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OVERPARTITIONS RELATED TO THE MOCK THETA FUNCTION $V_0(q)$

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Abstract

Recently, Brietzke, Silva and Sellers ['Congruences related to an eighth order mock theta function of Gordon and McIntosh', *J. Math. Anal. Appl.* **479** (2019), 62–89] studied the number $v_0(n)$ of overpartitions of n into odd parts without gaps between the nonoverlined parts, whose generating function is related to the mock theta function $V_0(q)$ of order 8. In this paper we first present a short proof of the 3-dissection for the generating function of $v_0(2n)$. Then we establish three congruences for $v_0(n)$ along certain progressions which are subsequences of the integers $4n + 3$.

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1. Introduction

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n. Let $p(n)$ denote the number of partitions of n. Ramanujan established the three congruences, for $n \geq 0$:

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$
$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Ramanujan also discovered the beautiful identity

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}. \tag{1.4}$$

We refer the reader to [5] for elementary proofs of the three congruences (1.1)–(1.3) and the identity (1.4). Inspired by Ramanujan's work on $p(n)$, there are many studies

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of the arithmetic properties of other partition functions whose generating functions are usually eta-quotients.

As introduced by Corteel and Lovejoy in [12], an overpartition of n is a partition of n for which the first occurrence of a number may be overlined. Overpartitions naturally arise in diverse areas of mathematics, including symmetric functions [6], representation theory [17] and algebraic number theory [8, 19]. Let $\overline{p}(n)$ denote the number of overpartitions of n. Since the overlined parts form a partition into distinct parts and the nonoverlined parts form an ordinary partition, the generating function for overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

Arithmetic properties for $\overline{p}(n)$ are studied, for example, in [11, 13, 16]. From the identity

$$\sum_{n=0}^{\infty} \frac{(1+q)(1+q^2)\cdots(1+q^n)}{(1-q)(1-q^2)\cdots(1-q^n)} q^{1+2+\cdots+n} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}$$

due to Lebesgue [18], we see that the number of overpartitions of n without gaps between the nonoverlined parts equals the number of partitions of n with even parts distinct. There are many Ramanujan-type congruences satisfied by this restricted partition function (see, for example, [2]).

Recently, Andrews *et al.* [1] studied the partition functions $p_\omega(n)$ and $p_\nu(n)$ associated to the third-order mock theta functions $\omega(q)$ and $\nu(q)$. Later, congruences for $p_\omega(n)$ and $p_\nu(n)$ were investigated (see [3, 22]). Congruences satisfied by $p_\omega(an+b)$ and $p_\nu(an+b)$ are usually derived from their generating functions and identities between certain mock theta functions.

More recently, Brietzke *et al.* [7] considered overpartitions related to a mock theta function of order 8. Let $v_0(n)$ be the number of overpartitions of n into odd parts without gaps between the nonoverlined parts. Setting $v_0(0) = 1$, it is easy to see that

$$\sum_{n=0}^{\infty} v_0(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{(-q;q^2)_n}{(q;q^2)_n} q^{1+3+\cdots+(2n-1)},$$

where

$$(a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}).$$

Brietzke, Silva and Sellers observed that the generating function of $v_0(n)$ is related to the mock theta function $V_0(q)$ of order 8:

$$\sum_{n=0}^{\infty} v_0(n) q^n = \frac{V_0(q) + 1}{2}. \tag{1.5}$$

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By using the identity (2.4) between $V_0(q)$ and a mock theta function $B(q)$ of order 2 and an identity between $B(q)$ and $B(-q)$, Brietzke *et al.* [7] obtained the generating functions for $v_0(2n)$ and $v_0(4n+1)$ which are eta-quotients. They then established several Ramanujan-type congruences modulo small powers of 2, 3 and 5. Mao [20] established the generating functions of $v_0(8n+2)$ and $v_0(8n+5)$ by using the theory of mock modular forms and Zwegers's results on Appell–Lerch sums. In this paper, we will establish the following Ramanujan-type identities.

THEOREM 1.1. *We have*

$$\sum_{n=0}^{\infty} v_0(6n)q^n = \frac{1}{2}\left(\frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^4(q^{12};q^{12})_\infty}{(q;q)_\infty^8(q^4;q^4)_\infty^4(q^6;q^6)_\infty^4} + 1\right),$$

$$\sum_{n=0}^{\infty} v_0(6n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}{(q;q)_\infty^4(q^{12};q^{12})_\infty}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 3\frac{(q^2;q^2)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^4;q^4)_\infty}. \tag{1.7}$$

REMARK 1.2. Brietzke *et al.* [7] proved (1.6) and (1.7), but our proof is simpler.

The main goal of this paper is to prove congruences for $v_0(n)$ along certain progressions which are subsequences of the integers $4n+3$, including an unexpected congruence modulo 7. It should be noted that the generating function of $v_0(4n+3)$ cannot be written as an eta-quotient which is related to Appell–Lerch sums.

THEOREM 1.3. *For* $n \geq 0$,

$$v_0(12n+7) \equiv 0 \pmod 3,$$
$$v_0(28n+k) \equiv 0 \pmod 7, \quad \textit{where } k = 11, 15, 23.$$

THEOREM 1.4. *For* $n \geq 0$,

$$v_0(40n+35) \equiv 0 \pmod 5.$$

The rest of the paper is organised as follows. In Section 2 we introduce some preliminary results. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorems 1.3 and 1.4.

2. Preliminaries

Recall Ramanujan's theta function $\varphi(q)$ defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

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From [5, Equation (1.3.13)],

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \tag{2.1}$$

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}. \tag{2.2}$$

The following two lemmas are essential to prove Theorem 1.1.

LEMMA 2.1 [15, Equations (34.1.24) and (34.1.25)]. *We have*

$$\begin{aligned} 3\varphi(-q^3)^2 + \varphi(-q)^2 &= 4\frac{(q;q)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}, \\ 3\varphi(-q^3)^2 - \varphi(-q)^2 &= 2\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty}{(q;q)_\infty^2(q^4;q^4)_\infty^3(q^6;q^6)_\infty^3}. \end{aligned} \tag{2.3}$$

LEMMA 2.2 [4, page 49]; [16, Theorem 1]. *We have*

$$\varphi(-q) = \varphi(-q^9) - 2q\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2}{(q^6;q^6)_\infty(q^9;q^9)_\infty},$$

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8(q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6}.$$

Lemmas 2.3 and 2.4 will be useful to establish the generating functions for $v_0(2n+1)$ and $v_0(4n+3)$.

LEMMA 2.3 [21, Equations (3) and (6)]. *Let*

$$B(q) = \sum_{n=0}^{\infty}\frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}$$

denote the mock theta function of order 2*. Then*

$$V_0(q) = (-q^2;q^4)_\infty^4(q^8;q^8)_\infty + 2qB(q^2). \tag{2.4}$$

To obtain the generating function of $v_0(4n+3)$, we need to rewrite $B(q)$ as an Appell–Lerch sum. The Appell–Lerch sum $m(x,q,z)$ is given by

$$m(x,q,z) = -\frac{z}{(z;q)_\infty(q/z;q)_\infty(q;q)_\infty}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{r(r+1)/2}z^r}{1-xzq^r}.$$

LEMMA 2.4 [10, Equation (3.3)]. *We have*

$$-m(1,q^4,q^3) = \frac{1}{4}\frac{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^3}{(q;q)_\infty^4(q^8;q^8)_\infty} - \frac{(q^4;q^4)_\infty}{2(q^8;q^8)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{2n(n+1)}}{1+q^{4n}}.$$

Finally, to finish the proof of Theorem 1.3, we need the following congruences.

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Lemma 2.5 [9, Theorems 7.2 and 7.3]. *Let*

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^n}.$$

Then

$$c(3n+1) \equiv 0 \pmod 3,$$
$$c(7n+k) \equiv 0 \pmod 7, \quad \textit{where } k = 2, 3, 5.$$

3. Proof of Theorem 1.1

From (1.5) and Lemma 2.3,

$$\sum_{n=0}^{\infty} v_0(2n)q^n = \frac{1}{2} + \frac{1}{2}(-q;q^2)_\infty^4 (q^4;q^4)_\infty = \frac{1}{2} + \frac{1}{2}\frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^4 (q^4;q^4)_\infty^3}.$$

Invoking (2.1) and (2.3),

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(2n)q^n &= \frac{1}{2} + \frac{1}{4}\frac{(q^6;q^6)_\infty^3}{(q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty} \cdot \frac{1}{\varphi(-q)} \cdot (3\varphi(-q^3)^2 - \varphi(-q)^2)\\ &= \frac{1}{2} + \frac{1}{4}\frac{(q^6;q^6)_\infty^3}{(q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty} \cdot \Big(3\varphi(-q^3)^2 \sum_{n=0}^{\infty} \overline{p}(n)q^n - \varphi(-q)\Big).\end{aligned}$$

By Lemma 2.2,

$$\sum_{n=0}^{\infty} v_0(6n)q^n = \frac{1}{2} + \frac{1}{4}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty}\Big(3\varphi(-q)^2 \cdot \frac{(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^6}{(q;q)_\infty^8 (q^6;q^6)_\infty^3} - \varphi(-q^3)\Big),$$

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(6n+2)q^n &= \frac{1}{4}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty}\\ &\quad \times \Big(6\varphi(-q)^2 \cdot \frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^3}{(q;q)_\infty^7} + 2\frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}\Big),\end{aligned}$$

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = \frac{1}{4}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty} \cdot 12\varphi(-q)^2 \cdot \frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^3}{(q;q)_\infty^6}$$

and

$$\sum_{n=0}^{\infty} v_0(6n)q^n = \frac{1}{2} + \frac{1}{4}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty} \cdot \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^2}{(q;q)_\infty^4 (q^6;q^6)_\infty} \cdot (3\varphi(-q^3)^2 - \varphi(-q)^2),$$

$$\sum_{n=0}^{\infty} v_0(6n+2)q^n = \frac{1}{4}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty} \cdot 2\frac{(q^2;q^2)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty^3 (q^3;q^3)_\infty} \cdot (3\varphi(-q^3)^2 + \varphi(-q)^2),$$

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 3\frac{(q^2;q^2)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^4;q^4)_\infty}.$$

Applying Lemma 2.1, completes the proof.

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4. Proofs of Theorems 1.3 and 1.4

We first establish the generating function of $\overline{v}_0(4n+3)$.

THEOREM 4.1. *We have*

$$\sum_{n=0}^{\infty} \overline{v}_0(4n+3)q^{n+1} = \frac{(q^2;q^2)_\infty^{17}}{4(q;q)_\infty^8(q^4;q^4)_\infty^8} - \frac{1}{2}\sum_{n=0}^{\infty} c(n)q^{2n}. \tag{4.1}$$

PROOF. From (1.5) and Lemma 2.3,

$$\sum_{n=0}^{\infty} \overline{v}_0(2n+1)q^n = B(q).$$

By [14, Equation (5.2)],

$$B(q) = -q^{-1}m(1,q^4,q^3).$$

Applying Lemma 2.4,

$$\sum_{n=0}^{\infty} \overline{v}_0(2n+1)q^{n+1} = \frac{1}{4}\frac{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^3}{(q;q)_\infty^4(q^8;q^8)_\infty} - \frac{(q^4;q^4)_\infty}{2(q^8;q^8)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{2n(n+1)}}{1+q^{4n}}.$$

Invoking the identity

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}} \tag{4.2}$$

from [7, Lemma 2.2], we see that

$$\sum_{n=0}^{\infty} \overline{v}_0(4n+3)q^{n+1} = \frac{1}{4}\frac{(q;q)_\infty^6(q^2;q^2)_\infty^3}{(q^4;q^4)_\infty}\cdot\frac{(q^2;q^2)_\infty^{14}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^4} - \frac{(q^2;q^2)_\infty}{2(q^4;q^4)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)}}{1+q^{2n}},$$

and the desired result follows. □

THEOREM 4.2. *We have*

$$\sum_{n=0}^{\infty} \overline{v}_0(8n+3)q^n = 2\frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^7}. \tag{4.3}$$

PROOF. Substituting (4.2) into (4.1) and extracting those terms from both sides of (4.1) that involve only the powers q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{v}_0(8n+3)q^{2n+1} = \frac{1}{4}\frac{(q^2;q^2)_\infty^{17}}{(q^4;q^4)_\infty^8}\cdot 8q\frac{(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^{24}},$$

which is equivalent to the desired result. □

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PROOF OF THEOREM 1.3. From Theorem 4.1 and Lemma 2.5, to prove the desired results, it suffices to prove the congruences

$$d(3n+2) \equiv 0 \pmod 3,$$
$$d(7n+k) \equiv 0 \pmod 7, \quad \text{for } k = 3, 4, 6,$$

where

$$\sum_{n=0}^{\infty} d(n)q^n := \frac{(q^2;q^2)_\infty^{17}}{(q;q)_\infty^8 (q^4;q^4)_\infty^8}.$$

It is easy to see that

$$\sum_{n=0}^{\infty} d(n)q^n \equiv \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \cdot \frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^2} \pmod 3$$

and, from (2.2),

$$\sum_{n=0}^{\infty} d(n)q^n \equiv \frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^2} \sum_{n=-\infty}^{\infty} q^{n^2} \pmod 3.$$

Since $n^2 \equiv 0, 1 \pmod 3$, we immediately deduce that $d(3n+2) \equiv 0 \pmod 3$.

Since

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty},$$

it follows that

$$\sum_{n=0}^{\infty} d(n)q^n \equiv (-q;-q)_\infty \cdot \frac{(q^{14};q^{14})_\infty^2}{(q^7;q^7)_\infty (q^{28};q^{28})_\infty} \pmod 7.$$

Applying Euler's pentagonal number theorem [5, Equation (1.3.18)],

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2},$$

and since $n(3n+1)/2 \equiv 0, 1, 2, 5 \pmod 7$, we deduce that

$$d(7n+3) \equiv d(7n+4) \equiv d(7n+6) \equiv 0 \pmod 7.$$

This completes the proof. □

PROOF OF THEOREM 1.4. From (4.3),

$$\sum_{n=0}^{\infty} v_0(8n+3)q^n \equiv 2(q;q)_\infty^3 (q^2;q^2)_\infty^3 \cdot \frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty^2} \pmod 5.$$

By Jacobi's identity [5, page 14],

$$(q;q)_\infty^3 (q^2;q^2)_\infty^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)}.$$

Since $m(m+1)/2$ can only be congruent to $0, 1$ or 3 modulo 5, it follows at once that $m(m+1)/2 + n(n+1) \equiv 4 \pmod 5$ only if $m(m+1)/2 \equiv 3 \pmod 5$, which implies that $m \equiv 2 \pmod 5$ and $2m+1 \equiv 0 \pmod 5$. Thus, the coefficients of q^{5n+4} in $(q;q)_\infty^3 (q^2;q^2)_\infty^3$ are all divisible by 5 and $v_0(40n+35) \equiv 0 \pmod 5$. This completes the proof. □

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