the dispersion is crucial for getting rid of the well-preparedness assumption when studying the convergence towards the incompressible Euler system. More precisely, it was shown that on averaging over time, the compressible and acoustic parts vanish when the Mach number goes to zero. In our case we will take advantage of the Strichartz estimates, firstly to deal with the ill-prepared case and secondly to improve the lifespan by a judicious combination with the special structure of the vorticity for the axisymmetric flows. The results concerning Strichartz estimates that we will use here are well-known in the literature and are discussed only briefly. Nevertheless, we will give more details about their applications for the isentropic Euler system.

We will start with rewriting the system (1.1) with the aid of the free wave propagator

$$\begin{cases}\partial_t v_\varepsilon+\dfrac{1}{\varepsilon}\nabla c_\varepsilon=f_\varepsilon:=-v_\varepsilon\cdot\nabla v_\varepsilon-c_\varepsilon\nabla c_\varepsilon\\[2mm] \partial_t c_\varepsilon+\dfrac{1}{\varepsilon}\mathrm{div}\, v_\varepsilon=g_\varepsilon:=-v_\varepsilon\cdot\nabla c_\varepsilon-c_\varepsilon\mathrm{div}\, v_\varepsilon.\end{cases}\tag{4.1}$$

Let $\mathcal{Q}$ be the operator $\mathcal{Q}v:=\nabla\Delta^{-1}\mathrm{div}\, v$ which is nothing but the compressible part of the velocity v. Set $|\mathrm{D}|:=\sqrt{-\Delta}$; then by simple computations we show that the quantity

$$\Gamma_\epsilon:=\mathcal{Q}v_\varepsilon-i\nabla\,|\mathrm{D}|^{-1}\,c_\varepsilon$$

satisfies the wave equation

$$\partial_t\Gamma_\epsilon+\frac{i}{\varepsilon}|\mathrm{D}|\Gamma_\epsilon=\mathcal{Q}f_\varepsilon-i\nabla\,|\mathrm{D}|^{-1}\,g_\varepsilon.\tag{4.2}$$

Analogously, the quantity

$$\gamma_\epsilon:=|\mathrm{D}|^{-1}\,\mathrm{div}\, v_\varepsilon+i\,c_\varepsilon$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

satisfies the wave equation

$$\partial_t \gamma_\epsilon + \frac{i}{\varepsilon}|\mathrm{D}|\gamma_\epsilon = |\mathrm{D}|^{-1}\operatorname{div} f_\varepsilon + i\, g_\varepsilon. \tag{4.3}$$

We will use the following Strichartz estimates which can be found for instance in [6, 17].

Lemma 3. *Let φ be a complex solution of the wave equation*

$$\partial_t \varphi + \frac{i}{\varepsilon}|\mathrm{D}|\varphi = F.$$

Then for every $s \in \mathbb{R}$, $r > 2$ there exists C such that for every $T > 0$ we have

$$\|\varphi\|_{L^r_T \dot{B}^{s-\frac{3}{2}+\frac{1}{r}}_{\infty,1}} \leqslant C\varepsilon^{\frac{1}{r}}\big(\|\varphi(0)\|_{\dot{B}^s_{2,1}} + \|F\|_{L^1_T \dot{B}^s_{2,1}}\big).$$

As a consequence we get the following Strichartz estimates for both compressible and acoustic parts.

Corollary 2. *Let $r > 2$ and $v_{0,\varepsilon}, c_{0,\varepsilon} \in B^{\frac{5}{2}-\frac{1}{r}}_{2,1}$. Then the solutions of the system (1.1) satisfy for all $T > 0$ and $0 < \varepsilon \leqslant 1$,*

$$\|\mathcal{Q}v_\varepsilon\|_{L^r_T \dot{B}^0_{\infty,1}} + \|c_\varepsilon\|_{L^r_T \dot{B}^0_{\infty,1}} \leqslant C_0 \varepsilon^{\frac{1}{r}}(1+T)e^{CV_\varepsilon(T)}$$

with C_0 a constant depending on r and the norm $\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{\frac{5}{2}-\frac{1}{r}}_{2,1}}$ and

$$V_\varepsilon(T) = \int_0^T \big(\|\nabla v_\varepsilon(t)\|_{L^\infty} + \|\nabla c_\varepsilon(t)\|_{L^\infty}\big)\,dt.$$

Proof. Applying Lemma 3 to the equation (4.2) with $r > 2$ and $s = \frac{3}{2} - \frac{1}{r}$ we get

$$\|\Gamma_\varepsilon\|_{L^r_T \dot{B}^0_{\infty,1}} \leqslant C\varepsilon^{\frac{1}{r}}\left(\|\Gamma_\varepsilon(0)\|_{\dot{B}^{\frac{3}{2}-\frac{1}{r}}_{2,1}} + \|\mathcal{Q}f_\varepsilon - i\nabla\,|\mathrm{D}|^{-1} g_\varepsilon\|_{L^1_T \dot{B}^{\frac{3}{2}-\frac{1}{r}}_{2,1}}\right).$$

Since the operators $\mathcal{Q}$ and $\nabla\,|\mathrm{D}|^{-1}$ are homogeneous of order 0 then they are continuous on the homogeneous Besov spaces and thus we get

$$\begin{aligned}\|\Gamma_\varepsilon\|_{L^r_T \dot{B}^0_{\infty,1}} &\leqslant C\varepsilon^{\frac{1}{r}}\left(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{\dot{B}^{\frac{3}{2}-\frac{1}{r}}_{2,1}} + \|(f_\varepsilon, g_\varepsilon)\|_{L^1_T \dot{B}^{\frac{3}{2}-\frac{1}{r}}_{2,1}}\right)\\ &\leqslant C\varepsilon^{\frac{1}{r}}\left(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{\frac{3}{2}-\frac{1}{r}}_{2,1}} + \|(f_\varepsilon, g_\varepsilon)\|_{L^1_T B^{\frac{3}{2}-\frac{1}{r}}_{2,1}}\right).\end{aligned}$$

We point out that we have used above the embedding $B^s_{2,1} \hookrightarrow \dot{B}^s_{2,1}$, for $s > 0$. To estimate the terms f_ε and g_ε we will use the following law product that can be easily proven by using Bony's decomposition: for $s > 0$,

$$\|f\partial_j g\|_{B^s_{2,1}} \lesssim \|f\|_{L^\infty}\,\|g\|_{B^{s+1}_{2,1}} + \|g\|_{L^\infty}\,\|f\|_{B^{s+1}_{2,1}}\,.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Therefore we find

$$\|(f_\varepsilon, g_\varepsilon)\|_{B_{2,1}^{\frac{3}{2}-\frac{1}{r}}} \leqslant C\,\|(v_\varepsilon, c_\varepsilon)\|_{L^\infty}\,\|(v_\varepsilon, c_\varepsilon)\|_{B_{2,1}^{\frac{5}{2}-\frac{1}{r}}}\,.$$

Combining this estimate with the Sobolev embedding $B_{2,1}^{\frac{5}{2}-\frac{1}{r}} \hookrightarrow L^\infty$ (true for $r \geqslant 1$), we get

$$\|(f_\varepsilon, g_\varepsilon)\|_{B_{2,1}^{\frac{3}{2}-\frac{1}{r}}} \leqslant C\,\|(v_\varepsilon, c_\varepsilon)\|^2_{B_{2,1}^{\frac{5}{2}-\frac{1}{r}}}\,.$$

Applying Proposition 1 yields

$$\|(v_\varepsilon, c_\varepsilon)\|_{L_T^\infty B_{2,1}^{\frac{5}{2}-\frac{1}{r}}} \leqslant C\,\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{5}{2}-\frac{1}{r}}}\,e^{CV_\varepsilon(T)}.$$

Thus we obtain for $r > 2$,

$$\begin{aligned}\|\Gamma_\varepsilon\|_{L_T^r \dot{B}^0_{\infty,1}} &\leqslant C\varepsilon^{\frac{1}{r}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{3}{2}-\frac{1}{r}}} + \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|^2_{B_{2,1}^{\frac{5}{2}-\frac{1}{r}}}\,Te^{CV_\varepsilon(T)}\Big)\\ &\leqslant C_0\varepsilon^{\frac{1}{r}}(1+T)e^{CV_\varepsilon(T)}.\end{aligned}$$

Since the real part of Γ_ε is the compressible part of v_ε, then

$$\|\mathcal{Q}v_\varepsilon\|_{L_T^r \dot{B}^0_{\infty,1}} \leqslant C_0\varepsilon^{\frac{1}{r}}(1+T)e^{CV_\varepsilon(T)}.$$

In the same way we prove a similar result for γ_ε:

$$\|\gamma_\varepsilon\|_{L_T^r \dot{B}^0_{\infty,1}} \leqslant C_0\varepsilon^{\frac{1}{r}}(1+T)e^{CV_\varepsilon(T)}.$$

This gives the desired estimate for $\|c_\varepsilon\|_{L_T^r \dot{B}^0_{\infty,1}}$. □

Proposition 2. *Let* $(v_{0,\varepsilon}, c_{0,\varepsilon})$ *be an* H^s*-bounded family with* $s > \frac{5}{2}$ *and* $v_\varepsilon, c_\varepsilon \in \mathcal{C}([0, T_\varepsilon); H^s)$ *be the maximal solution of the system (1.1). Then for every* $r > 2$, $0 \leqslant T < T_\varepsilon$ *and* $\varepsilon \in]0, 1]$, *we have*

$$\|\mathrm{div}\, v_\varepsilon\|_{L_T^1 B^0_{\infty,1}} + \|\nabla c_\varepsilon\|_{L_T^1 B^0_{\infty,1}} \leqslant C_0\varepsilon^{\frac{2s-5}{r(2s-3)}}(1+T^2)e^{CV_\varepsilon(T)}.$$

Proof. Applying Lemma 4 with $\varphi = \mathcal{Q}v_\varepsilon$ and using the identity $\mathrm{div}\,\mathcal{Q}v_\varepsilon = \mathrm{div}\, v_\varepsilon$, we get

$$\begin{aligned}\|\mathrm{div}\, v_\varepsilon\|_{L_T^1 B^0_{\infty,1}} &\leqslant CT^{1-\frac{2s-5}{r(2s-3)}}\,\|\mathcal{Q}v_\varepsilon\|^{\frac{2s-5}{2s-3}}_{L_T^r L^\infty}\,\|\mathcal{Q}v_\varepsilon\|^{\frac{2}{2s-3}}_{L_T^\infty H^s}\\ &\leqslant CT^{1-\frac{2s-5}{r(2s-3)}}\,\|\mathcal{Q}v_\varepsilon\|^{\frac{2s-5}{2s-3}}_{L_T^r \dot{B}^0_{\infty,1}}\,\|v_\varepsilon\|^{\frac{2}{2s-3}}_{L_T^\infty H^s}\,.\end{aligned}$$

We have used in the last inequality the embedding $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$. Now, combining this estimate with Proposition 1 and Corollary 2,

$$\begin{aligned}\|\mathrm{div}\, v_\varepsilon\|_{L_T^1 B^0_{\infty,1}} &\leqslant C_0\varepsilon^{\frac{2s-5}{r(2s-3)}}\,(1+T)^{1+\frac{2s-5}{2s-3}(1-\frac{1}{r})}\,e^{CV_\varepsilon(t)}\\ &\leqslant C_0\varepsilon^{\frac{2s-5}{r(2s-3)}}(1+T^2)e^{CV_\varepsilon(t)}.\end{aligned}$$

Similarly we obtain an analogous estimate for the acoustic part $\|\nabla c_\varepsilon\|_{L_T^1 B^0_{\infty,1}}$. □

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

5. Axisymmetric flows

In this section we intend to establish some preliminary results concerning the axisymmetric geometry for the compressible flows. First, we prove the persistence in time of this geometry when it is initially prescribed and second we analyze the structure and the dynamics of the vorticity. We end this section with some useful *a priori* estimates.

5.1. Persistence of the geometry

The study of axisymmetric flows was initiated by Ladyzhenskaya [22] and Ukhovskii and Yudovich [32] for both incompressible Euler and Navier–Stokes equations. This study has been recently extended to other models of incompressible fluid dynamics like stratified Euler and Navier–Stokes systems [1, 19]. First of all we will show the compatibility of this geometry with the model (1.1) but we need, before this, to give a precise definition of axisymmetric vector fields.

Definition 2.

- We say that a vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$ is axisymmetric if it satisfies

$$\mathcal{R}_{-\alpha}\{v(\mathcal{R}_\alpha x)\} = v(x), \quad \forall \alpha \in [0, 2\pi], \forall x \in \mathbb{R}^3,$$

where $\mathcal{R}_\alpha$ denotes the rotation of axis (Oz) and with angle θ. An axisymmetric vector field v is called without swirl if its angular component vanishes, which is equivalent to the fact that v takes the form

$$v(x) = v^r(r, z)e_r + v^z(r, z)e_z, \quad x = (x_1, x_2, z), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

where $\left(e_r, e_\theta, e_z\right)$ is the cylindrical basis of $\mathbb{R}^3$ and the components v^r and v^z do not depend on the angular variable.

- A scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ is called axisymmetric if the vector field $x \mapsto f(x)e_z$ is axisymmetric, which means that

$$f(\mathcal{R}_\alpha x) = f(x), \quad \forall x \in \mathbb{R}^3, \forall \alpha \in [0, 2\pi].$$

This is equivalent to saying that f depends only on r and z.

Now we will prove the persistence of this geometry for the system (1.1).

Proposition 3. *Let $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be a smooth axisymmetric initial data without swirl. Then the associated maximal solution of (1.1) remains axisymmetric.*

Proof. For the sake of the simplicity we will remove the subscript ε from our notation. We set

$$v_\alpha(t, x) = \mathcal{R}_{-\alpha}\{v(t, \mathcal{R}_\alpha x)\} \quad \text{and} \quad c_\alpha(t, x) = c(t, \mathcal{R}_\alpha x).$$

Our goal is to show that (v_α, c_α) solves the system (1.1). First of all, we claim that

$$(v_\alpha \cdot \nabla v_\alpha)(t, x) = \mathcal{R}_{-\alpha}\{(v \cdot \nabla v)(t, \mathcal{R}_\alpha x)\}. \tag{5.1}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Indeed, obvious computations yield

$$\begin{aligned}(v_\alpha\cdot\nabla v_\alpha)(t,x) &= \sum_{i=1}^{3}\big(\mathcal{R}_{-\alpha}\{v(t,\mathcal{R}_\alpha x)\}\big)^i\,\partial_i\{\mathcal{R}_{-\alpha}\,v(t,\mathcal{R}_\alpha x)\}\\ &= \sum_{i=1}^{3}\big(\mathcal{R}_{-\alpha}\{v(t,\mathcal{R}_\alpha x)\}\big)^i\,\mathcal{R}_{-\alpha}\partial_i\{v(t,\mathcal{R}_\alpha x)\}\\ &= \mathcal{R}_{-\alpha}\sum_{i=1}^{3}\big(\mathcal{R}_{-\alpha}\{v(t,\mathcal{R}_\alpha x)\}\big)^i\,\partial_i\{v(t,\mathcal{R}_\alpha x)\}\\ &:= \mathcal{R}_{-\alpha}w.\end{aligned}$$

From the formula

$$\partial_i\,\{v(t,\mathcal{R}_\alpha x)\}^j = \big(\mathcal{R}_{-\alpha}\{(\nabla v^j)(t,\mathcal{R}_\alpha x)\}\big)^i$$

and using the fact that the rotations preserve the Euclidean scalar product we obtain

$$\begin{aligned}w^j &= \sum_{i=1}^{3}\big(\mathcal{R}_{-\alpha}\{v(t,\mathcal{R}_\alpha x)\}\big)^i\,\big(\mathcal{R}_{-\alpha}\{(\nabla v^j)(\mathcal{R}_\alpha x)\}\big)^i\\ &= (v\cdot\nabla v^j)(t,\mathcal{R}_\alpha x).\end{aligned}$$

Therefore we get

$$(v_\alpha\cdot\nabla v_\alpha)(t,x) = \mathcal{R}_{-\alpha}\{(v\cdot\nabla v)(t,\mathcal{R}_\alpha x)\}.$$

Now if f is a scalar function then

$$\mathcal{R}_{-\alpha}\{(\nabla f)(\mathcal{R}_\alpha x)\} = \nabla\{f(\mathcal{R}_\alpha x)\}. \tag{5.2}$$

Using this identity and (5.1) we prove that (v_α, c_α) satisfies the first equation of (1.1). It remains to prove that this couple of functions satisfies also the second equation of (1.1). We write, according to the identity (5.2) and from the fact that $\mathcal{R}_\alpha$ is an isometry,

$$\begin{aligned}(v_\alpha\cdot\nabla c_\alpha)(t,x) &= \sum_{i=1}^{3}\mathcal{R}_{-\alpha}\{v(t,\mathcal{R}_\alpha x)\}\cdot\mathcal{R}_{-\alpha}\{\nabla c\}(t,\mathcal{R}_\alpha x)\\ &= \{v\cdot\nabla c\}(t,\mathcal{R}_\alpha x).\end{aligned}$$

It remains to check that

$$\operatorname{div} v_\alpha(t,x) = \{\operatorname{div} v\}(t,\mathcal{R}_\alpha x).$$

Indeed, set $\mathcal{R}_{-\alpha} := (a_{ij})_{1\leqslant i,j\leqslant 3}$; then since $\mathcal{R}_\alpha^\star = \mathcal{R}_{-\alpha}$, we find

$$\begin{aligned}\operatorname{div} v_\alpha(t,x) &= \sum_{i,j=1}^{3}\partial_i\{a_{ij}v^j(t,\mathcal{R}_\alpha x)\}\\ &= \sum_{i,k,j=1}^{3}a_{ij}a_{ik}\{\partial_k v^j\}(t,\mathcal{R}_\alpha x)\end{aligned}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

$$= \sum_{k,j=1}^{3} \delta_{kj}\{\partial_k v^j\}(t, \mathcal{R}_\alpha x)$$
$$= \{\operatorname{div} v\}(t, \mathcal{R}_\alpha x).$$

Finally, we deduce that (v_α, c_α) satisfies the same equations as (v, c) and by the uniqueness of the solutions we get $v_\alpha = v, c_\alpha = c$ for every $\alpha \in [0, 2\pi]$, and thus the solution is axisymmetric. To achieve the proof it remains to show that the angular component v^θ of the velocity v is zero. By direct computations using the axisymmetry of the solution (v, c) we get

$$\partial_t v^\theta + v \cdot \nabla v^\theta + \frac{v^r}{r} v^\theta = 0.$$

It follows from the maximum principle and Gronwall's inequality that

$$\|v^\theta(t)\|_{L^\infty} \leqslant \|v_0^\theta\|_{L^\infty} e^{\|v^r/r\|_{L^1_t L^\infty}}.$$

Therefore if $v_0^\theta \equiv 0$, then $v^\theta(t) \equiv 0$ everywhere and the solution is defined. □

Remark 4. From the above computations we get the following assertions:

(1) If c is a scalar axisymmetric function, then its gradient ∇c is an axisymmetric vector field.

(2) If v is an axisymmetric vector field, then its divergence $\operatorname{div} v$ is an axisymmetric scalar function.

5.2. Dynamics of the vorticity

We will start with recalling some algebraic properties of the axisymmetric vector fields and in particular we will discuss the special structure of the vorticity of the system (1.1). First, we make some general statements: let $w = w^r(r, z)e_r + w^\theta(r, z)e_\theta + w^z(r, z)e_z$ and $v = v^r(r, z)e_r + v^z(r, z)e_z$ be two smooth vector fields; then

$$w \cdot \nabla = w^r \partial_r + \frac{w^\theta}{r}\partial_\theta + w^z \partial_z, \quad \operatorname{div} v = \partial_r v^r + \frac{v^r}{r} + \partial_z v^z. \tag{5.3}$$

Easy computations show that the vorticity $\Omega := \operatorname{curl}\ v$ of the vector field v takes the form,

$$\Omega = (\partial_z v^r - \partial_r v^z)e_\theta.$$

Now let us study, under this special geometry, the dynamics of the vorticity of the system (1.1). Denote by $\Omega_\varepsilon = (\partial_z v_\varepsilon^r - \partial_r v_\varepsilon^z)e_\theta := \Omega_\varepsilon^\theta e_\theta$ the vorticity of v_ε. Then applying the curl operator to the velocity equation yields

$$\partial_t \Omega_\varepsilon + \operatorname{curl}(v_\varepsilon \cdot \nabla v_\varepsilon) = 0.$$

By straightforward computations we get the identity

$$\operatorname{curl}(v_\varepsilon \cdot \nabla v_\varepsilon) = v_\varepsilon \cdot \nabla \Omega_\varepsilon - \Omega_\varepsilon \cdot \nabla v_\varepsilon + \Omega_\varepsilon \operatorname{div} v_\varepsilon.$$

Therefore we get

$$\partial_t \Omega_\varepsilon + v_\varepsilon \cdot \nabla \Omega_\varepsilon + \Omega_\varepsilon \operatorname{div} v_\varepsilon = \Omega_\varepsilon \cdot \nabla v_\varepsilon$$

Now, since $\Omega_\varepsilon = \Omega_\varepsilon^\theta e_\theta$ and by (5.3), it follows that the stretching term takes the form

$$\begin{aligned}\Omega_\varepsilon \cdot \nabla v_\varepsilon &= \Omega_\varepsilon^\theta \frac{1}{r}\partial_\theta (v_\epsilon^r e_r + v_\epsilon^z e_z)\\ &= \Omega_\varepsilon^\theta \,\frac{v_\epsilon^r}{r}\, e_\theta\\ &= \frac{v_\epsilon^r}{r}\Omega_\varepsilon.\end{aligned}$$

Consequently the vorticity equation becomes

$$\partial_t \Omega_\varepsilon + v_\varepsilon \cdot \nabla \Omega_\varepsilon + \Omega_\varepsilon \operatorname{div} v_\varepsilon = \frac{v_\epsilon^r}{r}\Omega_\varepsilon. \tag{5.4}$$

Thus the quantity $\frac{\Omega_\varepsilon}{r}$ is governed by the following equation:

$$(\partial_t + v_\varepsilon \cdot \nabla + \operatorname{div} v_\varepsilon)\left(\frac{\Omega_\varepsilon}{r}\right) = 0. \tag{5.5}$$

We observe that this equation is analogous to the vorticity equation in dimension 2. In the case of incompressible axisymmetric flows the quantity $\frac{\Omega_\varepsilon}{r}$ is just transported by the flow and this gives new conservation laws that lead to the global existence of smooth solutions. However in our case we cannot get global estimates due to the presence of $\operatorname{div} v_\varepsilon$, yet we have already seen that this quantity is damped by higher oscillations and thus we can expect that the time lifespan can grow when the Mach number becomes small. This will be clearly discussed in later sections.

5.3. Geometric properties

When we deal with the critical regularities, it seems that the use of just equation (5.5) is not sufficient for our study and so we need more refined properties of the vorticity. We start with the following results.

Proposition 4. *Let $v = (v^1, v^2, v^3)$ be a smooth axisymmetric vector field without swirl. Then the following assertions hold true.*

(1) *The vector $\omega = \nabla \times v = (\omega^1, \omega^2, \omega^3)$ satisfies $\omega(x) \times e_\theta(x) = (0, 0, 0)$. In particular, we have for every $(x_1, x_2, z) \in \mathbb{R}^3$,*

$$\omega^3 = 0, \quad x_1\omega^1(x_1, x_2, z) + x_2\omega^2(x_1, x_2, z) = 0 \quad \textit{and}$$
$$\omega^1(x_1, 0, z) = \omega^2(0, x_2, z) = 0.$$

(2) *For every $q \geqslant -1$, $\Delta_q u$ is axisymmetric without swirl and*

$$(\Delta_q u^1)(0, x_2, z) = (\Delta_q u^2)(x_1, 0, z) = 0.$$

(3) *Let Ω be a divergence-free vector field such that $\Omega(x) \times e_\theta(x) = 0$. Then for $q \geqslant -1$ we have*

$$\Delta_q \Omega(x) \times e_\theta(x) = 0.$$

Proof. The results (1) and (2) are proved in for example [1]. It remains to prove the last assertion. First, it is easy to check that $(\Delta_q\Omega)^3 = \Delta_q(\Omega^3)$ and thus the last component of $\Delta_q\Omega$ is zero, and consequently our claim reduces to the following identity:

$$x_1\Delta_q\Omega^1 + x_2\Delta_q\Omega^2 = 0.$$

Using the Fourier transform this is equivalent to

$$\partial_1\big(\varphi(2^{-q}|\xi|)\widehat{\Omega^1}(\xi)\big) + \partial_2\big(\varphi(2^{-q}|\xi|)\widehat{\Omega^2}(\xi)\big) = 0.$$

Straightforward computations and the fact that $\Omega^3 \equiv 0$ yield

$$\begin{aligned}\partial_1\big(\varphi(2^{-q}|\xi|)\widehat{\Omega^1}(\xi)\big) + \partial_2\big(\varphi(2^{-q}|\xi|)\widehat{\Omega^2}(\xi)\big) &= 2^{-q}\,|\xi|^{-1}\,\varphi'(2^{-q}|\xi|)\big(\xi^1\widehat{\Omega^1}(\xi) + \xi^2\widehat{\Omega^2}(\xi)\big)\\ &\quad + \varphi(2^{-q}|\xi|)\big(\partial_1\widehat{\Omega^1}(\xi) + \partial_2\widehat{\Omega^2}(\xi)\big)\\ &= -i2^{-q}\,|\xi|^{-1}\,\varphi'(2^{-q}|\xi|)\widehat{\operatorname{div}\Omega}(\xi)\\ &\quad - i\varphi(2^{-q}|\xi|)\mathcal{F}\big(x_1\Omega^1 + x_2\Omega^2\big)(\xi)\\ &= 0.\end{aligned}$$

The last identity is an easy consequence from the hypotheses $\operatorname{div}\Omega = 0$ and $\Omega \times e_\theta = 0$. □

The next result deals with some properties of the flow ψ associated with a time-dependent axisymmetric vector field $(t,x) \mapsto v(t,x)$. It is defined as follows through the integral equation:

$$\psi(t,x) = x + \int_0^t v\big(\tau, \psi(\tau,x)\big)d\tau.$$

It is well-known from the theory of differential equation that if I is an interval and $v \in L^1(I, \mathrm{Lip})$ then the flow is well-defined on the full interval I. Denote by $\mathcal{G}$ the group of rotations with axis (Oz), that is

$$\mathcal{G} := \big\{\mathcal{R}_\theta, \theta \in [0, 2\pi]\big\}.$$

Now we will prove the following result.

Proposition 5. *Let $(t,x) \mapsto v(t,x)$ be a time-dependent smooth axisymmetric vector field without swirl and $(t,x) \mapsto \psi(t,x)$ its flow. Then the following results hold true.*

(1) *For all $\mathcal{R}_\theta \in \mathcal{G}$ we have*

$$\mathcal{R}_\theta(\psi(t,x)) = \psi(t, \mathcal{R}_\theta x), \quad \forall x \in \mathbb{R}^3.$$

(2) *For every $x \in \mathbb{R}^3, t \geqslant 0$ we have*

$$\psi(t,x)\cdot e_\theta(x) = 0.$$

(3) *For every t, the vector field $x \mapsto v(t, \psi(t,x))$ is axisymmetric without swirl.*

(4) *For all $q \in \mathbb{N}$ we have*

$$S_q\big(v(t)\circ\psi(t)\big)(x)\cdot e_\theta(x) = 0, \quad \forall x \in \mathbb{R}^3.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Proof. (1) We set $\psi_\theta(t,x) := \mathcal{R}_\theta\psi(t,x)$. Thus differentiating with respect to t we get

$$\partial_t\psi_\theta(t,x) = \mathcal{R}_\theta\big(v(t,\psi(t,x))\big).$$

Since v is axisymmetric then

$$\mathcal{R}_\theta\big(v(t,X)\big) = v(t,\mathcal{R}_\theta X), \quad \forall X \in \mathbb{R}^3$$

and consequently

$$\partial_t\psi_\theta(t,x) = v\big(t,\mathcal{R}_\theta(\psi(t,x))\big) = v(t,\psi_\theta(t,x)), \quad \psi_\theta(0,x) = \mathcal{R}_\theta x.$$

It is easy to see that $(t,x) \mapsto \psi(t,\mathcal{R}_\theta x)$ satisfies the same differential equation as ψ_θ and thus by uniqueness we get the desired identity.

(2) This result, which means that the trajectory of a given particle x remains in the vertical plane, was proved in a proposition in [2].

(3) Let $\mathcal{R}_\theta \in \mathcal{G}$. Since v is axisymmetric, then using (1) of Proposition 5 yields

$$\begin{aligned}\mathcal{R}_\theta\big(v(t,\psi(t,x))\big) &= v(t,\mathcal{R}_\theta(\psi(t,x))) \\ &= v(t,\psi(t,\mathcal{R}_\theta x)).\end{aligned}$$

It remains to show that the angular component of the vector field $x \mapsto v(t,\psi(t,x))$ is zero. Since the angular component of v is zero, then

$$v(t,X)\cdot e_\theta(X) = 0, \quad \forall X \in \mathbb{R}^3$$

and by (2) of Proposition 5 we have $e_\theta(\psi(t,x)) = \pm e_\theta(x)$. It follows that the angular component of the vector field $x \mapsto v(t,\psi(t,x))$ vanishes and consequently it is axisymmetric without swirl.

(4) Combining the preceding result with part (2) of Proposition 4 we obtain that the vector field $x \mapsto S_q\big(v(t)\circ\psi(t)\big)(x)$ is also axisymmetric without swirl and consequently its angular component is zero. □

The end of this section is devoted to the study of some geometric properties of a compressible transport equation which models the vorticity equation:

$$(\mathrm{CT})\begin{cases}\partial_t\Omega + (v\cdot\nabla)\Omega + \Omega\,\mathrm{div}\,v = \Omega\cdot\nabla v,\\ \Omega_{|t=0} = \Omega_0.\end{cases}$$

We will assume that v is axisymmetric and the unknown function $\Omega = (\Omega^1,\Omega^2,\Omega^3)$ is a vector field. The following result describes the persistence of some initial geometric conditions of the solution Ω.

Proposition 6. *Let $T > 0$ and v be an axisymmetric vector field without swirl and belonging to the space $L^1([0,T],\mathrm{Lip}(\mathbb{R}^3))$. Denote by Ω the unique solution of* (CT) *corresponding to smooth initial data Ω_0. Then we have the following properties:*

(1) *If* $\mathrm{div}\,\Omega_0 = 0$, *then* $\mathrm{div}\,\Omega(t) = 0, \forall\, t \in [0,T]$.

(2) *If* $\Omega_0 \times e_\theta = 0$, *then we have*

$$\forall t \in [0,T], \qquad \Omega(t)\times e_\theta = 0.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Consequently, $\Omega^1(t, x_1, 0, z) = \Omega^2(t, 0, x_2, z) = 0,$ *and*

$$\partial_t \Omega + (v \cdot \nabla)\Omega + \Omega \operatorname{div} v = \frac{v^r}{r}\Omega.$$

Proof. (1) We apply the divergence operator to the equation in (V):

$$\partial_t \operatorname{div} \Omega + \operatorname{div}(u \cdot \nabla\Omega + \Omega \operatorname{div} u) = \operatorname{div}(\Omega \cdot \nabla u).$$

Straightforward computations yield

$$\operatorname{div}(u \cdot \nabla\Omega) = u \cdot \nabla \operatorname{div} \Omega + \sum_{i,j=1}^{3} \partial_i u^j \, \partial_j \Omega^i$$
$$\operatorname{div}(\Omega \operatorname{div} u) = \operatorname{div} \Omega \operatorname{div} u + \Omega \cdot \nabla \operatorname{div} u$$
$$\operatorname{div}(\Omega \cdot \nabla u) = \Omega \cdot \nabla \operatorname{div} u + \sum_{i,j=1}^{3} \partial_i u^j \, \partial_j \Omega^i.$$

Thus, the quantity $\operatorname{div} \Omega$ satisfies the equation

$$\partial_t \operatorname{div} \Omega + u \cdot \nabla \operatorname{div} \Omega + \operatorname{div} \Omega \operatorname{div} u = 0.$$

Using the maximum principle and Gronwall inequality we get

$$\|\operatorname{div} \Omega(t)\|_{L^\infty} \leqslant \|\operatorname{div} \Omega_0\|_{L^\infty} \, e^{\|\operatorname{div} u\|_{L^1_t L^\infty}}.$$

Therefore if $\operatorname{div} \Omega_0 = 0$ then for every time, $\Omega(t)$ remains incompressible.

(2) We denote by $(\Omega^r, \Omega^\theta, \Omega^z)$ the coordinates of Ω in the cylindrical basis. It is obvious that $\Omega^r = \Omega \cdot e_r$. Recall that in cylindrical coordinates the operator $u \cdot \nabla$ has the form

$$u \cdot \nabla = u^r \partial_r + \frac{1}{r} u^\theta \partial_\theta + u^z \partial_z = u^r \partial_r + u^z \partial_z.$$

We have used in the last equality the fact that for axisymmetric flows the angular component is zero. Hence we get

$$\begin{aligned}(u \cdot \nabla\Omega) \cdot e_r &= u^r \partial_r \Omega \cdot e_r + u^z \partial_z \Omega \cdot e_r \\ &= (u^r \partial_r + u^z \partial_z)(\Omega \cdot e_r) \\ &= u \cdot \nabla \Omega^r,\end{aligned}$$

where we use $\partial_r e_r = \partial_z e_r = 0$. Now it remains to compute $(\Omega \cdot \nabla u) \cdot e_r$. By straightforward computations we get

$$\begin{aligned}(\Omega \cdot \nabla u) \cdot e_r &= \Omega^r \, \partial_r u \cdot e_r + \frac{1}{r} \Omega^\theta \, \partial_\theta u \cdot e_r + \Omega^3 \, \partial_3 u \cdot e_r \\ &= \Omega^r \partial_r u^r + \Omega^3 \partial_3 u^r.\end{aligned}$$

Thus the component Ω^r obeys the equation

$$\partial_t \Omega^r + u \cdot \nabla \Omega^r + \operatorname{div} u \, \Omega^r = \Omega^r \partial_r u^r + \Omega^3 \partial_3 u^r.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

From the maximum principle we deduce

$$\|\Omega^r(t)\|_{L^\infty} \leqslant \int_0^t \big(\|\Omega^r(\tau)\|_{L^\infty} + \|\Omega^3(\tau)\|_{L^\infty}\big)\,\|\nabla u(\tau)\|_{L^\infty}\,d\tau.$$

On the other hand the component Ω^3 satisfies the equation

$$\partial_t\Omega^3 + u\cdot\nabla\Omega^3 + \mathrm{div}\,u\,\Omega^3 = \Omega^3\partial_3 u^3 + \Omega^r\partial_r u^3.$$

This leads to

$$\|\Omega^3(t)\|_{L^\infty} \leqslant \int_0^t \big(\|\Omega^3(\tau)\|_{L^\infty} + \|\Omega^r(\tau)\|_{L^\infty}\big)\,\|\nabla u(\tau)\|_{L^\infty}\,d\tau.$$

Combining these estimates and using Gronwall's inequality we obtain for every $t\in\mathbb{R}_+$, $\Omega^3(t) = \Omega^r(t) = 0$, which is the desired result.

Under these assumptions the stretching term becomes

$$\begin{aligned}\Omega\cdot\nabla u &= \frac{1}{r}\Omega^\theta\partial_\theta(u^r e_r)\\ &= \frac{1}{r}u^r\Omega^\theta e_\theta = \frac{1}{r}u^r\Omega. \quad \square\end{aligned}$$

5.4. Some *a priori* estimates

Our goal in this section is to establish some *a priori* estimates that will be used later for both cases of critical regularities and cases of subcritical regularities. Our result reads as follows.

Proposition 7. *Let $(v_\varepsilon, c_\varepsilon)$ be a smooth axisymmetric solution of the system (1.1). Then the following estimates hold true.*

(1) *Denote by Ω_ε the vorticity of v_ε ; then for $t\geqslant 0$ and $p\in[1,\infty]$ we have*

$$\Big\|\frac{\Omega_\varepsilon}{r}(t)\Big\|_{L^p} \leqslant \Big\|\frac{\Omega_\varepsilon^0}{r}\Big\|_{L^p} e^{(1-\frac{1}{p})\,\|\mathrm{div}\,v_\varepsilon\|_{L^1_tL^\infty}}$$

and for $1<p<\infty$, $q\in[1,+\infty]$,

$$\Big\|\frac{\Omega_\varepsilon}{r}(t)\Big\|_{L^{p,q}} \leqslant C\Big\|\frac{\Omega_\varepsilon^0}{r}\Big\|_{L^{p,q}} e^{(1-\frac{1}{p})\,\|\mathrm{div}\,v_\varepsilon\|_{L^1_tL^\infty}}.$$

(2) *For $t\geqslant 0$, we have*

$$\Big\|\frac{v_\varepsilon^r}{r}\Big\|_{L^\infty} \lesssim \|v_\varepsilon\|_{L^2} + \|\mathrm{div}\,v_\varepsilon\|_{B^0_{\infty,1}} + \Big\|\frac{\Omega_\varepsilon^0}{r}\Big\|_{L^{3,1}} e^{\|\mathrm{div}\,v_\varepsilon\|_{L^1_tL^\infty}}.$$

(3) *For $t\geqslant 0$, we have*

$$\|\Omega_\varepsilon(t)\|_{L^\infty} \leqslant \|\Omega_\varepsilon^0\|_{L^\infty} e^{C\|v_\varepsilon\|_{L^1_tL^2}} e^{C\big\{\|\mathrm{div}\,v_\varepsilon\|_{L^1_tB^0_{\infty,1}} + t\,\|\Omega_\varepsilon^0/r\|_{L^{3,1}}\,e^{\|\mathrm{div}\,v_\varepsilon\|_{L^1_tL^\infty}}\big\}}.$$

Proof. (1) To prove this result we will use the characteristic method. Let ψ_ε denote the flow associated with the velocity v_ε and defined by the integral equation

$$\psi_\varepsilon(t,x) = x + \int_0^t v_\varepsilon(\tau,\psi_\varepsilon(\tau,x))d\tau.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Set $f_\varepsilon(t,x)=\frac{\Omega_\varepsilon}{r}(t,\psi(t,x))$; then it is easy to see from equation (5.5) that

$$\partial_t f_\varepsilon(t,x)+(\operatorname{div} v_\varepsilon)(t,\psi(t,x))f_\varepsilon(t,x)=0.$$

Thus we get easily

$$f_\varepsilon(t,x)=f_\varepsilon(0,x)\,e^{-\int_0^t(\operatorname{div} v_\varepsilon)(\tau,\psi(\tau,x))d\tau}. \tag{5.6}$$

For $p=+\infty$ we get

$$\left\|\frac{\Omega_\varepsilon}{r}(t)\right\|_{L^\infty}\leqslant\left\|\frac{\Omega_\varepsilon^0}{r}\right\|_{L^\infty}e^{\|\operatorname{div} v_\varepsilon\|_{L^1_tL^\infty}}.$$

Now for $p\in[1,+\infty[$ we will use the identity

$$\det\nabla\psi_\varepsilon(t,x)=e^{\int_0^t(\operatorname{div} v_\varepsilon)(\tau,\psi(\tau,x))d\tau}$$

and thus we get by (5.6)

$$\begin{aligned}\left\|\frac{\Omega_\varepsilon}{r}(t)\right\|_{L^p}^p&=\|f_\varepsilon(t)\circ\psi_\varepsilon^{-1}(t,\cdot)\|_{L^p}^p\\&=\int_{\mathbb{R}^3}|f_\varepsilon(t,x)|^p\det\nabla\psi_\varepsilon(t,x)dx\\&=\int_{\mathbb{R}^3}|f_\varepsilon(0,x)|^p\,e^{(1-p)\int_0^t(\operatorname{div} v_\varepsilon)(\tau,\psi(\tau,x))}d\tau\\&\leqslant\left\|\frac{\Omega_\varepsilon^0}{r}\right\|_{L^p}^p e^{(p-1)\,\|\operatorname{div} v_\varepsilon\|_{L^1_tL^\infty}}.\end{aligned}$$

The estimates in Lorentz spaces are a direct consequence of the real interpolation argument.

(2) The estimate of the quantity $\|\frac{v_\varepsilon^r}{r}\|_{L^1_tL^\infty}$ will require the use of some special properties of axisymmetric flows. First, we split the velocity into compressible and incompressible parts:

$$v_\varepsilon=\mathcal{P}v_\varepsilon+\nabla\Delta^{-1}\operatorname{div} v_\varepsilon.$$

We point out that in this decomposition both vector fields are also axisymmetric. Indeed, from Remark 4, the scalar function $\operatorname{div} v_\varepsilon$ is an axisymmetric scalar function and $\Delta^{-1}\operatorname{div} v_\varepsilon$ too. Again from Remark 4, the vector field $\nabla\Delta^{-1}\operatorname{div} v_\varepsilon$ is axisymmetric. Now obvious computations yield

$$\begin{aligned}v_\varepsilon^r&=(\mathcal{P}v_\varepsilon)^r+(\nabla\Delta^{-1}\operatorname{div} v_\varepsilon)\cdot e_r\\&=(\mathcal{P}v_\varepsilon)^r+\partial_r\Delta^{-1}\operatorname{div} v_\varepsilon.\end{aligned} \tag{5.7}$$

Therefore we deduce

$$\frac{v_\varepsilon^r}{r}=\frac{(\mathcal{P}v_\varepsilon)^r}{r}+\frac{\partial_r}{r}\Delta^{-1}\operatorname{div} v_\varepsilon\cdot$$

To estimate the second term of the preceding identity we make use of the algebraic identity described in Lemma 7 and deal with the action of the operator $\frac{\partial_r}{r}\Delta^{-1}u$ over

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

axisymmetric functions:

$$\begin{aligned}\Big\|\frac{\partial_r}{r}\Delta^{-1}\mathrm{div}\, v_\varepsilon\Big\|_{L^\infty} &\leqslant \sum_{i,j=1}^{2}\|\mathcal{R}_{ij}\mathrm{div}\, v_\varepsilon\|_{L^\infty}\\ &\leqslant \sum_{i,j=1}^{2}\|\mathcal{R}_{ij}\mathrm{div}\, v_\varepsilon\|_{\dot B^0_{\infty,1}}\\ &\leqslant C\,\big\|\mathrm{div}\, v_\varepsilon\big\|_{\dot B^0_{\infty,1}}\\ &\leqslant C\,\|v_\varepsilon\|_{L^2}+\big\|\mathrm{div}\, v_\varepsilon\big\|_{B^0_{\infty,1}}.\end{aligned}$$

We have used the embedding $\dot B^0_{\infty,1}\hookrightarrow L^\infty$ combined with the continuity of Riesz transforms on the homogeneous Besov spaces. Furthermore, in the last line we use Bernstein inequalities in order to bound low frequencies:

$$\sum_{j\in\mathbb{Z}_-}\|\dot\Delta_j\mathrm{div}\, v_\varepsilon\|_{L^\infty}\leqslant C\,\|v_\varepsilon\|_{L^2}\sum_{j\in\mathbb{Z}_-}2^{\frac52 j}\leqslant C\,\|v_\varepsilon\|_{L^2}.$$

Now let us come back to (5.7) and look at the incompressible term. Since $\mathcal{P}v_\varepsilon$ is axisymmetric and satisfies furthermore $\mathrm{div}\,\mathcal{P}v_\varepsilon=0$ and $\mathrm{curl}\,\mathcal{P}v_\varepsilon=\Omega_\varepsilon$, then we can use the following inequality proven by Shirota and Yanagisawa [28]:

$$\Big|\frac{(\mathcal{P}v_\varepsilon)^r}{r}(x)\Big|\leqslant C\int_{\mathbb{R}^3}\frac{|\frac{\Omega_\varepsilon^\theta}{r}(y)|}{|x-y|^2}dy.$$

Now since $\frac{1}{|\cdot|^2}$ belongs to the Lorentz space $L^{3,\infty}$, then the usual convolution laws yield

$$\Big\|\frac{(\mathcal{P}v_\varepsilon)^r}{r}(t)\Big\|_{L^\infty}\lesssim\Big\|\frac{\Omega_\varepsilon}{r}(t)\Big\|_{L^{3,1}}.$$

According to the first part of Proposition 7 we have

$$\Big\|\frac{\Omega_\varepsilon}{r}(t)\Big\|_{L^{3,1}}\leqslant C\Big\|\frac{\Omega_\varepsilon^0}{r}\Big\|_{L^{3,1}}e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}}.$$

Combining these estimates we get

$$\Big\|\frac{v_\varepsilon^r}{r}\Big\|_{L^\infty}\lesssim\|v_\varepsilon\|_{L^2}+\|\mathrm{div}\, v_\varepsilon\|_{B^0_{\infty,1}}+\Big\|\frac{\Omega_\varepsilon^0}{r}\Big\|_{L^{3,1}}e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}}. \tag{5.8}$$

(3) Applying the maximum principle to the vorticity equation (5.4) and using Gronwall's inequality we get

$$\|\Omega_\varepsilon(t)\|_{L^\infty}\leqslant\|\Omega_\varepsilon(0)\|_{L^\infty}\,e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}+\|\frac{v_\varepsilon^r}{r}\|_{L^1_tL^\infty}}. \tag{5.9}$$

Plugging the estimate (5.8) into (5.9) gives

$$\|\Omega_\varepsilon(t)\|_{L^\infty}\leqslant\|\Omega_\varepsilon^0\|_{L^\infty}\,e^{C\|v_\varepsilon\|_{L^1_tL^2}}e^{C\left\{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tB^0_{\infty,1}}+t\,\|\Omega_\varepsilon^0/r\|_{L^{3,1}}\,e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}}\right\}}.\quad\square$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

6. Subcritical regularities

The main goal of this section is to prove Theorem 1; however we shall limit our analysis to the estimate of the lifespan of the solutions and to the rigorous justification of the low Mach number limit. For example we will not deal with the construction of a maximal solution, which is classical and was done for instance in [20].

6.1. The lower bound of T_ε

We will show how the combination of Strichartz estimates with the special structure of axisymmetric flows allows us to improve the estimate of the time lifespan T_ε in the case of ill-prepared initial data. We shall prove the following result.

Proposition 8. *Let $s > \frac{5}{2}$ and assume that*

$$\sup_{\varepsilon\leqslant 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^s} < +\infty.$$

Then the system (1.1) admits a unique solution $(v_\varepsilon, c_\varepsilon) \in C([0, T_\varepsilon]; H^s)$ such that

$$T_\varepsilon \geqslant C_0 \log\log\log(\varepsilon^{-1}).$$

Moreover, there exists $\sigma > 0$ such that for every $T \in [0, T_\varepsilon]$,

$$\|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}} + \|\nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} \leqslant C_0 \varepsilon^\sigma$$

and

$$\|\Omega_\varepsilon(T)\|_{L^\infty} \leqslant C_0 e^{C_0 T}, \quad \|\nabla v_\varepsilon\|_{L^1_T L^\infty} \leqslant C_0 e^{\exp\{C_0 T\}},$$
$$\|(v_\varepsilon, c_\varepsilon)(T)\|_{H^s} \leqslant C_0 e^{e^{\exp\{C_0 T\}}}.$$

Proof. According to Proposition 7 we have

$$\|\Omega_\varepsilon(t)\|_{L^\infty} \leqslant \|\Omega^0_\varepsilon\|_{L^\infty} \, e^{C\|v_\varepsilon\|_{L^1_t L^2}} e^{C\left\{\|\operatorname{div} v_\varepsilon\|_{L^1_t B^0_{\infty,1}} + t\, \|\Omega^0_\varepsilon/r\|_{L^{3,1}}\, e^{\|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}\right\}}.$$

To bound $\|\Omega^0_\varepsilon/r\|_{L^{3,1}}$, we use the fact that $\Omega^0_\varepsilon(0,0,z) = 0$ (see Proposition 4), combined with the Taylor formula:

$$\Omega^0_\varepsilon(x_1, x_2, z) = \int_0^1 \Big(x_1 \partial_{x_1} \Omega^0_\varepsilon(\tau x_1, \tau x_2, z) + x_2 \partial_{x_2} \Omega^0_\varepsilon(\tau x_1, \tau x_2, z)\Big) d\tau.$$

Applying (2.6) with a homogeneity argument yields

$$\begin{aligned}\|\Omega^0_\varepsilon/r\|_{L^{3,1}} &\lesssim \int_0^1 \|\nabla \Omega^0_\varepsilon(\tau\cdot, \tau\cdot, \cdot)\|_{L^{3,1}}\, d\tau \\ &\lesssim \|\nabla\Omega^0_\varepsilon\|_{L^{3,1}} \int_0^1 \tau^{-\frac{2}{3}} d\tau \\ &\lesssim \|\nabla\Omega^0_\varepsilon\|_{L^{3,1}}\,.\end{aligned}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

According to the embedding $H^{s-2} \hookrightarrow L^{3,1}$, for $s > \frac{5}{2}$ one has

$$\begin{aligned} \|\Omega_\varepsilon^0/r\|_{L^{3,1}} &\lesssim \|\nabla \Omega_\varepsilon^0\|_{H^{s-2}} \\ &\lesssim \|\Omega_\varepsilon^0\|_{H^{s-1}} \\ &\lesssim \|v_{0,\varepsilon}\|_{H^s}. \end{aligned}$$

From Proposition 1 we get

$$\|\Omega_\varepsilon(t)\|_{L^\infty} \leqslant C_0 e^{C_0(1+t)\exp\left\{\|\mathrm{div}\, v_\varepsilon\|_{L^1_t B^0_{\infty,1}}\right\}}. \tag{6.1}$$

Using Lemma 5 and Proposition 1 we get

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \lesssim \|v_\varepsilon(t)\|_{L^2} + \|\mathrm{div}\, v_\varepsilon(t)\|_{B^0_{\infty,1}} + C_0 \|\Omega_\varepsilon(t)\|_{L^\infty}\big(1 + V_\varepsilon(t)\big).$$

We recall that

$$V_\varepsilon(t) = \int_0^t \|(\nabla v_\varepsilon, \nabla c_\varepsilon)(\tau)\|_{L^\infty}\, d\tau.$$

Integrating over time and using Propositions 1 and 2, we obtain

$$\begin{aligned} \|\nabla v_\varepsilon\|_{L^1_T L^\infty} &\lesssim \|v_\varepsilon\|_{L^1_T L^2} + \|\mathrm{div}\, v_\varepsilon\|_{L^1_T B^0_{\infty,1}} + C_0 \int_0^T \|\Omega_\varepsilon(t)\|_{L^\infty}\big(1 + V_\varepsilon(t)\big) dt \\ &\leqslant C_0 T e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_T L^\infty}} + C_0 \alpha_\varepsilon(T) e^{C V_\varepsilon(T)} + C_0 \int_0^T \|\Omega_\varepsilon(t)\|_{L^\infty}\big(1 + V_\varepsilon(t)\big) dt, \end{aligned}$$

with

$$\alpha_\varepsilon(T) := C_0 \varepsilon^{\frac{2s-5}{r(2s-3)}}(1 + T^2).$$

Using again Proposition 2 we obtain

$$V_\varepsilon(T) \leqslant C_0 \alpha_\varepsilon(T) e^{C V_\varepsilon(T)} + C_0 T e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}} + C_0 \int_0^T \|\Omega_\varepsilon(t)\|_{L^\infty}(1 + V_\varepsilon(t) dt.$$

Thus Gronwall's lemma combined with (6.1) and Proposition 2 yield

$$\begin{aligned} V_\varepsilon(T) &\leqslant C_0\, e^{C_0 \int_0^T \|\Omega_\varepsilon(t)\|_{L^\infty} dt}\Big(\alpha_\varepsilon(T) e^{C V_\varepsilon(T)} + T e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}\Big) \\ &\leqslant e^{e^{C_0(1+T)\exp\left\{\|\mathrm{div}\, v_\varepsilon\|_{L^1_T B^0_{\infty,1}}\right\}}}\Big(\alpha_\varepsilon(T) e^{C V_\varepsilon(T)} + T e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}\Big) \\ &\leqslant e^{e^{C_0(1+T)\exp\left\{\|\mathrm{div}\, v_\varepsilon\|_{L^1_t B^0_{\infty,1}}\right\}}}\Big(\alpha_\varepsilon(T)\, e^{C V_\varepsilon(T)} + 1\Big) \\ &\leqslant e^{e^{C_0(1+T)\exp\left\{\alpha_\varepsilon(T) e^{C V_\varepsilon(T)}\right\}}}. \end{aligned} \tag{6.2}$$

We choose T_ε such that

$$\log\left(\frac{1}{\alpha_\varepsilon(T_\varepsilon)}\right) > C e^{e^{C_0\, e\,(1+T_\varepsilon)}}. \tag{6.3}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Then we claim that for every $t \in [0, T_\varepsilon]$,

$$CV_\varepsilon(t) \leqslant \log\left(\frac{1}{\alpha_\varepsilon(T_\varepsilon)}\right). \tag{6.4}$$

Indeed, we set $I_{T_\varepsilon} := \left\{t \in [0, T_\varepsilon]; CV_\varepsilon(t) \leqslant \log\left(\frac{1}{\alpha_\varepsilon(T_\varepsilon)}\right)\right\}$. First this set is nonempty since $0 \in I_{T_\varepsilon}$. By the continuity of $t \mapsto V_\varepsilon(t)$, the set I_{T_ε} is closed and thus to prove that I_{T_ε} coincides with $[0, T_\varepsilon]$ it suffices to show that I_{T_ε} is an open set. Let $t \in I_{T_\varepsilon}$; then using (6.2) and (6.4) we get

$$\begin{aligned} CV_\varepsilon(t) &\leqslant Ce^{e^{C_0(1+t)\exp\left\{\alpha_\varepsilon(t)e^{CV_\varepsilon(t)}\right\}}} \\ &\leqslant Ce^{e^{C_0(1+t)\exp\left\{\frac{\alpha_\varepsilon(t)}{\alpha_\varepsilon(T_\varepsilon)}\right\}}} \\ &\leqslant Ce^{e^{C_0\, e\,(1+t)}} \\ &< \log\left(\frac{1}{\alpha_\varepsilon(T_\varepsilon)}\right). \end{aligned}$$

This proves that t is in the interior of I_{T_ε} and thus I_{T_ε} is an open set of $[0, T_\varepsilon]$. Consequently we conclude that $I_{T_\varepsilon} = [0, T_\varepsilon]$. Now we choose precisely T_ε such that

$$C_0 e(1 + T_\varepsilon) = \log\log\log(\varepsilon^{-\theta}), \quad \text{with } \theta > 0,$$

and then the assumption (6.3) is satisfied whenever

$$\varepsilon^{\frac{2s-5}{r(2s-3)}}(1 + T_\varepsilon^2) < \varepsilon^{C\theta}.$$

This last condition is satisfied for small values of ε when $C\theta < \frac{2s-5}{r(2s-3)}$. Now inserting (6.4) into (6.2) we get for $T \in [0, T_\varepsilon]$,

$$V_\varepsilon(T) \leqslant e^{e^{C_0 e(1+T)}}. \tag{6.5}$$

Plugging this into the estimate of Proposition 2 we obtain for $T \in [0, T_\varepsilon]$,

$$\begin{aligned} \|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}} + \|\nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} &\leqslant C_0 e^{Ce^{\exp\{C_0 e(1+T)\}}} \varepsilon^{\frac{2s-5}{r(2s-3)}} \\ &\leqslant C\, \varepsilon^{\frac{2s-5}{r(2s-3)} - C\theta} \\ &\leqslant C\varepsilon^\sigma, \quad \text{with } \sigma = \frac{2s-5}{r(2s-3)} - C\theta. \end{aligned} \tag{6.6}$$

From Corollary 2 and by choosing θ sufficiently small we obtain for every $r \in]2, +\infty[$,

$$\begin{aligned} \|\mathcal{Q}v_\varepsilon\|_{L^r_T L^\infty} + \|c_\varepsilon\|_{L^r_T L^\infty} &\leqslant C_0 \varepsilon^{\frac{1}{r}}(1 + T)e^{CV_\varepsilon(t)} \\ &\leqslant C_0 \varepsilon^{\frac{1}{r} - C\theta} \log\log\log(\varepsilon^{-\theta}) \\ &\leqslant C_0 \varepsilon^{\sigma'}, \quad \sigma' > 0. \end{aligned}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

We point out that the use of the Hölder inequality and the slow growth of T_ε allows us to get the following: for every $r \in [1, +\infty[$,

$$\|\mathcal{Q}v_\varepsilon\|_{L^r_T L^\infty} + \|c_\varepsilon\|_{L^r_T L^\infty} \leqslant C_0 \varepsilon^{\sigma'}, \quad \sigma' > 0. \tag{6.7}$$

Inserting (6.6) into (6.1) leads to

$$\|\Omega_\varepsilon(T)\|_{L^\infty} \leqslant C_0 e^{C_0 T}.$$

To estimate the solutions of (1.1) in Sobolev norms we use Proposition 1 combined with the Lipschitz bound (6.5)

$$\|(v_\varepsilon, c_\varepsilon)(T)\|_{H^s} \leqslant C_0 e^{e^{\exp\{C_0 T\}}}. \quad \square$$

6.2. The incompressible limit

Our task now is to prove the second part of Theorem 1. More precisely we will establish the following result.

Proposition 9. *Let $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be an H^s-bounded family of axisymmetric initial data with $s > \frac{5}{2}$, that is*

$$\sup_{0<\varepsilon\leqslant 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^s} < +\infty.$$

Assume that there exists $v_0 \in H^s$ such that

$$\lim_{\varepsilon\to 0} \|\mathcal{P}v_{0,\varepsilon} - v_0\|_{L^2} = 0.$$

Then the family $(\mathcal{P}v_\varepsilon)_\varepsilon$ tends to the solution v of the incompressible Euler system (1.2) associated with the initial data v_0 : for every $T > 0$,

$$\lim_{\varepsilon\to 0} \|\mathcal{P}v_\varepsilon - v\|_{L^\infty_T L^2} = 0.$$

However, the compressible and acoustic parts tend to zero: for every $r \in [1, +\infty[, T > 0$,

$$\lim_{\varepsilon\to 0} \|(\mathcal{Q}v_\varepsilon, c_\varepsilon)\|_{L^r_T L^\infty} = 0.$$

Proof. The last assertion concerning the compressible and acoustic parts of the fluid is a direct consequence of (6.7). Now we intend to show that for every time $T > 0$ the family $(\mathcal{P}v_\varepsilon)_\varepsilon$ is a Cauchy sequence in $L^\infty_T L^2$ for small values of ε and this will allow us to prove strong convergence, when ε tends to zero, of the incompressible parts to a vector-valued function v which is a solution of the incompressible Euler equations. For this purpose, let $\varepsilon > \varepsilon' > 0$ and set $\zeta_{\varepsilon,\varepsilon'} = v_\varepsilon - v_{\varepsilon'}$ and $w_{\varepsilon,\varepsilon'} = \mathcal{P}v_\varepsilon - \mathcal{P}v_{\varepsilon'}$. Applying Leray's projector $\mathcal{P}$ to the first equation of (1.1) yields

$$\partial_t \mathcal{P}v_\varepsilon + \mathcal{P}(v_\varepsilon \cdot \nabla v_\varepsilon) = 0.$$

It follows that

$$\partial_t w_{\varepsilon,\varepsilon'} + \mathcal{P}(v_\varepsilon \cdot \nabla \zeta_{\varepsilon,\varepsilon'}) + \mathcal{P}(\zeta_{\varepsilon,\varepsilon'} \cdot \nabla v_{\epsilon'}) = 0.$$

Taking the L^2 inner product of this equation with $w_{\varepsilon,\varepsilon'}$ and using the fact that Leray's projector is an involutive self-adjoint operator we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_{\varepsilon,\varepsilon'}(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^3}(v_\varepsilon\cdot\nabla\zeta_{\varepsilon,\varepsilon'})\mathcal{P}w_{\varepsilon,\varepsilon'}dx - \int_{\mathbb{R}^3}(\zeta_{\varepsilon,\varepsilon'}\cdot\nabla v_{\epsilon'})\mathcal{P}w_{\varepsilon,\varepsilon'}dx\\
&= -\int_{\mathbb{R}^3}(v_\varepsilon\cdot\nabla(w_{\varepsilon,\varepsilon'}+\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}))\,w_{\varepsilon,\varepsilon'}dx - \int_{\mathbb{R}^3}((w_{\varepsilon,\varepsilon'}+\mathcal{Q}\zeta_{\varepsilon,\varepsilon'})\cdot\nabla v_{\epsilon'})w_{\varepsilon,\varepsilon'}dx\\
&= \frac{1}{2}\int_{\mathbb{R}^3}|w_{\varepsilon,\varepsilon'}|^2\ \mathrm{div}\ v_\varepsilon dx - \int_{\mathbb{R}^3}(w_{\varepsilon,\varepsilon'}\cdot\nabla v_{\epsilon'})w_{\varepsilon,\varepsilon'}dx\\
&\quad -\int_{\mathbb{R}^3}\big(v_\varepsilon\cdot\nabla\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}+\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}\cdot\nabla v_{\epsilon'}\big)w_{\varepsilon,\varepsilon'}dx\\
&\leqslant \big(\|\mathrm{div}\, v_\varepsilon(t)\|_{L^\infty}+\|\nabla v_{\varepsilon'}(t)\|_{L^\infty}\big)\,\|w_{\varepsilon,\varepsilon'}(t)\|_{L^2}^2\\
&\quad +\Big(\|v_\varepsilon(t)\|_{L^2}\,\|\nabla\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}(t)\|_{L^\infty}+\|\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}(t)\|_{L^\infty}\,\|\nabla v_{\varepsilon'}(t)\|_{L^2}\Big)\,\|w_{\varepsilon,\varepsilon'}(t)\|_{L^2}.
\end{aligned}$$

Integrating over time and using Gronwall's lemma, we obtain

$$\|w_{\varepsilon,\varepsilon'}(t)\|_{L^2}\leqslant\big(\|w^0_{\varepsilon,\varepsilon'}\|_{L^2}+F_{\varepsilon,\varepsilon'}(t)\big)e^{\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}+\|\nabla v_{\varepsilon'}\|_{L^1_tL^\infty}}$$

with

$$F_{\varepsilon,\varepsilon'}(t)=\|v_\varepsilon\|_{L^\infty_tL^2}\,\|\nabla\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}\|_{L^1_tL^\infty}+\|\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}\|_{L^1_tL^\infty}\,\|\nabla v_{\varepsilon'}\|_{L^\infty_tL^2}.$$

Splitting $\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}$ into low and high frequencies we get

$$\|\nabla\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}\|_{L^1_tL^\infty}\lesssim\|\mathrm{div}\zeta_{\varepsilon,\varepsilon'}\|_{L^1_tB^0_{\infty,1}}+\|\mathcal{Q}\zeta_{\varepsilon,\varepsilon'}\|_{L^1_tL^\infty}.$$

Therefore combining Propositions 1 and 8 with (7.47) we get for some $\eta>0$,

$$\sup_{t\in[0,T_\varepsilon]}F_{\varepsilon,\varepsilon'}(t)\leqslant C_0\varepsilon^\eta.$$

According to Proposition 8 we obtain

$$\|w_{\varepsilon,\varepsilon'}(t)\|_{L^2}\leqslant C_0e^{e^{\exp\{C_0t\}}}\big(\|w^0_{\varepsilon,\varepsilon'}\|_{L^2}+\varepsilon^\eta\big).$$

Thus we deduce that $(\mathcal{P}v_\varepsilon)_\varepsilon$ is a Cauchy sequence in the Banach space $L^\infty_TL^2$ and therefore it converges strongly to some v which belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^s)$. This last claim is a consequence of the uniform bound in H^s. It remains now to prove that v is a solution for the incompressible Euler system with initial data $v^0:=\lim_{\varepsilon\to0}\mathcal{P}v_{0,\varepsilon}$. The passage to the limit in the linear part $(\partial_tv_\varepsilon)_\varepsilon$ is easy to carry out by using the convergence in the weak sense. For the nonlinear term we split the velocity into its compressible and incompressible parts:

$$v_\varepsilon\cdot\nabla v_\varepsilon=\mathcal{P}v_\varepsilon\cdot\nabla\mathcal{P}v_\varepsilon+\mathcal{P}v_\varepsilon\cdot\nabla\mathcal{Q}v_\varepsilon+\mathcal{Q}v_\varepsilon\cdot\nabla v_\varepsilon.$$

By virtue of Propositions 1 and 8 and (6.7) we get

$$\begin{aligned}
\|\mathcal{P}v_\varepsilon\cdot\nabla\mathcal{Q}v_\varepsilon\|_{L^1_TL^2}&\leqslant\|v_\varepsilon\|_{L^\infty_TL^2}\,\|\nabla\mathcal{Q}v_\varepsilon\|_{L^1_TL^\infty}\\
&\leqslant C_0\big(\|\mathcal{Q}v_\varepsilon\|_{L^1_TL^\infty}+\|\mathrm{div}\, v_\varepsilon\|_{L^1_TB^0_{\infty,1}}\big)\\
&\leqslant C_0\varepsilon^\eta.
\end{aligned}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

For the last term we combine (6.7) with Proposition 8:

$$\begin{aligned}\|\mathcal{Q}v_\varepsilon\cdot\nabla v_\varepsilon\|_{L^1_TL^2}&\leqslant\|v_\varepsilon\|_{L^\infty_TH^1}\,\|\mathcal{Q}v_\varepsilon\|_{L^1_TL^\infty}\\&\leqslant C_0\varepsilon^\eta.\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to0}\|v_\varepsilon\cdot\nabla v_\varepsilon-\mathcal{P}v_\varepsilon\cdot\nabla\mathcal{P}v_\varepsilon\|_{L^1_TL^2}=0.$$

The strong convergence of $\mathcal{P}v_\varepsilon$ to v in $L^\infty_TL^2$ allows us to claim that

$$\lim_{\varepsilon\to0}\|\mathcal{P}v_\varepsilon\otimes\mathcal{P}v_\varepsilon-v\otimes v\|_{L^1_TL^1}=0.$$

Consequently, the family $\mathcal{P}(v_\varepsilon\cdot\nabla v_\varepsilon)$ converges strongly in some Banach space to $\mathcal{P}(v\cdot\nabla v)$ and thus we deduce that v satisfies the incompressible Euler system. □

7. Critical regularities

In this section we will study the system (1.1) with initial data lying in the critical Besov spaces $B^{\frac52}_{2,1}$. There are some additional difficulties compared to the subcritical case that we can summarize in two points. The first one concerns the Beale–Kato–Majda criterion which is inapplicable and the second one is related to the extension of the interpolation argument used in Proposition 2 to the critical regularity. We will start with a strong version of Theorem 2 in which we require the initial data to be in a more regular space of type $B^{\frac52,\Psi}_{2,1}$. We are able to prove the desired result for any slow growth of Ψ. Roughly speaking, to get the incompressible limit and quantify the lower bound of the lifespan, the function Ψ must only increase to infinity. The study of the case $\Psi\equiv1$ which is the subject of Theorem 2 will be deduced from Corollary 1 and Theorem 3.

Theorem 3. *Let $\Psi\in\mathcal{U}_\infty$ and $\{(v_{0,\varepsilon},c_{0,\varepsilon})_{0<\varepsilon\leqslant1}\}$ be a $B^{\frac52,\Psi}_{2,1}$-bounded family of axisymmetric initial data, that is*

$$\sup_{0<\varepsilon\leqslant1}\|(v_{0,\varepsilon},c_{0,\varepsilon})\|_{B^{\frac52,\Psi}_{2,1}}<+\infty.$$

Then the system (1.1) has a unique solution $(v_\varepsilon,c_\varepsilon)\in C([0,T_\varepsilon[;B^{\frac52,\Psi}_{2,1})$, with

$$T_\varepsilon\geqslant C_0\log\log\log\Phi(\varepsilon):=\tilde T_\varepsilon,\quad \textit{if}\ 0<\varepsilon<<1,$$

where

$$\Phi(x):=x^{\frac{1}{2r}}+\frac{1}{\Psi\big(\log(\frac{1}{x^{\frac{1}{2r}}})\big)},\quad x\in]0,1]$$

and r is a free parameter belonging to $]2,+\infty[$. Moreover we have for small values of ε,

$$\|(\operatorname{div}v_\varepsilon,\nabla c_\varepsilon)\|_{L^1_{\tilde T_\varepsilon}L^\infty}\leqslant C_0\Phi^{\frac13}(\varepsilon).\tag{7.1}$$

Assume in addition that the incompressible parts $(\mathcal{P}v_{0,\varepsilon})$ converge in L^2 to some v_0. Then the incompressible parts of the solutions tend to the Kato solution of the *system (1.2):*

$$\mathcal{P}v_\varepsilon \to v \quad in\ L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2).$$

Before giving the proof we will state some remarks.

Remarks 2. (1) When Ψ has at most an exponential growth at infinity, that is $\Psi(x) \leqslant e^{Cx}$, then according to Theorem 3 we get that

$$T_\varepsilon \geqslant C_0 \log\log\left|\log \Psi\left(\log\left(\frac{1}{\varepsilon^{\frac{1}{2r}}}\right)\right)\right|.$$

We observe that in the case where Ψ has exactly an exponential growth $\Psi(x) = 2^{ax}$ then the Besov space $B^{s,\Psi}_{2,1}$ reduces to the Besov space $B^{s+a}_{2,1}$. With this choice the regularity of the initial data in Theorem 3 becomes subcritical and what we obtain is in agreement with Theorem 1; we get in particular the same result for the lower bound of the lifespan.

(2) When Ψ has the polynomial growth $\Psi(x) = (x+2)^\alpha, \alpha > 0$, then it is clear that $\Psi \in \mathcal{U}_\infty$ and moreover

$$T_\varepsilon \geqslant C_0 \log\log\log\log\frac{1}{\varepsilon}.$$

For the proof of Theorem 3 we shall examine only the estimate of the lower bound of T_ε and the estimate (7.1). The proof of the incompressible limit can be performed in a similar way to the proof of Theorem 1. We point out that the analysis in the critical case is rather difficult compared to that for the subcritical case and we distinguish two main difficulties that one has to face.

The first one is related to the criticality of the Strichartz estimate for $|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_t L^\infty}$. This quantity has the same scale as the critical Besov space $B^{\frac{5}{2}}_{2,1}$, and to circumvent this problem we move to a space $B^{\frac{5}{2},\Psi}_{2,1}$ that is slightly smoother but that is very close to $B^{\frac{5}{2}}_{2,1}$.

The second difficulty that one has to deal with has a close relation with the Beale–Kato–Majda criterion, which is not known to work for critical regularities. In other words, the bound of $\|\Omega_\varepsilon(t)\|_{L^\infty}$ is not sufficient for propagating initial regularities and thus one should estimate a stronger norm $\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}}$ instead. This is the hard part of the proof and the geometric structure of the vorticity will play a significant role, especially the results discussed before in Propositions 5 and 6.

7.1. Proof of Theorem 2

The proof of Theorem 2 follows from Theorem 3 combined with Corollary 1. Indeed, from the assumption that

$$\sum_q 2^{\frac{5}{2}q} \sup_{0\leqslant\varepsilon\leqslant1} \|(\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})\|_{L^2} < +\infty$$

and using Corollary 1, we conclude the existence of $\Psi \in \mathcal{U}_\infty$ such that the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon\leqslant1}$ is uniformly bounded in $B^{\frac{5}{2},\Psi}_{2,1}$. Now we can just use Theorem 3.

Now we intend to give the proof of Theorem 3, which will be done in several steps.

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

7.2. The logarithmic estimate

Now let us examine a little further some interesting properties of the compressible transport model given by

$$\text{(CT)}\begin{cases}\partial_t\Omega+(u\cdot\nabla)\Omega+\Omega\,\text{div}\,u=\Omega\cdot\nabla u\\ \Omega_{|t=0}=\Omega_0.\end{cases}$$

This is the model governing the vorticity dynamics for the system (1.1) and it is worthwhile to study the linear growth of the norm $B^0_{\infty,1}$ which is crucial for the analysis of the critical case, as has already been pointed out. Our main result reads as follows.

Theorem 4. *Let $p\in[1,+\infty[$, u be an axisymmetric smooth vector field without swirl and Ω be a smooth vector-valued solution of the equation (CT). We assume that Ω_0 satisfies*

$$\text{div}\,\Omega_0=0\quad\textit{and}\quad\Omega_0\times e_\theta=0.$$

Then

$$\|\Omega(t)\|_{B^0_{\infty,1}}\leqslant C\,\|\Omega_0\|_{B^0_{\infty,1}}\,e^{\|u^r/r\|_{L^1_tL^\infty}}\big(1+e^{C\,\|\nabla u\|_{L^1_tL^\infty}}\,\|\text{div}\,u\|^2_{L^1_tB^{\frac{3}{p}}_{p,1}}\big)$$
$$\times\Big(1+\int_0^t\|u(\tau)\|_{\text{Lip}}\,d\tau\Big).$$

with the notation $\|u\|_{\text{Lip}}:=\|u\|_{L^\infty}+\|\nabla u\|_{L^\infty}$ and C depends only on p.

Proof. First of all, we point out that for the incompressible case, that is $\text{div}\,u=0$, this result was proved in [1] by using a tricky splitting of Ω combined with some geometric aspects of axisymmetric flows. We shall use here the same approach as was used by [1]; however the lack of incompressibility brings about more technical difficulties that we are unable to circumvent without using some refined geometric properties of axisymmetric vector fields. The results previously examined in Propositions 4 and 5 are crucial at different steps of the proof.

Before going further into detail, we will first discuss the main idea of the proof. We localize the initial data in frequency and consider the solution of the same problem (CT). Then by linearity we can rebuild the solution of the initial problem by a superposition argument. The main information that we shall establish in order to get the logarithmic estimate is a frequency decay property. To be more precise, let $q\geqslant-1$ and denote by $\tilde\Omega_q$ the unique global vector-valued solution of the problem

$$\begin{cases}\partial_t\tilde\Omega_q+(u\cdot\nabla)\tilde\Omega_q+\tilde\Omega_q\,\text{div}\,u=\tilde\Omega_q\cdot\nabla u\\ \tilde\Omega_{q|t=0}=\Delta_q\Omega_0.\end{cases}\tag{7.2}$$

Since $\text{div}\,\Delta_q\Omega_0=\Delta_q\text{div}\,\Omega_0=0$, then it follows from Proposition 6 that $\text{div}\,\tilde\Omega_q(t,x)=0$. Using the linearity of the problem and the uniqueness of the solutions we get the following decomposition:

$$\Omega(t,x)=\sum_{q\geqslant-1}\tilde\Omega_q(t,x).\tag{7.3}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

We wish now to prove a frequency decay for $\tilde{\Omega}_q(t)$. More precisely we will establish the following estimate: let ψ be the flow associated with u and F be the function defined by

$$F(t,x) = \int_0^t (\operatorname{div} u)\big(\tau, \psi(\tau)\circ\psi^{-1}(t,x)\big)d\tau.$$

Then for $p \in [1,6[$ and for every $j, q \geqslant -1$ we have

$$\|\Delta_j(e^{F(t)}\tilde{\Omega}_q(t))\|_{L^\infty} \leqslant C2^{-\frac{1}{2}|j-q|}\,\|\Delta_q\Omega_0\|_{L^\infty}\exp\big\{C\big(\|\nabla u\|_{L^1_tL^\infty} + \|\operatorname{div} u\|_{L^1_tB^{\frac{3}{p}}_{p,1}}\big)\big\}. \tag{7.4}$$

In the language of Besov spaces, this estimate is equivalent to

$$\|e^{F(t)}\tilde{\Omega}_q(t))\|_{B^{\pm\frac{1}{2}}_{\infty,\infty}} \leqslant C\,\|\Delta_q\Omega_0\|_{B^{\pm\frac{1}{2}}_{\infty,\infty}}\exp\big\{C\big(\|\nabla u\|_{L^1_tL^\infty} + \|\operatorname{div} u\|_{L^1_tB^{\frac{3}{p}}_{p,1}}\big)\big\}. \tag{7.5}$$

We remark that in the statement of Theorem 4 we require p to belong to $[1,+\infty[$ which is not the case in the above claim (7.5). To do so, we should work with a regularity of size $\eta \in]0,1[$ instead of $\frac{1}{2}$ and the condition on p will be $-\eta + \frac{3}{p} > 0$. Before going further into detail we shall recall some classical properties for the flow ψ. It is defined by the integral equation

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau,x))d\tau,\ x\in\mathbb{R}^3, t\geqslant 0.$$

The following estimates for the flow and its inverse ψ^{-1} are well-known:

$$\|\nabla\psi^{\pm 1}(t)\|_{L^\infty} \leqslant e^{\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau},\quad \|\nabla\big(\psi(\tau)\circ\psi^{-1}(t)\big)\|_{L^\infty} \leqslant e^{|\int_\tau^t\|\nabla v(\tau)\|_{L^\infty}d\tau|}. \tag{7.6}$$

The propagation of the negative regularity is much easier than that of the positive one, which needs more elaborate analysis. We apply first the Proposition 10, yielding

$$\begin{aligned} e^{-CV_p(t)}\,\|\tilde{\Omega}_q(t)\|_{B^{-\frac{1}{2}}_{\infty,\infty}} \lesssim \|\Delta_q\Omega_0\|_{B^{-\frac{1}{2}}_{\infty,\infty}} &+ \int_0^t e^{-CV_p(\tau)}\,\|\tilde{\Omega}^1_q\operatorname{div} u\|_{B^{-\frac{1}{2}}_{\infty,\infty}}d\tau \\ &+ \int_0^t e^{-CV_p(\tau)}\,\|\tilde{\Omega}_q\cdot\nabla u(\tau)\|_{B^{-\frac{1}{2}}_{\infty,\infty}}d\tau. \end{aligned} \tag{7.7}$$

with the following notation:

$$V_p(t) := \|u\|_{L^1_t\mathrm{Lip}} + \|\operatorname{div} u\|_{L^1_tB^{\frac{3}{p}}_{p,1}},\qquad p\in[1,6[;\qquad \|u\|_{\mathrm{Lip}} := \|u\|_{L^\infty} + \|\nabla u\|_{L^\infty}.$$

The first integral term of the right-hand side can be estimated as follows: for $p \in [1,6[$,

$$\int_0^t e^{-CV_p(\tau)}\,\|\tilde{\Omega}^1_q\operatorname{div} u\|_{B^{-\frac{1}{2}}_{\infty,\infty}}d\tau \leqslant C\int_0^t e^{-CV_p(\tau)}\,\|\operatorname{div} u\|_{B^{\frac{3}{p}}_{p,1}}\|\tilde{\Omega}^1_q\|_{B^{-\frac{1}{2}}_{\infty,\infty}}d\tau.$$

We have used in the preceding estimate the following law product: for $s<0$, $p\in[1,+\infty[$ such that $s+\frac{3}{p}>0$, we have

$$\|fg\|_{B^s_{\infty,\infty}} \leqslant C\,\|f\|_{B^{\frac{3}{p}}_{p,1}}\|g\|_{B^s_{\infty,\infty}}. \tag{7.8}$$

The proof of this estimate can be achieved for example through the use of Bony's decomposition. Let us now come back to (7.7) and examine the last integral term. Using

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Bony's decomposition, we get

$$\begin{aligned}\|\tilde{\Omega}_q\cdot\nabla u\|_{B^{-\frac12}_{\infty,\infty}} &\leqslant \|T_{\tilde{\Omega}_q}\cdot\nabla u\|_{B^{-\frac12}_{\infty,\infty}}+\|T_{\nabla u}\cdot\tilde{\Omega}_q\|_{B^{-\frac12}_{\infty,\infty}}\\ &\quad+\|\mathcal{R}\big(\tilde{\Omega}_q\cdot\nabla,u\big)\|_{B^{-\frac12}_{\infty,\infty}}\\ &\lesssim \|\nabla u\|_{L^\infty}\,\|\tilde{\Omega}_q\|_{B^{-\frac12}_{\infty,\infty}}+\|\mathcal{R}\big(\tilde{\Omega}_q\cdot\nabla,u\big)\|_{B^{-\frac12}_{\infty,\infty}}.\end{aligned}$$

Since $\operatorname{div}\tilde{\Omega}_q(t)=0$, then the remainder term can be treated as follows:

$$\begin{aligned}\|\mathcal{R}\big(\tilde{\Omega}_q\cdot\nabla,u\big)\|_{B^{-\frac12}_{\infty,\infty}} &= \|\operatorname{div}\mathcal{R}\big(\tilde{\Omega}_q\otimes,u\big)\|_{B^{-\frac12}_{\infty,\infty}}\\ &\lesssim \sup_k 2^{\frac{k}{2}}\sum_{j\geqslant k-3}\|\Delta_j\tilde{\Omega}_q\|_{L^\infty}\,\|\widetilde{\Delta}_j u\|_{L^\infty}\\ &\lesssim \|\tilde{\Omega}_q\|_{B^{-\frac12}_{\infty,\infty}}\,\|u\|_{B^1_{\infty,\infty}}\\ &\lesssim \|\tilde{\Omega}_q\|_{B^{-\frac12}_{\infty,\infty}}\,\|u\|_{\mathrm{Lip}}\end{aligned}$$

where we have used the embedding $\mathrm{Lip}\hookrightarrow B^1_{\infty,\infty}$. Combining these estimates yields

$$\|\tilde{\Omega}_q\cdot\nabla u\|_{B^{-\frac12}_{\infty,\infty}}\lesssim\|\tilde{\Omega}_q\|_{B^{-\frac12}_{\infty,\infty}}\,\|u\|_{\mathrm{Lip}}. \tag{7.9}$$

Now by inserting this estimate into (7.7) we obtain

$$\begin{aligned}e^{-CV_p(t)}\,\|\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}} &\lesssim \|\Delta_q\Omega_0\|_{B^{-\frac12}_{\infty,\infty}}\\ &\quad+\int_0^t\big(\|u(\tau)\|_{\mathrm{Lip}}+\|\operatorname{div}u\|_{B^{\frac3p}_{p,1}}\big)e^{-CV_p(\tau)}\,\|\tilde{\Omega}_q(\tau)\|_{B^{-\frac12}_{\infty,\infty}}\,d\tau.\end{aligned}$$

Hence we obtain by Gronwall's inequality

$$\|\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}}\leqslant C\,\|\Delta_q\Omega_0\|_{B^{-\frac12}_{\infty,\infty}}\,e^{CV_p(t)}. \tag{7.10}$$

Now we claim that for any $p\in[1,6[$ we have

$$\|e^{F(t)}\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}}\leqslant C\,\|\Delta_q\Omega_0\|_{B^{-\frac12}_{\infty,\infty}}\,e^{CV_p(t)}. \tag{7.11}$$

Indeed, by (7.8) we get for $p\in[1,6[$,

$$\begin{aligned}\|e^{F(t)}\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}} &\leqslant \|\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}}+\big\|\big(e^{F(t)}-1\big)\tilde{\Omega}_q(t)\big\|_{B^{-\frac12}_{\infty,\infty}}\\ &\lesssim \|\tilde{\Omega}_q(t)\|_{B^{-\frac12}_{\infty,\infty}}+\big\|e^{F(t)}-1\big\|_{B^{\frac3p}_{p,1}}\,\big\|\tilde{\Omega}_q(t)\big\|_{B^{-\frac12}_{\infty,\infty}}.\end{aligned} \tag{7.12}$$

We shall use the following estimate:

$$\begin{aligned}\big\|e^{F(t)}-1\big\|_{B^{\frac3p}_{p,1}} &\lesssim \big\|F(t)\big\|_{B^{\frac3p}_{p,1}}\,e^{C\,\|F(t)\|_{L^\infty}}\\ &\lesssim \big\|F(t)\big\|_{B^{\frac3p}_{p,1}}\,e^{C\,\|\operatorname{div}u\|_{L^1_tL^\infty}}\end{aligned} \tag{7.13}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

which can be deduced for instance from the following result: there exists $C>0$ such that for $u\in B^{\frac{3}{p}}_{p,1}$ and $n\in\mathbb{N}$ we have

$$\|u^n\|_{B^{\frac{3}{p}}_{p,1}}\leqslant C^{n-1}\,\|u\|^{n-1}_{L^\infty}\,\|u\|_{B^{\frac{3}{p}}_{p,1}}.$$

To estimate $F(t)$ we will use the following composition law (see for instance [31]): for $s\in]0,1[,\,p,r\in[1,\infty],f\in B^s_{p,r}$ and ψ a diffeomorphism, we then have

$$\|f\circ\psi\|_{B^s_{p,r}}\leqslant C\,\|f\|_{B^s_{p,r}}\big(1+\|\nabla\psi\|^s_{L^\infty}\big).$$

Applying this result combined with (7.6) we get for $p\in]3,\infty[$,

$$\begin{aligned}\|F(t)\|_{B^{\frac{3}{p}}_{p,1}}&\leqslant\int_0^t\|\operatorname{div}u(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\big(1+\|\nabla\big(\psi(\tau,\psi^{-1}(t,\cdot))\big)\|^{\frac{3}{p}}_{L^\infty}\big)d\tau\\&\lesssim\int_0^t\|\operatorname{div}u(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\,e^{\int_\tau^t\|\nabla u(s)\|_{L^\infty}\,ds}d\tau.\end{aligned}$$

Combining this estimate with (7.13) yields

$$\big\|e^{F(t)}-1\big\|_{B^{\frac{3}{p}}_{p,1}}\leqslant Ce^{C\,\|\nabla u\|_{L^1_tL^\infty}}\int_0^t\|\operatorname{div}u(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\,d\tau.\tag{7.14}$$

Putting together this estimate, (7.12) and (7.10) , we obtain

$$\begin{aligned}\|e^{F(t)}\tilde{\Omega}_q(t)\|_{B^{-\frac{1}{2}}_{\infty,\infty}}&\lesssim\|\tilde{\Omega}_q(t)\|_{B^{-\frac{1}{2}}_{\infty,\infty}}\Big(1+e^{C\,\|\nabla u\|_{L^1_tL^\infty}}\,\|\operatorname{div}u\|_{L^1_tB^{\frac{3}{p}}_{p,1}}\Big)\\&\lesssim\|\Delta_q\Omega_0\|_{B^{-\frac{1}{2}}_{\infty,\infty}}\,e^{CV_p(t)}\Big(1+e^{C\,\|\nabla u\|_{L^1_tL^\infty}}\,\|\operatorname{div}u\|_{L^1_tB^{\frac{3}{p}}_{p,1}}\Big)\\&\leqslant C\,\|\Delta_q\Omega_0\|_{B^{-\frac{1}{2}}_{\infty,\infty}}\,e^{CV_p(t)}.\end{aligned}$$

This completes the proof of (7.5) in the case of the negative regularity. It remains to show the estimate for $B^{\frac{1}{2}}_{\infty,\infty}$, which is the hard part of the proof. But before giving precise discussions of the difficulties, we shall rewrite in a suitable way the stretching term of equation (7.2). For this purpose we use Proposition 4(3), leading to $(\Delta_q\Omega_0)\times e_\theta=0$. Now according to Proposition 6(2), this geometric property is conserved through time, that is $\tilde{\Omega}_q(t)\times e_\theta=0$, and furthermore equation (7.2) becomes

$$\begin{cases}\partial_t\tilde{\Omega}_q+(u\cdot\nabla)\tilde{\Omega}_q+\tilde{\Omega}_q\operatorname{div}u=\dfrac{u^r}{r}\tilde{\Omega}_q\\ \tilde{\Omega}_{q|t=0}=\Delta_q\Omega_0.\end{cases}\tag{7.15}$$

Applying the maximum principle and using Gronwall's inequality we obtain

$$\|\tilde{\Omega}_q(t)\|_{L^\infty}\leqslant\|\Delta_q\Omega_0\|_{L^\infty}\,e^{\|(\operatorname{div}u,u^r/r)\|_{L^1_tL^\infty}}.\tag{7.16}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

From the geometric constraint $\tilde{\Omega}_q(t) \times e_\theta = 0$ we see that the solution $\tilde{\Omega}_q$ has two components in the Cartesian basis, $\tilde{\Omega}_q = (\tilde{\Omega}_q^1, \tilde{\Omega}_q^2, 0)$, and the mathematical analysis will be the same for both components. Thus we will just focus on the treatment of the first component $\tilde{\Omega}_q^1$.

From the identity $\frac{u^r}{r} = \frac{u^1}{x_1} = \frac{u^2}{x_2}$, which is an easy consequence of $u^\theta = 0$, it is clear that the function $\tilde{\Omega}_q^1$ satisfies the equation

$$\begin{cases} \partial_t \tilde{\Omega}_q^1 + (u \cdot \nabla)\tilde{\Omega}_q^1 + \tilde{\Omega}_q^1 \operatorname{div} u = u^2 \dfrac{\tilde{\Omega}_q^1}{x_2}, \\ \tilde{\Omega}^1_{q|t=0} = \Delta_q \Omega_0^1. \end{cases}$$

Now we shall discuss the main difficulties encountered when we wish to propagate positive regularities and explain how to circumvent these difficulties. In the work of [1], the velocity is divergence-free and thus the persistence of the regularity $B^s_{\infty,\infty}, 0 < s \leqslant 1$, is guaranteed by the special structure of the stretching term $\frac{v^r}{r}\tilde{\Omega}_q$ where the axisymmetry of the velocity plays a central role. In our case the velocity is not incompressible and the new term $\tilde{\Omega}_q \operatorname{div} u$ does not in general belong to Besov spaces $B^s_{\infty,\infty}, s > 0$, because in the context of critical regularities the quantity $\operatorname{div} u$ is only allowed to be in L^∞ or in any Banach space with the same scaling. Our idea is to filter the compressible part leading to a new function governed by an equation where all the terms have positive regularity. But we need to be careful with the filtration procedure; its commutation with the transport part leads to a bad term and that is why we have to extract the compressible part after using the Lagrangian coordinates. For simplicity, we will use the following notation: we denote by $t \mapsto \big(x_1(t), x_2(t), x_3(t)\big) = \psi(t, x)$ the path of the individual particle located initially at the position x. Now, we introduce some new functions:

$$\eta_q(t, x) = e^{G(t,x)} \tilde{\Omega}_q(t, \psi(t, x)),$$

and

$$G(t, x) = \int_0^t (\operatorname{div} u)(\tau, \psi(\tau, x)) d\tau, \qquad U(t, x) = u(t, \psi(t, x)).$$

As before, we shall restrict attention to the treatment of the first component of $\tilde{\Omega}_q$; the second one can be treated in a similar way. It is not hard to see that the function η_q^1 satisfies the equation

$$\partial_t \eta_q^1(t, x) = U^2(t, x) \frac{\eta_q^1(t, x)}{x_2(t)}, \qquad \eta_q(0, x) = \Delta_q \Omega_0^1(x). \tag{7.17}$$

Accordingly, we deduce from this equation that the quantity $t \mapsto \frac{\eta_q^1(t,x)}{x_2(t)}$ is conserved. Indeed, differentiating this function with respect to t, then using the equation for the

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

flow, we find

$$\partial_t\left(\frac{\eta_q^1(t,x)}{x_2(t)}\right)=\frac{x_2(t)\partial_t\eta_q^1(t,x)-\eta_q^1(t,x)x_2'(t)}{x_2^2(t)}$$
$$=\frac{U^2(t,x)\eta_q^1(t,x)-\eta_q^1(t,x)u^2(t,\psi(t,x))}{x_2^2(t)}$$
$$=0.$$

It follows that

$$\frac{\eta_q^1(t,x)}{x_2(t)}=\frac{\Delta_q\Omega_0^1(x)}{x_2}. \tag{7.18}$$

Now, we are going to estimate η_q^1 in the Besov space $B^{\frac12}_{\infty,\infty}$ and thus we prove the remaining case of (7.5) concerning the positive regularity. Applying Taylor's formula to equation (7.17) and using (7.18) we obtain

$$\|\eta_q^1(t)\|_{B^{\frac12}_{\infty,\infty}}\leqslant\|\eta_q^1(0)\|_{B^{\frac12}_{\infty,\infty}}+\int_0^t\left\|U^2(\tau)\frac{\eta_q^1(\tau)}{x_2(\tau)}\right\|_{B^{\frac12}_{\infty,\infty}}d\tau$$
$$\leqslant\|\Delta_q\Omega_0^1\|_{B^{\frac12}_{\infty,\infty}}+\int_0^t\left\|U^2(\tau)\frac{\eta_q^1(\tau)}{x_2(\tau)}\right\|_{B^{\frac12}_{\infty,\infty}}d\tau. \tag{7.19}$$

To estimate the integrand in the right-hand side in the preceding estimate, we shall make use of the paradifferential calculus through Bony's decomposition. So, we get by definition and from the triangular inequality,

$$\left\|U^2\frac{\eta_q^1}{x_2(t)}\right\|_{B^{\frac12}_{\infty,\infty}}\leqslant\left\|T_{\frac{\eta_q^1}{x_2(t)}}U^2\right\|_{B^{\frac12}_{\infty,\infty}}+\left\|T_{U^2}\frac{\eta_q^1}{x_2(t)}\right\|_{B^{\frac12}_{\infty,\infty}}+\left\|\mathcal{R}\left(U^2,\frac{\eta_q^1}{x_2(t)}\right)\right\|_{B^{\frac12}_{\infty,\infty}}.$$

The estimation of the first paraproduct and that of the remainder term are easy compared to that of the second paraproduct, for which we will use some sophisticated analysis. Now for the first paraproduct we write by definition and by the continuity of the Fourier cut-offs,

$$\left\|T_{\frac{\eta_q^1}{x_2(t)}}U^2\right\|_{B^{\frac12}_{\infty,\infty}}=\sup_{k\geqslant-1}2^{\frac{k}{2}}\sum_{j\in\mathbb{N}}\|\Delta_k(S_{j-1}(\eta_q^1/x_2(t))\Delta_jU^2)\|_{L^\infty}$$
$$=\sup_{k\geqslant-1}2^{\frac{k}{2}}\sum_{|j-k|\leqslant4}\|\Delta_k(S_{j-1}(\eta_q^1/x_2(t))\Delta_jU^2)\|_{L^\infty}$$
$$\lesssim\sup_{j\in\mathbb{N}}2^{\frac{j}{2}}\,\|\Delta_jU^2\|_{L^\infty}\,\|S_{j-1}(\eta_q^1/x_2(t)\|_{L^\infty}$$
$$\lesssim\sup_{j\in\mathbb{N}}2^{j}\,\|\Delta_jU^2\|_{L^\infty}\sup_{j\in\mathbb{N}}2^{-\frac{j}{2}}\left\|S_{j-1}\left(\frac{\eta_q^1}{x_2(t)}\right)\right\|_{L^\infty}$$
$$\lesssim\|\nabla U\|_{L^\infty}\left\|\frac{\eta_q^1}{x_2(t)}\right\|_{B^{-\frac12}_{\infty,\infty}}, \tag{7.20}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

where we have used the Bernstein inequality in the last line. Now, using the identity (7.18) we get

$$\begin{aligned}\Big\|\frac{\eta_q^1}{x_2(t)}\Big\|_{B^{-\frac12}_{\infty,\infty}} &= \Big\|\frac{\Delta_q\Omega_0^1}{x_2}\Big\|_{B^{-\frac12}_{\infty,\infty}}\\ &\lesssim 2^{\frac12 q}\,\|\Delta_q\Omega_0\|_{L^\infty}\approx\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}.\end{aligned}$$

The proof of the last inequality can be achieved as follows: applying Proposition 4(3), we get the identity $\Delta_q\Omega_0^1(x_1,0,x_3)=0$. This yields, in view of Taylor's expansion,

$$\Delta_q\Omega_0^1(x_1,x_2,x_3)=x_2\int_0^1(\partial_{x_2}\Delta_q\Omega_0^1)(x_1,\tau x_2,x_3)d\tau$$

and hence we obtain, by Lemma 8 and the Bernstein inequality,

$$\begin{aligned}\Big\|\frac{\Delta_q\Omega_0^1}{x_2}\Big\|_{B^{-\frac12}_{\infty,\infty}} &\leqslant \int_0^1\|(\partial_2\Delta_q\Omega_0^1)(\cdot,\tau\cdot,\cdot)\|_{B^{-\frac12}_{\infty,\infty}}d\tau\\ &\leqslant C\,\|\partial_2\Delta_q\Omega_0^1\|_{B^{-\frac12}_{\infty,\infty}}\int_0^1\tau^{-\frac12}d\tau\\ &\leqslant C\,\|\Delta_q\Omega_0^1\|_{B^{\frac12}_{\infty,\infty}}.\end{aligned}\tag{7.21}$$

Therefore we get, by (7.20),

$$\Big\|T_{\frac{\eta_q^1}{x_2(t)}}U^2\Big\|_{B^{\frac12}_{\infty,\infty}}\lesssim\|\nabla U\|_{L^\infty}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}.$$

Now according to the Leibniz formula and (7.6) we find

$$\begin{aligned}\|\nabla U(t)\|_{L^\infty} &\leqslant \|\nabla u(t)\|_{L^\infty}\,\|\nabla\psi(t)\|_{L^\infty}\\ &\leqslant \|\nabla u(t)\|_{L^\infty}\,e^{\|\nabla u\|_{L^1_tL^\infty}}.\end{aligned}\tag{7.22}$$

Hence it follows that

$$\begin{aligned}\Big\|T_{\frac{\eta_q^1}{x_2(\cdot)}}U^2\Big\|_{L^1_tB^{\frac12}_{\infty,\infty}} &\leqslant C\,\|\nabla u\|_{L^1_tL^\infty}\,e^{\|\nabla u\|_{L^1_tL^\infty}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}\\ &\leqslant Ce^{C\,\|\nabla u\|_{L^1_tL^\infty}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}.\end{aligned}\tag{7.23}$$

To estimate the remainder term we use its definition combined with (7.21):

$$\begin{aligned}\Big\|\mathcal{R}\Big(U^2,\frac{\eta_q^1}{x_2(t)}\Big)\Big\|_{B^{\frac12}_{\infty,\infty}} &\lesssim \sup_{k\geqslant-1}\sum_{j\geqslant k-3}2^{\frac k2}\,\|\Delta_jU^2\|_{L^\infty}\,\Big\|\widetilde{\Delta}_j\Big(\frac{\eta_q^1}{x_2}\Big)\Big\|_{L^\infty}\\ &\lesssim \|U\|_{B^1_{\infty,\infty}}\,\Big\|\frac{\eta_q^1}{x_2(t)}\Big\|_{B^{-\frac12}_{\infty,\infty}}\\ &\lesssim \|U\|_{B^1_{\infty,\infty}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}.\end{aligned}\tag{7.24}$$

Straightforward computations combined with the embedding $\mathrm{Lip}(\mathbb{R}^3) \hookrightarrow B^1_{\infty,\infty}$ imply that

$$\begin{aligned}\|U(t)\|_{B^1_{\infty,\infty}} &\lesssim \|U(t)\|_{L^\infty} + \|\nabla U(t)\|_{L^\infty}\\ &\lesssim \|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty}\,\|\nabla\psi(t)\|_\infty\\ &\lesssim \|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty}\, e^{\|\nabla u\|_{L^1_tL^\infty}}\\ &\lesssim \|u(t)\|_{\mathrm{Lip}}\, e^{\|u\|_{L^1_t\mathrm{Lip}}}.\end{aligned}$$

Plugging this estimate into (7.24) yields

$$\left\|\mathcal{R}\left(U^2, \frac{\eta_q^1}{x_2(\cdot)}\right)\right\|_{L^1_tB^{\frac12}_{\infty,\infty}} \leqslant Ce^{C\|u\|_{L^1_t\mathrm{Lip}}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}. \tag{7.25}$$

Now let us examine the second paraproduct which is more subtle and requires some special properties of the axisymmetric vector field u. First we write by definition

$$\left\|T_{U^2}\frac{\eta_q^1}{x_2}\right\|_{B^{\frac12}_{\infty,\infty}} \lesssim \sup_{j\in\mathbb{N}} 2^{\frac{j}{2}}\left\|S_{j-1}U^2\Delta_j\left(\frac{\eta_q^1}{x_2(t)}\right)\right\|_{L^\infty}.$$

At first sight the term $\frac{\eta_q^1}{x_2(t)}$ appears to involve the derivative more than we require, and the best configuration is obtained by transferring this derivative into the velocity U^2. To do this we will use the commutation between the frequency cut-off operator Δ_j and the singular multiplication by $\frac{1}{x_2}$. According to the identity (7.18) we write

$$\begin{aligned}S_{j-1}U^2(x)\Delta_j\left(\frac{\eta_q^1(t,x)}{x_2(t)}\right) &= S_{j-1}U^2(x)\Delta_j\left(\frac{\Delta_q\Omega_0^1}{x_2}\right)\\ &= \frac{S_{j-1}U^2(x)}{x_2}\,\Delta_j\big(\Delta_q\Omega_0^1\big) + S_{j-1}U^2(x)\left[\Delta_j, \frac{1}{x_2}\right](\Delta_q\Omega_0^1)\\ &:= \mathrm{I}_j(t,x) + \mathrm{II}_j(t,x).\end{aligned}$$

The estimate of the first term can be obtained as follows:

$$\|\mathrm{I}_j(t)\|_{L^\infty} \lesssim \left\|\frac{S_{j-1}U^2(t,x)}{x_2}\right\|_{L^\infty}\|\Delta_j\Delta_q\Omega_0^1\|_{L^\infty}.$$

Using Proposition 5(4) we get $S_{j-1}U^2(x_1,0,x_3) = 0$, and thus by Taylor's formula and (7.22) we obtain

$$\begin{aligned}\left\|\frac{S_{j-1}U^2(t,x)}{x_2}\right\|_{L^\infty} &\lesssim \|\nabla S_{j-1}U^2\|_{L^\infty}\\ &\lesssim \|\nabla U^2\|_{L^\infty}\\ &\lesssim \|\nabla u\|_{L^\infty}\, e^{C\|\nabla u\|_{L^1_tL^\infty}}.\end{aligned} \tag{7.26}$$

Since $\Delta_j\Delta_q\omega_0 = 0$ for $|j-q| \geqslant 2$, we then deduce

$$\sup_j 2^{\frac{j}{2}}\,\|\mathrm{I}_j(t)\|_{L^\infty} \leqslant C\,\|\nabla u(t)\|_{L^\infty}\, e^{C\|\nabla u\|_{L^1_tL^\infty}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}. \tag{7.27}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

For the commutator term II_j we write by definition

$$\mathrm{II}_j(t,x) = \frac{S_{j-1}U^2(t,x)}{x_2}\; 2^{3j}\int_{\mathbb{R}^3} h(2^j(x-y))(x_2-y_2)\frac{\Delta_q\Omega_0^1(y)}{y_2}dy$$
$$= 2^{-j}\frac{S_{j-1}U^2(x)}{x_2}\; 2^{3j}\tilde h(2^j\cdot)\star\Big(\frac{\Delta_q\Omega_0^1}{y_2}\Big)(x)$$

with $\tilde h(x) = x_2 h(x)$. Now we claim that for every $f\in\mathcal{S}'$ we have

$$2^{3j}\tilde h(2^j\cdot)\star f = \sum_{|j-k|\leqslant 1} 2^{3j}\tilde h(2^j\cdot)\star\Delta_k f.$$

Indeed, we have $\widehat{\tilde h}(\xi) = i\partial_{\xi_2}\widehat h(\xi) = i\partial_{\xi_2}\varphi(\xi)$. It follows that $supp\,\widehat{\tilde h}\subset supp\,\varphi$. So we get $2^{3j}\tilde h(2^j\cdot)\star\Delta_k f = 0$, for $|j-k|\geqslant 2$. This leads, in view of the Young inequality, (7.26) and (7.21), to

$$\sup_{j\in\mathbb{N}} 2^{\frac{j}{2}}\,\|\mathrm{II}_j\|_{L^\infty} \lesssim \sup_{j\in\mathbb{N}}\Big\|\frac{S_{j-1}U^2}{x_2}\Big\|_{L^\infty}\sum_{|j-k|\leqslant 1}2^{-\frac{k}{2}}\Big\|\Delta_k\Big(\frac{\Delta_q\Omega_0^1}{x_2}\Big)\Big\|_{L^\infty}$$
$$\lesssim \|\nabla U\|_{L^\infty}\Big\|\frac{\Delta_q\Omega_0^1}{x_2}\Big\|_{B^{-\frac12}_{\infty,\infty}}$$
$$\lesssim \|\nabla u(t)\|_{L^\infty}\, e^{\|\nabla u\|_{L^1_tL^\infty}}\,\|\Delta_q\Omega_0\|_{B^{\frac12}_{\infty,\infty}}. \tag{7.28}$$

Putting together (7.27) and (7.28) yields

$$\Big\|T_{U^2}\frac{\eta_q^1}{x_2(\cdot)}\Big\|_{L^1_tB^{\frac12}_{\infty,\infty}} \leqslant Ce^{C\|\nabla u\|_{L^1_tL^\infty}}\,\|\Delta_q\Omega_0^1\|_{B^{\frac12}_{\infty,\infty}}. \tag{7.29}$$

Finally, combining (7.23), (7.25) and (7.29) we get

$$\Big\|U^2\frac{\eta_q^1}{x_2(\cdot)}\Big\|_{L^1_tB^{\frac12}_{\infty,\infty}} \lesssim Ce^{C\|u\|_{L^1_t\mathrm{Lip}}}\,\|\Delta_q\Omega_0^1\|_{B^{\frac12}_{\infty,\infty}}.$$

Inserting this estimate into (7.19) we obtain

$$\|\eta_q^1(t)\|_{B^{\frac12}_{\infty,\infty}} \leqslant Ce^{C\|u\|_{L^1_t\mathrm{Lip}}}\,\|\Delta_q\Omega_0^1\|_{B^{\frac12}_{\infty,\infty}}. \tag{7.30}$$

Next, we shall show how to derive the estimate (7.5) from the preceding one. First, we observe that

$$e^{F(t,x)}\tilde\Omega_q(t,x) = \eta_q(t,\psi^{-1}(t,x)).$$

This will be combined with the following classical composition law:

$$\|f\circ\psi^{-1}\|_{B^{\frac12}_{\infty,\infty}} \leqslant C\,\|f\|_{B^{\frac12}_{\infty,\infty}}\big(1+\|\nabla\psi^{-1}\|^{\frac12}_{L^\infty}\big)$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

leading according to (7.6) and (7.30) to

$$\begin{aligned}\|\eta_q^1(t,\psi^{-1}(t,\cdot))\|_{B^{\frac{1}{2}}_{\infty,\infty}} &\leqslant C\,\|\eta_q^1\|_{B^{\frac{1}{2}}_{\infty,\infty}}\, e^{C\,\|\nabla u\|_{L^1_t L^\infty}}\\ &\leqslant C\,\|\Delta_q\Omega_0^1\|_{B^{\frac{1}{2}}_{\infty,\infty}}\, e^{C\,\|u\|_{L^1_t \mathrm{Lip}}}.\end{aligned}$$

This achieves the proof of (7.5). According to (7.4), its local version reads as follows:

$$\|\Delta_j(e^{F(t)}\tilde{\Omega}_q(t))\|_{L^\infty} \leqslant C2^{-\frac{1}{2}|j-q|}\,\|\Delta_q\Omega_0\|_{L^\infty}\, e^{CV_p(t)}. \tag{7.31}$$

On the other hand, coming back to the equation (7.17), taking the L^∞-estimate and using Gronwall's inequality, we get

$$\begin{aligned}\|e^{F(t)}\tilde{\Omega}_q(t)\|_{L^\infty} &= \|\eta_q(t)\|_{L^\infty}\\ &\leqslant \|\Delta_q\Omega_0\|_{L^\infty}\, e^{\int_0^t \|(U^2(\tau)/x_2(\tau)\|_{L^\infty}\,d\tau}.\end{aligned}$$

Now, since ψ is a homeomorphism and $u^2/x_2 = u^r/r$, then

$$\|(U^2(\tau)/x_2(\tau)\|_{L^\infty} = \|(u^r/r)(\tau)\|_{L^\infty}$$

and thus we obtain

$$\|e^{F(t)}\tilde{\Omega}_q(t)\|_{L^\infty} \leqslant \|\Delta_q\Omega_0\|_{L^\infty}\, e^{\|\frac{u^r}{r}\|_{L^1_t L^\infty}}. \tag{7.32}$$

Set $\zeta(t,x) = e^{F(t,x)}\Omega(t,x)$; then by the linearity of the problem (7.2) we get the decomposition

$$\zeta(t,x) = \sum_{q\geqslant -1} e^{F(t,x)}\tilde{\Omega}_q(t,x).$$

Let $N\in\mathbb{N}$ be an integer that will be fixed later; then using (7.31) and (7.32) we find

$$\begin{aligned}\|\zeta(t)\|_{B^0_{\infty,1}} &\leqslant \sum_{j,q\geqslant -1}\|\Delta_j\big(e^{F(t)}\tilde{\Omega}_q(t)\big)\|_{L^\infty}\\ &\leqslant \sum_{|j-q|\leqslant N}\|\Delta_j\big(e^{F(t)}\tilde{\Omega}_q(t)\big)\|_{L^\infty} + \sum_{|j-q|>N}\|\Delta_j\big(e^{F(t)}\tilde{\Omega}_q\big)(t)\|_{L^\infty}\\ &\lesssim N\,\|\Omega_0\|_{B^0_{\infty,1}}\, e^{\|u^r/r\|_{L^1_t L^\infty}} + 2^{-\frac{1}{2}N}\,\|\Omega_0\|_{B^0_{\infty,1}}\, e^{CV_p(t)}.\end{aligned}$$

Now we choose $N\approx V_p(t)$; then

$$\|\zeta(t)\|_{B^0_{\infty,1}} \leqslant C\,\|\Omega_0\|_{B^0_{\infty,1}}\, e^{\|u^r/r\|_{L^1_t L^\infty}}\Big(1+\|\operatorname{div} u\|_{L^1_t B^{\frac{3}{p}}_{p,1}} + \int_0^t \|u(\tau)\|_{\mathrm{Lip}}\,d\tau\Big). \tag{7.33}$$

Our next object is to obtain an estimate for $\|\Omega(t)\|_{B^0_{\infty,1}}$ from ζ. For this purpose we develop computations similar to (7.12) and (7.13). We start with the obvious identity

$$\Omega(t,x) = \zeta(t,x) + (e^{-F(t,x)}-1)\zeta(t,x),$$

Using the law product for $p\in[1,\infty[$

$$\|uv\|_{B^0_{\infty,1}} \leqslant C\,\|u\|_{B^0_{\infty,1}}\,\|v\|_{B^{\frac{3}{p}}_{p,1}},$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

we obtain

$$\|\Omega(t)\|_{B^0_{\infty,1}} \lesssim \|\zeta(t)\|_{B^0_{\infty,1}} \Big(1 + \|e^{-F(t)} - 1\|_{B^{\frac{3}{p}}_{p,1}}\Big).$$

We get, similarly to (7.14), for $p \in]3, +\infty[$,

$$\|e^{-F(t)} - 1\|_{B^{\frac{3}{p}}_{p,1}} \leqslant C e^{C\|\nabla u\|_{L^1_t L^\infty}} \int_0^t \|\operatorname{div} u(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \, d\tau.$$

Combining these estimates with (7.33) we find

$$\begin{aligned}
\|\Omega(t)\|_{B^0_{\infty,1}} &\lesssim \|\zeta(t)\|_{B^0_{\infty,1}} \Big(1 + e^{C\|\nabla u\|_{L^1_t L^\infty}} \int_0^t \|\operatorname{div} u(\tau)\|_{B^{\frac{3}{p}}_{p,1}} \, d\tau\Big) \\
&\leqslant C\|\Omega_0\|_{B^0_{\infty,1}} e^{\|u^r/r\|_{L^1_t L^\infty}} \Big(1 + e^{C\|\nabla u\|_{L^1_t L^\infty}} \|\operatorname{div} u\|^2_{L^1_t B^{\frac{3}{p}}_{p,1}}\Big) \\
&\quad \times \Big(1 + \int_0^t \|u(\tau)\|_{\mathrm{Lip}} \, d\tau\Big).
\end{aligned}$$

This achieves the proof of Theorem 4. □

7.3. The lower bound of T_ε

The object of this section is to give the proof of Theorem 3, but for the sake of conciseness we will restrict attention to the lower bound of T_ε. At the same time we will derive the Strichartz estimates involved, which form the cornerstone for the proof of the low Mach number limit, and to avoid redundancy we will omit the proof of this latter point which can be achieved analogously to the one for the subcritical case.

Proof. Using Lemma 5 we get

$$\|v_\varepsilon(t)\|_{\mathrm{Lip}} \lesssim \|v_\varepsilon(t)\|_{L^2} + \|\operatorname{div} v_\varepsilon(t)\|_{B^0_{\infty,1}} + \|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}}.$$

Integrating over time, using Propositions 1 and 2 we find

$$\begin{aligned}
\|v_\varepsilon\|_{L^1_T \mathrm{Lip}} &\lesssim \|v_\varepsilon\|_{L^1_T L^2} + \|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}} + \|\Omega_\varepsilon\|_{L^1_T B^0_{\infty,1}} \\
&\leqslant C_0(1+T)\, e^{\|\operatorname{div} v_\varepsilon\|_{L^1_t B^0_{\infty,1}}} + \|\Omega_\varepsilon\|_{L^1_T B^0_{\infty,1}}.
\end{aligned}$$

Let

$$V_\varepsilon(T) := \|v_\varepsilon\|_{L^1_T \mathrm{Lip}} + \|\nabla c_\varepsilon\|_{L^1_T L^\infty}$$

then using again Proposition 2 we obtain

$$V_\varepsilon(T) \leqslant C_0(1+T) e^{\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_t B^0_{\infty,1}}} + \|\Omega_\varepsilon\|_{L^1_T B^0_{\infty,1}}. \tag{7.34}$$

Now, according to Theorem 4 we have

$$\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}} \leqslant C_0 e^{\|v^r_\varepsilon/r\|_{L^1_t L^\infty}} \Big(1 + e^{C V_\varepsilon(T)} \|\operatorname{div} v_\varepsilon\|^2_{L^1_t B^{\frac{3}{p}}_{p,1}}\Big) \Big(1 + \int_0^t \|v_\varepsilon(\tau)\|_{\mathrm{Lip}} \, d\tau\Big). \tag{7.35}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

By virtue of Propositions 7 and 1, we get

$$\|v_\varepsilon^r/r\|_{L_t^1L^\infty} \leqslant C\,\|v_\varepsilon\|_{L_T^1L^2} + \|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}} + C_0Te^{\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}}}$$
$$\leqslant C_0(1+T)\,e^{C\,\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}}}. \tag{7.36}$$

In the inequality (7.35) we need to estimate $\|\mathrm{div}\, v_\varepsilon\|_{L_t^1B^{\frac{3}{p}}_{p,1}}$. For this purpose we will interpolate this norm between the energy estimate and the dispersive one. More precisely, for $p \in]2,+\infty[$ we get by interpolation

$$\|\mathrm{div}\, v_\varepsilon\|_{B^{\frac{3}{p}}_{p,1}} \leqslant C\,\|\mathrm{div}\, v_\varepsilon\|^{\frac{2}{p}}_{B^{\frac{3}{2}}_{2,1}}\,\|\mathrm{div}\, v_\varepsilon\|^{1-\frac{2}{p}}_{B^0_{\infty,1}}.$$

Integrating over time and combining the Hölder inequality with Proposition 1 we find

$$\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^{\frac{3}{p}}_{p,1}} \leqslant CT^{\frac{2}{p}}\,\|\mathrm{div}\, v_\varepsilon\|^{\frac{2}{p}}_{L_T^\infty B^{\frac{3}{2}}_{2,1}}\,\|\mathrm{div}\, v_\varepsilon\|^{1-\frac{2}{p}}_{L_T^1B^0_{\infty,1}}$$
$$\leqslant C_0T^{\frac{2}{p}}\,e^{CV_\varepsilon(T)}\,\|\mathrm{div}\, v_\varepsilon\|^{1-\frac{2}{p}}_{L_T^1B^0_{\infty,1}}. \tag{7.37}$$

Putting together (7.35)–(7.37) and taking $p=4$ yields

$$\|\Omega_\varepsilon(t)\|_{B^0_{\infty,1}} \leqslant C_0\exp\Big\{C_0(1+T)\,e^{C\,\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}}}\Big\}\Big(1+e^{CV_\varepsilon(T)}\,\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}}\Big)\big(1+V_\varepsilon(T)\big)$$
$$\leqslant C_0\exp\Big\{C_0(1+T)\,e^{Ce^{CV_\varepsilon(T)}\,\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}}}\Big\}\big(1+V_\varepsilon(T)\big).$$

Combining this estimate with (7.34) and applying Gronwall's inequality we obtain

$$V_\varepsilon(T) \leqslant C_0e^{\exp\Big\{C_0(1+T)\,e^{Ce^{CV_\varepsilon(T)}\,\|(\mathrm{div}\, v_\varepsilon,\nabla c_\varepsilon)\|_{L_T^1B^0_{\infty,1}}}\Big\}}. \tag{7.38}$$

Let $N\in\mathbb{N}^*$ be an integer that will be fixed later. Using the dyadic decomposition combined with the Bernstein inequality and property (1) in Definition 1 we get

$$\|\mathrm{div}\, v_\varepsilon\|_{B^0_{\infty,1}} = \|\mathrm{div}\,\mathcal{Q}v_\varepsilon\|_{B^0_{\infty,1}}$$
$$= \sum_{q<N}\|\Delta_q\mathrm{div}\,\mathcal{Q}v_\varepsilon\|_{L^\infty} + \sum_{q\geqslant N}\|\Delta_q\mathrm{div}\,\mathcal{Q}v_\varepsilon\|_{L^\infty}$$
$$\lesssim \sum_{q<N}2^q\,\|\Delta_q\mathcal{Q}v_\varepsilon\|_{L^\infty} + \frac{1}{\Psi(N)}\sum_{q\geqslant N}2^{\frac{5}{2}q}\Psi(q)\,\|\Delta_q\mathcal{Q}v_\varepsilon\|_{L^2}$$
$$\lesssim 2^N\,\|\mathcal{Q}v_\varepsilon\|_{L^\infty} + \frac{1}{\Psi(N)}\,\|v_\varepsilon\|_{B^{\frac{5}{2},\Psi}_{2,1}}.$$

Integrating over time yields

$$\|\mathrm{div}\, v_\varepsilon\|_{L_T^1B^0_{\infty,1}} \lesssim 2^N\,\|\mathcal{Q}v_\varepsilon\|_{L_T^1L^\infty} + \frac{1}{\Psi(N)}\,\|v_\varepsilon\|_{L_T^1B^{\frac{5}{2},\Psi}_{2,1}}.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Using Proposition 1, Corollary 2 and the Hölder inequalities we get for $r > 2$,

$$\begin{aligned}\|\mathrm{div}\, v_\varepsilon\|_{L^1_T B^0_{\infty,1}} &\leqslant C_0(1+T^{2-\frac{1}{r}})e^{CV_\varepsilon(T)}\left(2^N\varepsilon^{\frac{1}{r}} + \frac{1}{\Psi(N)}\right)\\ &\leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\left(2^N\varepsilon^{\frac{1}{r}} + \frac{1}{\Psi(N)}\right).\end{aligned}$$

Similarly we get

$$\|\nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} \leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\left(2^N\varepsilon^{\frac{1}{r}} + \frac{1}{\Psi(N)}\right).$$

Thus

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}} \leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\left(2^N\varepsilon^{\frac{1}{r}} + \frac{1}{\Psi(N)}\right).$$

We take N such that

$$e^N \approx \varepsilon^{-\frac{1}{2r}}$$

which leads for small ε to

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}} \leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\left(\varepsilon^{\frac{1}{2r}} + \frac{1}{\Psi\left(\log\left(\frac{1}{\varepsilon^{\frac{1}{2r}}}\right)\right)}\right).$$

To simplify the notation we introduce the function

$$\Phi(x) = x^{\frac{1}{2r}} + \frac{1}{\Psi\left(\log\left(\frac{1}{x^{\frac{1}{2r}}}\right)\right)}, \quad x \in]0,1[.$$

We observe that since $\Psi \in \mathcal{U}_\infty$ and it satisfies the property (ii) of Definition 1, then

$$\lim_{\varepsilon\to 0} \Phi(\varepsilon) = 0. \tag{7.39}$$

Thus we can rewrite the preceding estimate in the form

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}} \leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\Phi(\varepsilon). \tag{7.40}$$

This gives

$$e^{CV_\varepsilon(T)}\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}} \leqslant C_0(1+T^2)e^{CV_\varepsilon(T)}\Phi(\varepsilon).$$

Plugging this estimate into (7.38) yields

$$CV_\varepsilon(T) \leqslant e^{e^{C_0(1+T)\exp\{C_0(1+T^2)\Phi(\varepsilon)\, e^{CV_\varepsilon(T)}\}}}. \tag{7.41}$$

We choose T_ε such that

$$e^{e^{\exp\{2C_0(1+T_\varepsilon)\}}} = \Phi^{-\frac{1}{2}}(\varepsilon). \tag{7.42}$$

Then we claim that for small ε and for every $t \in [0, T_\varepsilon]$,

$$e^{CV_\varepsilon(t)} \leqslant \Phi^{-\frac{1}{2}}(\varepsilon). \tag{7.43}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Indeed, we set

$$I_{T_\varepsilon} := \Big\{ t \in [0, T_\varepsilon];\, e^{CV_\varepsilon(t)} \leqslant \Phi^{-\frac{1}{2}}(\varepsilon) \Big\}.$$

First this set is nonempty since $0 \in I_{T_\varepsilon}$. By the continuity of $t \mapsto V_\varepsilon(t)$, the set I_{T_ε} is closed, and thus to prove that I_{T_ε} coincides with $[0, T_\varepsilon]$ it suffices to show that I_{T_ε} is an open set. Let $t \in I_{T_\varepsilon}$; then using (7.41) we get for small ε,

$$e^{CV_\varepsilon(t)} \leqslant e^{e^{\exp\{C_0(1+T)\exp\{C_0(1+T^2)\Phi^{\frac{1}{2}}(\varepsilon)\}\}}}. \tag{7.44}$$

From (7.42) we get for small values of ε,

$$T_\varepsilon \approx c \log\log\log \Phi^{-\frac{1}{2}}(\varepsilon) \tag{7.45}$$

for some constant c and thus

$$\lim_{\varepsilon\to 0}(1+T_\varepsilon^2)\Phi^{\frac{1}{2}}(\varepsilon) = \lim_{\varepsilon\to 0} \Phi^{\frac{1}{2}}(\varepsilon)\log^2\log\log(\Phi^{-\frac{1}{2}}(\varepsilon)) = 0.$$

Therefore for small ε and for every $t \in [0, T_\varepsilon]$ we have $e^{C_0(1+t^2)\Phi^{\frac{1}{2}}(\varepsilon)} < 2$, and consequently from (7.44) and (7.42) we find

$$\begin{aligned} e^{CV_\varepsilon(t)} &< e^{e^{\exp\{2C_0(1+T)\}}} \\ &< \Phi^{-\frac{1}{2}}(\varepsilon). \end{aligned}$$

This proves that t is in the interior of I_{T_ε} and thus I_{T_ε} is an open set of $[0, T_\varepsilon]$. Consequently we conclude that $I_{T_\varepsilon} = [0, T_\varepsilon]$.

Now inserting (7.43) into (7.41) we get for $T \in [0, T_\varepsilon]$,

$$V_\varepsilon(T) \leqslant C_0 e^{\exp C_0 T}. \tag{7.46}$$

Plugging (7.43) into the estimate (7.40) we obtain for $T \in [0, T_\varepsilon]$ and for small ε,

$$\begin{aligned} \|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}} + \|\nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} &\leqslant C_0(1+T^2)\Phi^{\frac{1}{2}}(\varepsilon) \\ &\leqslant C_0 \Phi^{\frac{1}{3}}(\varepsilon). \end{aligned}$$

From Corollary 2 and the above estimates we obtain for every $r \in]2, +\infty[$ and for small values of ε,

$$\begin{aligned} \|\mathcal{Q} v_\varepsilon\|_{L^r_T L^\infty} + \|c_\varepsilon\|_{L^r_T L^\infty} &\leqslant C_0 \varepsilon^{\frac{1}{r}}(1+T)e^{CV_\varepsilon(t)} \\ &\leqslant C_0 \varepsilon^{\frac{1}{r}} \Phi^{-1}(\varepsilon) \\ &\leqslant C_0 \varepsilon^{\frac{1}{2r}}. \end{aligned}$$

We point out that the use of the Hölder inequality and the slow growth of T allow us to get the following: for every $r \in [1, +\infty[$,

$$\|\mathcal{Q} v_\varepsilon\|_{L^r_T L^\infty} + \|c_\varepsilon\|_{L^r_T L^\infty} \leqslant C_0 \varepsilon^{\sigma'}, \quad \sigma' > 0. \tag{7.47}$$

Inserting (7.47) into (6.1) leads to

$$\|\Omega_\varepsilon(T)\|_{L^\infty} \leqslant C_0 e^{C_0 T}.$$

To estimate the solutions of (1.1) in Sobolev norms we use Proposition 1 combined with the Lipschitz bound (7.46)

$$\|(v_\varepsilon, c_\varepsilon)(T)\|_{B_{2,1}^{\frac{5}{2},\Psi}} \leqslant C_0 e^{e^{\exp\{C_0 T\}}}. \quad \square$$

Acknowledgements. The author would like to thank J.-Y. Chemin, I. Gallagher and D. Li for fruitful discussions about the critical case.

Appendix.

This last section is devoted to the proof of some lemmata used in the preceding sections.

Lemma 4. *Let $s > \frac{5}{2}$ and $\varphi \in L^\infty([0,T]; H^s(\mathbb{R}^3))$; then we have*

$$\|\nabla\varphi\|_{L^1_T B^0_{\infty,1}} \leqslant C T^{1-\frac{2s-5}{r(2s-3)}} \|\varphi\|_{L^r_T L^\infty}^{\frac{2s-5}{2s-3}} \|\varphi\|_{L^\infty_T H^s}^{\frac{2}{2s-3}}.$$

Proof. To prove this inequality we will use a frequency interpolation argument. We split the function φ into its dyadic blocks: $\varphi = \sum_{q\geqslant -1} \Delta_q\varphi$, and we take an integer N that will be fixed later. Then by the Bernstein inequality we obtain

$$\begin{aligned}
\|\nabla\varphi\|_{B^0_{\infty,1}} &= \sum_{q=-1}^{N} \|\Delta_q \nabla\varphi\|_{L^\infty} + \sum_{q>N} \|\Delta_q \nabla\varphi\|_{L^\infty} \\
&\leqslant C 2^N \|\varphi\|_{L^\infty} + C \sum_{q>N} 2^{\frac{5}{2}q} \|\Delta_q\varphi\|_{L^2} \\
&\leqslant C 2^N \|\varphi\|_{L^\infty} + C 2^{-N(s-\frac{5}{2})} \|\varphi\|_{H^s}.
\end{aligned}$$

Now we choose N such that

$$2^{N(s-\frac{3}{2})} \approx \frac{\|\varphi\|_{H^s}}{\|\varphi\|_{L^\infty}},$$

and this gives

$$\|\nabla\varphi\|_{L^\infty} \leqslant C \|\varphi\|_{H^s}^{\frac{2}{2s-3}} \|\varphi\|_{L^\infty}^{\frac{2s-5}{2s-3}}.$$

Now to get the desired estimate, it suffices to use the Hölder inequality in the time variable. $\square$

The next lemma deals with a logarithmic estimate in the compressible context.

Lemma 5. *The following assertions hold true:*

(1) *The subcritical case: let v be a vector field belonging to $H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$ and denote by $\Omega = \nabla \wedge v$ its vorticity; then we have*

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\mathrm{div}\, v\|_{B^0_{\infty,1}} + \|\Omega\|_{L^\infty} \log(e + \|v\|_{H^s}).$$

(2) *The critical case: let v be a vector field belonging to $B^1_{\infty,1} \cap L^2$; then*

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\mathrm{div}\, v\|_{B^0_{\infty,1}} + \|\Omega\|_{B^0_{\infty,1}}.$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Proof. We split v into compressible and incompressible parts: $v = \mathcal{Q}v + \mathcal{P}v$. Then

$$\operatorname{curl} v = \operatorname{curl} \mathcal{P}v.$$

Thus the incompressible part $\mathcal{P}v$ has the same vorticity as the total velocity and thus we can use the Brezis–Gallouët logarithmic estimate

$$\begin{aligned}\|\nabla \mathcal{P}v\|_{L^\infty} &\lesssim \|\mathcal{P}v\|_{L^2} + \|\Omega\|_{L^\infty} \log(e + \|\mathcal{P}v\|_{H^s})\\ &\lesssim \|v\|_{L^2} + \|\Omega\|_{L^\infty} \log(e + \|v\|_{H^s}).\end{aligned}$$

It remains to estimate the Lipschitz norm of the compressible part $\mathcal{Q}v$. Using the Bernstein inequality for low frequency,

$$\begin{aligned}\|\nabla \mathcal{Q}v\|_{L^\infty} &= \|\nabla^2 \Delta^{-1} \operatorname{div} v\|_{L^\infty}\\ &\leqslant \|\Delta_{-1}\nabla^2\Delta^{-1}\operatorname{div} v\|_{L^\infty} + \sum_{q\in\mathbb{N}} \|\Delta_q \nabla^2 \Delta^{-1} \operatorname{div} v\|_{L^\infty}\\ &\lesssim \|v\|_{L^2} + \sum_{q\in\mathbb{N}} \|\Delta_q \operatorname{div} v\|_{L^\infty}\\ &\lesssim \|v\|_{L^2} + \|\operatorname{div} v\|_{B^0_{\infty,1}}.\end{aligned}$$

This concludes the proof of the logarithmic estimate. □

We will establish the following.

Lemma 6. *Let $s > 0$ and $\Psi \in \mathcal{U}$; see Definition 1. Then we have the following commutator estimate:*

$$\sum_{q\geqslant -1} 2^{qs}\Psi(q)\|[\Delta_q, v\cdot\nabla]u\|_{L^2} \lesssim \|\nabla v\|_{L^\infty}\|u\|_{B^{s,\Psi}_{2,1}} + \|\nabla u\|_{L^\infty}\|v\|_{B^{s,\Psi}_{2,1}}.$$

Proof. By using Bony's decomposition, we can split the commutator into three parts:

$$\begin{aligned}[\Delta_q, v\cdot\nabla]u &= [\Delta_q, T_v\cdot\nabla]u + [\Delta_q, T_{\nabla\cdot}\cdot v]u + [\Delta_q, \mathcal{R}(v,\nabla)]u\\ &= \mathrm{I}_q + \mathrm{II}_q + \mathrm{III}_q.\end{aligned}$$

We start with the estimate of the first term I. Then by definition,

$$\|[\Delta_q, T_v\cdot\nabla]u\|_{L^2} \leqslant \sum_{|j-q|\leqslant 4} \|[\Delta_q, S_{j-1}v\cdot\nabla]\Delta_j u\|_{L^2}.$$

We can now use the following commutator inequality (for a proof see for example [10]):

$$\|[\Delta_q, a]b\|_{L^p} \leqslant C2^{-q}\|\nabla a\|_{L^\infty}\|a\|_{L^p}$$

which yields, in view of the Bernstein inequality,

$$\begin{aligned}\|[\Delta_q, T_v\cdot\nabla]u\|_{L^2} &\leqslant C\sum_{|j-q|\leqslant 4} 2^{-q}\|\nabla S_{j-1}v\|_{L^\infty}\|\nabla\Delta_j u\|_{L^2}\\ &\lesssim C\|\nabla v\|_{L^\infty}\sum_{|j-q|\leqslant 4} 2^{j-q}\|\Delta_j u\|_{L^2}\\ &\lesssim \|\nabla v\|_{L^\infty}\sum_{|j-q|\leqslant 4}\|\Delta_j u\|_{L^2}.\end{aligned}$$

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Multiplying this last inequality by $2^{qs}\Psi(q)$, summing over q, and using the property (2) in Definition 1, we get

$$\sum_{q\geqslant -1} 2^{qs}\Psi(q)\,\|\mathrm{I}_q\|_{L^2} \lesssim \|\nabla v\|_{L^\infty} \sum_{|j-q|\leqslant 4} \frac{\Psi(q)}{\Psi(j)}\big(2^{js}\Psi(j)\|\Delta_j u\|_{L^2}\big)$$
$$\lesssim \|\nabla v\|_{L^\infty}\,\|u\|_{B^{s,\Psi}_{2,1}}\,.$$

Similarly, the second term II_q is estimated as follows:

$$\sum_{q\geqslant -1} 2^{qs}\Psi(q)\,\|\mathrm{II}_q\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\,\|v\|_{B^{s,\Psi}_{2,1}}\,.$$

What is left is to estimate the remainder term. From the definition and since the Fourier transform of $[\Delta_q, \Delta_j v]\nabla$ is supported in a ball of radius 2^q, we then obtain

$$\|[\Delta_q, \mathcal{R}(v,\nabla)]u\|_{L^2} \leqslant \sum_{j\geqslant q-4} \|[\Delta_q, \Delta_j v]\nabla\widetilde{\Delta}_j u\|_{L^2}\,.$$

To estimate the term inside the sum we do not need to use the structure of the commutator. Applying the Hölder and Bernstein inequalities yields

$$\|[\Delta_q, \mathcal{R}(v,\nabla)]u\|_{L^2} \lesssim \sum_{j\geqslant q-4} \|\Delta_j v\|_{L^2}\,\|\nabla\widetilde{\Delta}_j u\|_{L^\infty}$$
$$\lesssim \|\nabla u\|_{L^\infty} \sum_{j\geqslant q-4} \|\Delta_j v\|_{L^2}\,.$$

Multiplying this last inequality by $2^{qs}\Psi(q)$, summing over q, and using the Fubini identity and the property (1) of Definition 1, we get

$$\sum_{q\geqslant -1} 2^{qs}\Psi(q)\|[\Delta_q, \mathcal{R}(v,\nabla)]u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \sum_{q\geqslant -1}\sum_{j\geqslant q-4} 2^{qs}\Psi(q)\,\|\Delta_j u\|_{L^2}$$
$$\lesssim \|\nabla u\|_{L^\infty} \sum_{j\geqslant -1} \|\Delta_j u\|_{L^2} \sum_{q\leqslant j+4} 2^{qs}\Psi(q)$$
$$\lesssim \|\nabla u\|_{L^\infty}\,\|u\|_{B^{s,\Psi}_{2,1}}\,.$$

This concludes the proof of Lemma 6. □

The following proposition describes the propagation of Besov regularity for the transport equation.

Proposition 10. *Let u be a smooth vector field, not necessarily of zero divergence. Let f be a smooth solution of the transport equation*

$$\partial_t f + u\cdot\nabla f = g, \quad f_{|t=0} = f_0,$$

such that $f_0 \in B^s_{p,r}(\mathbb{R}^3)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+; B^s_{p,r})$. Then the following assertions hold true:

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

(1) *Let* $p, r \in [1, \infty]$ *and* $s \in]0, 1[$ *; then*

$$\|f(t)\|_{B^s_{p,r}} \leqslant Ce^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\,\|g(\tau)\|_{B^s_{p,r}}\,d\tau\Big),$$

where $V(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau$ *and* C *is a constant depending on* s.

(2) *Let* $s \in]-1, 0]$, $r \in [1, +\infty]$ *and* $p \in [2, +\infty]$ *with* $s + \frac{3}{p} > 0$ *; then*

$$\|f(t)\|_{B^s_{p,r}} \leqslant Ce^{CV_p(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV_p(\tau)}\,\|g(\tau)\|_{B^s_{p,r}}\,d\tau\Big),$$

with $V_p(t) = \|\nabla u\|_{L^1_t L^\infty} + \|\mathrm{div}\, u\|_{L^1_t B^{\frac{3}{p}}_{p,\infty}}$.

Proof. (1) This estimate is classical; see for example [10] for the Hölderian case—and a similar proof works also for Besov spaces.

(2) The proof for a general case can be found in for example [11] and for the convenience of the reader we will give here the complete proof for our special case. We start by localizing the equation in frequency, leading to

$$\begin{aligned}\partial_t \Delta_q f + (u\cdot\nabla)\Delta_q f &= \Delta_q g + (u\cdot\nabla)\Delta_q f - \Delta_q\big(u\cdot\nabla f\big)\\ &= \Delta_q g - [\Delta_q, u\cdot\nabla]f.\end{aligned}$$

Taking the L^p norm, the zero divergence of the flow then gives

$$\|\Delta_q f(t)\|_{L^p} \leqslant \|\Delta_q f_0\|_{L^p} + \int_0^t \|\Delta_q g\|_{L^p}\,d\tau + \int_0^t \big\|[\Delta_q, u\cdot\nabla]f\big\|_{L^p}\,d\tau. \qquad (\mathrm{A}\,1)$$

From Bony's decomposition, the commutator may be decomposed as follows:

$$[\Delta_q, u\cdot\nabla]f = [\Delta_q, T_u\cdot\nabla]f + [\Delta_q, T_{\nabla\cdot}\cdot u]f + [\Delta_q, \mathcal{R}(u,\nabla)]f.$$

For the paraproducts we do not need the structure for the velocity, so using the same proof as for Lemma 6 we find for $s < 1$,

$$\|[\Delta_q, T_u\cdot\nabla]f + [\Delta_q, T_{\nabla\cdot}\cdot u]f\|_{L^p} \leqslant C\,\|\nabla u\|_{L^\infty}\,\|f\|_{B^s_{p,r}}\,c_q 2^{-qs},$$

with $(c_q)_{\ell^r} = 1$. As regards the remainder term, we write

$$\begin{aligned}[\Delta_q, \mathcal{R}(u,\nabla)]f &= \sum_j [\Delta_q, \Delta_j u^i \partial_i]\tilde{\Delta}_j f\\ &= \partial_i \sum_{j\geqslant q-4}[\Delta_q, \Delta_j u^i]\tilde{\Delta}_j f - \sum_{j\geqslant q-4}[\Delta_q, \Delta_j \mathrm{div}\, u]\tilde{\Delta}_j f\\ &= \mathrm{I}_q + \mathrm{II}_q.\end{aligned}$$

Like in the proof of Lemma 6, we get for $s > -1$,

$$\|\mathrm{I}_q\|_{L^p} \leqslant C\,\|\nabla u\|_{L^\infty}\,\|f\|_{B^s_{p,r}}\,c_q 2^{-qs}, \quad (c_q)_{\ell^r} = 1.$$

For the second term, since the Fourier transform of $[\Delta_q, \Delta_j \mathrm{div}\, u]\tilde{\Delta}_j f$ is supported in a disc of size 2^q, then using Bernstein and Hölder inequalities we obtain

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

for $p \geqslant 2$,

$$\|[\Delta_q, \Delta_j \text{div}\, u]\tilde{\Delta}_j f\|_{L^p} \leqslant C2^{\frac{3}{p}q} \|[\Delta_q, \Delta_j \text{div}\, u]\tilde{\Delta}_j f\|_{L^{\frac{p}{2}}}$$
$$\leqslant C2^{\frac{3}{p}q} \|\Delta_j \text{div}\, u\|_{L^p} \|\tilde{\Delta}_j f\|_{L^p}.$$

Therefore

$$2^{qs} \|\text{II}_q\|_{L^p} \leqslant C2^{\frac{3}{p}q} \sum_{j\geqslant q-4} \|\tilde{\Delta}_j f\|_{L^p} \|\Delta_j \text{div}\, u\|_{L^p}$$
$$\leqslant C \|\text{div}\, v\|_{B^{\frac{3}{p}}_{p,\infty}} \sum_{j\geqslant q-4} 2^{(q-j)(s+\frac{3}{p})}\big(2^{js} \|\tilde{\Delta}_j f\|_{L^p}\big).$$

Now since $s + \frac{3}{p} > 0$, we can then use the Young inequality to deduce that

$$\Big(2^{qs} \|\text{II}_q\|_{L^p}\Big)_{\ell^r} \leqslant C \|\text{div}\, v\|_{B^{\frac{3}{p}}_{p,\infty}} \|f\|_{B^s_{p,r}}$$

Combining these estimates yields for $-\min(1, \frac{3}{p}) < s \leqslant 0$,

$$\Big(2^{qs} \|[\Delta_q, u\cdot\nabla]u\|_{L^p}\Big)_{\ell^r} \leqslant C \|\text{div}\, v\|_{B^{\frac{3}{p}}_{p,\infty}} \|f\|_{B^s_{p,r}}.$$

Plugging this estimate into (A 1) we get

$$\|f(t)\|_{B^s_{p,r}} \lesssim \|f(0)\|_{B^s_{p,r}} + \int_0^t V_p(t)\, \|f(\tau)\|_{B^s_{p,r}}\, d\tau + \int_0^t \|g(\tau)\|_{B^s_{p,r}}\, d\tau,$$

with

$$V_p(t) := \|\nabla u\|_{L^\infty} + \|\text{div}\, u\|_{B^{\frac{3}{p}}_{p,\infty}}.$$

To get the desired estimate it suffices to use Gronwall's lemma. □

The last point that we will discuss concerns the action of the operator $(\partial_r/r)\Delta^{-1}$ on axisymmetric functions. We will show that its restriction to this class of functions behaves like Riesz transforms. This study was done before in [19], and for the convenience of the reader we will give the complete proof here.

Lemma 7. *We have for every axisymmetric smooth scalar function u,*

$$(\partial_r/r)\Delta^{-1}u(x) = \frac{x_2^2}{r^2}\mathcal{R}_{11}u(x) + \frac{x_1^2}{r^2}\mathcal{R}_{22}u(x) - 2\frac{x_1 x_2}{r^2}\mathcal{R}_{12}u(x), \tag{A 2}$$

with $\mathcal{R}_{ij} = \partial_{ij}\Delta^{-1}$.

Proof. We set $f = \Delta^{-1}u$; then we can show from the Biot–Savart law that f is also axisymmetric. Hence we get by using polar coordinates that

$$\partial_{11}f + \partial_{22}f = (\partial_r/r)f + \partial_{rr}f. \tag{A 3}$$

where

$$\partial_r = \frac{x_1}{r}\partial_1 + \frac{x_2}{r}\partial_2.$$

By using this expression of ∂_r, we obtain

$$\partial_{rr} = \left(\frac{x_1}{r}\partial_1 + \frac{x_2}{r}\partial_2\right)^2 = \partial_r\left(\frac{x_1}{r}\right)\partial_1 + \partial_r\left(\frac{x_2}{r}\right)\partial_2 + \frac{x_1^2}{r^2}\partial_{11} + \frac{x_2^2}{r^2}\partial_{22} + \frac{2x_1x_2}{r^2}\partial_{12}.$$
$$= \frac{x_1^2}{r^2}\partial_{11} + \frac{x_2^2}{r^2}\partial_{22} + \frac{2x_1x_2}{r^2}\partial_{12}$$

since

$$\partial_r\left(\frac{x_i}{r}\right) = 0, \quad \forall i \in \{1, 2\}.$$

This yields by using (A 3) that

$$\frac{\partial_r}{r} f = \left(1 - \frac{x_1^2}{r^2}\right)\partial_{11} f + \left(1 - \frac{x_2^2}{r^2}\right)\partial_{22} f - \frac{2x_1x_2}{r^2}\partial_{12} f$$
$$= \frac{x_2^2}{r^2}\partial_{11} f + \frac{x_1^2}{r^2}\partial_{22} f - \frac{2x_1x_2}{r^2}\partial_{12} f.$$

To get (A 2), it suffices to replace f by $\Delta^{-1}u$. □

The following result describes the anisotropic dilatation in Besov spaces.

Lemma 8. *Let* $s \in]-1, +\infty[\setminus\{0\}$, $f : \mathbb{R}^3 \to \mathbb{R}$ *be a function belonging to* $B^s_{\infty,\infty}$, *and define* $f_\lambda(x_1, x_2, x_3) = f(\lambda x_1, x_2, x_3)$. *Then, there exists an absolute constant* $C > 0$ *such that for all* $\lambda \in]0, 1[$,

$$\|f_\lambda\|_{B^s_{\infty,\infty}} \leqslant C(1 + \lambda^s)\, \|f\|_{B^s_{\infty,\infty}}\,.$$

Proof. Letting $q \geqslant -1$, we define $f_{q,\lambda} = (\Delta_q f)_\lambda$. From the definition we have

$$\|f_\lambda\|_{B^s_{\infty,\infty}} = \|\Delta_{-1} f_\lambda\|_{L^\infty} + \sup_{j\in\mathbb{N}} 2^{js}\, \|\Delta_j f_\lambda\|_{L^\infty}\,.$$

For $j, q \in \mathbb{N}$, the Fourier transform of $\Delta_j f_{q,\lambda}$ is supported in the set

$$\left\{|\xi_1| + |\xi'| \approx 2^j \text{ and } \lambda^{-1}|\xi_1| + |\xi'| \approx 2^q\right\},$$

where $\xi' = (\xi_2, \xi_3)$. A direct consideration shows that this set is empty if $2^q \lesssim 2^j$ or $2^{j-q} \lesssim \lambda$. Thus we get for an integer n_1,

$$\|f_\lambda\|_{B^s_{\infty,\infty}} \lesssim \|\Delta_{-1} f_\lambda\|_{L^\infty} + \sum_{\substack{q-n_1+\log_2\lambda\leqslant j\\ j\leqslant q+n_1}} 2^{js}\, \|\Delta_j f_{q,\lambda}\|_{L^\infty}$$
$$\lesssim \|\Delta_{-1} f_\lambda\|_{L^\infty} + \sum_{\substack{q-n_1+\log_2\lambda\leqslant j\\ j\leqslant q+n_1}} 2^{(j-q)s} 2^{qs}\, \|f_q\|_{L^\infty}$$
$$\lesssim \|\Delta_{-1} f_\lambda\|_{L^\infty} + \|f\|_{B^s_{\infty,\infty}} \sum_{-n_1+\log_2\lambda}^{n_1} 2^{js}$$
$$\lesssim \|\Delta_{-1} f_\lambda\|_{L^\infty} + \|f\|_{B^s_{\infty,\infty}}\, \lambda^s.$$

Let us now turn to the estimate for the low frequency $\|\Delta_{-1} f_\lambda\|_{L^\infty}$. We observe that the Fourier transform of $\Delta_{-1} f_{q,\lambda}$ vanishes when $q \geqslant n_0$ where n_0 is an absolute integer.

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

Consequently we get

$$\begin{aligned}\|\Delta_{-1} f_\lambda\|_{L^\infty} &= \sum_{q=-1}^{n_0} \|\Delta_{-1} f_{q,\lambda}\|_{L^\infty}\\ &\leqslant C \sum_{q=-1}^{n_0} \|\Delta_q f\|_{L^\infty}\\ &\leqslant C\,\|f\|_{B^s_{\infty,\infty}}.\end{aligned}$$

This completes the proof of the lemma. □

References

1. H. Abidi, T. Hmidi and S. Keraani, On the global well-posedness for the axisymmetric Euler equations, *Math. Ann.* **347**(1) (2010), 15–41.
2. H. Abidi, T. Hmidi and S. Keraani, On the global regularity of axisymmetric Navier–Stokes–Boussinesq system, *Discrete Contin. Dyn. Syst.* **29**(3) (2011), 737–756.
3. T. Alazard, Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions, *Adv. Differential Equations* **10**(1) (2005), 19–44.
4. S. Alinhac, Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux, *Invent. Math.* **111**(3) (1993), 627–670.
5. K. Asano, On the incompressible limit of the compressible Euler equation, *Japan J. Appl. Math.* **4**(3) (1987), 455–488.
6. H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften, Volume 343. p. xvi+523 pp (Springer, Heidelberg, 2011).
7. J. T. Beale, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equations, *Comm. Math. Phys.* **94** (1984), 61–66.
8. J. Bergh and J. Löfström, *Interpolation spaces. An introduction.* (Springer-Verlag, 1976).
9. J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Ann. de l'Éc. Norm. Supér.* **14** (1981), 209–246.
10. J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, Volume 14. (The Clarendon Press Oxford University Press, New York, 1998), Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
11. Q. Chen, C. Miao and Z. Zhang, Well-posedness in critical spaces for the compressible Navier–Stokes equations with density dependent viscosities, *Rev. Mat. Iberoam.* **26**(3) (2010), 915–946.
12. R. Danchin, Zero Mach number limit in critical spaces for compressible Navier–Stokes equations, *Ann. Sc. Éc. Norm. Supér. (4)* **35**(1) (2002), 27–75.
13. B. Desjardins and E. Grenier, Low Mach number limit of viscous compressible flows in the whole space, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455**(1986) (1999), 2271–2279.
14. B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi, Incompressible limit for solutions of the isentropic Navier–Stokes equations with Dirichlet boundary conditions, *J. Math. Pures Appl. (9)* **78**(5) (1999), 461–471.
15. A. Dutrifoy and T. Hmidi, The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data, *Comm. Pure Appl. Math.* **57**(9) (2004), 1159–1177.

https://doi.org/10.1017/S1474748012000746 Published online by Cambridge University Press

16. I. Gallagher, Résultats récents sur la limite incompressible. Séminaire Bourbaki. Volume 2003/2004. Astérisque No. 299 (2005), Exp. No. 926, vii, 29–57.
17. J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation, *J. Funct. Anal.* **133**(1) (1995), 50–68.
18. M. Grassin, Global smooth solutions to Euler equations for a perfect gas, *Indiana Univ. Math. J.* **47**(4) (1998), 1397–1432.
19. T. Hmidi and F. Rousset, Global well-posedness for the Euler–Boussinesq system with axisymmetric data, *J. Funct. Anal.* **260**(3) (2011), 745–796.
20. S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* **34**(4) (1981), 481–524.
21. S. Klainerman and A. Majda, Compressible and incompressible fluids, *Comm. Pure Appl. Math.* **35**(5) (1982), 629–651.
22. O. A. Ladyzhenskaya, Unique solvability in large of a three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry, *Zap. Nauchn. Sem. LOMI* **7** (1968), 155–177.
23. C.-K. Lin, On the incompressible limit of the slightly compressible viscous fluid flows, in *Nonlinear waves (Sapporo, 1995)*, GAKUTO Internat. Ser. Math. Sci. Appl., Volume 10. pp. 277–282. (Gakkōtosho, Tokyo, 1997).
24. P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid, *J. Math. Pures Appl. (9)* **77**(6) (1998), 585–627.
25. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, Volume 53. (Springer-Verlag, New York, 1984).
26. G. Métivier and S. Schochet, The incompressible limit of the non-isentropic Euler equations, *Arch. Ration. Mech. Anal.* **158**(1) (2001), 61–90.
27. M. A. Rammaha, Formation of singularities in compressible fluids in two-space dimensions, *Proc. Amer. Math. Soc.* **107**(3) (1989), 705–714.
28. T. Shirota and T. Yanagisawa, Note on global existence for axially symmetric solutions of the Euler system, *Proc. Japan Acad. Ser. A Math. Sci.* **70**(10) (1994), 299–304.
29. D. Serre, Solutions classiques globales des équations d'Euler pour un fluide parfait compressible, *Ann. Inst. Fourier (Grenoble)* **47**(1) (1997), 139–153.
30. Thomas C. Sideris, Formation of singularities in three-dimensional compressible fluids, *Comm. Math. Phys.* **101**(4) (1985), 475–485.
31. H. Triebel, *Theory of function spaces*, Monographs in Mathematics, Volume 78. (Birkhäuser Verlag, Basel, 1983).
32. M. R. Ukhovskii and V. I. Yudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, *Prikl. Mat. Mekh.* **32**(1) (1968), 59–69.
33. S. Ukai, The incompressible limit and the initial layer of the compressible Euler equation, *J. Math. Kyoto Univ.* **26**(2) (1086), 323–331.