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ANALYTIC REDUCIBILITY OF RESONANT COCYCLES TO A NORMAL FORM

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Abstract We consider systems of quasi-periodic linear differential equations associated to a 'resonant' frequency vector ω, namely, a vector whose coordinates are not linearly independent over $\mathbb{Z}$. We give sufficient conditions that ensure that a small analytic perturbation of a constant system is analytically conjugate to a 'resonant cocycle'. We also apply our results to the non-resonant case: we obtain sufficient conditions for reducibility.

Keywords: normal forms; quasi-periodic cocycles; reducibility; resonances; small divisors

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1. Introduction

Let $\theta = (\theta_1, \ldots, \theta_d)$ be the coordinates on the torus $\mathbb{T}^d$. Let $x = (x_1, \ldots, x_n)$ be coordinates on $\mathbb{R}^n$. Let $\omega \in \mathbb{R}^d$, and let $A_0 = (a_{i,j})_{1\leqslant i,j\leqslant n}$ be an $n \times n$-matrix. Let us consider the following vector field on $\mathbb{T}^d \times \mathbb{R}^n$:

$$D := \omega\frac{\partial}{\partial\theta} + A_0 x\frac{\partial}{\partial x}.$$

Here, the notation $\omega\frac{\partial}{\partial\theta}$ stands for $\sum_{i=1}^{d}\omega_i\frac{\partial}{\partial\theta_i}$, and $A_0 x\frac{\partial}{\partial x}$ stands for the vector field $\sum_{1\leqslant i,j\leqslant n} a_{i,j}x_j\frac{\partial}{\partial x_i}$.

We consider a linear perturbation of D of the form

$$X := D + R = \omega\frac{\partial}{\partial\theta} + A(\theta)x\frac{\partial}{\partial x}, \tag{1}$$

where $A = A_0 + a(\theta)$, a being an analytic matrix-valued function on $\mathbb{T}^d$ with zero mean value: $\hat{a}(0) = 0$. This corresponds to the differential equation

$$\dot\theta = \omega$$
$$\dot x = A(\theta)x,$$

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or, in other words, it corresponds to the linear differential equations with quasi-periodic coefficients

$$\frac{dx}{dt}(t) = A(\theta_0 + t\omega)x(t) \tag{2}$$

for some initial $\theta_0 \in \mathbb{T}^d$. On this vector field, there is a natural action by linear diffeomorphisms over the torus: if $X : \mathbb{T}^d \to gl_n(\mathbb{C})$ is a matrix-valued function with zero mean value, we can consider the (linear) change of variables $Z = (\mathrm{Id}_n + X(\theta))Y$ over the torus, which conjugates (2) to

$$\frac{dx}{dt}(t) = A'(\theta_0 + t\omega)x(t), \tag{3}$$

where A' is the matrix-valued function satisfying

$$DX(\theta)\cdot\omega = A(\theta)(\mathrm{Id}_n + X(\theta)) - (\mathrm{Id}_n + X(\theta))A'(\theta). \tag{4}$$

This can readily be seen by differentiating the quantity

$$Z(\theta_0 + t\omega) = (\mathrm{Id}_n + X(\theta_0 + t\omega))Y(\theta_0 + t\omega)$$

with respect to t. We will denote $DX(\theta)\cdot\omega$ by $\partial_\omega X(\theta)$.

We are interested in the problem of analytic classification with respect to such a group action. A case of particular interest is the one for which the cocycle A can be transformed to a *constant* matrix. In that situation, we say that A is *reducible*. The regularity class of the transformations used is important. This reducibility phenomenon is well known for cocycles over the circle (Floquet theory; see [4, Chapter 26]). When the base is a higher-dimensional torus, the situation is much more complicated. The frequencies $\omega = (\omega_1, \dots, \omega_d)$ are usually assumed to be linearly independent over $\mathbb{Z}$. In that case, they are also assumed to satisfy a 'Diophantine' condition.

Most known results consider the case where the 'fiber' has dimension $n = 2$. In that case, the rotation number [10] of the cocycle can be defined. The rotation number is a very useful quantity, since its arithmetical properties determine whether the cocycle can be reduced or not: in the analytic category, [7] proved reducibility for Schrödinger cocycles with a large energy or a small potential, under a Siegel-type Diophantine condition on the frequency vector and on the rotation number; a similar result was obtained in [6] under weaker arithmetical conditions, namely, Brjuno–Rüssmann conditions, on the frequencies and on the rotation number. Non-perturbative versions of Eliasson's results were also obtained: the article [3] gives a description of Schrödinger cocycles with a recurrent Diophantine frequency vector for almost every energy.

In a higher-dimensional fiber, reducibility results and reducibility criteria were obtained in [11, 12] when the fiber is in a compact Lie group: in particular, reducibility in full measure for a one-parameter family of cocycles under non-degeneracy conditions and the equivalence between reducibility and precompactness of the iterates. The authors of [9] proved a similar result in the non-compact case. Almost reducibility of cocycles in any dimension, that is, the possibility of conjugating them to a cocycle arbitrarily close to a constant, was also proved in a perturbative framework in [8] and improved in [5]. However, positive reducibility results still seem to depend on the existence of a rotation number, or on strong spectral assumptions as in [13].

The previously mentioned articles all require Diophantine or Brjuno conditions on the frequency vector. In [15] and in [2], the case of a Liouvillean frequency vector was considered. However, the results are only stated for a one-dimensional or two-dimensional frequency vector.

In this article, we shall consider analytic cocycles over a torus $\mathbb{T}^d$ with a *fiber of any dimension* n. We consider them as 'small' perturbations of constant cocycles. We consider the general case where the frequencies might be linearly dependent over $\mathbb{Z}$; that is, there are *resonances*. We cannot expect such a general cocycle to be reducible over the torus. We shall give sufficient conditions that ensure that the cocycle is analytically conjugate to an analytic *resonant cocycle*, that is, a cocycle in which Fourier modes $e^{i\langle k,\theta\rangle}$ depend only on the resonances relations of the frequencies, namely, $\langle k, \omega\rangle = 0$. In particular, a resonant cocycle is constant along resonant trajectories of the torus.

Our first result concerns perturbations of a diagonal matrix with separated spectrum, and can be viewed as a generalization of [1, 13].

Theorem 1.1. *Let A_0 be a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ having distinct real parts, and let $C = \min_{j\neq k} \left|\mathrm{Re}\,\lambda_j - \mathrm{Re}\,\lambda_k\right| > 0$. Assume that ω is very weak with exponent $R > 0$ (see Definition 2.8). Let $r > R$, and let a be a C^ω_r matrix-valued function with zero mean value. There exists $\varepsilon_0(r, n, C)$ such that, if $|a|_r \leqslant \varepsilon_0$, then the system whose coefficients are $A_0 + a$ can be analytically conjugated to a system whose coefficient matrix is resonant and diagonal.*

Corollary 1.2. *Under the assumptions of the previous theorem, equation (1) is analytically reducible to a constant cocycle on each level set $\cap_i\{\theta \mid \langle m_i, \theta\rangle = constant_i\}$.*

Corollary 1.3. *Under the assumptions of the previous theorem, and if there are no resonance relations (i.e., $\mathcal{R} = \{0\}$), then the cocycle A is analytically reducible.*

This result is due to Adrianova [1] and to Mitropolskii and Samoilenko [13] when the frequencies satisfy a Siegel-type small divisors condition. In our result, we only require a much weaker condition (even much weaker than the Brjuno–Rüssmann condition).

The second main result considers triangular perturbations of a constant system.

Theorem 1.4. *Let S be a diagonal matrix which satisfies a second Melnikov condition away from the resonances (see Definition 2.9). Let F be a C^ω_r matrix-valued function, with upper-triangular and nilpotent values. There exists $\varepsilon_0(n, d, \omega, S, r)$ such that, if $|F - \hat{F}(0)|_r \leqslant \varepsilon_0$, then the system whose coefficients are given by $S + F$ can be analytically conjugated to a system whose coefficients matrix is resonant, upper triangular, and commuting with S.*

As a consequence, we obtain a decomposition into analytic invariant subbundles of the initial cocycles defined by the unperturbed constant cocycle.

Next, we will consider strongly commuting perturbations, i.e., perturbations which commute with the constant part, and which also are such that their various Fourier modes all commute with each other (see Definition 5.1).

Theorem 1.5. *Assume that the vector ω is very weak with exponent $R > 0$. Let $A_0 \in gl(n, \mathbb{C})$. Let $r > R$, and let $F \in C_r^\omega$ be matrix valued and strongly commuting, and such that all Fourier modes of F commute with A_0. There exists $C(\omega, A_0, r, n, d)$ such that, if $|F|_r \leqslant C$, then the system $A_0 + F$ is reducible to a resonant system $A_\infty(\theta)$ which commutes with A_0.*

Notice that, under this strong algebraic condition on the perturbation, no second Melnikov condition is needed, since most small divisors will be avoided.

In the last section, we shall consider similar statements under assumptions that the cocycles belongs to a Lie subalgebra of matrices. We shall also apply our results to a non-resonant situation to obtain sufficient conditions that ensure analytic reducibility. Our proofs are based on a Newton–KAM scheme whose convergence is due to the control of the small divisors of the initial unperturbed constant cocycle.

2. Notation and assumptions

Definition 2.1. The vector ω is said to be *resonant* if there exists $m_1, \dots m_s \in \mathbb{Z}^d$ linearly independent over $\mathbb{Q}$ such that $\langle m_i, \omega\rangle = 0, i = 1, \dots, s$, and if $\langle k, \omega\rangle = \omega_i$ then $k = E_i + \sum_{j=1}^s l_j m_j$, where the l_j are integers and E_i is the ith unit vector of $\mathbb{Z}^d$. We shall denote by $\mathcal{R}$ the $\mathbb{Z}$-module generated by the m_i. We define an equivalence relation $\sim$ on $\mathbb{Z}^d$ by

$$k_1 \sim k_2 \Leftrightarrow \exists l \in \mathbb{Z}^s, \quad k_1 = k_2 + l \cdot m \Leftrightarrow k_1 - k_2 \in \mathcal{R}.$$

The equivalence class of an integer vector v will be denoted by $\langle v\rangle$.

Remark 2.2. A sum is defined on equivalence classes as follows: $\langle u\rangle + \langle v\rangle = \langle u + v\rangle$; this is well defined, since $u \sim u', v \sim v' \Rightarrow u + v \sim u' + v'$.

Definition 2.3. A function f defined on the torus is said to be *resonant* if its Fourier series has only modes which are proportional to the m_j, i.e., $f = \sum_{l\in\mathbb{Z}^s} f_l e^{i\langle l\cdot m,\theta\rangle}$, where $l \cdot m$ stands for $l_1 m_1 + \cdots + l_s m_s$. In that case, we shall write f_{res} to notify this fact.

Definition 2.4. Let $\mathcal{H}_{\langle v\rangle}$ be the subspace of functions f defined on the torus whose elements only have Fourier modes in $\langle v\rangle$.

Remark 2.5. For all functions f continuous on the torus, there exists a unique decomposition $f = f_{\text{res}} + f_{nr}$ where f_{res} is resonant and f_{nr} does not have any harmonics in the lattice $\mathcal{R}$ generated by the m_j. More generally, there is a natural decomposition of the space of continuous functions on the torus into subspaces $\mathcal{H}_{\langle v\rangle}$: $f = \sum_{\langle v\rangle\in\mathbb{Z}^d/\sim} f_{\langle v\rangle}$. With this notation, f_{res} coincides with $f_{\langle 0\rangle}$. Note that $\mathcal{H}_{\langle v\rangle} \cdot \mathcal{H}_{\langle v'\rangle} \subset \mathcal{H}_{\langle v+v'\rangle}$.

Definition 2.6. For a function f defined on $\mathbb{T}^d$ (be it matrix valued or scalar valued), if the Fourier modes of f belong to a finite number of equivalence classes, then we shall denote by $\deg F$ the *degree* of f, i.e., the smallest integer such that all equivalence classes of Fourier modes of f have a representative $v = (v_1, \dots, v_d)$ with length $|v_1| + \cdots + |v_d| \leqslant \deg F$.

When necessary, for a matrix-valued function F, we will denote by F_{diag} its diagonal part.

We will consider the Banach space C_r^{ω} of (matrix-valued) functions which are analytic on a neighborhood $\{|\mathrm{Im}\,\theta| \leqslant r\}$ of the torus such that

$$|f|_r := \sum_{\langle v\rangle \in \mathbb{Z}^d/\sim} |f_{\langle v\rangle}|_r < +\infty, \quad |f_{\langle v\rangle}|_r := \sup_{|\mathrm{Im}\,\theta| \leqslant r} \|f_{\langle v\rangle}(\theta)\|,$$

where $\|\cdot\|$ denotes the operator norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. We notice that this norm is sub-multiplicative, i.e., $|fg|_r \leqslant |f|_r |g|_r$.

Assumption 2.7. The function A in equation (1) is analytic in a neighborhood of the torus (in fact, in a complex strip around the real d-dimensional plane), so there exists $r > 0$ such that $A \in C_r^{\omega}$.

Definition 2.8. We shall say that the frequency vector ω is *very weak* if there exist $R > 0$, $K > 0$ such that, for all $k \in \mathbb{Z}^d$ with $\langle k, \omega\rangle \neq 0$,

$$|\langle k, \omega\rangle| \geqslant K e^{-R|k|}. \tag{5}$$

If this holds, the number R is simply called the exponent of ω.

This kind of very weak condition also appeared in a different context in [14].

Definition 2.9. We shall say that A_0 is a *Melnikov matrix* or that it satisfies the second Melnikov condition away from the resonances if there exist a constant κ' and a strictly increasing differentiable function $g' : [1, +\infty[\to \mathbb{R}^{*+}$ satisfying

$$\int_1^{\infty} \frac{\ln(g'(t))}{t^2}\, dt < +\infty \tag{6}$$

such that, for all pairs (α, β) of eigenvalues of A_0, and all $m' \in \mathbb{Z}^d$ such that $i\langle m', \omega\rangle - \alpha + \beta \neq 0$,

$$|\alpha - \beta - i\langle m', \omega\rangle| \geqslant \frac{\kappa'}{g'(|m'|)}. \tag{7}$$

We shall also say that A_0 is Melnikov up to order $N \in \mathbb{N}$ if equation (7) holds for all m' with $|m'| \leqslant N$.

Remark 2.10. If A_0 is a Melnikov matrix, then in particular ω satisfies a Brjuno–Rüssmann arithmetical condition away from the resonances, which is weaker than Siegel's Diophantine condition, and was used for instance in [6].

3. Analytic conjugation to a normal form: the separated diagonal case

Let us make an extra assumption.

Assumption 3.1. $A_0 = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ is diagonal with eigenvalues having distinct real parts.

The main result of this section is the following result, which corresponds to Theorem 1.1 above.

Theorem 3.2. *Assume that ω is very weak with exponent $R > 0$. Let $r > R$, and let $A = A_0 + a$ be a C^ω_r-analytic cocycle (1), where a is a zero-mean-valued matrix function. There exists ε_0 depending only on A_0, n such that, if $|a|_r \leqslant \varepsilon_0$, then there exists an analytic transformation conjugating (1) to a normal form in a neighborhood of the torus:*

$$NF = \omega \frac{\partial}{\partial \theta} + D_{\text{res}}(\theta) x \frac{\partial}{\partial x},$$

where D_{res} is a resonant diagonal matrix-valued function.

We shall prove Theorem 3.2 in two steps: first, the system can be diagonalized without any arithmetical assumption on ω; second, such a diagonal system can be conjugated to a resonant one under the very mild arithmetical assumption (5) on ω (it is much weaker than the Brjuno–Rüssmann condition contained in Definition 2.9). The precise statements are as follows.

Proposition 3.1. *Suppose that A_0 has eigenvalues with distinct real parts and that a is a matrix-valued function, analytic on the torus, with zero mean value. Then there exists ε_0 depending only on A_0, n such that, if $|a|_r \leqslant \varepsilon_0$, then there exists an analytic transformation conjugating the system whose coefficient matrix is $A_0 + a$ to a diagonal system.*

Proposition 3.1 will be proved in the following sections. The second step is much easier.

Proposition 3.2. *Let D be a C^ω_r-analytic function on the torus whose values are diagonal matrices; if there exist $r' > 0, C' > 0$ such that, for all $k \in \mathbb{Z}^d$ with $\langle k, \omega \rangle \neq 0$,*

$$|\langle k, \omega \rangle| \geqslant C' e^{-|k|(r-r')},$$

then the system with coefficient matrix D can be $C^\omega_{r'}$-conjugated to a resonant diagonal system.

Proof. Consider the system with analytic coefficients given by D, that is,

$$\begin{cases} \dot{x}_1(\theta, t) = D_{1,1}(\theta + t\omega) x_1(\theta, t) \\ \vdots \\ \dot{x}_n(\theta, t) = D_{n,n}(\theta + t\omega) x_n(\theta, t). \end{cases} \tag{8}$$

Each line can be solved separately and immediately by

$$x_j(\theta, t) = \exp\left(\int_0^t D_{j,j}(\theta + s\omega)\, ds\right) x_j(\theta, 0).$$

By setting $D_{j,j}(\theta) = D_{j,j,\text{res}}(\theta) + D_{j,j,nr}(\theta)$, one has

$$x_j(\theta, t) = \exp(t D_{j,j,\text{res}}(\theta)) \exp\left(\int_0^t D_{j,j,nr}(\theta + s\omega)\, ds\right) x_j(\theta, 0).$$

Indeed, we have

$$D_{j,j,\mathrm{res}}(\theta) = \sum_{l\in\mathbb{Z}^s} D_{j,j,l} e^{i\langle l\cdot m,\theta\rangle},$$

where $l \cdot m$ stands for $l_1 m_1 + \cdots + l_s m_s$. Therefore, $D_{j,j,\mathrm{res}}(\theta + s\omega) = D_{j,j,\mathrm{res}}(\theta)$. One can write explicitly

$$\int_0^t D_{j,j,nr}(\theta + s\omega) = \sum_{k\in\mathbb{Z}^d} \widehat{D_{j,j,nr}}(k) \frac{e^{i\langle k,\theta+t\omega\rangle} - e^{i\langle k,\theta\rangle}}{i\langle k,\omega\rangle}.$$

Letting now $D_{\mathrm{res}}(\theta) = \mathrm{diag}(D_{j,j,\mathrm{res}}(\theta))$ and letting Z be the diagonal-valued function given by

$$Z_{j,j}(\theta) = \exp\left(\sum_{k\in\mathbb{Z}^d,\langle k,\omega\rangle\neq 0} \widehat{D_{j,j}}(k) \frac{e^{i\langle k,\theta\rangle}}{i\langle k,\omega\rangle}\right)$$

gives the desired conjugation:

$$X(\theta,t) = Z(\theta + t\omega) e^{tD_{\mathrm{res}}(\theta)} Z(\theta)^{-1}.$$

The question therefore reduces to finding an arithmetical condition on ω under which Z is analytic. Since D is assumed to be analytic, for every k,

$$|\widehat{D_{j,j}}(k)| \leqslant |D_{j,j}|_r e^{-|k|r};$$

thus Z is analytic whenever there exist $r' > 0$ and $C > 0$ such that, for all $k \in \mathbb{Z}^d$, $\langle k,\omega\rangle \neq 0$,

$$e^{-|k|r} |\langle k,\omega\rangle|^{-1} \leqslant C e^{-|k|r'},$$

that is to say,

$$|\langle k,\omega\rangle| \geqslant C' e^{-|k|(r-r')}.$$

□

3.1. Cohomological equation and iteration process

The proof of Proposition 3.1 is based on the following.

Proposition 3.3. *Let $A : \mathbb{T}^d \to gl(n,\mathbb{C})$ be a diagonal perturbation of A_0. Assume that $\inf_{|\mathrm{Im}\,\theta|\leqslant r} |\mathrm{Re}\, A(\theta)_{j,j} - \mathrm{Re}\, A(\theta)_{k,k}| \geqslant \delta > 0$ for all $v \in \mathbb{Z}^d$ and all $j \neq k$. Let $F : \mathbb{T}^d \to gl(n,\mathbb{C})$ be a C_r^ω-analytic matrix-valued function. There exists a constant C_n only depending on n such that, if*

$$|A - A_0|_r < \frac{\delta}{8C_n},$$

then there exists a solution $X : \mathbb{T}^d \to gl(n,\mathbb{C})$ of the equation

$$\partial_\omega X(\theta) = [A(\theta), X(\theta)] + F(\theta) - F_{\mathrm{diag}}(\theta), \tag{9}$$

where ∂_ω stands for the derivative in the direction ω.

Moreover, X is analytic and belongs to C_r^ω, with the estimate

$$|X|_r \leqslant \frac{4C_n |F|_r}{3\delta - 8C_n |A - A_0|_r} \leqslant \frac{2C_n}{\delta} |F|_r.$$

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Proof. Equation (9) is equivalent to

$$(\mathcal{D}+\mathcal{R})(X)(\theta) = F(\theta) - F_{\text{diag}}(\theta), \tag{10}$$

where $\mathcal{D} : X \mapsto \partial_\omega X - [A_{\text{res}}, X]$ and $\mathcal{R} : X \mapsto -[A_{nr}, X]$. One has the bound $|\mathcal{R}X|_r \leqslant 2|A - A_0|_r |X|_r$. Concerning $\mathcal{D}$, an equation of the form $\mathcal{D}X = G - G_{\text{diag}}$ is equivalent to its decomposition along the $\mathcal{H}_{\langle v\rangle}$: for all $\langle v\rangle \in \mathbb{Z}^d/\sim$ such that $\langle v, \omega\rangle \neq 0$,

$$i\langle v, \omega\rangle X_{\langle v\rangle}(\theta) = [A_{\text{res}}(\theta), X_{\langle v\rangle}(\theta)] + G_{\langle v\rangle}(\theta) - G_{\langle v\rangle,\text{diag}}(\theta). \tag{11}$$

By decomposing by matrix coefficients, we obtain, if $j = k$, that $(G - G_{\text{diag}})_{\langle v\rangle,j,j} = 0$, and we can set $X_{\langle v\rangle}(\theta)_{j,j} := 0$. For $j \neq k$, and all v, θ,

$$X_{\langle v\rangle}(\theta)_{j,k} = \frac{G_{\langle v\rangle}(\theta)_{j,k}}{i\langle v, \omega\rangle - A(\theta)_{j,j,\text{res}} + A(\theta)_{k,k,\text{res}}}.$$

If $\langle v\rangle \in \mathbb{Z}^d/\sim$ is such that $\langle v, \omega\rangle = 0$, then, for $j = k$, we select $X_{\langle v\rangle}(\theta)_{j,j} := 0$. For $j \neq k$, we have

$$X_{\langle v\rangle}(\theta)_{j,k} = \frac{G_{\langle v\rangle}(\theta)_{j,k}}{-A(\theta)_{j,j,\text{res}} + A(\theta)_{k,k,\text{res}}}.$$

Since, for all $j \neq k$ and for all $v \in \mathbb{Z}^d$, we have

$$\inf_{|\operatorname{Im}\theta|\leqslant r} |A(\theta)_{j,j,\text{res}} - A(\theta)_{k,k,\text{res}} - i\langle v, \omega\rangle| \geqslant \inf_{|\operatorname{Im}\theta|\leqslant r} |\operatorname{Re} A(\theta)_{j,j,\text{res}} - \operatorname{Re} A(\theta)_{k,k,\text{res}}|,$$

and since we have

$$A(\theta)_{j,j,\text{res}} = A(\theta)_{j,j} + \big(A(\theta)_{j,j,\text{res}} - A(\theta)_{j,j}\big)$$

and

$$\sup_{|\operatorname{Im}\theta|\leqslant r} |A(\theta)_{j,j,\text{res}} - A(\theta)_{j,j}| \leqslant |A(\theta)_{j,j,\text{res}} - A(\theta)_{j,j}|_r \leqslant C_n |A_0 - A(\theta)|_r,$$

we have

$$\inf_{|\operatorname{Im}\theta|\leqslant r} |A(\theta)_{j,j,\text{res}} - A(\theta)_{k,k,\text{res}} - i\langle v, \omega\rangle| \geqslant \delta - \frac{C_n\delta}{4C_n} = \delta\frac{3}{4}$$

as well as

$$|X_{\langle v\rangle\ j,k}|_r = \sup_{|\operatorname{Im}\theta|\leqslant r} |X_{\langle v\rangle\ j,k}(\theta)| \leqslant \frac{4}{3} \sup_{|\operatorname{Im}\theta|\leqslant r} \frac{|G_{\langle v\rangle,j,k}(\theta)|}{\delta} \leqslant \frac{4}{3} C_n \frac{|G_{\langle v\rangle}|_r}{\delta}.$$

Therefore, summing in $\langle v\rangle$ and by definition of the analytic weighted norm,

$$|X_{j,k}|_r \leqslant \frac{4}{3} C_n \frac{|G|_r}{\delta}.$$

Therefore, the operator $\mathcal{D}$ is invertible, and one has the bound $|\mathcal{D}^{-1}|_{C_r^\omega \to C_r^\omega} \leqslant \frac{4C_n}{3\delta}$. Therefore, if $|A - A_0|_r < \frac{\delta}{8C_n}$, then

$$\begin{aligned}|(\mathcal{D}+\mathcal{R})^{-1}|_r &\leqslant |\mathcal{D}^{-1}|_r |(\mathrm{Id} + \mathcal{D}^{-1}\mathcal{R})^{-1}|_r \\ &\leqslant \frac{4C_n}{3\delta} \sum_{j\geqslant 0} \left(\frac{4C_n}{3\delta} 2|A - A_0|_r\right)^k \\ &\leqslant \frac{4C_n}{3\delta - 8C_n|A - A_0|_r} \leqslant \frac{2C_n}{\delta},\end{aligned}$$

and thus equation (9) has a solution X satisfying $|X|_r \leqslant \frac{2C_n}{\delta}|F|_r$. □

Proposition 3.4 (Induction argument). *Let $A : \mathbb{T}^d \to gl(n, \mathbb{C})$ be a diagonal analytic perturbation of A_0. Let δ and $\varepsilon < \frac{\delta}{4C_n}$ be positive numbers. Let $F : \mathbb{T}^d \to gl(n, \mathbb{C})$ belong to C_r^ω with*

$$|F|_r \leqslant \varepsilon. \tag{12}$$

Let C_n be the constant defined in the assumptions of Proposition 3.3. Assume that A satisfies

$$\inf_{|\mathrm{Im}\,\theta| \leqslant r} |\mathrm{Re}\, A(\theta)_{j,j} - \mathrm{Re}\, A(\theta)_{k,k}| \geqslant \delta, \tag{13}$$

for all $v \in \mathbb{Z}^d$, and that

$$|A - A_0|_r \leqslant \frac{\varepsilon}{2} \leqslant \frac{\delta}{8C_n}. \tag{14}$$

There exists $A' : \mathbb{T}^d \to gl(n, \mathbb{C})$, a diagonal C_r^ω-analytic perturbation of A_0, there exists $F' : \mathbb{T}^d \to gl(n, \mathbb{C})$, which belongs to C_r^ω, and there exists $Z : \mathbb{T}^d \to GL(n, \mathbb{C})$, which belongs to C_r^ω, such that, for all $\theta \in \mathbb{T}^d$,

$$\partial_\omega Z(\theta) = (A(\theta) + F(\theta))Z(\theta) - Z(\theta)(A'(\theta) + F'(\theta))$$

and

$$|F'|_r \leqslant \frac{8C_n}{\delta} |F|_r^2.$$

Moreover,

$$|Z - I|_r \leqslant \frac{2C_n}{\delta} |F|_r, \tag{15}$$

and A' satisfies

$$\inf_{|\mathrm{Im}\,\theta| \leqslant r} |\mathrm{Re}\, A'(\theta)_{j,j} - \mathrm{Re}\, A'(\theta)_{k,k}| \geqslant \delta - 2C_n\varepsilon \tag{16}$$

for all $v \in \mathbb{Z}^d$, as well as

$$|A' - A_0|_r \leqslant \frac{\varepsilon}{2} + \varepsilon. \tag{17}$$

Proof. Let X be the solution of $\partial_\omega X = [A, X] + F - F_{\mathrm{diag}}$ given by Proposition 3.3, and let us set $Z := I + X$. Then, according to (4), we have

$$\partial_\omega Z(\theta) = (A(\theta) + F(\theta))Z(\theta) - Z(\theta)(A'(\theta) + F'(\theta)),$$

where

$$\begin{aligned} A' &:= A + F_{\mathrm{diag}} \\ F' &:= (I + X)^{-1} F X + \sum_{l \geqslant 1} (-X)^l F_{\mathrm{diag}}. \end{aligned}$$

By definition, A' is diagonal, and the estimate (17) comes directly from (14). According to (13), applying Proposition 3.3, we obtain

$$|X|_r \leqslant \frac{2C_n}{\delta} |F|_r \leqslant \frac{1}{2},$$

where C_n only depends on n (which also gives the estimate (15)).

Thus we have

$$|F'|_r \leqslant |(I+X)^{-1}(FX - XF_{\rm diag})|_r$$
$$\leqslant \left(\sum_{l\geqslant 0}|X|_r^l\right)\frac{4C_n}{\delta}|F|_r^2 \tag{18}$$
$$\leqslant \frac{8C_n}{\delta}|F|_r^2. \tag{19}$$

Moreover, for any scalar $f\in C_r^\omega$, we have $\sup_{|\mathrm{Im}\,\theta|\leqslant r}|f(\theta)|\leqslant |f|_r$. As a consequence, we have, for $|\mathrm{Im}\,\theta|\leqslant r$, and for all $j\neq k$,

$$|\mathrm{Re}\,A'(\theta)_{j,j}-\mathrm{Re}\,A'(\theta)_{k,k}| \geqslant (|\mathrm{Re}\,A(\theta)_{j,j}-\mathrm{Re}\,A(\theta)_{k,k}|) - (|\mathrm{Re}(F(\theta))_{j,j}|+|\mathrm{Re}(F(\theta))_{k,k}|)$$
$$\geqslant (|\mathrm{Re}\,A(\theta)_{j,j}-\mathrm{Re}\,A(\theta)_{k,k}|) - (|(F)_{j,j}|_r+|(F)_{k,k}|_r).$$

Hence, we have

$$\inf_{|\mathrm{Im}\,\theta|\leqslant r}|\mathrm{Re}\,A'(\theta)_{j,j}-\mathrm{Re}\,A'(\theta)_{k,k}|\geqslant \delta-2C_n\varepsilon.$$ □

3.2. Proof of Proposition 3.1

We prove this by iterating Proposition 3.4 and constructing a sequence of changes of variables, all C_r^ω-analytic, conjugating the system A_0+a to something which is arbitrarily close to the system given in the statement.

Let C_n be the constant defined in Proposition 3.3. Let $\delta=\delta_0$ and $\varepsilon=\varepsilon_0\leqslant 1/2$ be such that

$$\varepsilon_0\leqslant\varepsilon_0^{1/2}\leqslant\frac{\delta_0}{8C_n}$$

and

$$\min_{j\neq k}|\mathrm{Re}\,A_{0,j,j}-\mathrm{Re}\,A_{0,k,k}|\geqslant\delta_0.$$

Let us set $\varepsilon_k:=\varepsilon_0^{(3/2)^k}$ as well as

$$\delta_k:=\delta_{k-1}-2C_n\varepsilon_{k-1}=\delta_0-2C_n\sum_{j=0}^{k-1}\varepsilon_j,\quad k\geqslant 1.$$

Notice that, if ε is small enough, we have $\delta_k>16C_n\varepsilon_0$, for all $k\geqslant 0$. Indeed, for $k\geqslant 1$, we have $\varepsilon_0^{(3/2)^k}\leqslant\varepsilon_0^k$, so

$$\sum_{j\geqslant 0}\varepsilon_j\leqslant\varepsilon_0+\frac{\varepsilon_0}{1-\varepsilon_0}\leqslant 3\varepsilon_0.$$

Therefore, we have, if ε_0 is small enough,

$$\delta_k\geqslant 2C_n(4\varepsilon_0^{1/2}-3\varepsilon_0)\geqslant 16C_n\varepsilon_0.$$

Assumptions 2.7 and 3.1 make it possible to apply Proposition 3.4 with $A=A_0$ and $F=a$: if $|a|_r\leqslant\varepsilon_0$, we obtain a conjugation Z_1 to a new vector field D_1+R_1

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with $D_1 = \omega\frac{\partial}{\partial\theta} + A_1(\theta)x\frac{\partial}{\partial x}$, where $A_1(\theta)$ is diagonal with $\delta_1 = \delta_0 - 2C_n\varepsilon_0$-separated spectrum, and R_1 is C^ω_r-analytic on the torus with

$$|R_1|_r \leqslant 8\frac{C_n}{\delta_0}|a|_r^2 \leqslant \varepsilon_0^{3/2} = \varepsilon_1$$

and

$$|A_1 - A_0|_r \leqslant \frac{3}{2}\varepsilon_0 \leqslant \frac{\delta_1}{2C_n}.$$

The change of variable Z_1 is itself C^ω_r-analytic on the torus, and $\frac{\varepsilon_0^{1/2}}{4}$-close to the identity, since

$$\frac{2C_n}{\delta_0}|a|_r \leqslant \frac{\varepsilon_0^{1/2}}{4}.$$

Now suppose that we have a change of variables Z_k which conjugates the vector field $D + R$ given by (1) to $D_k + R_k$, where $D_k(\theta) = \omega\frac{\partial}{\partial\theta} + A_k(\theta)x\frac{\partial}{\partial x}$ and A_k is diagonal with δ_k-separated spectrum, with the estimates

$$|R_k|_r \leqslant \varepsilon_k,$$

$$|A_k - A_0|_r \leqslant \sum_{j=0}^{k-1}\varepsilon_j < 2\varepsilon_0 \leqslant \frac{\delta_k}{8C_n}$$

and

$$|\tilde{Z}_k - \mathrm{Id}|_r \leqslant 2C\sum_{j=0}^{k-1}\varepsilon_j^{1/2}.$$

We remark that, if ε_0 is small enough, then, for all $k \geqslant 1$, we have $\varepsilon_k^{1/2} < \frac{\delta_k}{8C_n}$. Indeed, we have

$$\frac{\delta_k}{8C_n} \geqslant 2C_n\varepsilon_0 \geqslant \varepsilon_0^{(1/2)(3/2)^k}.$$

Thus, one can apply again Proposition 3.4 to conjugate $D_k + R_k$ by some change of variable Z_k such that

$$|Z_k - \mathrm{Id}|_r \leqslant \frac{2C_n}{\delta_k}\varepsilon_k \leqslant \varepsilon_k^{1/2}$$

to a vector field $D_{k+1} + R_{k+1}$ with analogous properties:

$$D_{k+1}(\theta) = \omega\frac{\partial}{\partial\theta} + A_{k+1}(\theta)x\frac{\partial}{\partial x},$$

where A_{k+1} is diagonal, with δ_{k+1}-separated spectrum,

$$|R_{k+1}|_r \leqslant \frac{8C_n}{\delta_k}\varepsilon_k^2 \leqslant \varepsilon_{k+1},$$

$$|A_{k+1} - A_0|_r \leqslant |A_k - A_0|_r + \varepsilon_k \leqslant \sum_{j=0}^{k}\varepsilon_j \leqslant \frac{\delta_{k+1}}{8C_n}$$

so $D + R$ is conjugated to $D_{k+1} + R_{k+1}$ by $\tilde{Z}_{k+1} = Z_k\tilde{Z}_k$.

In C^ω_r, R_k tends to 0 and $\{D_k\}_k$ is a Cauchy sequence; thus it converges to a limit D_∞ which is diagonal and analytic. Since $\tilde{Z}_k$ is C^ω_r-analytic for all k and defines a Cauchy sequence, it has an analytic limit Z_∞ conjugating $D + R$ to D_∞. □

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4. Analytic reduction of an upper-triangular perturbation

In this section, we assume that $A_0 = S + M$, where $S = \mathrm{diag}(\lambda_j)$ is diagonal and M is upper triangular and commutes with S. This assumption is not restrictive, since A_0 can be conjugated from the start to its Jordan normal form. In § 6, we will see how to preserve real structures while still using the Jordan normal form.

The main result of this section corresponds to Theorem 1.4 above.

Theorem 4.1. *Let $A_0 = S + M \in gl(n, \mathbb{C})$, where $S = \mathrm{diag}(\lambda_j)$ is diagonal, M is nilpotent and upper triangular, and $[M, S] = 0$. Let $F \in C^\omega_r$ be upper-triangular valued. Suppose that S is Melnikov (see Definition 2.9). There exists $\varepsilon_0(n, d, \kappa', g', A_0, r)$ such that, if $|F|_r \leqslant \varepsilon_0$, then there exist $r_\infty > 0$ and $Z_\infty \in C^\omega_{r_\infty}$, and an upper-triangular resonant $R_\infty \in C^\omega_{r_\infty}$ such that $\partial_\omega Z_\infty = (A_0 + F)Z_\infty - Z_\infty R_\infty$.*

Remark 4.2. If A_0 is not in Jordan normal form, the assumption on F is that it is conjugate to an upper-triangular-valued function by the conjugation which takes A_0 to its Jordan normal form.

4.1. Cohomological equation and iteration process

Definition 4.3. Let $l \leqslant n$, and let $\mathcal{F}_l$ be the set of $n \times n$-matrices $M = (m_{i,j})_{1\leqslant i,j\leqslant n}$ such that $m_{i,j} = 0$ if $j < i + l$. We have $\mathcal{F}_n = \{0\}$.

Let us start with an elementary lemma.

Lemma 4.4. *We have $[\mathcal{F}_l, \mathcal{F}_p] \subset \mathcal{F}_{p+l}$.*

Proof. Let $M \in \mathcal{F}_l$ and $N \in \mathcal{F}_p$. Then, we have

$$[N, M]_{i,j} := \sum_{k=1}^{n} n_{i,k} m_{k,j} - \sum_{k=1}^{n} m_{i,k} n_{k,j} = \sum_{k=i+l}^{n} n_{i,k} m_{k,j} - \sum_{k=i+p}^{n} m_{i,k} n_{k,j}.$$

In the first sum, $m_{k,j} = 0$ if $j < k + p$, so the sum is zero if $j < (i + l) + p \leqslant k + p$. By the same argument, the second sum is zero for $j < (i + p) + l \leqslant k + l$. □

We will now give a lemma (appearing in [8] for the non-resonant case) which shows that, up to a simple analytic change of variables, one can assume that $S = \mathrm{diag}(\lambda_j)$ satisfies

$$i\langle v, \omega\rangle - \lambda_k + \lambda_l = 0 \Rightarrow i\langle v, \omega\rangle = 0.$$

Lemma 4.5. *Let $S = \mathrm{diag}(\lambda_j)$ be a diagonal matrix, and let N commute with S. There exists an analytic change of variables Φ and a diagonal matrix $\tilde{S}$ commuting with N such that*

$$\partial_\omega \Phi(\theta) = (S + N)\Phi(\theta) - \Phi(\theta)(\tilde{S} + N)$$

and every pair $(\tilde{\lambda}_j, \tilde{\lambda}_k)$ of eigenvalues of $\tilde{S}$ satisfies

$$i\langle v, \omega\rangle - \tilde{\lambda}_k + \tilde{\lambda}_l = 0 \Rightarrow i\langle v, \omega\rangle = 0. \tag{20}$$

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Proof. Suppose that there exist indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ such that $\lambda_{i_1} = \cdots = \lambda_{i_r}$, $\lambda_{j_1} = \cdots = \lambda_{j_s}$ and, for some v such that $\langle v, \omega \rangle \neq 0$, one has $\lambda_{i_1} - \lambda_{j_1} - i\langle v, \omega \rangle = 0$. Let $\Phi^1(\theta)$ be the diagonal matrix with diagonal coefficients $(\Phi_1^1(\theta), \ldots, \Phi_n^1(\theta))$ such that $\Phi_{i_1}^1(\theta) = \cdots = \Phi_{i_r}^1(\theta) = e^{i\langle v, \theta \rangle}$ and all other diagonal coefficients are equal to one. Then

$$\partial_\omega \Phi^1(\theta) = D \cdot \Phi^1(\theta),$$

where D is the diagonal matrix with coefficients $D_{i_1,i_1} = \cdots = D_{i_r,i_r} = i\langle v, \omega \rangle$ and all other coefficients equal to 0. By construction, Φ^1 commutes with $S + N$ and so does D; therefore

$$\partial_\omega \Phi^1(\theta) = (S + N)\Phi^1(\theta) - \Phi^1(\theta)(S - D + N),$$

and the coefficients (λ_j^1) of $S - D$ are the same as those of S, except that $\lambda_{i_1}, \ldots, \lambda_{i_r}$ are shifted to $\lambda_{i_1}^1 = \lambda_{i_1} - i\langle v, \omega \rangle, \ldots, \lambda_{i_r}^1 = \lambda_{i_r} - i\langle v, \omega \rangle$, so $\lambda_{i_1}^1 = \cdots = \lambda_{i_r}^1 = \lambda_{j_1}^1 = \cdots = \lambda_{j_s}^1$. Therefore, the change of variables has merged two groups of identical coefficients into one.

In the new matrix $S - D$, if another resonance appears, one can perform a similar change of variables in order to merge the two groups of coefficients which are in resonance with each other. Thus, in a finite number of steps (at most $n - 1$), a new diagonal matrix $\tilde{S}$ is obtained, which commutes with N, and in which all pairs of coefficients are either identical to or non-resonant with each other. □

From now on, we will assume that the eigenvalues of S satisfy (20). In the following statement, which gives the solution of the cohomological equation, recall that $T^N F$ stands for a truncation that preserves the equivalence classes of Fourier modes.

Proposition 4.6. *Let $S = \mathrm{diag}(\lambda_j)$ be a constant Melnikov diagonal matrix. Let $R(\theta) = S + M(\theta)$, where $M(\theta)$ is resonant and upper triangular. Assume that, for all $1 \leqslant k, l \leqslant n$ and for all $v \in \mathbb{Z}^d$,*

$$i\langle v, \omega \rangle - \lambda_k + \lambda_l = 0 \Rightarrow i\langle v, \omega \rangle = 0$$

and

$$|M|_r \leqslant 2\|A_0\| + 2\varepsilon_0,$$

where $\varepsilon_0 = \frac{\kappa'}{2n\|A_0\|^n}$. Then, for every $N \geqslant 1$, and if $F \in C_r^\omega$ is upper triangular, there is an upper-triangular $X \in C_r^\omega$ such that

$$\partial_\omega X = [R, X] + T^N F_{nr} \tag{21}$$

and

$$|X|_r \leqslant \frac{C''(n)(\|A_0\| + 1)}{\kappa'^{n+1}} N^d g'(N)^{n+1} |F|_r,$$

where $C''(n)$ only depends on n. Moreover, if $T^N F_{nr}$ is upper triangular, then so is X.

Proof. Developing (21) into the subspaces $\mathcal{H}_{\langle v \rangle}$ yields

$$(\mathcal{L}_{\langle v \rangle} + \mathcal{N}(\theta)) X_{\langle v \rangle}(\theta) = T^N F_{\langle v \rangle}(\theta),$$

where $\mathcal{L}_{\langle v \rangle} : gl(n, \mathbb{C}) \to gl(n, \mathbb{C}), X \mapsto i\langle v, \omega \rangle X - [S, X]$ and $\mathcal{N}(\theta) : X \mapsto -[M(\theta), X]$. Since M is upper triangular, $M \in \mathcal{F}_1$. According to the previous lemma, $\mathcal{N}(\mathcal{F}_k) \subset \mathcal{F}_{k+1}$. Hence, the operator $\mathcal{N}$ is nilpotent.

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If $i\langle v,\omega\rangle-(\lambda_k-\lambda_l)\neq 0$ for all $1\leqslant k,l\leqslant n$, then $\mathcal{L}_{\langle v\rangle}$ is invertible, and this equation amounts to

$$(\mathrm{Id}+\mathcal{L}_{\langle v\rangle}^{-1}\mathcal{N}(\theta))X_{\langle v\rangle}=\mathcal{L}_{\langle v\rangle}^{-1}T^N F_{\langle v\rangle}(\theta). \tag{22}$$

One has the estimate

$$\|\mathcal{L}_{\langle v\rangle}^{-1}\|\leqslant\frac{C(n)}{\min_{j,k}|i\langle v,\omega\rangle+\lambda_j-\lambda_k|}\leqslant\frac{C(n)g'(|v|)}{\kappa'}.$$

Each $\mathcal{F}_k$ is left invariant by $\mathcal{L}_{\langle v\rangle}$. Therefore, $\mathcal{L}_{\langle v\rangle}^{-1}\mathcal{N}(\mathcal{F}_k)\subset\mathcal{F}_{k+1}$. Hence, the restriction to $\mathcal{F}_k$ of $\mathcal{L}_{\langle v\rangle}^{-1}\mathcal{N}$ is nilpotent. Hence, if $T^N F_{\langle v\rangle}(\theta)$ belongs to $\mathcal{F}_k$, so does $X_{\langle v\rangle}$, and there is a $p\leqslant n$ such that $(I+\mathcal{L}_{\langle v\rangle}^{-1}\mathcal{N}(\theta))^{-1}=\sum_{l=0}^{p}(\mathcal{L}_{\langle v\rangle}^{-1}\mathcal{N}(\theta))^l$. Therefore,

$$|X_{\langle v\rangle}|_r\leqslant\sum_{l=0}^{n}\|\mathcal{L}_{\langle v\rangle}^{-1}\|^{l+1}|\mathcal{N}|_r^l|T^N F_{\langle v\rangle}|_r,$$

where $|\mathcal{N}|_r=\sup_{X\in C_r^\omega}\frac{|\mathcal{N}X|_r}{|X|_r}$ (so $|\mathcal{N}|_r\leqslant 2n|M|_r$: recall that $|M|_r=\sup_\theta|M(\theta)|$, since M is resonant). Let us set $X_{\mathrm{res}}:=0$ and $X:=X_{\mathrm{res}}+\sum_{\langle v\rangle/\langle v,\omega\rangle\neq 0}X_{\langle v\rangle}$; then X is a solution of (21), and

$$\begin{aligned}|X|_r&\leqslant C(n)\sum_{\langle v\rangle}\sum_{l=0}^{n}\left(\frac{\kappa'}{g'(|v|)}\right)^{-(l+1)}|M|_r^l|T^N F_{\langle v\rangle}|_r\\&\leqslant C'(n)|M|_r^n\frac{N^d g'(N)^{n+1}}{\kappa'^{n+1}}|F|_r\\&\leqslant\frac{C''(n)(\|A_0\|+1)}{\kappa'^{n+1}}N^d g'(N)^{n+1}|F|_r.\end{aligned} \tag{23}$$

□

Proposition 4.7. *Let S be a constant Melnikov diagonal matrix. Let $R(\theta)=S+M(\theta)$, where $M(\theta)$ is resonant and upper triangular. Let $\varepsilon>0$, and let $F\in C_r^\omega$ be upper triangular with $|F|_r\leqslant\varepsilon$. Let $r'\in(0,r)$ be defined by*

$$\frac{|\ln\varepsilon|}{r-r'}=g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon}}\right), \tag{24}$$

where $g''(x)=x^d g'(x)^{n+1}$ and $C''(n)$ is given by Proposition 4.6. Suppose also that

$$\varepsilon\leqslant\frac{1}{4C_N^2}(r-r')^{2d+2}, \tag{25}$$

where $C_N=N^d+dN^{d-1}+\cdots+d!$. Then there exists $Z\in C_r^\omega$ such that

$$\partial_\omega Z=(R+F)Z-Z(R'+F'),$$

where F' is upper triangular and $R'(\theta)=S+M'(\theta)$ with M' resonant and upper triangular. We have $|F'|_{r'}\leqslant\varepsilon^{3/2}$, and $|Z-I|_r\leqslant\sqrt{\varepsilon}$.

Proof. Let N be such that $\frac{1}{2}\varepsilon^{-1/2} = \frac{C''(n)(\|A_0\|+1)}{\kappa'^{n+1}} N^d g'(N)^{n+1}$. By assumption (24) on r', $e^{-N(r-r')} = \varepsilon$. By Proposition 4.6, there exists $X \in C_r^\omega$ such that $\partial_\omega X = [R, X] + T^N F_{nr}$ and $|X|_r < 1$. Let $Z = I + X$. Then

$$\partial_\omega Z = [R, Z] + T^N F_{nr} = (R+F)Z - Z(R + Z^{-1}FZ - Z^{-1}T^N F_{nr})$$

(note that Z is invertible, since $|X|_r < 1$). Let $R' = R + T^N F_{\text{res}}$. Then

$$R + Z^{-1}FZ - Z^{-1}T^N F_{nr} = R' + Z^{-1}FZ - Z^{-1}T^N F_{nr} - T^N F_{\text{res}}.$$

Let $F' = Z^{-1}FZ - Z^{-1}T^N F_{nr} - T^N F_{\text{res}}$; thus

$$\begin{aligned} F' &= \sum_{l\geqslant 0}(-X)^l F(I+X) - \sum_{l\geqslant 0}(-X)^l T^N F_{nr} - T^N F_{\text{res}} \\ &= F - T^N F + \sum_{l\geqslant 1}(-X)^l F(I+X) - \sum_{l\geqslant 1}(-X)^l T^N F_{nr} + FX. \end{aligned} \tag{26}$$

Now

$$|F - T^N F|_{r'} \leqslant \sum_{|k|>N} \|\hat{F}(k)\| e^{|k|r'} \leqslant \sum_{|k|>N} |F|_r e^{-|k|(r-r')} \leqslant \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')}$$

(where $C_N = N^d + dN^{d-1} + \cdots + d!$), and thus

$$\begin{aligned} |F'|_{r'} &\leqslant \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')} + 6|F|_r |X|_r \\ &\leqslant \frac{C_N |F|_r}{(r-r')^{d+1}} e^{-N(r-r')} + \frac{C''(n)(\|A_0\|+1)}{\kappa'^{n+1}} N^d g'(N)^{n+1} |F|_r^2 \\ &\leqslant \frac{C_N |F|_r \varepsilon}{(r-r')^{d+1}} + \frac{1}{2}|F|_r^{3/2} \end{aligned} \tag{27}$$

(the last inequality holds because of the choice of the parameter N). By assumption (25), one has $|F'|_{r'} \leqslant \varepsilon^{3/2}$.

Moreover, if F is upper triangular, then so is X; thus F' is as well, since it is a product of upper-triangular matrices. As required, $R'(\theta) = S + M'(\theta)$, where $M'(\theta) = M(\theta) + F_{\text{res}}^N(\theta)$ is resonant and upper triangular. □

We need the following (trivial) lemma.

Lemma 4.8. *Let $g''(x) = x^d g'(x)^{n+1}$. If g'' satisfies*

$$\frac{|\ln \varepsilon|}{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon}}\right)} \geqslant (4C^2\varepsilon)^{\frac{1}{2d+2}}, \tag{28}$$

and if $r' \in (0, r)$ is defined by

$$\frac{|\ln \varepsilon|}{r - r'} = g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon}}\right), \tag{29}$$

then

$$\varepsilon \leqslant \frac{1}{4C^2}(r-r')^{2d+2}. \tag{30}$$

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The next lemma shows that assumption (28) is indeed not a restriction.

Lemma 4.9. *Let $C > 0$. Let g' be the approximation function given in Definition 2.9. Let $\alpha \in (0,1)$, and, for all $\varepsilon > 0$, let $\phi(\varepsilon) = \frac{-C\ln\varepsilon}{\varepsilon^\alpha}$. For all $t \in [1,+\infty[$, let $\tilde{g}(t) = \max(g'(t), \frac{1}{\phi^{-1}(t)^{1/4}})$. Then:*

(a) *for all $k \in \mathbb{Z}^d$ such that $\langle k,\omega\rangle \neq 0$, $|\langle k,\omega\rangle| \geqslant \frac{\kappa}{\tilde{g}(|k|)}$;*

(b) *$\int_1^\infty \frac{\ln\tilde{g}(t)}{t^2}\,dt < +\infty$;*

(c) *for all $\varepsilon \in (0,1/e)$, $\varepsilon^{1/4}\tilde{g}\left(\frac{C|\ln\varepsilon|}{\varepsilon^\alpha}\right) \geqslant 1$.*

Proof. Since, for all t, $\tilde{g}(t) \geqslant g(t)$, property (a) follows (6). Property (c) holds by definition of $\tilde{g}$, since $\tilde{g}(t) \geqslant \frac{1}{\phi^{-1}(t)^{1/4}}$ for all $t \geqslant 1$, and in particular for $t = \phi(\varepsilon)$, $\varepsilon \in (0,1/e)$. The property (b) comes from the following computation:

$$\begin{aligned}
\int_1^\infty \frac{\ln\tilde{g}(t)}{t^2}\,dt &\leqslant \int_1^\infty \frac{\ln g'(t)}{t^2}\,dt + \int_1^\infty -\frac{1}{4}\frac{\ln\phi^{-1}(t)}{t^2}\,dt \\
&\leqslant \int_1^\infty \frac{\ln g'(t)}{t^2}\,dt - \frac{1}{4}\int_{\phi^{-1}(1)}^0 \frac{\ln\varepsilon}{\phi(\varepsilon)^2}\phi'(\varepsilon)\,d\varepsilon \\
&\leqslant \int_1^\infty \frac{\ln g'(t)}{t^2}\,dt - \frac{1}{4}\int_{\phi^{-1}(1)}^0 \frac{\varepsilon^{2\alpha}\ln\varepsilon}{(C\ln\varepsilon)^2}\left(\frac{-C\varepsilon^{\alpha-1} + C\ln\varepsilon\,\alpha\varepsilon^{\alpha-1}}{\varepsilon^{2\alpha}}\right)d\varepsilon \\
&\leqslant \int_1^\infty \frac{\ln g'(t)}{t^2}\,dt - \frac{1}{4}\int_{\phi^{-1}(1)}^0 \frac{-\varepsilon^{\alpha-1}}{C\ln\varepsilon} + \frac{\alpha\varepsilon^{\alpha-1}}{C}\,d\varepsilon.
\end{aligned}$$

The second term is finite since $1-\alpha \in (0,1)$, and the first is finite by (6). □

The Brjuno–Rüssmann assumption on ω gives a control on the loss of analyticity.

Lemma 4.10. *There exists $\varepsilon_0 > 0$ which depends only on n, d, r, κ', g' such that, if, for all $k \in \mathbb{N}$, $\varepsilon_k = \varepsilon_0^{(3/2)^k}$ and $r_{k+1} = r_k - \frac{|\ln\varepsilon_k|}{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}\right)}$ with $r_0 = r$, then (r_k) has a positive limit r_∞.*

Proof. One has

$$r - \sum_{k\geqslant 1}(r_k - r_{k+1}) = r - \sum_k \frac{|\ln\varepsilon_k|}{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}\right)},$$

and by the change of variables $g''(Y) = \frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}$,

$$r - \sum_{k\geqslant 1}(r_k - r_{k+1}) \geqslant r - \int_{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_0}}\right)}^\infty \frac{C\ln g''(Y)}{Y}\,dY,$$

where $C = \frac{2\sqrt{\varepsilon_0}}{|\ln\varepsilon_0|}$. Therefore $r_\infty > 0$ if ε_0 is small enough as a function of r, g'', κ', n, A_0. □

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4.2. Proof of Theorem 4.1

Let us define the following sequences: for all $k \in \mathbb{N}$,

$$\varepsilon_k = \varepsilon_0^{(3/2)^k}$$
$$r_{k+1} = r_k - \frac{|\ln \varepsilon_k|}{g''^{-1}\left(\frac{\kappa'^{n+1}}{C''(n)(\|A_0\|+1)\sqrt{\varepsilon_k}}\right)}$$

(by Lemma 4.10, r_k has a positive limit). Let N_k be defined by the relation

$$\frac{1}{2}\varepsilon_k^{-1/2} = \frac{C''(n)(\|A_0\|+1)}{\kappa'^{n+1}} N_k^d g'(N_k)^{n+1}.$$

Assumption (28) of Lemma 4.8 is satisfied; thus one can apply Proposition 4.7. Applying Proposition 4.7 with R being the constant map equal to $S+M$, one obtains $Z \in C^\omega_{r_1}$ conjugating $R+F$ to R_1+F_1, where

- $|F_1|_{r_1} \leqslant \varepsilon_1$;
- R_1 is resonant, $R_1(\theta) = S + M_1(\theta)$, where $M_1(\theta)$ is upper triangular;
- F_1 is upper triangular;
- $|Z - I|_{r_1} \leqslant \sqrt{\varepsilon_1}$.

Now suppose, by induction, that $A_0 + F$ is conjugated by $Z_k \in C^\omega_{r_k}$ to $R_k + F_k$ with the following properties:

- $|F_k|_{r_k} \leqslant \varepsilon_k$;
- $R_k(\theta) = S + M_k(\theta)$, where M_k is resonant with upper-triangular values;
- F_k has upper-triangular values;
- $|Z_k - I|_{r_k} \leqslant 2\sum_{l=1}^k \sqrt{\varepsilon_l}$.

Then, applying again Proposition 4.7 (by means of Lemma 4.8), one has $Z_{k+1} \in C^\omega_{r_{k+1}}$ conjugating $A_0 + F$ to $R_{k+1} + F_{k+1}$ with

- $|F_{k+1}|_{r_{k+1}} \leqslant \varepsilon_{k+1}$;
- $R_{k+1}(\theta) = S + M_{k+1}(\theta)$, where M_{k+1} is resonant with upper-triangular values;
- F_{k+1} has upper-triangular values;
- $|Z_{k+1} - I|_{r_{k+1}} \leqslant 2\sum_{l=1}^{k+1} \sqrt{\varepsilon_l}$.

Thus, for all $k \in \mathbb{N}$, the system $A_0 + F$ is conjugated, in $C^\omega_{\frac{r}{2}}$, to $R_k + F_k$, where R_k is resonant, $|F_k|_{\frac{r}{2}} \leqslant \varepsilon_k$, and $|Z_k - I|_{r_k} \leqslant 2$.

By Lemma 4.10, r_k has a strictly positive limit r_∞. Let R_∞ be a limit point, in $C^\omega_{r_\infty}$, of $(R_k)_{k\in\mathbb{N}}$ (thus R_∞ is a resonant map); let (k_l) be a sequence such that R_{k_l} tends to R_∞, and let Z_∞ be a limit point, in $C^\omega_{r_\infty}$, of the subsequence Z_{k_l}. Then $\partial_\omega Z_\infty = (A_0 + F)Z_\infty - Z_\infty R_\infty$. □

5. Analytic reduction of a strongly commuting perturbation

Definition 5.1. Let $F \in C^\omega_r$; F is strongly commuting if, for all equivalence classes $v, v' \in \mathbb{Z}^d/\sim$, one has $[F_{\langle v\rangle}, F_{\langle v'\rangle}] = 0$.

The aim of this section is to prove the following result, which corresponds to Theorem 1.5 above.

Theorem 5.2. *Let $F \in C^{\omega}_r$ be strongly commuting. Assume that there exist $C > 0, 0 < R < r$ such that, for all $k \in \mathbb{Z}^d$ with $\langle k, \omega\rangle \neq 0$,*

$$|\langle k, \omega\rangle| \geqslant Ce^{-|k|R}. \tag{31}$$

Then the system with coefficient matrix F is analytically reducible to a resonant system.

Proof. The solution can be written as

$$X(t,\theta) = \exp\left(\int_0^t F(\theta + s\omega)\, ds\right) X(0,\theta).$$

The strong commutation assumption implies that the solution of the initial system can be directly computed and written in a reduced form:

$$\begin{aligned} X(t,\theta) &= e^{tF_{\text{res}}(\theta)} \prod_{\langle v\rangle \in \mathbb{Z}^d/\sim, \langle v,\omega\rangle \neq 0} \exp\left(\frac{e^{it\langle v,\omega\rangle}-1}{i\langle v,\omega\rangle} \sum_{k\in\mathbb{Z}^d, k\in\langle v\rangle} \hat{F}_k e^{i\langle k,\theta\rangle}\right) \\ &= e^{tF_{\text{res}}(\theta)} \prod_{\langle v\rangle \in \mathbb{Z}^d/\sim, \langle v,\omega\rangle \neq 0} \exp\left(\frac{e^{it\langle v,\omega\rangle}-1}{i\langle v,\omega\rangle} F_{\langle v\rangle}(\theta)\right) \\ &= Z(\theta + t\omega) e^{tF_{\text{res}}(\theta)} Z(\theta)^{-1}, \end{aligned} \tag{32}$$

where $Z(\theta) = \prod_{\langle v,\omega\rangle\neq 0} \exp\left(\frac{F_{\langle v\rangle}(\theta)}{i\langle v,\omega\rangle}\right)$. Thus we seek an arithmetical condition on ω under which Z is analytic. This will hold if the function

$$Y(\theta) := \sum_{\langle v,\omega\rangle\neq 0} \frac{F_{\langle v\rangle}(\theta)}{i\langle v,\omega\rangle}$$

is itself analytic. Now

$$Y(\theta) = \sum_{k\in\mathbb{Z}^d, \langle k,\omega\rangle\neq 0} \frac{\hat{F}(k)e^{i\langle k,\theta\rangle}}{i\langle k,\omega\rangle};$$

thus $\hat{Y}(k) = \frac{\hat{F}(k)}{i\langle k,\omega\rangle}$ if $\langle k,\omega\rangle \neq 0$. Therefore Y is $C^{\omega}_{r'}$-analytic if there is $C > 0$ such that, for all k with $\langle k,\omega\rangle \neq 0$,

$$\|\hat{Y}(k)\| \leqslant Ce^{-|k|r'},$$

which holds if

$$|F|_r e^{-|k|(r-r')} \leqslant C|\langle k,\omega\rangle|.$$

Now this holds if ω satisfies condition (31) with $R \leqslant r - r'$. □

6. Preservation of Lie structures

Assuming that the initial system takes its values in a Lie algebra $\mathfrak{g}$ among $gl(n,\mathbb{R})$, $sl(n,\mathbb{C})$ or $sp(n,\mathbb{R})$ (for n even), a slight modification of the proofs will make the

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reduced system have its values in the same Lie algebra and the reducing transformation have its values in the corresponding Lie group G.

First, notice that the homological equations (9) and (21) have a solution in the Lie algebra where the coefficients A, R, and F have their values. This was used in [5, Proposition 2.8] in the non-resonant case, and it works identically even if the frequency vector ω is resonant, since by construction the solution is unique (since resonances are removed from the right-hand side of the homological equation). This comes from the fact that the Lie algebras considered here are defined by an equation of the form $L(F) = 0$, where L is a linear operator on matrices, and such that, for all X, whenever A is in the Lie algebra, either $L([A, X]) = 0$ (for instance if L is the trace operator) or $L([A, X]) = [A, L(X)]$ $\big($for instance if $L(X) = J(X^*J + JX)$ where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)\big)$.

Then one has to define a change of variables which takes its values in the Lie group. The change of variables defined above was $I + X$; one would have to take $\exp(X)$ instead. Since, at the first order, $I + X$ and $\exp(X)$ coincide, the difference in the estimates will be quadratic; thus the new estimates will change only by a universal constant and will not prevent the convergence of the KAM scheme.

Also, the structure will be preserved by integration in the proofs of Proposition 3.2 and Theorem 5.2: integration preserves the zero trace property, commutes with the operator $X \mapsto X^*J + JX$, and preserves the space of integrable real functions.

Therefore, Theorem 3.2 can be restated as follows.

Theorem 6.1. *Assume that ω is very weak with exponent $R > 0$, and that the eigenvalues of A_0 have distinct real parts. Let $r > R$. Let $A = A_0 + a$ be a C^ω_r-analytic cocycle (1) with values in $\mathfrak{g}$, where a is a zero-mean-valued function satisfying $|a|_r \leqslant \varepsilon_0(n, A_0)$. Then there exists a convergent transformation with values in G conjugating (1) to a normal form in a neighborhood of the torus:*

$$NF = \omega \frac{\partial}{\partial \theta} + D_{\text{res}}(\theta) x \frac{\partial}{\partial x},$$

where D_{res} is a resonant diagonal matrix-valued function with values in $\mathfrak{g}$.

Theorem 5.2 becomes the following.

Theorem 6.2. *Assume that ω is very weak with exponent R. Let $r > R$. Let $A_0 \in \mathfrak{g}$. Let $F \in C^\omega_r$ be $\mathfrak{g}$-valued, strongly commuting, and such that, for all v, θ, $[A_0, F_{\langle v \rangle}(\theta)] = 0$. There exists $C(A_0, n)$ such that, if $|F|_r \leqslant C$, then the system $A_0 + F$ is reducible, by a G-valued transformation, to a resonant system $A_\infty(\theta)$ which commutes with A_0 and has its values in $\mathfrak{g}$.*

In § 4, it was assumed that the constant part A_0 was in a Jordan normal form, and it was said that this assumption is not restrictive. In order to remain in $gl(n, \mathbb{R})$, however, two more arguments are needed.

First of all, the Lie structure can slightly affect the way Lemma 4.5 is applied. Indeed, if $S + N$ is the Jordan normal form of a real matrix, one will have to preserve the pairs of eigenvalues which are complex conjugate, and therefore it will be necessary to double the period. While doubling the period, new resonances might appear, namely $\langle k, \omega \rangle$ with

$k \in \frac{1}{2}\mathbb{Z}^d$. However, after doubling the period a finite number of times (at most $n-1$ times), all resonances will be deleted. So we will find the transformation not defined on the original torus but rather on a 2^{n-1} covering.

While preserving the real structure, this way of eliminating the resonances also produces a transformation with determinant 1; thus the structure of $SL(n,\mathbb{R})$ is also preserved.

On the other hand, let P be such that $A_0 = P(S+N)P^{-1}$, where S is diagonal and N is nilpotent. Assume that PRP^{-1} and PFP^{-1} are real valued. If X is a solution of (21), then PXP^{-1} is solution of $\partial_\omega PXP^{-1} = [PRP^{-1}, PXP^{-1}] + PT^N F_{nr} P^{-1}$, and thus it is real. The change of variables $Pe^X P^{-1}$ is thus also real, and its iteration is real (recall that taking e^X instead of $I+X$ does not essentially change the estimates). The estimates are not changed except by the constant $\|P^{-1}\| \cdot \|P\|$, since the change of basis is unchanged through the iteration. Finally, the system with constant part A_0 can be reduced by a real-valued change of variables, and the reduced system is then automatically real.

Moreover, the trace is invariant by matrix conjugation. Thus, Theorem 4.1 can be restated as follows.

Theorem 6.3. *Let $\mathfrak{g} = gl(n,\mathbb{R})$ or $\mathfrak{g} = sl(n,\mathbb{R})$. Let $A_0 = P(S+M)P^{-1} \in \mathfrak{g}$, where $S =$ $\mathrm{diag}(\lambda_j)$ is diagonal, M is nilpotent and upper triangular, and $[M,S]=0$. Let $F \in C^\omega_r(\mathbb{T}^d,\mathfrak{g})$ be such that $P^{-1}FP$ is upper-triangular valued. Assume that S is Melnikov and that $|F|_r \leqslant \varepsilon_0(n,d,\kappa',g',A_0,r)$. Then there exist an $r_\infty > 0$, a $Z_\infty \in C^\omega_{r_\infty}(2^{n-1}\mathbb{T}^d, G)$, and a resonant $R_\infty \in C^\omega_{r_\infty}(2^{n-1}\mathbb{T}^d, G)$, such that $P^{-1}RP$ has upper-triangular values and such that $\partial_\omega Z_\infty = (A_0+F)Z_\infty - Z_\infty R_\infty$.*

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