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The slice Burnside ring and the section Burnside ring of a finite group

Serge Bouc

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Abstract

This paper introduces two new Burnside rings for a finite group G, called *the slice Burnside ring* and *the section Burnside ring*. They are built as Grothendieck rings of the category of morphisms of G-sets and of Galois morphisms of G-sets, respectively. The well-known results on the usual Burnside ring, concerning ghost maps, primitive idempotents, and description of the prime spectrum, are extended to these rings. It is also shown that these two rings have a natural Green biset functor structure. The functorial structure of unit groups of these rings is also discussed.

1. Introduction

This paper introduces two variations on Burnside rings of finite groups, called the *slice Burnside ring* and the *section Burnside ring*. Both are built as Grothendieck rings of some category of *morphisms* of finite G-sets, instead of the category of finite G-sets used to build the usual Burnside ring. The difference between these two new Burnside rings is that the slice Burnside ring is built from arbitrary morphisms of finite G-sets, whereas the section Burnside ring uses only *Galois morphisms* of finite G-sets.

It turns out that most of the well-known properties of the Burnside ring extend to the slice Burnside ring and to the section Burnside ring: both are commutative rings, which are free of finite rank as $\mathbb{Z}$-modules. There is an analogue of Burnside's theorem: both of these rings embed in a product of copies of the integers via a *ghost map*, and this map has a finite cokernel. After tensoring with $\mathbb{Q}$, both rings become split semisimple $\mathbb{Q}$-algebras, and explicit formulas for their primitive idempotents can be stated. The prime spectrum of both rings can also be described, and Dress's characterization of solvable groups in terms of the connectedness of the spectrum of the Burnside ring can be generalized as well. Finally, both constructions have a natural biset functor structure, for which they become *Green biset functors*.

A major exception in this list of generalizable properties concerns unit groups: it can be shown that, unlike the case of the usual Burnside ring, the correspondence sending a finite group to the unit group of either the slice or the section Burnside ring cannot be endowed with a biset functor structure. This is due to the lack of a suitable *tensor induction* for these rings.

This paper is divided into two parts and an appendix. The first part, consisting of §§ 2–8, is devoted to the slice Burnside functor. Section 2 recalls the basic definitions and properties on the category of morphisms of G-sets. Section 3 introduces the slice Burnside functor and its Green biset functor structure. Section 4 is devoted to the definition of and main result on the ghost map. In § 5 the explicit formulas for the primitive idempotents of the slice Burnside

algebra over $\mathbb{Q}$ are stated. Section 6 gives a characterization of the image of the ghost map. Then §7 considers the prime spectrum of the slice Burnside ring. Finally, in §8, it is shown that tom Dieck's theorem, building on Dress's characterization of solvable groups, can be extended to the slice Burnside ring: namely, the Feit–Thompson theorem is equivalent to the fact that the only units in the slice Burnside ring of a group of odd order are ± 1. But unlike the case of the usual Burnside ring, the unit group of the slice Burnside ring cannot be endowed with a biset functor structure.

The second part of the paper is devoted to the section Burnside ring, and it is organized similarly. Section 9 introduces Galois morphisms and states their general properties. In particular, a left adjoint functor to the forgetful functor from Galois morphisms to arbitrary morphisms of G-sets is described, which attaches to any morphism of G-sets a canonical Galois morphism. Section 10 considers the section Burnside functor, with its Green biset functor structure. Section 11 deals with the ghost map for the section Burnside ring. Section 12 states the formulas for the primitive idempotents of the section Burnside $\mathbb{Q}$-algebra, and §13 gives a characterization of the ghost map. Section 14 considers the prime spectrum of the section Burnside ring. In §15, the results on unit groups from §8 are extended to the unit group of the section Burnside ring. In particular, it is shown that this unit group cannot be endowed with a biset functor structure.

The appendix deals with the functorial structure of the unit group of the slice Burnside ring and the section Burnside ring: it is possible to define biset functor operations for these unit groups, but only for *left inert* bisets. This gives two interesting examples of somehow natural biset functors *without induction*. The last result of the appendix is the explicit computation of this unit group in the case of an abelian group.

Part I. The slice Burnside ring

2. Morphisms of G-sets

DEFINITION 2.1. Let G be a group. If $f : X \to Y$ and $f' : X' \to Y'$ are morphisms of G-sets, a *morphism* from f to f' is a pair of morphisms of G-sets $\alpha : X \to X'$ and $\beta : Y \to Y'$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
X' & \xrightarrow{\ f'\ } & Y'
\end{array}
$$

is commutative.

Morphisms of morphisms of G-sets can be composed in the obvious way. This composition endows the class of morphisms of G-sets with a category structure, denoted by G-Mor.

PROPOSITION 2.2. *The disjoint union of G-sets induces a coproduct*

$$(X \xrightarrow{\ f\ } Y, X' \xrightarrow{\ f'\ } Y') \mapsto (X \sqcup X' \xrightarrow{\ f \sqcup f'\ } Y \sqcup Y')$$

in the category G-Mor.

Similarly, the direct product of G-sets, with diagonal G-action, induces a product

$$(X \xrightarrow{\ f\ } Y, X' \xrightarrow{\ f'\ } Y') \mapsto (X \times X' \xrightarrow{\ f \times f'\ } Y \times Y')$$

in the category G-Mor.

Proof. For any morphism of G-sets, the bijections

$$\operatorname{Hom}_{G\text{-}\mathsf{Set}}(X \sqcup X', A) \cong \operatorname{Hom}_{G\text{-}\mathsf{Set}}(X, A) \times \operatorname{Hom}_{G\text{-}\mathsf{Set}}(X', A)$$

induce obvious bijections between

$$\operatorname{Hom}_{G\text{-}\mathsf{Mor}}((X \sqcup X') \xrightarrow{f \sqcup f'} (Y \sqcup Y'), A \xrightarrow{\alpha} B)$$

and

$$\operatorname{Hom}_{G\text{-}\mathsf{Mor}}(X \xrightarrow{f} Y, A \xrightarrow{\alpha} B) \times \operatorname{Hom}_{G\text{-}\mathsf{Mor}}(X' \xrightarrow{f'} Y', A \xrightarrow{\alpha} B).$$

These bijections are obviously functorial in G-Mor.

Similarly, the bijections

$$\operatorname{Hom}_{G\text{-}\mathsf{Set}}(A, X \times X') \cong \operatorname{Hom}_{G\text{-}\mathsf{Set}}(A, X) \times \operatorname{Hom}_{G\text{-}\mathsf{Set}}(A, X')$$

induce obvious bijections between

$$\operatorname{Hom}_{G\text{-}\mathsf{Mor}}(A \xrightarrow{\alpha} B, (X \times X') \xrightarrow{f \times f'} (Y \times Y'))$$

and

$$\operatorname{Hom}_{G\text{-}\mathsf{Mor}}(A \xrightarrow{\alpha} B, X \xrightarrow{f} Y) \times \operatorname{Hom}_{G\text{-}\mathsf{Mor}}(A \xrightarrow{\alpha} B, X' \xrightarrow{f'} Y').$$

These bijections are obviously functorial in G-Mor. $\qquad\square$

3. The slice Burnside functor

DEFINITION AND NOTATION 3.1. Let G be a group. A *slice* of G is a pair (T, S) of subgroups of G with $T \geqslant S$. A *section* of G is a slice (T, S) with $S \trianglelefteq T$.

Let $\Pi(G)$ denote the set of slices of G, and $\Sigma(G)$ denote the set of sections of G.

When $(T, S) \in \Pi(G)$, denote by $G/S \to G/T$ the projection morphism.

DEFINITION 3.2. Let G be a finite group. *The slice Burnside group $\Xi(G)$ of G* is the quotient of the free abelian group on the set of isomorphism classes $[X \xrightarrow{f} Y]$ of morphisms of finite G-sets, by the subgroup generated by elements of the form

$$[(X_1 \sqcup X_2) \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f(X_1)] - [X_2 \xrightarrow{f_2} f(X_2)],$$

whenever $X \xrightarrow{f} Y$ is a morphism of finite G-sets with a decomposition $X = X_1 \sqcup X_2$ as a disjoint union of G-sets, where $f_1 = f_{|X_1}$ and $f_2 = f_{|X_2}$.

When $f : X \to Y$ is a morphism of finite G-sets, let $\pi(f)$ denote the image in $\Xi(G)$ of the isomorphism class of f. If $S \leqslant T$ are subgroups of G, set $\langle T, S \rangle_G = \pi(G/S \to G/T)$.

LEMMA 3.3. (i) $\pi(\emptyset \to \emptyset) = 0$.

(ii) *Let $X \xrightarrow{f} Y$ be a morphism of finite G-sets. Then*

$$\pi(X \xrightarrow{f} Y) = \pi(X \xrightarrow{f} f(X)).$$

(iii) *Let $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ be morphisms of finite G-sets. Then*

$$\pi((X \sqcup X') \xrightarrow{f \sqcup f'} (Y \sqcup Y')) = \pi(X \xrightarrow{f} Y) + \pi(X' \xrightarrow{f'} Y').$$

Proof. For assertion (i), set $e = \pi(\emptyset \to \emptyset)$. Since the morphism $\emptyset \sqcup \emptyset \to \emptyset$ is isomorphic to $\emptyset \to \emptyset$, it follows that $e + e = e$, hence $e = 0$.

For assertion (ii), writing $X = X \sqcup \emptyset$ gives

$$\pi(X \xrightarrow{f} Y) = \pi(X \xrightarrow{f} f(X)) + \pi(\emptyset \to \emptyset) = \pi(X \xrightarrow{f} f(X)).$$

For assertion (iii),

$$
\begin{aligned}
\pi((X \sqcup X') \xrightarrow{f \sqcup f'} (Y \sqcup Y')) &= \pi(X \xrightarrow{f} (Y \sqcup Y')) + \pi(X' \xrightarrow{f'} (Y \sqcup Y')) \\
&= \pi(X \xrightarrow{f} f(X)) + \pi(X' \xrightarrow{f'} f'(X')) \\
&= \pi(X \xrightarrow{f} Y) + \pi(X' \xrightarrow{f'} Y'),
\end{aligned}
$$

where the first equality follows from the defining relations of $\Xi(G)$, and the other ones from assertion (ii). $\qquad\square$

LEMMA 3.4. *Let $f : X \to Y$ be a morphism of finite G-sets. Then in the group $\Xi(G)$,*

$$\pi(X \xrightarrow{f} Y) = \sum_{x \in [G \backslash X]} \langle G_{f(x)}, G_x \rangle_G$$

Proof. Indeed $X \cong \bigsqcup_{x \in [G \backslash X]} G/G_x$, and the image $f(G \cdot x)$ of the G-orbit of x is equal to the G-orbit of $f(x)$. Moreover, the morphisms $f_{|G \cdot x} : G \cdot x \to G \cdot f(x)$ and $G/G_x \to G/G_{f(x)}$ are isomorphic. The lemma is thus proved. $\qquad\square$

COROLLARY 3.5. *The group $\Xi(G)$ is generated by the elements $\langle T, S \rangle_G$, where (T, S) runs through a set $[\Pi(G)]$ of representatives of conjugacy classes of slices of G.*

Proof. Indeed, the morphisms $G/{}^g S \to G/{}^g T$ and $G/S \to G/T$ are isomorphic, for any $g \in G$, and any slice (T, S) of G. $\qquad\square$

Remark 3.6. It will be shown in Theorem 4.6 that this generating set is actually a basis of $\Xi(G)$.

PROPOSITION 3.7. *The product of morphisms induces a commutative unital ring structure on $\Xi(G)$. The identity element for multiplication is the image of the class $[\bullet \to \bullet]$, where $\bullet$ denotes a G-set of cardinality 1.*

Proof. If we can show that the product of morphisms induces a well-defined bilinear product $\Xi(G) \times \Xi(G) \to \Xi(G)$, it will be clear that this product is associative, commutative, and admits $[\bullet \to \bullet]$ as an identity element. Hence the only point to check is that the product preserves the defining relations of $\Xi(G)$. This is clear, since if $g : Z \to T$ is a morphism of finite G-sets, and if $X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$ is a morphism, setting $X = X_1 \sqcup X_2$, the domain of the morphism

$$h : Z \times X \xrightarrow{g \times (f_1 \sqcup f_2)} T \times Y$$

has a disjoint union decomposition $Z \times X = (Z \times X_1) \sqcup (Z \times X_2)$, and moreover, the restriction of $g \times f$ to $Z \times X_1$ is $g \times f_1$. Thus

$$
\begin{aligned}
\pi(h) &= \pi((Z \times X_1) \xrightarrow{g \times f_1} (g(Z) \times f_1(X_1))) + \pi((Z \times X_2) \xrightarrow{g \times f_2} (g(Z) \times f_2(X_2))) \\
&= \pi((Z \times X_1) \xrightarrow{g \times f_1} (T \times f_1(X_1))) + \pi((Z \times X_2) \xrightarrow{g \times f_2} (T \times f_2(X_2))),
\end{aligned}
$$

where the last equality follows from Lemma 3.3. $\qquad\square$

PROPOSITION 3.8. *Let (T, S) and (Y, X) be slices of G. Then in $\Xi(G)$,*

$$\langle T, S \rangle_G \langle Y, X \rangle_G = \sum_{g \in [S \backslash G / X]} \langle T \cap {}^g Y, S \cap {}^g X \rangle_G.$$

Proof. Indeed

$$(G/S) \times (G/X) \cong \bigsqcup_{g \in [S \backslash G / X]} G/(S \cap {}^g X)$$

via the map (from right to left) sending $u(S \cap {}^g X)$ to (uS, ugX), for $u \in G$. The image of (S, gX) by the map $(G/S) \times (G/X) \to (G/T) \times (G/Y)$ is the pair (T, gY), whose stabilizer in G is $T \cap {}^g Y$. The result now follows from Lemma 3.4. □

THEOREM 3.9. (i) *Let G and H be finite groups, and let U be a finite (H, G)-biset. The functor*

$$(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)$$

from G-Mor to H-Mor induces a group homomorphism

$$\Xi(U) : \Xi(G) \to \Xi(H).$$

(ii) *The correspondence $G \mapsto \Xi(G)$ is a Green biset functor.*

Proof. For assertion (i), the only thing to check is that the defining relations of $\Xi(G)$ are mapped to relations in $\Xi(H)$. But if

$$X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y$$

is a morphism of finite G-sets, then

$$U \times_G (X_1 \sqcup X_2) \cong (U \times_G X_1) \sqcup (U \times_G X_2).$$

Moreover, the image of the map $U \times_G f_1$ is equal to $U \times_G f_1(X_1)$. It follows that the relation

$$[X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f_1(X_1)] - [X_2 \xrightarrow{f_2} f_2(X_2)]$$

in $\Xi(G)$ is mapped to the relation

$$[UX_1 \sqcup UX_2 \xrightarrow{Uf_1 \sqcup Uf_2} UY] - [UX_1 \xrightarrow{Uf_1} (Uf_1)(UX_1)] - [UX_2 \xrightarrow{Uf_2} (Uf_2)(UX_2)],$$

where $U \times_G$ is abbreviated to U.

It is now clear that the correspondence sending a finite group G to $\Xi(G)$ and a finite (H, G)-biset U to $\Xi(U)$ endows Ξ with a biset functor structure (see [Bou10]).

Moreover, if G, G' are finite groups, if $f : X \to Y$ is a morphism of finite G-sets and $f' : X' \to Y'$ is a morphism of finite G'-sets, then

$$f \times f' : X \times X' \to Y \times Y'$$

is a morphism of $G \times G'$-sets. This induces a product

$$\Xi(G) \times \Xi(G') \to \Xi(G \times G'),$$

which is associative in the obvious sense. Moreover, the morphism $\bullet \to \bullet$ of $\mathbf{1}$-sets is obviously an identity element for this product, up to identification $G \times \mathbf{1} = G$.

Finally, if G, G', H, H' are finite groups, if U is a finite (H, G)-biset, if U' is a finite (H', G')-biset, it is clear that the morphisms

$$(U \times U') \times_{G \times G'} (f \times f')$$

and
$$(U \times_G f) \times (U' \times_{G'} f')$$
are isomorphic morphisms of $(H \times H')$-sets. Thus Ξ is a Green biset functor (see [Bou10, §8.5]). □

PROPOSITION 3.10. *Let G and H be finite groups, and U be a finite (H, G)-biset. If $(T, S) \in \Pi(G)$, then*
$$U \times_G \langle T, S \rangle_G = \sum_{u \in [H \backslash U / S]} \langle {}^u T, {}^u S \rangle_H$$
(where ${}^u X = \{h \in H \mid \exists x \in X, hu = ux\}$, for $X \leqslant G$).

Proof. Indeed $U \times_G (G/S) \cong U/S$, and the stabilizer in H of uS is equal to ${}^u S$. The result follows from Lemma 3.4. □

The following proposition clarifies the links existing between the usual Burnside functor and the slice Burnside functor.

PROPOSITION 3.11. (i) *Let G be a finite group. The correspondence sending the morphism $X \xrightarrow{f} Y$ of finite G-sets to the G-set X induces a unital ring homomorphism s_G from $\Xi(G)$ to the Burnside ring $B(G)$.*

(ii) *The correspondence sending the finite G-set X to the identity morphism of X induces a unital ring homomorphism $i_G : B(G) \to \Xi(G)$, such that $s_G \circ i_G = \mathrm{Id}_{B(G)}$.*

(iii) *As G varies, the morphisms s_G and i_G define morphisms of Green biset functors $s : \Xi \to B$ and $i : B \to \Xi$, such that $s \circ i = \mathrm{Id}_B$. In particular, i is injective and s is surjective.*

Proof. This is straightforward from the definitions. □

4. Slices and ghost map

Notation 4.1. Let $S \leqslant T$ be subgroups of G. If $X \xrightarrow{f} Y$ is a morphism of finite G-sets, set
$$\phi_{T,S}(X \xrightarrow{f} Y) = |\mathrm{Hom}_{G\text{-}\mathsf{Mor}}(G/S \to G/T, X \xrightarrow{f} Y)|.$$

Notation 4.2. Define a relation $\preceq$ on the set $\Pi(G)$ by
$$(T, S) \preceq (Y, X) \Leftrightarrow (T \leqslant Y \text{ and } S \leqslant X).$$

The relation $\preceq$ is an order relation on $\Pi(G)$.

LEMMA 4.3. *With this notation,*
$$\phi_{T,S}(X \xrightarrow{f} Y) = |f^{-1}(Y^T)^S|.$$

In particular, for any $A \leqslant B \leqslant G$,
$$\phi_{T,S}(G/A \to G/B) = |\{g \in G/A \mid (T^g, S^g) \preceq (B, A)\}|.$$

Proof. The morphisms of G-sets from G/S to X are in one-to-one correspondence with the set X^S of fixed points of S on X: the morphism associated to $x \in X^S$ is defined by $gS \mapsto gx$. Similarly, the homomorphisms of G-sets from G/T to Y are in one-to-one correspondence with Y^T. Hence the set
$$\mathrm{Hom}_{G\text{-}\mathsf{Mor}}(G/S \to G/T, X \xrightarrow{f} Y)$$

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is in one-to-one correspondence with the set of pairs $(x, y) \in X^S \times Y^T$ such that $f(x) = y$, i.e. with the elements x of $f^{-1}(Y^T)^S$. $\qquad \square$

COROLLARY 4.4. *Let $(T, S) \in \Pi(G)$ and p be a prime number. If P is a p-subgroup of $N_G(T, S)$, then*

$$\phi_{T,S} \equiv \phi_{PT,PS} \pmod{p}.$$

Proof. Indeed for any morphism of finite G-sets $X \xrightarrow{f} Y$, the set $f^{-1}(Y^T)^S$ is invariant by $N_G(T, S)$, thus, as P is a p-group,

$$|f^{-1}(Y^T)^S| \equiv |f^{-1}(Y^T)^{PS}| \pmod{p},$$

and moreover, $f^{-1}(Y^T)^{PS} = f^{-1}(Y^{PT})^{PS}$. $\qquad \square$

PROPOSITION 4.5. *Let $S \leqslant T$ be subgroups of G. Then the map $\phi_{T,S}$ induces a ring homomorphism $\Xi(G) \to \mathbb{Z}$, still denoted by $\phi_{T,S}$.*

Proof. Since the product of morphisms

$$(X \xrightarrow{f} Y, X' \xrightarrow{f'} Y') \mapsto ((X \times X') \xrightarrow{f \times f'} (Y \times Y'))$$

is a product in the category G-Mor, it follows that

$$\phi_{T,S}((X \times X') \xrightarrow{f \times f'} (Y \times Y')) = \phi_{T,S}(X \xrightarrow{f} Y)\phi_{T,S}(X' \xrightarrow{f'} Y').$$

Also $\phi_{T,S}(\bullet \to \bullet) = 1$. The only thing to check is that $\phi_{T,S}$ induces a well-defined map $\Xi(G) \to \mathbb{Z}$, i.e. that the defining relations of $\Xi(G)$ are mapped to 0 by $\phi_{T,S}$. First, by Lemma 4.3, for any morphism of G-sets $f : X \to Y$,

$$\phi_{T,S}(X \xrightarrow{f} Y) = \phi_{T,S}(X \xrightarrow{f} f(X)).$$

Now

$$
\begin{aligned}
\phi_{T,S}((X_1 \sqcup X_2) \xrightarrow{f_1 \sqcup f_2} Y) &= |(f_1 \sqcup f_2)^{-1}(Y^T)^S| \\
&= |X_1 \cap (f_1 \sqcup f_2)^{-1}(Y^T)^S| + |X_2 \cap (f_1 \sqcup f_2)^{-1}(Y^T)^S| \\
&= |f_1^{-1}(Y^T)^S| + |f_2^{-1}(Y^T)^S| \\
&= \phi_{T,S}(X_1 \xrightarrow{f_1} Y) + \phi_{T,S}(X_2 \xrightarrow{f_2} Y) \\
&= \phi_{T,S}(X_1 \xrightarrow{f_1} f(X_1)) + \phi_{T,S}(X_2 \xrightarrow{f_2} f(X_2)).
\end{aligned}
$$

This completes the proof. $\qquad \square$

THEOREM 4.6. *The group $\Xi(G)$ is a free abelian group, with basis the set of elements $\langle T, S \rangle_G$, where (T, S) runs through a set $[\Pi(G)]$ of representatives of conjugacy classes of slices of G. Moreover, the map*

$$\Phi = \prod_{(T,S) \in [\Pi(G)]} \phi_{T,S} : \Xi(G) \to \prod_{(T,S) \in [\Pi(G)]} \mathbb{Z},$$

called the ghost map for $\Xi(G)$, is an injective ring homomorphism, with finite cokernel as morphism of abelian groups.

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Proof. By Lemma 3.5, the elements $\langle T, S\rangle_G$, for $(T, S) \in [\Pi(G)]$, generate $\Xi(G)$. Suppose that there is a non-zero linear combination in the kernel of Φ,

$$\Lambda = \sum_{(T,S)\in[\Pi(G)]} \lambda_{T,S}\langle T, S\rangle_G,$$

with integer coefficients $\lambda_{T,S} \in \mathbb{Z}$, for $(T, S) \in [\Pi(G)]$. Extend λ to a function $\Pi(G) \to \mathbb{Z}$, constant on conjugacy classes. Let (Y, X) be an element of $\Pi(G)$, maximal for the relation $\preceq$, such that $\lambda_{Y,X} \neq 0$. Then since by Lemma 4.3,

$$\phi_{Y,X}(G/S \to G/T) = |\{g \in G/S \mid (Y^g, X^g) \preceq (T, S)\}|,$$

it follows that

$$
\begin{aligned}
\phi_{Y,X}(\Lambda) &= \sum_{(T,S)\in[\Pi(G)]} \lambda_{T,S}\phi_{Y,X}(G/S \to G/T) \\
&= \lambda_{Y,X}\phi_{Y,X}(G/X \to G/Y) \\
&= \lambda_{Y,X}|N_G(X, Y)/X| = 0.
\end{aligned}
$$

Hence $\lambda_{Y,X} = 0$, and this contradiction shows that Φ is injective. In particular, the elements $\langle T, S\rangle_G$, for $(T, S) \in [\Pi(G)]$, form a $\mathbb{Z}$-basis of $\Xi(G)$. Thus Φ is an injective morphism between free abelian groups with the same finite rank, hence it has finite cokernel. $\qquad\square$

COROLLARY 4.7. *Set $\mathbb{Q}\Xi(G) = \mathbb{Q} \otimes_{\mathbb{Z}} \Xi(G)$ and $\mathbb{Q}\Phi = \mathbb{Q} \otimes_{\mathbb{Z}} \Phi$. Then*

$$\mathbb{Q}\Phi : \mathbb{Q}\Xi(G) \to \prod_{(T,S)\in[\Pi(G)]} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-algebras.

5. Slices and idempotents

By Corollary 4.7, the commutative $\mathbb{Q}$-algebra $\mathbb{Q}\Xi(G)$ is split semisimple. Its primitive idempotents are indexed by slices of G, up to conjugation: they are the inverse images under $\mathbb{Q}\Phi$ of the primitive idempotents of the algebra $\prod_{(T,S)\in[\Pi(G)]} \mathbb{Q}$.

Notation 5.1. If $(T, S) \in \Pi(G)$, denote by $\xi_{T,S}^G$ the unique element of $\mathbb{Q} \otimes_{\mathbb{Z}} \Xi(G)$ such that

$$\forall (Y, X) \in \Pi(G), \quad \mathbb{Q}\phi_{Y,X}(\xi_{T,S}^G) = \begin{cases} 1 & \text{if } (Y, X) =_G (T, S) \\ 0 & \text{otherwise.} \end{cases}$$

The set of elements $\xi_{T,S}^G$, for $(T, S) \in [\Pi(G)]$, is the set of primitive idempotents of $\mathbb{Q}\Xi(G)$.

THEOREM 5.2. *Let $(T, S) \in \Pi(G)$. Then*

$$\xi_{T,S}^G = \frac{1}{|N_G(T, S)|} \sum_{(V,U)\preceq(T,S)} |U|\mu_{\Pi}((V, U), (T, S))\langle V, U\rangle_G,$$

where μ_{Π} is the Möbius function of the poset $(\Pi(G), \preceq)$.

Proof [1]. Denote by $\bar{\Pi}(G)$ the set of orbits of G for its conjugacy action on $\Pi(G)$. Thus $\bar{\Pi}(G)$ is in one-to-one correspondence with $[\Pi(G)]$, and the map $\mathbb{Q}\Phi$ can also be viewed a $\mathbb{Q}$-algebra isomorphism from $\mathbb{Q}\Xi(G)$ to $\mathbb{Q}^{\bar{\Pi}(G)}$. The $\mathbb{Q}$-vector space $\mathbb{Q}\Xi(G)$ has a basis consisting of the elements $\langle V, U\rangle_G$, for $(V, U) \in [\Pi(G)]$. Let $\mathbb{Q}^{\Pi(G)}$ denote the $\mathbb{Q}$-vector space with basis $\Pi(G)$, and let $p : \mathbb{Q}^{\Pi(G)} \to \mathbb{Q}\Xi(G)$ denote the $\mathbb{Q}$-linear map sending $(V, U) \in \Pi(G)$ to $\langle V, U\rangle_G$.

Let $\beta_{T,S}^G$ denote the vector of the canonical basis of $\mathbb{Q}^{\bar{\Pi}(G)}$ indexed by the G-orbit of $(T, S) \in \Pi(G)$, and let $q : \mathbb{Q}^{\Pi(G)} \to \mathbb{Q}^{\bar{\Pi}(G)}$ denote the $\mathbb{Q}$-linear map sending $(T, S) \in \Pi(G)$ to $\beta_{T,S}^G$.

With this notation Lemma 4.3 shows that, for any $(V, U) \in \Pi(G)$,

$$\Phi \circ p((V, U)) = \sum_{(T,S)\in[\Pi(G)]} \phi_{T,S}(\langle V, U\rangle_G)\beta_{T,S}^G$$

$$= \sum_{(T,S)\in[\Pi(G)]} \frac{1}{|U|}|\{g \in G \mid (T, S) \preceq ({}^gV, {}^gU)\}|\beta_{T,S}^G$$

$$\Phi \circ p((V, U)) = \sum_{(T,S)\in\Pi(G)} \frac{|N_G(T, S)|}{|G||U|}|\{g \in G \mid (T^g, S^g) \preceq (V, U)\}|\beta_{T,S}^G$$

$$= \sum_{\substack{(T,S)\in\Pi(G) \\ g\in G \\ (T^g,S^g)\preceq(V,U)}} \frac{|N_G(T^g, S^g)|}{|G||U|}\beta_{T^g,S^g}^G.$$

But $|N_G(T^g, S^g)| = |N_G(T, S)|$ and $\beta_{T^g,S^g}^G = \beta_{T,S}^G$, for any $g \in G$. Thus

$$\Phi \circ p((V, U)) = \sum_{\substack{(T,S)\in\Pi(G) \\ (T,S)\preceq(V,U)}} \frac{|N_G(T, S)|}{|U|}\beta_{T,S}^G = \sum_{\substack{(T,S)\in\Pi(G) \\ (T,S)\preceq(V,U)}} \frac{|N_G(T, S)|}{|U|}q((T, S)).$$

This shows that if $\widetilde{\Phi}$ is the $\mathbb{Q}$-endomorphism of $\mathbb{Q}^{\Pi(G)}$ defined by

$$\widetilde{\Phi}((V, U)) = \sum_{\substack{(T,S)\in\Pi(G) \\ (T,S)\preceq(V,U)}} \frac{|N_G(T, S)|}{|U|}(T, S),$$

then $\Phi \circ p = q \circ \widetilde{\Phi}$. The matrix of the map $\widetilde{\Phi}$ is equal to the product $E \cdot J \cdot D$, where D is a diagonal matrix with diagonal coefficients $(|N_G(T, S)|)_{(T,S)\in\Pi(G)}$, E is a diagonal matrix with diagonal coefficients $(1/|U|)_{(V,U)\in\Pi(G)}$, and J is the incidence matrix of the order relation $\preceq$ on $\Pi(G)$. It follows that $\widetilde{\Phi}$ is invertible, with inverse equal to $D^{-1} \cdot J^{-1} \cdot E^{-1}$. Now the entries of the matrix J^{-1} are precisely the values of the Möbius function μ_Π of the poset $(\Pi(G), \preceq)$. It follows that

$$\xi_{T,S}^G = \Phi^{-1}(\beta_{T,S}^G)$$

$$= \Phi^{-1} \circ q((T, S)) = p \circ \widetilde{\Phi}^{-1}((T, S))$$

$$= \frac{1}{|N_G(T, S)|} \sum_{\substack{(V,U)\in\Pi(G) \\ (V,U)\preceq(T,S)}} |U|\mu_\Pi((V, U), (T, S))\langle V, U\rangle_G,$$

which completes the proof. $\qquad\square$

---

[1] As suggested by the referee, a direct computation using the formula for $\xi_{T,S}^G$ in Theorem 5.2 shows that if (Y, X) is a slice of G, then $\mathbb{Q}\phi_{Y,X}(\xi_{T,S}^G)$ is equal to 0 if (Y, X) and (T, S) are not G-conjugate, and to 1 otherwise. This gives a short proof of Theorem 5.2. The proof given here is longer, but it may have the advantage of explaining the origin of the formula for $\xi_{T,S}^G$, in particular why Möbius functions appear in this expression.

PROPOSITION 5.3. *Let $(\mathcal{X}, \leqslant)$ be a finite poset. Let $\Pi(\mathcal{X})$ denote the set of pairs (y, x) of elements of $\mathcal{X}$ such that $x \leqslant y$. Define a partial order $\preceq$ on $\Pi(\mathcal{X})$ by*

$$\forall (y, x), (t, z) \in \Pi(\mathcal{X}), \quad (y, x) \preceq (t, z) \Leftrightarrow y \leqslant t \text{ and } x \leqslant z.$$

Then the Möbius function μ_Π of the poset $(\Pi(\mathcal{X}), \preceq)$ can be computed as follows, for any $(y, x), (t, z) \in \Pi(\mathcal{X})$:

$$\mu_\Pi((y, x), (t, z)) = \begin{cases} \mu_\mathcal{X}(x, z) \mu_\mathcal{X}(y, t) & \text{if } x \leqslant z \leqslant y \leqslant t \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $\mu_\mathcal{X}$ is the Möbius function of the poset $(\mathcal{X}, \leqslant)$.

Proof. Let $m((y, x), (t, z))$ denote the expression defined by the right-hand side of (5.4). If $(y, x) \preceq (t, z)$, i.e. if $y \leqslant t$ and $x \leqslant z$, then

$$\sum_{\substack{(v,u) \in \Pi(\mathcal{X}) \\ (y,x) \preceq (v,u) \preceq (t,z)}} m((y, x), (v, u)) = \sum_{\substack{y \leqslant v \leqslant t \\ x \leqslant u \leqslant z \\ u \leqslant v, u \leqslant y}} \mu_X(x, u) \mu_X(y, v)$$

$$= \sum_{\substack{y \leqslant v \leqslant t \\ x \leqslant u \leqslant z \\ u \leqslant y}} \mu_X(x, u) \mu_X(y, v)$$

$$= \left(\sum_{y \leqslant v \leqslant t} \mu_\mathcal{X}(y, v) \right) \left(\sum_{\substack{x \leqslant u \leqslant z \\ u \leqslant y}} \mu_\mathcal{X}(x, u) \right).$$

The first factor $\sum_{y \leqslant v \leqslant t} \mu_\mathcal{X}(y, v)$ is equal to 0 if $y \neq t$, and to 1 if $y = t$. In this case, the second factor $\sum_{\substack{x \leqslant u \leqslant z \\ u \leqslant y}} \mu_\mathcal{X}(x, u)$ is equal to $\sum_{x \leqslant u \leqslant z} \mu_\mathcal{X}(x, u)$. This is equal to zero if $x \neq z$, and to 1 if $x = z$.

It follows that

$$\sum_{\substack{(v,u) \in \Pi(\mathcal{X}) \\ (y,x) \preceq (v,u) \preceq (t,z)}} m((y, x), (v, u))$$

is equal to 0 if $(y, x) \neq (t, z)$ and to 1 otherwise. The proposition follows. $\square$

Applying Proposition 5.3 to the poset of subgroups of G, ordered by inclusion of subgroups, gives the following corollary.

COROLLARY 5.5. *Let (V, U) and (T, S) be slices of G. Then*

$$\mu_\Pi((V, U), (T, S)) = \begin{cases} \mu(U, S) \mu(V, T) & \text{if } U \leqslant S \leqslant V \leqslant T \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

where μ is the Möbius function of the poset of subgroups of G. In particular, in $\Xi(G)$,

$$\xi_{T,S}^G = \frac{1}{|N_G(T, S)|} \sum_{U \leqslant S \leqslant V \leqslant T} |U| \mu(U, S) \mu(V, T) \langle V, U \rangle_G.$$

6. The image of the ghost map

The following characterization of the image of the ghost map is the analogue for the slice Burnside ring of a theorem of Dress [Dre69] on the ordinary Burnside ring.

THEOREM 6.1. *Let G be a finite group, and let $\mathsf{m} = (m_{T,S})_{(T,S) \in \Pi(G)}$ be a sequence of integers indexed by $\Pi(G)$, constant on G-conjugacy classes of slices. Then the sequence $[\mathsf{m}] = (m_{T,S})_{(T,S) \in [\Pi(G)]}$ of representatives lies in the image of the ghost map Φ if and only if, for any slice (T, S) of G,*

$$\sum_{g \in N_G(T,S)/S} m_{\langle gT \rangle, \langle gS \rangle} \equiv 0 \pmod{|N_G(T,S)/S|}.$$

Proof. Saying that $[\mathsf{m}]$ lies in the image of Φ is equivalent to saying that the element

$$s_{\mathsf{m}} = \sum_{(T,S) \in [\Pi(G)]} m_{T,S} \xi_{T,S}^G$$

of $\mathbb{Q}\Xi(G)$ lies in $\Xi(G)$, i.e. that it is a linear combination with integer coefficients of the elements $\langle V, U \rangle_G$, for $(V, U) \in [\Pi(G)]$. Now by Theorem 5.2,

$$
\begin{aligned}
s_{\mathsf{m}} &= \sum_{(T,S) \in \Pi(G)} \frac{|N_G(T,S)|}{|G|} m_{T,S} \xi_{T,S}^G \\
&= \frac{1}{|G|} \sum_{(T,S) \in \Pi(G)} m_{T,S} \sum_{(V,U) \preceq (T,S)} |U| \mu_{\Pi}((V,U),(T,S)) \langle V, U \rangle_G \\
&= \frac{1}{|G|} \sum_{(V,U) \in \Pi(G)} |U| \left(\sum_{(V,U) \preceq (T,S)} \mu_{\Pi}((V,U),(T,S)) m_{T,S} \right) \langle V, U \rangle_G \\
&= \sum_{(V,U) \in [\Pi(G)]} \frac{1}{|N_G(V,U)/U|} \left(\sum_{(V,U) \preceq (T,S)} \mu_{\Pi}((V,U),(T,S)) m_{T,S} \right) \langle V, U \rangle_G.
\end{aligned}
$$

Hence $s_{\mathsf{m}} \in \mathrm{Im}\ \Phi$ if and only if the number

$$\beta_{V,U} = \frac{1}{|N_G(V,U)/U|} \left(\sum_{(V,U) \preceq (T,S)} \mu_{\Pi}((V,U),(T,S)) m_{T,S} \right)$$

is an integer, for any slice (V, U) of G.

With this notation, for any $(Y, X) \in \Pi(G)$,

$$m_{Y,X} = \sum_{\substack{(V,U) \in \Pi(G) \\ (Y,X) \preceq (V,U)}} \beta_{V,U} |N_G(V,U)/U|.$$

Hence, setting $\sigma_{Y,X} = \sum_{g \in N_G(Y,X)/X} m_{\langle gY \rangle, \langle gX \rangle}$,

$$
\begin{aligned}
\sigma_{Y,X} &= \sum_{g \in N_G(Y,X)/X} \sum_{\substack{(V,U) \in \Pi(G) \\ (\langle gY \rangle, \langle gX \rangle) \preceq (V,U)}} \beta_{V,U} |N_G(V,U)/U| \\
&= \sum_{\substack{g \in N_G(Y,X) \\ (V,U) \in \Pi(G) \\ (\langle gY \rangle, \langle gX \rangle) \preceq (V,U)}} \frac{1}{|X|} \beta_{V,U} |N_G(V,U)/U| \\
&= \sum_{\substack{(V,U) \in \Pi(G) \\ (Y,X) \preceq (V,U) \\ g \in U \cap N_G(Y,X)}} \frac{1}{|X|} \beta_{V,U} |N_G(V,U)/U|
\end{aligned}
$$

$$= \sum_{\substack{(V,U)\in[N_G(Y,X)\backslash\Pi(G)] \\ (Y,X)\preceq(V,U)}} \frac{|N_G(Y,X)/X|}{|N_G(Y,X,V,U)|}\beta_{V,U}|N_G(V,U)/U||U\cap N_G(Y,X)|$$

$$= |N_G(Y,X)/X| \sum_{\substack{(V,U)\in[N_G(Y,X)\backslash\Pi(G)] \\ (Y,X)\preceq(V,U)}} \beta_{V,U}\frac{|N_G(V,U)||U\cap N_G(Y,X)|}{|U||N_G(Y,X,V,U)|}$$

$$= |N_G(Y,X)/X| \sum_{\substack{(V,U)\in[N_G(Y,X)\backslash\Pi(G)] \\ (Y,X)\preceq(V,U)}} \beta_{V,U}|N_G(V,U):U\cdot N_G(Y,X,V,U)|.$$

Setting $\alpha_{Y,X} = \sigma_{Y,X}/|N_G(Y,X)/X|$, it follows that

$$\alpha_{Y,X} = \sum_{\substack{(V,U)\in N_G(Y,X)\backslash\Pi(G) \\ (Y,X)\preceq(V,U)}} \beta_{V,U}|N_G(V,U):U\cdot N_G(Y,X,V,U)|.$$

Since $|N_G(V,U):U\cdot N_G(Y,X,V,U)| = 1$ if $(Y,X) = (V,U)$, it follows that the transition matrix from the scalars $\beta_{V,U}$ to the scalars $\alpha_{Y,X}$ is triangular, with integer coefficients, equal to 1 on the diagonal. Hence it is invertible over $\mathbb{Z}$, and the numbers $\beta_{V,U}$ are all integers, for $(V,U)\in\Pi(G)$, if and only if the numbers $\alpha_{Y,X}$ are all integers, for $(Y,X)\in\Pi(G)$. This completes the proof. □

7. Prime spectrum

Notation 7.1. Let p denote either 0 or a prime number.

– If $(T,S)\in\Pi(G)$, let $I_{T,S,p}$ be the prime ideal of $\Xi(G)$ defined as the kernel of the ring homomorphism

$$\Xi(G) \xrightarrow{\phi_{T,S}} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z},$$

where the right-hand-side map is the projection.

– Let $\Theta(G)$ denote the set of triples (T,S,p), where $(T,S)\in\Pi(G)$ is such that $|N_G(T,S)/S|\not\equiv 0 \pmod{p}$.

The group G acts on $\Theta(G)$, by ${}^g(T,S,p) = ({}^gT,{}^gS,p)$, for $g\in G$, and the ideal $I_{T,S,p}$ only depends on the G-orbit of (T,S,p). Conversely, we have the following proposition.

PROPOSITION 7.2. *Let I be a prime ideal of $\Xi(G)$, and $R = \Xi(G)/I$. Denote by $\phi:\Xi(G)\to R$ the projection map, and denote by $p\geqslant 0$ the characteristic of R. Then $R\cong\mathbb{Z}/p\mathbb{Z}$. Furthermore:*

(i) *if $p = 0$, there exists a slice (T,S) of G such that $\phi = \phi_{T,S}$, and (T,S) is unique up to G-conjugation, with this property;*

(ii) *if $p > 0$, there exists a slice (T,S) of G such that ϕ is the reduction modulo p of $\phi_{T,S}$ and $N_G(T,S)/S$ is a p'-group, and (T,S) is unique up to G-conjugation, with these properties.*

In particular, there exists a unique $(T,S,p)\in\Theta(G)$, up to conjugation, such that $I = I_{T,S,p}$.

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Proof. Let (T, S) be a slice of G, minimal for the relation $\preceq$, such that $\langle T, S \rangle_G \notin I$. Then by Proposition 3.8, for any $(Y, X) \in \Pi(G)$,

$$\langle T, S \rangle_G \langle Y, X \rangle_G = \sum_{g \in [S \backslash G / X]} \langle T \cap {}^g Y, S \cap {}^g X \rangle_G$$

$$\equiv \sum_{\substack{g \in G/X \\ S \leqslant {}^g X \\ T \leqslant {}^g Y}} \langle T, S \rangle_G \pmod{I}$$

$$= \phi_{T,S}(\langle Y, X \rangle_G) \langle T, S \rangle_G.$$

Since I is prime, it follows that $\langle Y, X \rangle_G - \phi_{T,S}(\langle Y, X \rangle_G) 1_{\Xi(G)} \in I$. In particular, $R = \Xi(G)/I$ is generated by the image of $1_{\Xi(G)}$, hence $R \cong \mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of R. Since R is an integral domain, the number p is either 0 or a prime.

(i) If $p = 0$, then $R = \mathbb{Z}$, and $\phi = \phi_{T,S}$. And if $(T', S') \in \Pi(G)$ is such that $\phi_{T,S} = \phi_{T',S'}$, then both $\phi_{T,S}(\langle T', S' \rangle_G)$ and $\phi_{T',S'}(\langle T, S \rangle_G)$ are non-zero. Then there exist elements $g, g' \in G$ such that $(T^g, S^g) \preceq (T', S')$ and $(T'^{g'}, S'^{g'}) \preceq (T, S)$, so (T, S) and (T', S') are conjugate in G.

(ii) If $p > 0$, then $R = \mathbb{Z}/p\mathbb{Z}$, and ϕ is equal to the reduction of $\phi_{T,S}$ modulo p. Since $\phi(\langle T, S \rangle_G) = |N_G(T, S)/S|$ is non-zero in R, it follows that $N_G(T, S)/S$ is a p'-group. If (T', S') is another slice of G such that ϕ is the reduction modulo p of $\phi_{T',S'}$, and $N_G(T', S')/S'$ is a p'-group, then

$$|N_G(T, S)/S| = \phi_{T,S}(\langle T, S \rangle_G) \equiv \phi_{T',S'}(\langle T, S \rangle_G) \pmod{p}.$$

This is non-zero. Similarly, $|N_G(T, S)/S| \equiv \phi_{T,S}(\langle T', S' \rangle_G) \pmod{p}$ is non-zero. In particular, $\phi_{T',S'}(\langle T, S \rangle_G)$ and $\phi_{T,S}(\langle T', S' \rangle_G)$ are both non-zero, and it follows as above that (T, S) and (T', S') are conjugate in G. $\qquad\square$

Notation 7.3. Let p be a prime number.

- Let $\Pi_p(G)$ denote the subset of $\Pi(G)$ consisting of the slices (T, S) such that $N_G(T, S)/S$ is a p'-group.
- For any $(T, S) \in \Pi(G)$, let $(T, S)\widehat{\;}_p$ denote the unique element (V, U) of $\Pi_p(G)$, up to conjugation, such that $I_{T,S,p} = I_{V,U,p}$.

PROPOSITION 7.4. *Let p be a prime number. If (T, S) is a slice of G, let $(T, S)_p^+$ denote a slice of the form (PT, PS) of G, where P is a Sylow p-subgroup of $N_G(T, S)$.*

Define inductively an increasing sequence (T_n, S_n) in $(\Pi(G), \preceq)$ by $(T_0, S_0) = (T, S)$, and $(T_{n+1}, S_{n+1}) = (T_n, S_n)_p^+$, for $n \in \mathbb{N}$. Then $(T, S)\widehat{\;}_p$ is conjugate to the largest term (T_∞, S_∞) of the sequence (T_n, S_n).

Proof. The proof of Proposition 7.2 shows that $(T, S)\widehat{\;}_p$ is a minimal element (V, U) of the poset $(\Pi(G), \preceq)$ such that

$$\phi_{T,S}(V, U) = |\{g \in G/U \mid (T^g, S^g) \preceq (V, U)\}| \not\equiv 0 \pmod{p}.$$

Thus one can assume that $(T, S) \preceq (V, U)$. But $\phi_{T,S} \equiv \phi_{PT,PS} \pmod{p}$ by Corollary 4.4, for any p-subgroup P of $N_G(T, S)$, hence one can also assume that $(T, S)_p^+ \preceq (V, U)$, and by induction, that $(T_\infty, S_\infty) \preceq (V, U)$. Moreover, $\phi_{T_\infty, S_\infty} \equiv \phi_{V,U} \pmod{p}$. As $N_G(T_\infty, S_\infty)/S_\infty$ is a p'-group, it follows that $(T_\infty, S_\infty) = (V, U)$, as was to be shown. $\qquad\square$

Remark 7.5. Let $(T, S) \in \Pi(G)$ and $(V, U) \in \Pi_p(G)$. It is easy to check, by induction on the integer n such that $(T_n, S_n) = (T_\infty, S_\infty)$, that $(T, S)\widehat{_p}$ is equal to (V, U) if and only if T is a subnormal subgroup of V, if S is a subnormal subgroup of U, if $|U : S|$ is a power of p, and if the set $T \cdot U$ is equal to V.

PROPOSITION 7.6. *Let $(T, S, p), (T', S', p')$ be elements of $\Theta(G)$. Then $I_{T',S',p'} \subseteq I_{T,S,p}$ if and only if:*

- *either $p' = p$ and the slices (T', S') and (T, S) are conjugate in G;*
- *or $p' = 0$ and $p > 0$, and the slices $(T', S')\widehat{_p}$ and (T, S) are conjugate in G.*

Proof. Set $I = I_{T,S,p}$ and $I' = I_{T',S',p'}$. Then $\Xi(G)/I' \cong \mathbb{Z}/p'\mathbb{Z}$ maps surjectively to $\Xi(G)/I \cong \mathbb{Z}/p\mathbb{Z}$. Thus if $p = p'$, this projection map is an isomorphism, hence $I = I'$ and the slices (T, S) and (T', S') are conjugate in G. And if $p \neq p'$, then $p' = 0$ and $p > 0$. The morphism $\phi_{T,S}$ is equal to the reduction modulo p of the morphism $\phi_{T',S'}$. In other words, $I_{T',S',p} = I_{T,S,p}$, hence (T, S) is conjugate to $(T', S')\widehat{_p}$. □

COROLLARY 7.7. *Let p be a prime number, and let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the set $\mathbb{Z} - p\mathbb{Z}$. Let $\Theta_p(G)$ denote the subset of $\Theta(G)$ consisting of triples $(T, S, 0)$, for $(T, S) \in \Pi(G)$, and (T, S, p), for $(T, S) \in \Pi_p(G)$.*

 (i) *The prime ideals of the ring $\mathbb{Z}_{(p)}\Xi(G)$ are the ideals $\mathbb{Z}_{(p)}I_{T,S,q}$, for $(T, S, q) \in \Theta_p(G)$.*

 (ii) *If $(T, S, q), (T', S', q') \in \Theta_p(G)$, then $\mathbb{Z}_{(p)}I_{T',S',q'} \subseteq \mathbb{Z}_{(p)}I_{T,S,q}$ if and only if:*
 - *either $q = q'$, and the slices (T, S) and (T', S') are conjugate in G;*
 - *or $q' = 0$, $q = p$, and the slices $(T', S')\widehat{_p}$ and (T, S) are conjugate in G.*

 (iii) *The connected components of the spectrum of $\mathbb{Z}_{(p)}\Xi(G)$ are indexed by the conjugacy classes of $\Pi_p(G)$. The component indexed by $(T, S) \in \Pi_p(G)$ consists of a unique maximal element $\mathbb{Z}_{(p)}I_{T,S,p}$, and of the ideals $\mathbb{Z}_{(p)}I_{T',S',0}$, where $(T', S') \in \Pi(G)$ is such that $(T', S')\widehat{_p}$ is conjugate to (T, S) in G.*

Proof. The prime ideals of $\mathbb{Z}_{(p)}\Xi(G)$ are of the form $\mathbb{Z}_{(p)}I$, where I is a prime ideal of $\Xi(G)$ such that $I \cap (\mathbb{Z} - p\mathbb{Z}) = \emptyset$. Equivalently, $I = I_{T,S,0}$ or $I = I_{T,S,p}$. This proves assertion (i). Now assertion (ii) follows from Proposition 7.6, and assertion (iii) follows from assertion (ii). □

COROLLARY 7.8. (i) *The primitive idempotents of the ring $\mathbb{Z}_{(p)}\Xi(G)$ are indexed by the conjugacy classes of $\Pi_p(G)$. The primitive idempotent $\eta_{V,U}^G$ indexed by $(V, U) \in \Pi_p(G)$ is equal to*

$$\eta_{V,U}^G = \sum_{\substack{(T,S) \in [\Pi(G)] \\ (T,S)\widehat{_p} =_G (T,S)}} \xi_{T,S}^G.$$

 (ii) *Let π be a set of prime numbers, and $\mathbb{Z}_{(\pi)}$ be the localization of $\mathbb{Z}$ relative to $\mathbb{Z} - \bigcup_{p \in \pi} p\mathbb{Z}$. Let $\mathcal{F}$ be a set of slices of G, invariant by G-conjugation, and $[\mathcal{F}]$ be a set of representatives of G-conjugacy classes of $\mathcal{F}$. Then the following conditions are equivalent:*
 (a) *The idempotent*

$$\xi_{\mathcal{F}}^G = \sum_{(T,S) \in [\mathcal{F}]} \xi_{T,S}^G$$

 of $\mathbb{Q}\Xi(G)$ lies in $\mathbb{Z}_{(\pi)}\Xi(G)$.

(b) *Let $(T, S) \in \Pi(G)$, and let P be a p-subgroup of $N_G(T, S)$, for some $p \in \pi$. Then $(T, S) \in \mathcal{F}$ if and only if $(PT, PS) \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a set of slices of G, invariant by G-conjugation, and $[\mathcal{F}]$ be a set of representatives of G-conjugacy classes of $\mathcal{F}$. The idempotent

$$\xi_{\mathcal{F}}^G = \sum_{(T,S) \in [\mathcal{F}]} \xi_{T,S}^G$$

of $\mathbb{Q}\Xi(G)$ lies in $\mathbb{Z}_{(p)}\Xi(G)$, for some prime p, if and only if there exists an integer m, not divisible by p, such that $u = m\xi_{\mathcal{F}}^G \in \Xi(G)$. Let $(T, S) \in \Pi(G)$, and let P be a p-subgroup of $N_G(T, S)$. The integer $\phi_{T,S}(u)$ is equal to m if $(T, S) \in \mathcal{F}$, and to 0 otherwise. Hence it is coprime to p if and only if $(T, S) \in \mathcal{F}$. Since $\phi_{T,S}$ and $\phi_{PT,PS}$ are congruent modulo p, it follows that $(T, S) \in \mathcal{F}$ if and only if $(PT, PS) \in \mathcal{F}$.

Hence if (T, S) and (T', S') are slices of G such that $(T, S)\widehat{}_p =_G (T', S')\widehat{}_p$, then $(T, S) \in \mathcal{F}$ if and only if $(T', S') \in \mathcal{F}$. Thus $\mathcal{F}$ is a disjoint union of sets of the form

$$E_{V,U} = \{(T, S) \in \Pi(G) \mid (T, S)\widehat{}_p =_G (V, U)\},$$

for some slices $(V, U) \in \Pi_p(G)$. In other words, the idempotent $\xi_{\mathcal{F}}^G$ is a sum of some idempotents $\eta_{V,U}^G$, for $(V, U) \in \Pi_p(G)$.

But the primitive idempotents of the ring $\mathbb{Z}_{(p)}\Xi(G)$ are in one-to-one correspondence with the connected components of its spectrum, which precisely are indexed by the conjugacy classes of $\Pi_p(G)$. It follows that $\xi_{\mathcal{F}}^G = \eta_{V,U}^G$ is equal to the idempotent corresponding to the component indexed by (V, U), for any $(V, U) \in \Pi_p(G)$. This proves assertion (i). This also proves assertion (ii) in the case where π consists of a single prime number.

For the general case, observe that $\xi_{\mathcal{F}}^G$ lies in $\mathbb{Z}_{(\pi)}\Xi(G)$ if and only if it lies in $\mathbb{Z}_{(p)}\Xi(G)$, for any $p \in \pi$. $\square$

THEOREM 7.9. *Let G be a finite group.*

(i) *Let $\sim$ denote the finest equivalence relation on the set $\Pi(G)$ such that for any $(T, S), (T', S') \in \Pi(G)$,*

$$\exists p, \quad (T, S)\widehat{}_p =_G (T', S')\widehat{}_p \Longrightarrow (T, S) \sim (T', S').$$

Then the primitive idempotents of $\Xi(G)$ are indexed by the equivalence classes of $\Pi(G)$ for the relation $\sim$. The idempotent $\xi_{\mathcal{C}}^G$ indexed by the component $\mathcal{C}$ is equal to

$$\xi_{\mathcal{C}}^G = \sum_{(T,S) \in [\mathcal{C}]} \xi_{T,S}^G,$$

where $[\mathcal{C}]$ is a set of representatives of G-conjugacy classes in $\mathcal{C}$.

(ii) *The prime spectrum of $\Xi(G)$ is connected if and only if G is solvable.*

Proof. Assertion (i) follows from Corollary 7.8, applied to the set π of all primes.

For assertion (ii), observe that the spectrum of $\Xi(G)$ is connected if and only if 1 is a primitive idempotent of $\Xi(G)$. This means that for any two slices (T, S) and (T', S') of G, there exists a sequence (T_i, S_i) of slices, for $i \in \{0, \dots, n\}$, and a sequence p_i of prime numbers,

for $i \in \{0, \ldots, n-1\}$, such that

$$(T, S) = (T_0, S_0) \overset{p_0}{\sim} (T_1, S_1) \overset{p_1}{\sim} \cdots \overset{p_{n-2}}{\sim} (T_{n-1}, S_{n-1}) \overset{p_{n-1}}{\sim} (T_n, S_n) = (T', S'),$$

where the notation $(Y, X) \overset{p}{\sim} (Y', X')$ means that $(Y, X)\widehat{_p} =_G (Y', X')\widehat{_p}$.

But clearly, if $(Y, X) \overset{p}{\sim} (Y', X')$, then the slices $(O^p(Y), O^p(X))$ and $(O^p(Y'), O^p(X'))$ are conjugate in G. Hence the slices $(D^\infty(Y), D^\infty(X))$ and $(D^\infty(Y'), D^\infty(X'))$ are conjugate in G, where $D^\infty(H)$ denotes the last term in the derived series of the group H.

If the spectrum of G is connected, taking $(T, S) = (G, G)$ and $(T', S') = (\mathbf{1}, \mathbf{1})$, it follows that $D^\infty(G) = \mathbf{1}$, i.e. that G is solvable.

Conversely, if G is solvable, let (T, S) be a slice of G. Then S is solvable, and there exists a prime p such that $O^p(S) < S$. Let P be a Sylow p-subgroup of S. Then $P \leqslant N_G(T, S)$ and $(T, S) = (PT, PO^p(S))$. Thus $(T, S) \overset{p}{\sim} (T, O^p(S))$. By induction, there is a sequence of prime numbers p_i, for $i \in \{1, \ldots, k\}$, such that

$$(T, S) = (T, S_0) \overset{p_0}{\sim} (T, S_1) \overset{p_1}{\sim} \cdots \overset{p_{k-1}}{\sim} (T, S_k) \overset{p_k}{\sim} (T, \mathbf{1}),$$

where $S_{i+1} = O^{p_i}(S_i)$.

Now T is solvable, so there exists a prime q such that $O^q(T) < T$. If Q is a Sylow q-subgroup of T, then $Q \leqslant N_G(T, \mathbf{1})$, and $(T, \mathbf{1}) \overset{q}{\sim} (O^q(T), \mathbf{1})$. Hence there exists a sequence of primes q_j, for $j \in \{0, \ldots, l\}$, such that

$$(T, \mathbf{1}) = (T_0, \mathbf{1}) \overset{q_0}{\sim} (T_1, \mathbf{1}) \overset{q_1}{\sim} \cdots \overset{q_{l-1}}{\sim} (T_l, \mathbf{1}) \overset{q_l}{\sim} (\mathbf{1}, \mathbf{1}),$$

where $T_{j+1} = O^{q_j}(T_j)$. This shows that the spectrum of $\Xi(G)$ is connected. $\qquad\square$

8. Unit group

8.1 When G is a finite group, denote by $\Xi(G)^\times$ the group of invertible elements of the ring $\Xi(G)$. It follows from Theorem 4.6 that the restricted ghost map yields an injective group homomorphism

$$\Phi^\times : \Xi(G)^\times \hookrightarrow \prod_{(T,S)\in\Pi(G)} \mathbb{Z}^\times.$$

The following lemma is a straightforward consequence of the existence of this injective group homomorphism.

LEMMA 8.2. *Let G be a finite group, and let $u \in \Xi(G)$. The following conditions are equivalent.*

    (i) $u \in \Xi(G)^\times$.
   (ii) $\phi_{T,S}(u) \in \{\pm 1\}$, *for any $(T, S) \in \Pi(G)$.*
  (iii) $u^2 = 1$.

In particular, $\Xi(G)^\times$ is a finite elementary abelian 2-group.

The main motivation in considering the group $\Xi(G)^\times$ lies in the following proposition, which extends a theorem of tom Dieck [tomD79, Proposition 1.5.1] about the unit group of the usual Burnside ring.

PROPOSITION 8.3. *The Feit–Thompson theorem is equivalent to the statement that, if G has odd order, then $\Xi(G)^\times = \{\pm 1\}$.*

Proof. The first observation is that for any finite group G, by Lemma 8.2, the correspondences $u \mapsto (1-u)/2$ and $e \mapsto 1-2e$ are mutually inverse bijections between $\Xi(G)^\times$ and the set of idempotents $e \in \mathbb{Q}\Xi(G)$ such that $2e \in \Xi(G)$.

Now Theorem 5.2 shows that $|G|e \in \Xi(G)$, for any idempotent e of $\mathbb{Q}\Xi(G)$. Hence if G has odd order, and if e is an idempotent of $\Xi(G)$ such that $2e \in \Xi(G)$, then $(2, |G|)e = e \in \Xi(G)$. Thus if $|G|$ is odd, the set $\Xi(G)^\times$ is in one-to-one correspondence with the set of idempotents of the ring $\Xi(G)$. By a standard argument from commutative ring theory, this set is in one-to-one correspondence with the set of connected components of the spectrum of $\Xi(G)$. It follows that if G has odd order, the spectrum of G is connected if and only if $\Xi(G)^\times = \{\pm 1\}$. By Theorem 7.9, this is equivalent to saying that G is solvable. $\qquad\square$

The following theorem is an analogue of Yoshida's characterization [Yos90] of the unit group of the usual Burnside ring.

THEOREM 8.4. *Let G be a finite group, and let* $\mathsf{m} = (m_{T,S})_{(T,S) \in \Pi(G)}$ *be a sequence of integers in $\{\pm 1\}$ indexed by $\Pi(G)$, constant on G-conjugacy classes of slices. Then the sequence $[m] = (m_{T,S})_{(T,S) \in [\Pi(G)]}$ of representatives lies in the image of the restricted ghost map $\Phi^\times$ if and only if, for any $(T, S) \in \Pi(G)$, the map*

$$g \in N_G(T,S)/S \mapsto m_{\langle gT\rangle, \langle gS\rangle}/m_{T,S} \in \{\pm 1\}$$

is a group homomorphism.

Proof. Let $X \xrightarrow{f} Y$ be a morphism of finite G-sets. It follows from Lemma 4.3 that, for any $(T, S) \in \Pi(G)$, the monoid of endomorphisms of $G/S \to G/T$ in the category $G\text{-Mor}$ is actually a group, isomorphic to $N_G(T, S)/S$. This group acts on the set of morphisms from $G/S \to G/T$ to $X \xrightarrow{f} Y$, by pre-composition: if

$$
\begin{array}{ccc}
G/S & \longrightarrow & G/T \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

is a morphism in $G\text{-Mor}$, and if $g \in N_G(T, S)$, the morphism ${}^g(\alpha, \beta) = ({}^g\alpha, {}^g\beta)$ is defined by $({}^g\alpha)(xS) = xgS$ and $({}^g\beta)(xT) = xgT$, for any $g \in T$. The morphism (α, β) is invariant under $g \in N_G(T, S)$ if and only if $\alpha(gS) = \alpha(S)$, i.e. if (α, β) factors as

$$
\begin{array}{ccc}
G/S & \longrightarrow & G/gT \\
\downarrow & & \downarrow \\
G/\langle gS\rangle & \longrightarrow & G/\langle gT\rangle \\
\downarrow{\scriptstyle \bar\alpha} & & \downarrow{\scriptstyle \bar\beta} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

It follows that the number of fixed points of $g \in N_G(T, S)/S$ on the set of homomorphisms from $G/S \to G/T$ to $X \xrightarrow{f} Y$, i.e. the value at gS of the corresponding permutation character $\theta_{T,S}$ of $N_G(T, S)/S$, is equal to $\phi_{\langle gT\rangle, \langle gS\rangle}(X \xrightarrow{f} Y)$. This shows more generally that, for any $u \in \Xi(G)$, the correspondence

$$\theta_{T,S} : g \in N_G(T,S)/S \mapsto \phi_{\langle gT\rangle, \langle gS\rangle}(u)$$

is a generalized character of the group $N_G(T, S)/S$.

Now if $u \in \Xi(G)^{\times}$, this generalized character has all its values in $\{\pm 1\}$. It follows that $\langle \theta_{T,S}, \theta_{T,S} \rangle_G = 1$, hence $\theta_{T,S}$ is, up to sign, equal to an irreducible character of $N_G(T,S)/S$ of degree one. Hence $\theta_{T,S}/\theta_{T,S}(1)$ is a group homomorphism from $N_G(T,S)/S$ to $\{\pm 1\}$. Thus if $\mathsf{m} \in \operatorname{Im}(\Phi^{\times})$, then the map $g \in N_G(T,S)/S \mapsto m_{\langle gT \rangle, \langle gS \rangle}/m_{T,S} \in \{\pm 1\}$ is a group homomorphism.

Conversely, if $\mathsf{m} = (m_{T,S})_{(T,S) \in \Pi(G)}$ is a G-invariant sequence with values in $\{\pm 1\}$, such that for any $(T,S) \in \Pi(G)$, the map $g \in N_G(T,S)/S \mapsto m_{\langle gT \rangle, \langle gS \rangle}/m_{T,S} \in \{\pm 1\}$ is a group homomorphism $\theta_{T,S}$, then

$$\sum_{g \in N_G(T,S)/S} m_{\langle gT \rangle, \langle gS \rangle} = m_{T,S} |N_G(T,S)/S| \langle \theta_{T,S}, 1 \rangle_G$$

is an integer multiple of $|N_G(T,S)/S|$. By Theorem 6.1, it follows that $[\mathsf{m}] = \Phi(u)$, for some $u \in \Xi(G)$. Then $u \in \Xi(G)^{\times}$, by Lemma 8.2, and this completes the proof. $\square$

8.5 In the case of the usual Burnside ring $B(G)$, the correspondence sending a finite group G to the unit group $B(G)^{\times}$ can be endowed with a biset functor structure, using *tensor induction* (see, for example, [Bou07, Proposition 5.5]). One may ask whether a similar structure exists for the group of units of the section Burnside ring.

PROPOSITION 8.6. *The correspondence sending a finite group G to $\Xi(G)^{\times}$ cannot be endowed with a biset functor structure.*

Proof. If such a structure exists, one may view it as a biset functor $\Xi^{\times}$ with values in $\mathbb{F}_2$-vector spaces. Obviously $\Xi^{\times}(\mathbf{1}) = \mathbb{F}_2$. This shows that there are two subfunctors $F_2 \subset F_1$ of $\Xi^{\times}$ such that F_1/F_2 is isomorphic to the simple functor $S_{\mathbf{1}, \mathbb{F}_2}$.

An easy computation (a special case of Theorem A.13 in the appendix) shows, moreover, that $\Xi^{\times}(C_2) \cong (\mathbb{F}_2)^3$, where C_2 is a group of order two. As $S_{\mathbf{1}, \mathbb{F}_2}(C_2) \cong (\mathbb{F}_2)^2$ (see, for example, [Bou10, Proposition 4.4.6]), this shows that either $(F/F_1)(C_2) \cong \mathbb{F}_2$ and $F_2(C_2) = \{0\}$, or $F(C_2) = F_1(C_2)$ and $F_2(C_2) \cong \mathbb{F}_2$. In the former case, there are subfunctors F_3 and F_4 of F with $F_1 \subseteq F_4 \subset F_3 \subseteq F$ and $F_3/F_4 \cong S_{C_2, \mathbb{F}_2}$. In the latter case, there are subfunctors $F_4 \subset F_3$ of F_2 such that $F_3/F_4 \cong S_{C_2, \mathbb{F}_2}$. In any case

$$\dim_{\mathbb{F}_2} \Xi^{\times}((C_2)^2) \geqslant \dim_{\mathbb{F}_2} S_{\mathbf{1}, \mathbb{F}_2}((C_2)^2) + \dim_{\mathbb{F}_2} S_{C_2, \mathbb{F}_2}((C_2)^2).$$

Now $\dim_{\mathbb{F}_2} S_{\mathbf{1}, \mathbb{F}_2}((C_2)^2) = 4$ by [Bou10, Proposition 4.4.6 or Corollary 10.5.6], and one can check that $\dim_{\mathbb{F}_2} S_{C_2, \mathbb{F}_2}((C_2)^2) = 5$ by inspection, using [Bou10, Proposition 4.4.6]. Thus $\dim_{\mathbb{F}_2} \Xi^{\times}((C_2)^2) \geqslant 9$.

When G is a finite abelian group, the unit group $\Xi(G)^{\times}$ can be determined explicitly (see Theorem A.13). In particular,

$$\Xi((C_2)^2)^{\times} \cong (\mathbb{F}_2)^7. \tag{8.7}$$

This contradiction shows that the biset functor $\Xi^{\times}$ cannot exist. $\square$

Remark 8.8. Boltje and Pfeiffer [BP07] have described an efficient algorithm to compute the unit group of the ordinary Burnside ring. A straightforward adaptation of this algorithm to the ring $\Xi(G)$ allows for a quick computation of the group $\Xi(G)^{\times}$, for not too large finite groups G, using GAP4 [GAP08]. These computations agree, in particular, with (8.7).

S. Bouc

Part II. The section Burnside ring

9. Galois morphisms of G-sets

DEFINITION 9.1. Let G be a group. A morphism $f : X \to Y$ of G-sets is *a Galois morphism* if for any $x, x' \in X$ such that $f(x) = f(x')$, there exists $\varphi \in \mathrm{Aut}_G(X)$ such that $f \circ \varphi = f$ and $\varphi(x) = x'$.

Example 9.2. Any injective morphism of G-sets is a Galois morphism, for trivial reasons.

PROPOSITION 9.3. *Let* $f : X \to Y$ *be a morphism of G-sets. The following conditions are equivalent.*

   (i) f *is a Galois morphism.*
   (ii) $\forall x, x' \in X,\ f(x) = f(x') \Longrightarrow G_x = G_{x'}.$

Proof. Suppose first that f is a Galois morphism. Then if $f(x) = f(x')$, there exists a G-automorphism φ of X such that $\varphi(x) = x'$. This implies that $G_x \leqslant G_{x'}$, hence $G_x = G_{x'}$ by symmetry. Thus condition (i) implies condition (ii).

Conversely, suppose that condition (ii) holds, and let $x, x' \in X$ with $f(x) = f(x')$. There are two cases.

   – Either x and x' are in the same G-orbit ω. Let $Z = X - \omega$. Define $\varphi : X \to X$ by $\varphi(t) = t$ if $t \in Z$ and $\varphi(ux) = ux'$ if $u \in G$. This is a well-defined G-automorphism of X, since $G_x = G_{x'}$. Clearly $\varphi(x) = x'$, and $f \circ \varphi(t) = f(t)$ if $t \in Z$. Moreover,

$$f \circ \varphi(ux) = f(ux') = uf(x') = uf(x) = f(ux)$$

for any $u \in G$. Hence $f \circ \varphi = f$.

   – If x and x' are in different G-orbits ω and ω', respectively, let $Z = X - (\omega \sqcup \omega')$. Define $\varphi : X \to X$ by $\varphi(t) = t$ if $t \in Z$, and $\varphi(ux) = ux'$ and $\varphi(ux') = ux$, for any $u \in G$. This is a well-defined G-automorphism of X, since $G_x = G_{x'}$. Clearly $\varphi(x) = x'$, and $f \circ \varphi(t) = f(t)$ if $t \in Z$. Moreover,

$$f \circ \varphi(ux) = f(ux') = uf(x') = uf(x) = f(ux),$$

for any $u \in G$. Also

$$f \circ \varphi(ux') = f(ux) = uf(x) = uf(x') = f(ux'),$$

for any $u \in G$. Hence $f \circ \varphi = f$.

It follows that f is a Galois morphism, and condition (ii) implies condition (i). $\square$

COROLLARY 9.4. *Let* $f : X \to Y$ *be a morphism of G-sets. Then f is a Galois morphism if and only if, for any $y \in f(X)$, there exists a normal subgroup N_y of G_y such that $G_x = N_y$ for any $x \in f^{-1}(y)$.*

Proof. Suppose first that f is a Galois morphism. If $x \in X$, set $y = f(x)$ and choose $g \in G_y$. Then $f(gx) = gy = y = f(x)$, thus $G_{gx} = {}^g G_x = G_x$, thus $G_x \trianglelefteq G_y$. Moreover, G_x does not depend on $x \in f^{-1}(y)$, by Proposition 9.3.

Conversely, suppose that for any $y \in f(X)$, there exists $N_y \trianglelefteq G_y$ such that $G_x = N_y$ for any $x \in f^{-1}(y)$. Then obviously $G_x = G_{x'} = N_y$ if $f(x) = f(x') = y$, so f is a Galois morphism, by Proposition 9.3. $\square$

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Remark 9.5. In particular, when (T, S) is a slice of G, the projection morphism $G/S \to G/T$ is a Galois morphism of G-sets if and only if $S \trianglelefteq T$, i.e. if (T, S) is a *section* of G.

COROLLARY 9.6. *Let* $f : X \to Y$ *be a Galois morphism of G-sets. If X_1 is a G-subset of X, the restricted morphism* $f_{|X_1} : X_1 \to f(X_1)$ *is a Galois morphism of G-sets.*

Proof. This is a straightforward consequence of Proposition 9.3. $\square$

LEMMA 9.7. *Let* $f : X \to Y$ *be a morphism of G-sets, and $j : Y \hookrightarrow Z$ be an injective morphism of G-sets. Then f is a Galois morphism if and only if $j \circ f$ is a Galois morphism.*

Proof. This is straightforward. $\square$

LEMMA 9.8. *Let*

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & Y \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\
Z & \xrightarrow[\ d\ ]{} & T
\end{array}
$$

be a cartesian square of G-sets. If c is a Galois morphism, then b is a Galois morphism.

Proof. Suppose that $x, x' \in X$ are such that $b(x) = b(x')$. Then

$$ca(x) = db(x) = db(x') = ca(x').$$

If c is a Galois morphism, it follows that $G_{a(x)} = G_{a(x')}$. Let $g \in G_x$. Then $g \in G_{a(x)} = G_{a(x')}$ and $g \in G_{b(x)} = G_{b(x')}$, thus

$$b(x') = gb(x') = b(gx'),$$
$$a(x') = ga(x') = a(gx').$$

It follows that $x' = gx'$, so $G_x \leqslant G_{x'}$, and $G_x = G_{x'}$ by symmetry. Hence b is a Galois morphism, by Proposition 9.3. $\square$

Remark 9.9. In particular, any morphism isomorphic to a Galois morphism (i.e. when a and d are isomorphisms) is a Galois morphism.

PROPOSITION 9.10. *A morphism $f : X \to Y_1 \sqcup Y_2$ is a Galois morphism if and only if the restricted morphisms $f^{-1}(Y_1) \to Y_1$ and $f^{-1}(Y_2) \to Y_2$ are Galois morphisms.*

Proof. Set $X_i = f^{-1}(Y_i)$, and denote by $f_i : X_i \to Y_i$ the restriction of f, for $i = 1, 2$. Assume first that f is a Galois morphism. If $x, x' \in X_1$ have the same image under f_1, then they have the same image under f, and $G_x = G_{x'}$. So f_1 is a Galois morphism, and f_2 is also a Galois morphism, by symmetry.

Conversely, suppose that f_1 and f_2 are Galois morphisms. If $x, x' \in X$ are such that $f(x) = f(x') \in Y_1$, then $x, x' \in X_1$ and $f_1(x) = f_1(x')$. Thus $G_x = G_{x'}$, as f_1 is a Galois morphism. If $f(x) = f(x') \in Y_2$, the argument is similar, with f_1 replaced by f_2. In any case $G_x = G_{x'}$, and f is a Galois morphism, by Proposition 9.3. $\square$

Let G and H be groups. A morphism $f : U \to U'$ of (H, G)-bisets is called a Galois morphism if it is a Galois morphism of $(H \times G^{\mathrm{op}})$-sets. Then we have the following proposition.

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PROPOSITION 9.11. *Let G, H and K be groups. If $f : U \to U'$ is a Galois morphism of (H, G)-bisets, and $g : V \to V'$ is a Galois morphism of (K, H)-bisets, then $g \times_H f : V \times_H U \to V' \times_H U'$ is a Galois morphism of (K, G)-bisets.*

Proof. Let $(v,_H u)$ and $(v',_H u')$ be elements of $V \times_H U$ with the same image under $g \times_H f$. This means that there exists $h \in H$ such that

$$g(v') = g(v)h = g(vh) \quad \text{and} \quad f(u') = h^{-1}f(u) = f(h^{-1}u).$$

As g is a Galois morphism, the stabilizers of v' and vh in $K \times H^{\mathrm{op}}$ are equal. Similarly, as f is a Galois morphism, the stabilizers of u' and $h^{-1}u$ in $H \times G^{\mathrm{op}}$ are equal.

Now let $(k, g) \in K \times G^{\mathrm{op}}$ such that $k(v',_H u')g^{-1} = (v',_H u')$. It means that there exists $a \in H$ such that $kv'a^{-1} = v'$ and $au'g^{-1} = u'$. Hence

$$kvha^{-1} = vh \quad \text{and} \quad ah^{-1}ug^{-1} = h^{-1}u.$$

It follows that

$$
\begin{aligned}
k(v,_H u)g^{-1} = (kv,_H ug^{-1}) &= (vhah^{-1},_H ug^{-1}) \\
&= (v,_H hah^{-1}ug^{-1}) = (v,_H hh^{-1}u) = (v,_H u).
\end{aligned}
$$

By symmetry, the stabilizers of $(v,_H u)$ and $(v',_H u')$ in $K \times G^{\mathrm{op}}$ are equal. Thus $(g \times_H f)$ is a Galois morphism of (K, G)-bisets. $\square$

COROLLARY 9.12. *Let G and H be finite groups, and U be an (H, G)-biset. If $f : X \to Y$ is a Galois morphism of G-sets, then*

$$U \times_G f : U \times_G X \to U \times_G Y \quad \text{and} \quad U \circ_G f : U \circ_G X \to U \circ_G Y$$

are Galois morphisms of H-sets.

Proof. The assertion on $U \times_G f$ is the special case of Proposition 9.11, with the following changes in the notation: the group G becomes the trivial group, the group H becomes G, the group K becomes H, the biset U becomes X, the biset V becomes U, and the map g becomes the identity map of U.

Now recall from [Bou96] that $U \circ_G X$ is the H-subset of $U \times_G X$ defined by

$$U \circ_G X = \{(u,_G x) \in U \times_G X \mid \forall g \in G, ug = u \implies gx = x\},$$

and that the map $U \circ_G f$ is the restriction of $U \times_G f$ to $U \circ_G X$. By Lemma 9.7 and Corollary 9.6, the morphism $U \circ_G f$ is also a Galois morphism of H-sets. $\square$

COROLLARY 9.13. *Let $f : X \to Y$ be a Galois morphism of G-sets. If H is a subgroup of G, the restriction $\mathrm{Res}_H^G f : \mathrm{Res}_H^G X \to \mathrm{Res}_H^G Y$ is a Galois morphism of H-sets.*

Proof. In Corollary 9.12, set $U = G$, viewed as an (H, G)-biset for left and right multiplication. $\square$

COROLLARY 9.14. *Let G and H be groups. If $f : X \to Y$ is a Galois morphism of G sets and $g : Z \to T$ is a Galois morphism of H-sets, then $f \times g : X \times Z \to Y \times T$ is a Galois morphism of $(G \times H)$-sets.*

Proof. Consider f as a morphism of $(G, \mathbf{1})$-bisets, and g as a morphism of $(\mathbf{1}, H^{\mathrm{op}})$-bisets. $\square$

Notation 9.15. Let $G\text{-}\mathsf{Mor}^{\mathrm{Gal}}$ denote the full subcategory of $G\text{-}\mathsf{Mor}$ consisting of Galois morphisms of G-sets.

Notation 9.16. Let $X \xrightarrow{f} Y$ be a morphism of G-sets. For $x \in X$, set
$$G_x^f = \langle G_z \mid \forall z \in X, f(z) = f(x) \rangle.$$
Let $\sim_f$ be the relation on X defined by
$$x \sim_f x' \Leftrightarrow \exists g \in G_x^f, \quad gx = x'.$$

LEMMA 9.17. *With this notation:*

(i) *the relation $\sim_f$ is an equivalence relation on X. Let X_f^{Gal} denote the set of equivalence classes, and let $\gamma_{X,f} : X \to X_f^{\mathrm{Gal}}$ denote the projection map;*

(ii) *if $x, x' \in X$ and $x \sim_f x'$, then $gx \sim_f gx'$ for any $g \in G$. Hence there exists a unique structure of G-set on X_f^{Gal} such that $\gamma_{X,f}$ is a morphism of G-sets;*

(iii) *there is a unique map $f^{\mathrm{Gal}} : X_f^{\mathrm{Gal}} \to Y$ such that the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ f\ \ } & Y \\
{\scriptstyle \gamma_{X,f}} \downarrow & \nearrow {\scriptstyle f^{\mathrm{Gal}}} & \\
X_f^{\mathrm{Gal}} & &
\end{array}
$$

is commutative.

Proof. For assertion (i), the relation $\sim_f$ is clearly reflexive, since $G_x \leqslant G_x^f$. Now if $x, x' \in X$ and $x \sim_f x'$, let $g \in G_x^f$ such that $gx = x'$. There exist $r \in \mathbb{N}$, elements $z_1, \ldots, z_r$ in X and elements $g_1, \ldots, g_r$ of G such that $g_i z_i = z_i$ and $f(z_i) = f(x)$, for $i = 1, \ldots, r$, and such that $g = g_1 \cdots g_r$. It follows that
$$f(x') = g_1 \cdots g_r f(x) = g_1 \cdots g_r f(z_r) = g_1 \cdots g_{r-1} f(g_r z_r) = g_1 \cdots g_{r-1} f(z_r)$$
$$= g_1 \cdots g_{r-1} f(x),$$
thus $f(x') = f(x)$, by induction on r. It follows that $G_{x'}^f = G_x^f$, hence that $x' \sim_f x$, since $x = g^{-1} x'$. So the relation $\sim_f$ is symmetric.

Finally, $\sim_f$ is transitive: if $x, x', x'' \in X$, if $x \sim_f x'$ and $x' \sim_f x''$, there exist $g \in G_x^f$ and $g' \in G_{x'}^f$ such that $gx = x'$ and $g'x' = x''$. But the previous argument shows that $G_x^f = G_{x'}^f = G_{x''}^f$. Hence $g'g \in G_x^f$ and $x \sim_f x''$, since $g'gx = x''$.

Assertion (ii) follows from the straightforward fact that $^g(G_x^f) = G_{gx}^f$ for any $x \in X$ and any $g \in G$. Assertion (iii) follows as well, since $x \sim_f x'$ implies that $f(x) = f(x')$. $\qquad\square$

PROPOSITION 9.18. *Let G be a finite group.*

(i) *Let $X \xrightarrow{f} Y$ be a morphism of G-sets. Then the morphism $X_f^{\mathrm{Gal}} \xrightarrow{f^{\mathrm{Gal}}} Y$ is a Galois morphism of G-sets.*

(ii) *If*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ f\ \ } & Y \\
{\scriptstyle \alpha} \downarrow & & \downarrow {\scriptstyle \beta} \\
A & \xrightarrow{\ \ a\ \ } & B
\end{array}
$$

is a morphism in the category G-Mor, and if $A \xrightarrow{\ a\ } B$ is a Galois morphism of G-sets, then there exists a unique morphism of G-sets $\widetilde{\alpha} : X_f^{\mathrm{Gal}} \to A$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle \gamma_{X,f}} & & \| \\
X_f^{\mathrm{Gal}} & \xrightarrow{\ f^{\mathrm{Gal}}\ } & Y \\
\downarrow{\scriptstyle \widetilde{\alpha}} & & \downarrow{\scriptstyle \beta} \\
A & \xrightarrow{\ a\ } & B
\end{array}
$$

(with α)

is commutative.

(iii) *The correspondence sending $X \xrightarrow{\ f\ } Y$ to $X_f^{\mathrm{Gal}} \xrightarrow{\ f^{\mathrm{Gal}}\ } Y$ is a functor from G-Mor to G-Mor$^{\mathrm{Gal}}$, and this functor is left adjoint to the forgetful functor G-Mor$^{\mathrm{Gal}} \to G$-Mor.*

Proof. When $x \in X$, let $\widetilde{x} = \gamma_{X,f}(x) \in X_f^{\mathrm{Gal}}$ denote its equivalence class for the relation $\sim_f$. Let $g \in G$. Then $g\widetilde{x} = \widetilde{x}$ if and only if $gx \sim_f x$, i.e. if there exists $h \in G_x^f$ such that $gx = hx$, or equivalently $h^{-1}g \in G_x$. Hence the stabilizer of $\widetilde{x}$ in G is equal to $G_x^f \cdot G_x = G_x^f$, since $G_x \leqslant G_x^f$. So if $x, x' \in X$ are such that $f^{\mathrm{Gal}}(\widetilde{x}) = f^{\mathrm{Gal}}(\widetilde{x}')$, i.e. if $f(x) = f(x')$, then the stabilizers G_x^f of $\widetilde{x}$ and $G_{x'}^f$ of $\widetilde{x}'$ are equal, since G_x^f depends only on $f(x)$. Assertion (i) follows, by Proposition 9.3.

Let $x, x' \in X$ such that $x \sim_f x'$. Then there exist $r \in \mathbb{N}$, elements $z_1, \ldots, z_r$ in X and elements $g_1, \ldots, g_r$ of G such that $g_i z_i = z_i$ and $f(z_i) = f(x)$, for $i = 1, \ldots, r$, and such that $g = g_1 \cdots g_r$. It follows that

$$\beta f(z_i) = a\alpha(z_i) = \beta f(x) = a\alpha(x),$$

for $i = 1, \ldots, r$. By Proposition 9.3, since $A \xrightarrow{\ a\ } B$ is a Galois morphism, this implies that $G_{\alpha(z_i)} = G_{\alpha(x)}$. Moreover, $G_z \leqslant G_{\alpha(z)}$ for any $z \in X$. Thus $g_i \in G_{\alpha(z_i)} = G_{\alpha(x)}$ for $i = 1, \ldots, r$. It follows that $g = g_1 \cdots g_r \in G_{\alpha(x)}$, hence $\alpha(x') = g\alpha(x) = \alpha(x)$. This shows the existence of a map $\widetilde{\alpha} : X_f^{\mathrm{Gal}} \to A$, sending the equivalence class of $x \in X$ for $\sim_f$ to $\alpha(x)$. Such a map is obviously unique, and it is a morphism of G-sets. This proves assertion (ii).

For assertion (iii), suppose that

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
X' & \xrightarrow{\ f'\ } & Y'
\end{array}
$$

is a morphism from $X \xrightarrow{\ f\ } Y$ to $X' \xrightarrow{\ f'\ } Y'$ in the category G-Mor. Then one can compose this morphism with the morphism

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f'\ } & Y' \\
\downarrow{\scriptstyle \gamma_{X',f'}} & & \| \\
X'^{\mathrm{Gal}}_{f'} & \xrightarrow{\ f'^{\mathrm{Gal}}\ } & Y'
\end{array}
$$

This yields a morphism from $X \xrightarrow{\ f\ } Y$ to $X'^{\mathrm{Gal}}_{f'} \xrightarrow{\ f'^{\mathrm{Gal}}\ } Y'$, which is a Galois morphism by assertion (i). By assertion (ii), this composition factors in a unique way through the morphism $X_f^{\mathrm{Gal}} \xrightarrow{\ f^{\mathrm{Gal}}\ } Y$. In other words, there is a unique morphism of G-sets $\widetilde{\alpha} : X_f^{\mathrm{Gal}} \to X'^{\mathrm{Gal}}_{f'}$ such that

the diagram
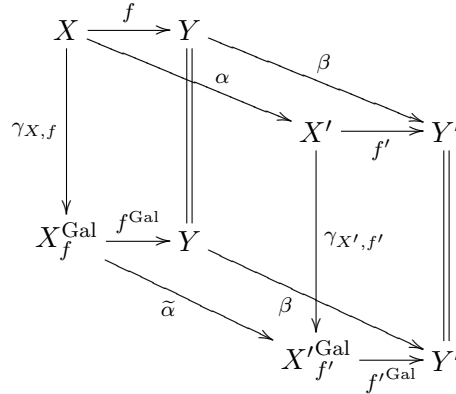
is commutative. Clearly, the map $(\alpha, \beta) \mapsto (\widetilde{\alpha}, \beta)$ endows the correspondence

$$(X \xrightarrow{f} Y) \mapsto (X_f^{\mathrm{Gal}} \xrightarrow{f^{\mathrm{Gal}}} Y)$$

with a functor structure from $G\text{-}\mathsf{Mor}$ to $G\text{-}\mathsf{Mor}^{\mathrm{Gal}}$. Moreover, for any Galois morphism $A \xrightarrow{a} B$, assertion (ii) yields a bijection

$$\mathrm{Hom}_{G\text{-}\mathsf{Mor}}(X \xrightarrow{f} Y, A \xrightarrow{a} B) \cong \mathrm{Hom}_{G\text{-}\mathsf{Mor}^{\mathrm{Gal}}}(X_f^{\mathrm{Gal}} \xrightarrow{f^{\mathrm{Gal}}} Y, A \xrightarrow{a} B),$$

and this bijection is clearly functorial with respect to $X \xrightarrow{f} Y$ and $A \xrightarrow{a} B$. Assertion (iii) follows. $\qquad\square$

Example 9.19. Let (T, S) be a slice of G, and f be the projection map from $X = G/S$ to $Y = G/T$. Then for $x = S \in X$, the group G_x^f is the group generated by the stabilizers G_{tS}, for $t \in T$. As $G_{tS} = {}^tS$, it follows that G_x^f is equal to the *normal closure* $S^{\trianglelefteq T}$ of S in T. In this case, moreover, the G-set X_f^{Gal} is isomorphic to $G/(S^{\trianglelefteq T})$, and the map f^{Gal} is the projection to G/T.

10. The section Burnside functor

PROPOSITION 10.1. *If* $X \xrightarrow{f} Y$ *and* $X' \xrightarrow{f'} Y'$ *are Galois morphisms of* G-*sets, then the morphisms* $X \sqcup X' \xrightarrow{f \sqcup f'} Y \sqcup Y'$ *and* $X \times X' \xrightarrow{f \times f'} Y \times Y'$ *are Galois morphisms of* G-*sets.*

Proof. The case of disjoint union follows from Proposition 9.10 and Corollary 9.6. Moreover, Proposition 9.14 shows that if $f : X \to Y$ and $f' : X' \to Y'$ are Galois morphisms of G-sets, then

$$f \times f' : X \times X' \to Y \times Y'$$

is a Galois morphism of $(G \times G)$-sets. By Corollary 9.13, the restriction of this morphism to the diagonal $G \leqslant G \times G$ is a Galois morphism of G-sets. $\qquad\square$

DEFINITION 10.2. Let G be a finite group. *The section Burnside group* $\Gamma(G)$ *of* G is the subgroup of the slice Burnside ring $\Xi(G)$ generated by the classes of Galois morphisms of G-sets.

By Proposition 10.1, the group $\Gamma(G)$ is actually a subring of $\Xi(G)$, called the section Burnside ring of G.

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LEMMA 10.3. *Let $f : X \to Y$ be a Galois morphism of finite G-sets. Then in the group $\Gamma(G)$,*

$$\pi(X \xrightarrow{f} Y) = \sum_{x \in [G \backslash X]} \langle G_{f(x)}, G_x \rangle_G$$

$$= \sum_{y \in [G \backslash Y]} |G_y \backslash f^{-1}(y)| \langle G_y, G_{x_y} \rangle_G,$$

where x_y is chosen in $f^{-1}(y)$, for each $y \in [G \backslash f(X)]$.

Proof. The first formula follows from Lemma 3.4. For the second, write

$$\pi(X \xrightarrow{f} Y) = \sum_{y \in [G \backslash Y]} \sum_{x \in [G_y \backslash f^{-1}(Y)]} \pi(G/G_x \to G/G_y),$$

and observe that $G_x = G_{x'}$ if $f(x) = f(x')$. □

COROLLARY 10.4. *The elements $\langle T, S \rangle_G$, where (T, S) runs through a set $[\Sigma(G)]$ of representatives of conjugacy classes of sections of G, form a basis of $\Gamma(G)$.*

Proof. Indeed, these elements generate $\Gamma(G)$, by Proposition 10.3, and they are linearly independent, by Theorem 4.6. □

Remark 10.5. This also shows that $\Gamma(G)$ is the quotient of the free abelian group on the set of isomorphism classes $[X \xrightarrow{f} Y]$ of Galois morphisms of finite G-sets, by the subgroup generated by elements of the form

$$[(X_1 \sqcup X_2) \xrightarrow{f_1 \sqcup f_2} Y] - [X_1 \xrightarrow{f_1} f(X_1)] - [X_2 \xrightarrow{f_2} f(X_2)],$$

whenever $X \xrightarrow{f} Y$ is a Galois morphism of finite G-sets with a decomposition $X = X_1 \sqcup X_2$ as a disjoint union of G-sets, where $f_1 = f_{|X_1}$ and $f_2 = f_{|X_2}$.

Remark 10.6. Since $\Gamma(G)$ is a subring of $\Xi(G)$, the product formula of Proposition 3.8,

$$\langle T, S \rangle_G \langle Y, X \rangle_G = \sum_{g \in [S \backslash G / X]} \langle T \cap {}^g Y, S \cap {}^g X \rangle_G,$$

also expresses the product of two sections $\langle T, S \rangle_G$ and $\langle Y, X \rangle_G$ of G. Other ring structures with the same additive group $\Gamma(G)$, but different multiplications, have been considered by Coşkun in [Cos09, Definition 2.2, Remark 2.3], as well as their functorial properties.

THEOREM 10.7. (i) *Let G and H be finite groups, and let U be a finite (H, G)-biset. The functor*

$$(X \xrightarrow{f} Y) \mapsto (U \times_G X \xrightarrow{U \times_G f} U \times_G Y)$$

from G-$\mathsf{Mor}^{\mathrm{Gal}}$ to H-$\mathsf{Mor}^{\mathrm{Gal}}$ induces a group homomorphism

$$\Gamma(U) : \Gamma(G) \to \Gamma(H).$$

(ii) *The correspondence $G \mapsto \Gamma(G)$ is a Green biset functor.*

Proof. This follows from Theorem 3.9 and Corollary 9.12. □

Remark 10.8. It follows from Remark 9.2 that the image of the morphism $i : B \to \Xi$ of Proposition 3.11 is actually contained in Γ. Thus i is a morphism of Green biset functors from B to Γ.

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11. Sections and ghost map

Notation 11.1. If (T, S) is a slice of G, denote by $\psi_{T,S} : \Gamma(G) \to \mathbb{Z}$ the restriction of the ring homomorphism $\phi_{T,S} : \Xi(G) \to \mathbb{Z}$.

LEMMA 11.2. *Let* $(T, S), (T', S') \in \Pi(G)$. *Then* $\psi_{T,S} = \psi_{T',S'}$ *if and only if the sections* $(T, S^{\trianglelefteq T})$ *and* $(T', S'^{\trianglelefteq T'})$ *are conjugate in* G *(recall that* $S^{\trianglelefteq T}$ *denotes the normal closure of* S *in* T). *In particular,* $\psi_{T,S} = \psi_{T,S^{\trianglelefteq T}}$.

Proof. This follows from Proposition 9.18 and Remark 9.19, but the following is a short direct proof. By Lemma 4.3, $\psi_{T,S} = \psi_{T',S'}$ if and only if for any section (V, U) of G,

$$|\{g \in G/U \mid (T, S)^g \preceq (V, U)\}| = |\{g \in G/U \mid (T', S')^g \preceq (V, U)\}|.$$

Taking $(V, U) = (T, S^{\trianglelefteq T})$ shows that there exists $g \in G$ such that $(T', S')^g \leqslant (T, S^{\trianglelefteq T})$. This implies that $(T', S'^{\trianglelefteq T'})^g \preceq (T, S^{\trianglelefteq T})$. Conversely, taking $(V, U) = (T', S'^{\trianglelefteq T'})$ shows that $(T', S'^{\trianglelefteq T'})^{g'} \preceq (T, S^{\trianglelefteq T})$, for some $g' \in G$. It follows that $(T', S'^{\trianglelefteq T'})^g = (T, S^{\trianglelefteq T})$. $\qquad\square$

THEOREM 11.3. *The map*

$$\Psi = \prod_{(T,S) \in [\Sigma(G)]} \psi_{T,S} : \Gamma(G) \to \prod_{(T,S) \in [\Sigma(G)]} \mathbb{Z},$$

called the ghost map *for* $\Gamma(G)$, *is an injective ring homomorphism, with finite cokernel as morphism of abelian groups.*

Proof. The proof is exactly the same as for Theorem 4.6. By Lemma 3.5, the elements $\langle T, S \rangle_G$, for $(T, S) \in [\Sigma(G)]$, generate $\Gamma(G)$. Suppose that there is a non-zero linear combination in the kernel of Ψ,

$$\Lambda = \sum_{(T,S) \in [\Sigma(G)]} \lambda_{T,S} \langle T, S \rangle_G,$$

with integer coefficients $\lambda_{T,S} \in \mathbb{Z}$, for $(T, S) \in [\Sigma(G)]$. Extend λ to a function $\Sigma(G) \to \mathbb{Z}$, constant on conjugacy classes. Let (Y, X) be an element of $\Sigma(G)$, maximal for the relation $\preceq$, such that $\lambda_{Y,X} \neq 0$. Then since, by Lemma 4.3,

$$\psi_{Y,X}(G/S \to G/T) = |\{g \in G/S \mid (Y^g, X^g) \preceq (T, S)\}|,$$

it follows that

$$\begin{aligned}
\psi_{Y,X}(\Lambda) &= \sum_{(T,S) \in [\Sigma(G)]} \lambda_{T,S} \psi_{Y,X}(G/S \to G/T) \\
&= \lambda_{Y,X} \psi_{Y,X}(G/X \to G/Y) \\
&= \lambda_{Y,X} |N_G(X, Y)/X| = 0.
\end{aligned}$$

Hence $\lambda_{Y,X} = 0$, and this contradiction shows that Ψ is injective (and, in particular, we recover the fact that the elements $\langle T, S \rangle_G$, for $(T, S) \in [\Sigma(G)]$, form a $\mathbb{Z}$-basis of $\Gamma(G)$). Thus Ψ is an injective morphism between free abelian groups with the same finite rank, hence it has finite cokernel. $\qquad\square$

COROLLARY 11.4. *Set* $\mathbb{Q}\Gamma(G) = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(G)$ *and* $\mathbb{Q}\Psi = \mathbb{Q} \otimes_{\mathbb{Z}} \Psi$. *Then*

$$\mathbb{Q}\Psi : \mathbb{Q}\Gamma(G) \to \prod_{(T,S) \in [\Sigma(G)]} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-*algebras.*

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12. Sections and idempotents

Corollary 11.4 shows that $\mathbb{Q}\Gamma(G)$ is a split semisimple commutative algebra. Its primitive idempotents are indexed by sections of G, up to conjugation: they are the inverse images under $\mathbb{Q}\Psi$ of the primitive idempotents of the algebra $\prod_{(T,S)\in[\Sigma(G)]}\mathbb{Q}$.

Notation 12.1. If $(T, S) \in \Sigma(G)$, denote by $\gamma_{T,S}^G$ the unique element of $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(G)$ such that

$$\forall (Y, X) \in \Sigma(G), \quad \mathbb{Q}\psi_{Y,X}(\gamma_{T,S}^G) = \begin{cases} 1 & \text{if } (Y, X) =_G (T, S) \\ 0 & \text{otherwise.} \end{cases}$$

The elements $\gamma_{T,S}^G$, for $(T, S) \in \Sigma(G)$, are the primitive idempotents of $\mathbb{Q}\Gamma(G)$.

THEOREM 12.2. *Let $\trianglelefteq$ denote the restriction of the relation $\preceq$ to $\Sigma(G)$. Then, for $(T, S) \in \Sigma(G)$,*

$$\gamma_{T,S}^G = \frac{1}{|N_G(T, S)|} \sum_{(V,U) \trianglelefteq (T,S)} |U| \mu_{\Sigma}((V, U), (T, S)) \langle V, U \rangle_G,$$

where μ_{Σ} is the Möbius function of the poset $(\Sigma(G), \trianglelefteq)$.

Proof. The proof is the same as the proof of Theorem 5.2. $\qquad\square$

PROPOSITION 12.3. *Let $(\mathcal{X}, \leqslant)$ be a finite poset. Let $\varphi : \mathcal{X} \to \mathcal{X}$ be a map of posets such that $\varphi \circ \varphi = \varphi$ and $x \leqslant \varphi(x)$ for any $x \in \mathcal{X}$. Then the Möbius function $\mu_{\mathcal{Y}}$ of the subposet $\mathcal{Y} = \varphi(\mathcal{X})$ of $\mathcal{X}$ is given by*

$$\forall y, z \in \mathcal{Y}, \quad \mu_{\mathcal{Y}}(y, z) = \sum_{\substack{y \leqslant u \in \mathcal{X} \\ \varphi(u)=z}} \mu_{\mathcal{X}}(y, u), \tag{12.4}$$

where $\mu_{\mathcal{X}}$ is the Möbius function of $(\mathcal{X}, \leqslant)$.

Proof. For $y, z \in \mathcal{Y}$, denote by $m(y, z)$ the right-hand side of equation (12.4). Then, for $y, t \in \mathcal{Y}$,

$$\sum_{\substack{z \in \mathcal{Y} \\ y \leqslant z \leqslant t}} m(y, z) = \sum_{\substack{z \in \mathcal{Y} \\ y \leqslant z \leqslant t}} \sum_{\substack{y \leqslant u \in \mathcal{X} \\ \varphi(u)=z}} \mu_{\mathcal{X}}(y, u)$$

$$= \sum_{(z,u) \in \mathcal{P}} \mu_{\mathcal{X}}(y, u),$$

where

$$\mathcal{P} = \{(z, u) \mid z \in \mathcal{Y}, u \in \mathcal{X}, y \leqslant u \leqslant \varphi(u) = z \leqslant t\}.$$

Set $\mathcal{Q} = \{u \in \mathcal{X} \mid y \leqslant u \leqslant t\}$. If $(z, u) \in \mathcal{P}$, then clearly $u \in \mathcal{Q}$. Conversely, if $u \in \mathcal{Q}$, then $(\varphi(u), u) \in \mathcal{P}$: indeed $\varphi(u) \in \mathcal{Y} = \varphi(\mathcal{X})$ and, moreover, $\varphi(u) \leqslant \varphi(t) = t$, since $t \in \varphi(\mathcal{X})$ and $\varphi \circ \varphi = \varphi$. Now the maps $(z, u) \in \mathcal{P} \mapsto u \in \mathcal{Q}$ and $u \in \mathcal{Q} \mapsto (\varphi(u), u) \in \mathcal{P}$ are mutual inverse bijections. It follows that

$$\sum_{\substack{z \in \mathcal{Y} \\ y \leqslant z \leqslant t}} m(y, z) = \sum_{\substack{u \in \mathcal{X} \\ y \leqslant u \leqslant t}} \mu_{\mathcal{X}}(y, u).$$

This is equal to 1 if $y = t$, and to zero otherwise. The proposition follows. $\qquad\square$

COROLLARY 12.5. *Let (V, U) and (T, S) be sections of G. Then*

$$\mu_{\Sigma(G)}((V, U), (T, S)) = \mu(V, T)\bigg(\sum_{\substack{U \leqslant X \leqslant V \\ X^{\trianglelefteq T} = S}} \mu(U, X)\bigg). \tag{12.6}$$

In particular, in $\mathbb{Q}\Gamma(G)$,

$$\gamma_{T,S}^G = \frac{1}{|N_G(T, S)|} \sum_{\substack{U \trianglelefteq V \leqslant T \\ U \leqslant X \leqslant V \\ X^{\trianglelefteq T} = S}} |U| \mu(U, X) \mu(V, T) \langle V, U \rangle_G.$$

Proof. For (12.6), apply Proposition 12.3 to the poset $\mathcal{X} = (\Pi(G), \preceq)$, and to the map $\varphi : \mathcal{X} \to \mathcal{X}$ defined by

$$\varphi((Y, X)) = (Y, X^{\trianglelefteq Y}),$$

and then use (5.6). Then substitute the value of $\mu_{\Sigma}((V, U), (T, S))$ in the formula of Theorem 12.2. $\qquad\square$

13. The image of the ghost map

The following is a characterization of the image of the ghost map for the section Burnside ring, similar to Theorem 6.1.

THEOREM 13.1. *Let G be a finite group. Let $\mathsf{m} = (m(T, S))_{(T,S) \in \Sigma(G)}$ be a sequence of integers indexed by $\Sigma(G)$, constant on G-conjugacy classes of sections. Then the sequence $[\mathsf{m}] = (m(T, S))_{(T,S) \in [\Sigma(G)]}$ of representatives lies in the image of the ghost map Ψ if and only if, for any section (T, S) of G,*

$$\sum_{g \in N_G(T,S)/S} m(\langle gT \rangle, \langle gS \rangle^{\trianglelefteq \langle gT \rangle}) \equiv 0 \pmod{|N_G(T, S)/S|}.$$

Proof. The proof is very similar to the proof of Theorem 6.1, with two differences. The first is that the poset $\Pi(G)$ of slices of G has to be replaced by the poset $\Sigma(G)$ of sections of G. The second is that the slice $(\langle gT \rangle, \langle gS \rangle)$ has to be changed to the corresponding section $(\langle gT \rangle, \langle gS \rangle^{\trianglelefteq gT})$ of G. Apart from these two differences, the proof goes through without changes. $\qquad\square$

14. Prime spectrum

Notation 14.1. Let p denote either 0 or a prime number.

- If $(T, S) \in \Sigma(G)$, denote by $J_{T,S,p}$ the prime ideal $I_{T,S,p} \cap \Gamma(G)$ of $\Gamma(G)$. In other words, $J_{T,S,p}$ is the kernel of the ring homomorphism

$$\Gamma(G) \xrightarrow{\psi_{T,S}} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z},$$

  where the right-hand-side map is the projection.

- Let $\Omega(G)$ denote the set of triples (T, S, p), where $(T, S) \in \Sigma(G)$ is such that $|N_G(T, S)/S| \not\equiv 0 \pmod{p}$.

The group G acts on $\Omega(G)$, by $^g(T, S, p) = (^gT, {}^gS, p)$, for $g \in G$, and the ideal $J_{T,S,p}$ only depends on the G-orbit of (T, S, p). Conversely, we have the following proposition.

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PROPOSITION 14.2. *Let I be a prime ideal of $\Gamma(G)$, and $R = \Gamma(G)/I$. Denote by $\psi : \Gamma(G) \to R$ the projection map, and denote by $p \geqslant 0$ the characteristic of R. Then $R \cong \mathbb{Z}/p\mathbb{Z}$. Furthermore:*

(i) *if $p = 0$, there exists a section (T, S) of G such that $\psi = \psi_{T,S}$, and (T, S) is unique up to G-conjugation, with this property;*

(ii) *if $p > 0$, there exists a section (T, S) of G such that ψ is the reduction modulo p of $\psi_{T,S}$ and $N_G(T, S)/S$ is a p'-group, and (T, S) is unique up to G-conjugation, with these properties.*

In particular, there exists a unique $(T, S, p) \in \Omega(G)$, up to conjugation, such that $I = J_{T,S,p}$.

Proof. The proof is exactly the same as the proof of Proposition 7.2, with slices replaced by sections. Let (T, S) be a section of G, minimal for the relation $\preceq$, such that $\langle T, S \rangle_G \notin I$. Then by Proposition 3.8, for any $(Y, X) \in \Sigma(G)$,

$$\langle T, S \rangle_G \langle Y, X \rangle_G = \sum_{g \in [S \backslash G / X]} \langle T \cap {}^g Y, S \cap {}^g X \rangle_G$$

$$\equiv \sum_{\substack{g \in G/X \\ S \leqslant {}^g X \\ T \leqslant {}^g Y}} \langle T, S \rangle_G \pmod{I}$$

$$= \psi_{T,S}(\langle Y, X \rangle_G) \langle T, S \rangle_G.$$

Since I is prime, it follows that $\langle Y, X \rangle_G - \psi_{T,S}(\langle Y, X \rangle_G) 1_{\Gamma(G)} \in I$. In particular, $R = \Gamma(G)/I$ is generated by the image of $1_{\Gamma(G)}$, hence $R \cong \mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of R. Since R is an integral domain, the number p is either 0 or a prime.

(i) If $p = 0$, then $R = \mathbb{Z}$, and $\psi = \psi_{T,S}$. And if $(T', S') \in \Sigma(G)$ is such that $\psi_{T,S} = \psi_{T',S'}$, then both $\psi_{T,S}(\langle T', S' \rangle_G)$ and $\psi_{T',S'}(\langle T, S \rangle_G)$ are non-zero. Then there exist elements $g, g' \in G$ such that $(T^g, S^g) \preceq (T', S')$ and $(T'^{g'}, S'^{g'}) \preceq (T, S)$, so (T, S) and (T', S') are conjugate in G.

(ii) If $p > 0$, then $R = \mathbb{Z}/p\mathbb{Z}$, and ψ is equal to the reduction of $\psi_{T,S}$ modulo p. Since $\psi(\langle T, S \rangle_G) = |N_G(T, S)/S|$ is non-zero in R, it follows that $N_G(T, S)/S$ is a p'-group. If (T', S') is another section of G such that ψ is the reduction modulo p of $\psi_{T',S'}$, and $N_G(T', S')/S'$ is a p'-group, then

$$|N_G(T, S)/S| = \psi_{T,S}(\langle T, S \rangle_G) \equiv \psi_{T',S'}(\langle T, S \rangle_G) \pmod{p}.$$

This is non-zero. Similarly, $|N_G(T, S)/S| \equiv \psi_{T,S}(\langle T', S' \rangle_G) \pmod{p}$ is non-zero. In particular, $\psi_{T',S'}(\langle T, S \rangle_G)$ and $\psi_{T,S}(\langle T', S' \rangle_G)$ are both non-zero, and it follows as above that (T, S) and (T', S') are conjugate in G. $\qquad\square$

Notation 14.3. Let p be a prime number.

- Let $\Sigma_p(G)$ denote the set of sections (T, S) of G such that $N_G(T, S)/S$ is a p'-group. In other words, $\Sigma_p(G) = \Sigma(G) \cap \Pi_p(G)$.

- For any $(T, S) \in \Sigma(G)$, let $(T, S)_p^{\widehat{}}$ denote the unique element (V, U) of $\Sigma_p(G)$, up to conjugation, such that $J_{T,S,p} = J_{V,U,p}$.

PROPOSITION 14.4. *Let p be a prime number. If (T, S) is a section of G, let $(T, S)_p^{\widehat{+}}$ denote a section of the form $(PT, (PS)^{\trianglelefteq PT})$ of G, where P is a Sylow p-subgroup of $N_G(T, S)$.*

Define inductively an increasing sequence (T_n, S_n) in $(\Sigma(G), \preceq)$ by $(T_0, S_0) = (T, S)$, and $(T_{n+1}, S_{n+1}) = (T_n, S_n)_p^{\widehat{+}}$, for $n \in \mathbb{N}$. Then $(T, S)_p^{\widehat{}}$ is conjugate to the largest term (T_∞, S_∞) of the sequence (T_n, S_n).

Proof. Again the proof is the same as the proof of Proposition 7.4. The section $(T, S)_p^{\widehat{\simeq}}$ is a minimal element (V, U) of the poset $(\Sigma(G), \preceq)$ such that

$$\psi_{T,S}(V, U) = |\{g \in G/U \mid (T^g, S^g) \preceq (V, U)\}| \not\equiv 0 \pmod{p}.$$

Thus one can assume that $(T, S) \preceq (V, U)$. But $\psi_{T,S} \equiv \psi_{PT,(PS)^{\lhd PT}} \pmod{p}$ by Corollary 4.4 and Lemma 11.2, for any p-subgroup P of $N_G(T, S)$, hence one can also assume that $(T, S)_p^+ \preceq (V, U)$, hence that $(T_\infty, S_\infty) \preceq (V, U)$, by induction. Moreover, $\psi_{T_\infty, S_\infty} \equiv \psi_{V,U} \pmod{p}$. As $N_G(T_\infty, S_\infty)/S_\infty$ is a p'-group, it follows that $(T_\infty, S_\infty) = (V, U)$, as was to be shown. $\square$

PROPOSITION 14.5. *Let* (T, S, p), (T', S', p') *be elements of* $\Omega(G)$. *Then* $J_{T',S',p'} \subseteq J_{T,S,p}$ *if and only if:*

- *either* $p' = p$ *and the sections* (T', S') *and* (T, S) *are conjugate in* G;
- *or* $p' = 0$ *and* $p > 0$, *and the sections* $(T', S')_p^{\widehat{\simeq}}$ *and* (T, S) *are conjugate in* G.

Proof. (See Proposition 7.6.) Set $J = J_{T,S,p}$ and $J' = J_{T',S',p'}$. Then $\Gamma(G)/J' \cong \mathbb{Z}/p'\mathbb{Z}$ maps surjectively to $\Gamma(G)/J \cong \mathbb{Z}/p\mathbb{Z}$. Thus if $p = p'$, this projection map is an isomorphism, hence $J = J'$ and the sections (T, S) and (T', S') are conjugate in G. And if $p \neq p'$, then $p' = 0$ and $p > 0$. The morphism $\psi_{T,S}$ is equal to the reduction modulo p of the morphism $\psi_{T',S'}$. In other words, $J_{T',S',p} = J_{T,S,p}$, hence (T, S) is conjugate to $(T', S')_p^{\widehat{\simeq}}$. $\square$

COROLLARY 14.6. *Let* p *be a prime number, and let* $\mathbb{Z}_{(p)}$ *be the localization of* $\mathbb{Z}$ *at the set* $\mathbb{Z} - p\mathbb{Z}$. *Let* $\Omega_p(G)$ *denote the subset of* $\Omega(G)$ *consisting of triples* $(T, S, 0)$, *for* $(T, S) \in \Sigma(G)$, *and* (T, S, p), *for* $(T, S) \in \Sigma_p(G)$.

(i) *The prime ideals of the ring* $\mathbb{Z}_{(p)}\Gamma(G)$ *are the ideals* $\mathbb{Z}_{(p)}J_{T,S,q}$, *for* $(T, S, q) \in \Omega_p(G)$.

(ii) *If* $(T, S, q), (T', S', q') \in \Theta_p(G)$, *then* $\mathbb{Z}_{(p)}J_{T',S',q'} \subseteq \mathbb{Z}_{(p)}J_{T,S,q}$ *if and only if:*
   - *either* $q = q'$, *and the sections* (T, S) *and* (T', S') *are conjugate in* G;
   - *or* $q' = 0$, $q = p$, *and the sections* $(T', S')_p^{\widehat{\simeq}}$ *and* (T, S) *are conjugate in* G.

(iii) *The connected components of the spectrum of* $\mathbb{Z}_{(p)}\Gamma(G)$ *are indexed by the conjugacy classes of* $\Sigma_p(G)$. *The component indexed by* $(T, S) \in \Sigma_p(G)$ *consists of a unique maximal element* $\mathbb{Z}_{(p)}J_{T,S,p}$, *and of the ideals* $\mathbb{Z}_{(p)}J_{T',S',0}$, *where* $(T', S') \in \Sigma(G)$ *is such that* $(T', S')_p^{\widehat{\simeq}}$ *is conjugate to* (T, S) *in* G.

Proof. (See Corollary 7.7.) The prime ideals of $\mathbb{Z}_{(p)}\Gamma(G)$ are of the form $\mathbb{Z}_{(p)}I$, where I is a prime ideal of $\Gamma(G)$ such that $I \cap (\mathbb{Z} - p\mathbb{Z}) = \emptyset$. Equivalently, $I = J_{T,S,0}$ or $I = J_{T,S,p}$. This proves assertion (i). Now assertion (ii) follows from Proposition 14.5, and assertion (iii) follows from assertion (ii). $\square$

COROLLARY 14.7. (i) *The primitive idempotents of the ring* $\mathbb{Z}_{(p)}\Gamma(G)$ *are indexed by the conjugacy classes of* $\Sigma_p(G)$. *The primitive idempotent* $\varepsilon_{V,U}^G$ *indexed by* $(V, U) \in \Sigma_p(G)$ *is equal to*

$$\varepsilon_{V,U}^G = \sum_{\substack{(T,S) \in [\Sigma(G)] \\ (T,S)_p^{\widehat{\simeq}} =_G (T,S)}} \gamma_{T,S}^G.$$

(ii) *Let* π *be a set of prime numbers, and* $\mathbb{Z}_{(\pi)}$ *be the localization of* $\mathbb{Z}$ *relative to* $\mathbb{Z} - \bigcup_{p \in \pi} p\mathbb{Z}$. *Let* $\mathcal{F}$ *be a set of sections of* G, *invariant by* G-*conjugation, and* $[\mathcal{F}]$ *be a set of representatives of* G-*conjugacy classes of* $\mathcal{F}$. *Then the following conditions are equivalent:*

(a) *the idempotent*

$$\gamma_{\mathcal{F}}^{G} = \sum_{(T,S)\in[\mathcal{F}]} \gamma_{T,S}^{G}$$

of $\mathbb{Q}\Gamma(G)$ *lies in* $\mathbb{Z}_{(\pi)}\Gamma(G)$;

(b) *let* $(T,S) \in \Sigma(G)$, *and let* P *be a* p-*subgroup of* $N_G(T,S)$, *for some* $p \in \pi$. *Then* $(T,S) \in \mathcal{F}$ *if and only if* $(PT, (PS)^{\trianglelefteq PT}) \in \mathcal{F}$.

Proof. The proof is almost identical to the proof of Corollary 7.8. Let $\mathcal{F}$ be a set of sections of G, invariant by G-conjugation, and $[\mathcal{F}]$ be a set of representatives of G-conjugacy classes of $\mathcal{F}$. The idempotent

$$\gamma_{\mathcal{F}}^{G} = \sum_{(T,S)\in[\mathcal{F}]} \gamma_{T,S}^{G}$$

of $\mathbb{Q}\Gamma(G)$ lies in $\mathbb{Z}_{(p)}\Gamma(G)$, for some prime p, if and only if there exists an integer m, not divisible by p, such that $u = m\gamma_{\mathcal{F}}^{G} \in \Gamma(G)$. Let $(T,S) \in \Sigma(G)$, and let P be a p-subgroup of $N_G(T,S)$. The integer $\psi_{T,S}(u)$ is equal to m if $(T,S) \in \mathcal{F}$, and to 0 otherwise. Hence it is coprime to p if and only if $(T,S) \in \mathcal{F}$. Since $\psi_{T,S}$ and $\psi_{PT,(PS)^{\trianglelefteq PT}}$ are congruent modulo p, it follows that $(T,S) \in \mathcal{F}$ if and only if $(PT, (PS)^{\trianglelefteq PT}) \in \mathcal{F}$.

Hence if (T,S) and (T',S') are sections of G such that $(T,S)\widehat{_p} =_G (T',S')\widehat{_p}$, then $(T,S) \in \mathcal{F}$ if and only if $(T',S') \in \mathcal{F}$. Thus $\mathcal{F}$ is a disjoint union of sets of the form

$$E_{V,U} = \{(T,S) \in \Sigma(G) \mid (T,S)\widehat{_p} =_G (V,U)\},$$

for some sections $(V,U) \in \Sigma_p(G)$. In other words, the idempotent $\gamma_{\mathcal{F}}^{G}$ is a sum of some idempotents $\varepsilon_{V,U}^{G}$, for $(V,U) \in \Sigma_p(G)$.

But the primitive idempotents of the ring $\mathbb{Z}_{(p)}\Gamma(G)$ are in one-to-one correspondence with the connected components of its spectrum, which precisely are indexed by the conjugacy classes of $\Sigma_p(G)$. It follows that $\gamma_{\mathcal{F}}^{G} = \varepsilon_{V,U}^{G}$ is equal to the idempotent corresponding to the component indexed by (V,U), for any $(V,U) \in \Sigma_p(G)$. This proves assertion (i). This also proves assertion (ii) in the case where π consists of a single prime number.

For the general case, observe that $\gamma_{\mathcal{F}}^{G}$ lies in $\mathbb{Z}_{(\pi)}\Gamma(G)$ if and only if it lies in $\mathbb{Z}_{(p)}\Gamma(G)$, for any $p \in \pi$. □

The following proposition is the analogue of Theorem 7.9 for the ring $\Gamma(G)$. Its statement is slightly simpler.

THEOREM 14.8. *Let* G *be a finite group.*

(i) *The primitive idempotents of* $\Gamma(G)$ *are indexed by the conjugacy classes of perfect subgroups of* G. *The idempotent* γ_H^G *indexed by the perfect subgroup* H *is equal to*

$$\gamma_H^G = \sum_{\substack{(T,S)\in[\Sigma(G)] \\ D^{\infty}(T)=_G H}} \gamma_{T,S}^G,$$

where $D^{\infty}(T)$ *denotes the last term in the derived series of* T.

(ii) *The prime spectrum of* $\Gamma(G)$ *is connected if and only if* G *is solvable.*

Proof. As in the proof of Theorem 7.9, let $\sim$ denote the finest equivalence relation on the set $\Sigma(G)$ such that, for any $(T,S), (T',S') \in \Sigma(G)$,

$$\exists p, (T,S)\widehat{_p} =_G (T',S')\widehat{_p} \Longrightarrow (T,S) \sim (T',S').$$

If we can show that

$$(T, S) \sim (T', S') \Leftrightarrow D^\infty(T) =_G D^\infty(T'),$$

then assertion (i) follows from Corollary 14.7, applied to the set π of all primes.

It is clear that if there exists a p-subgroup $P \leqslant N_G(T, S)$ such that (T', S') is conjugate to $(PT, (PS)^{\trianglelefteq PT})$, then T' is conjugate to PT, and $D^\infty(T')$ is conjugate to $D^\infty(T)$. By transitivity, for any $(T, S), (T', S') \in \Sigma(G)$, if $(T, S) \sim (T', S')$, then $D^\infty(T) =_G D^\infty(T')$.

To show the converse, is it enough to show that $(T, S) \sim (D^\infty(T), D^\infty(T))$, for any (T, S) in $\Sigma(G)$. Let p be any prime, and P be a Sylow p-subgroup of T. Then $(T, S)\widehat{}_p = (T, (PS)^{\trianglelefteq PT})\widehat{}_p$, hence $(T, S) \sim (T, S')$, where $S' = (PS)^{\trianglelefteq PT}$ is a normal subgroup of T, containing S, and of p'-index in T. Since p was arbitrary, it follows by induction that $(T, S) \sim (T, T)$. Now for any prime p, if P is a Sylow p-subgroup of T, then $P \leqslant N_G(O^p(T))$ and $T = PO^p(T)$. It follows that $(T, T)\widehat{}_p = (O^p(T), O^p(T))\widehat{}_p$. Again, since p is arbitrary, it follows that $(T, T) \sim (D^\infty(T), D^\infty(T))$, and this completes the proof of assertion (i).

Assertion (ii) follows easily since, by assertion (i), the spectrum of $\Gamma(G)$ is connected if and only if the trivial group is the only perfect subgroup of G, i.e. if G is solvable. $\qquad\square$

COROLLARY 14.9. *The images of the primitive idempotents of $B(G)$ by the morphism $i_G :$ $B(G) \to \Gamma(G)$ are the primitive idempotents of $\Gamma(G)$. In other words, if H is a perfect subgroup of G, then*

$$\gamma_H^G = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K| \mu(K, H) \langle K, K \rangle_G.$$

Proof. Indeed by a theorem of Dress [Dre69], the primitive idempotents of $B(G)$ are indexed by the conjugacy classes of perfect subgroups of G. As the morphism $i_G : B(G) \to \Gamma(G)$ is an injective unital ring homomorphism (see Remark 10.8 and Proposition 3.11), it follows that the primitive idempotents of $B(G)$ are mapped to primitive idempotents in $\Gamma(G)$, and that every primitive idempotent of $\Gamma(G)$ is in the image of $B(G)$. $\qquad\square$

15. Unit group

All the results of §8 about the unit group of the slice Burnside ring have an analogue for the group $\Gamma(G)^\times$ of the section Burnside ring of a finite group G.

First, the restricted ghost map yields an injective group homomorphism

$$\Psi^\times : \Gamma(G)^\times \hookrightarrow \prod_{(T,S) \in \Sigma(G)} \mathbb{Z}^\times.$$

The following lemma is a consequence of this result.

LEMMA 15.1. *Let G be a finite group, and let $u \in \Gamma(G)$. The following conditions are equivalent:*

(i) $u \in \Gamma(G)^\times$.

(ii) $\psi_{T,S}(u) \in \{\pm 1\}$, *for any $(T, S) \in \Sigma(G)$.*

(iii) $u^2 = 1$.

In particular, $\Gamma(G)^\times$ is a finite elementary abelian 2-group.

Second, the following proposition is an analogue of Proposition 8.3.

PROPOSITION 15.2. *The Feit–Thompson theorem is equivalent to the statement that, if G has odd order, then $\Gamma(G)^\times = \{\pm 1\}$.*

Proof. The proof is the same as that of Proposition 8.3: the argument uses only the formulas for the primitive idempotents of $\mathbb{Q}\Gamma(G)$ (Corollary 12.5), and the characterization of solvable groups by the connectedness of the prime spectrum of $\Gamma(G)$ (Proposition 14.8). $\qquad\square$

Third, the following theorem is an analogue of Yoshida's characterization [Yos90] of the unit group of the usual Burnside ring.

THEOREM 15.3. *Let G be a finite group, and let $\mathsf{m} = (m(T, S))_{(T,S)\in\Sigma(G)}$ be a sequence of integers in $\{\pm 1\}$ indexed by $\Sigma(G)$, constant on G-conjugacy classes of sections. Then the sequence $[m] = (m(T, S))_{(T,S)\in[\Sigma(G)]}$ of representatives lies in the image of the restricted ghost map $\Psi^\times$ if and only if, for any $(T, S) \in \Sigma(G)$, the map*

$$g \in N_G(T, S)/S \mapsto m(\langle gT\rangle, \langle gS\rangle^{\trianglelefteq\langle gT\rangle})/m(T, S) \in \{\pm 1\}$$

is a group homomorphism.

Proof. Again, the proof is the same as for Theorem 8.4: it only requires Theorem 13.1 and Lemma 11.2. $\qquad\square$

Finally, the correspondence $G \mapsto \Gamma(G)^\times$ is not a biset functor.

PROPOSITION 15.4. *The correspondence sending a finite group G to $\Gamma(G)^\times$ cannot be endowed with a biset functor structure.*

Proof. The proof of Proposition 8.6 goes through unchanged here: the argument uses computations for the trivial group, the group C_2, and the group $(C_2)^2$. All these groups are abelian, hence sections and slices coincide. $\qquad\square$

Appendix

A.1 Let G and H be finite groups, and U be a finite (H, G)-biset. Let U^{op} denote the *opposite biset*, i.e. the (G, H)-biset equal to U as a set, with actions reversed by taking inverses (i.e. $g \cdot u \cdot h$ [in U^{op}] $= h^{-1}ug^{-1}$ [in U], for $g \in G$, $u \in U$, and $h \in H$).

When X is a finite G-set, the set $\mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}}, X)$ is a finite H-set: if $\varphi : U^{\mathrm{op}} \to X$ is a morphism of G-sets, and if $h \in H$, then the morphism $h\varphi : U^{\mathrm{op}} \to X$ is defined by $(h\varphi)(u) = \varphi(h^{-1}u)$, for $u \in U$. This correspondence

$$T_U : X \mapsto \mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}}, X)$$

is actually a functor from the category G-set of finite G-sets to the category H-set. One can show (see, for example, [Bou10, § 11.2.13]) that this functor induces a map $t_U : B(G) \to B(H)$ between

the usual Burnside rings of G and H, called the *generalized tensor induction* with respect to U. This induction is not additive, but multiplicative (i.e. $t_U(ab) = t_U(a)t_U(b)$, for any $a, b \in B(G)$). It yields a biset functor structure on the unit group of the usual Burnside ring.

A natural question is whether this construction can be extended to the rings $\Xi(G)$ and $\Gamma(G)$: indeed, if $X \xrightarrow{f} Y$ is a morphism of finite G-sets, then the morphism $T_U(X \xrightarrow{f} Y)$,

$$\mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}}, X) \xrightarrow{\mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}},f)} \mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}}, Y),$$

is a morphism of finite H-sets. Does this correspondence induce a map $\Xi(G) \to \Xi(H)$ or a map $\Gamma(G) \to \Gamma(H)$? In other words:

(Q1) Are the defining relations of $\Xi(G)$ preserved by T_U?

(Q2) Does T_U map a Galois morphism to a Galois morphism?

The answer to these two questions has to be no in general, for otherwise T_U would yield a biset functor structure on the unit groups of the slice Burnside ring and the section Burnside ring, contradicting Propositions 8.6 and 15.4. But the answer is yes with the additional assumption that the biset U be *left inert*, according to the following definition.

DEFINITION A.2. An (H, G)-biset U is called *left inert* if $Hu \subseteq uG$, for any $u \in U$, in other words if H acts trivially on the set of orbits U/G.

Example A.3. – If U is transitive as a right G-set, then U is left inert, since $|U/G| = 1$. In particular, the identity (G, G)-biset G is left inert. Conversely, if U is left inert, then each right orbit uG, for $u \in U$, is an (H, G)-biset, so U is a disjoint union of (H, G)-bisets which are transitive as right G-sets.

– The disjoint union of two left inert (H, G)-bisets is left inert. Any sub-biset of a left inert biset is left inert.

– Left inert bisets can be composed: if G, H, and K are groups, if U is a left inert (H, G)-biset, and if V is a left inert (K, H)-biset, then $V \times_H U$ is a left inert (K, G)-biset: indeed, for $u \in U$ and $v \in V$,

$$K(v,_H u) = (Kv,_H u) \subseteq (vH,_H u) = (v,_H Hu) \subseteq (v,_H uG) = (v,_H u)G,$$

where $(v,_H u)$ denotes the image of (v, u) in $V \times_H U$.

A.4 The following proposition deals with question (Q2) above, in the case of a left inert biset.

PROPOSITION A.5. *Let* $X \xrightarrow{f} Y$ *be a Galois morphism of G-sets. If U is a left inert (H, G)-biset, then the morphism $T_U(X \xrightarrow{f} Y)$ is a Galois morphism of H-sets.*

Proof. Let $\varphi, \psi \in T_U(X) = \mathrm{Hom}_{G\text{-}\mathsf{set}}(U^{\mathrm{op}}, X)$, having the same image in $T_U(Y)$. This means that $f \circ \varphi = f \circ \psi$, i.e. that $f(\varphi(u)) = f(\psi(u))$, for any $u \in U$. Since f is a Galois morphism, it follows that $G_{\varphi(u)} = G_{\psi(u)}$, for any $u \in U$.

Suppose that $h \in H$ stabilizes φ. This means that $\varphi(h^{-1}u) = \varphi(u)$, for any $u \in U$. But since $Hu \subseteq uG$, there exists $g \in G$ (depending on u) such that $hu = ug$, i.e. $h^{-1}u = ug^{-1}$. Then

$$\varphi(u) = \varphi(h^{-1}u) = \varphi(ug^{-1}) = g\varphi(u),$$

hence $g \in G_{\varphi(u)}$. It follows that

$$\psi(u) = g\psi(u) = \psi(ug^{-1}) = \psi(h^{-1}u) = (h\psi)(u).$$

Hence h stabilizes ψ, and by symmetry φ and ψ have the same stabilizer in H. This shows that $T_U(X \xrightarrow{f} Y)$ is a Galois morphism of H-sets. $\qquad\square$

Remark A.6. The following example shows that Lemma A.5 is no longer true without the hypothesis that U is left inert. Suppose that $G = \mathbf{1}$, and that H is non-trivial. Let $U = H$, viewed as an (H, G)-biset. Then U is not left inert. Let X be a set of cardinality 2, let Y be a set of cardinality 1, and let $f : X \to Y$ be the unique map. Then f is a Galois morphism of G-sets, for trivial reasons.

In this case $\mathrm{Hom}_G(U^{\mathrm{op}}, X)$ is isomorphic to the set $\mathbf{2}^H$ of subsets of H, on which H acts by left translation. The set $\mathrm{Hom}_G(U^{\mathrm{op}}, Y)$ is a set $\bullet$ of cardinality 1, and the map $T_U(f)$ is the only possible map $\mathbf{2}^H \to \bullet$. In particular, all the elements of $\mathbf{2}^H$ have the same image. But the subset $\{1\}$ of H has a trivial stabilizer, whereas the subset H of H has stabilizer equal to H. Thus $T_U(X \xrightarrow{f} Y)$ is not a Galois morphism of H-sets, by Proposition 9.3.

Remark A.7. The example given in Remark A.6 also shows that the answer to question (Q1) is no, in general. Indeed, keeping the same notation, the image of the morphism $X \xrightarrow{f} Y$ in $\Xi(G)$ is equal to the sum of two copies of the image of $Y \xrightarrow{\mathrm{Id}} Y$. In other words, in $\Xi(G) \cong \mathbb{Z}$,

$$\pi(X \xrightarrow{f} Y) = 2 = \pi(X \xrightarrow{\mathrm{Id}} X).$$

But $T_U(X \xrightarrow{f} Y)$ is isomorphic to $\mathbf{2}^H \to \bullet$, hence

$$\pi(T_U(X \xrightarrow{f} Y)) = \sum_{\substack{A \subseteq H \\ A \bmod H}} \langle H, H_A \rangle_H \tag{A8}$$

by Lemma 3.4, where the summation runs over a set of representatives of *subsets* of H, up to translation by H, and H_A denotes the stabilizer in H of such a subset A. In particular, $H_A = H$ if and only if $A = H$ or $A = \emptyset$, hence if K is a subgroup of H, the only term of the form $\langle K, K \rangle_H$ in the right-hand side of equation (A8) is $\langle H, H \rangle_H$, with coefficient 2.

On the other hand, by Lemma 3.4 again,

$$\pi(X \xrightarrow{\mathrm{Id}} X) = \sum_{\substack{A \subseteq H \\ A \bmod H}} \langle H_A, H_A \rangle_H.$$

As there are proper non-empty subsets A of H, there are some terms of the form $\langle K, K \rangle_H$ with non-zero coefficient in this summation, for some subgroups $K < H$. Hence $\pi(X \xrightarrow{f} Y) \neq \pi(X \xrightarrow{\mathrm{Id}} X)$, thus T_U does not preserve the defining relations of $\Xi(G)$.

A.9 The following proposition answers question (Q1) in the case of left inert bisets.

Proposition A.10. (i) *Let G and H be finite groups, and U be a finite left inert (H, G)-biset. Then T_U induces a well-defined map $t_U : \Xi(G) \to \Xi(H)$, such that $t_U(\Gamma(G)) \subseteq \Gamma(H)$. Moreover, t_U only depends on the isomorphism class of the biset U.*

(ii) *The map t_U is multiplicative, i.e. $t_U(ab) = t_U(a)t_U(b)$, for any $a, b \in \Xi(G)$. Moreover, $t_U(1_{\Xi(G)}) = 1_{\Xi(H)}$. In particular, t_U restricts to group homomorphisms $\Xi(G)^\times \to \Xi(H)^\times$ and $\Gamma(G)^\times \to \Gamma(H)^\times$.*

(iii) *If U and U' are finite left inert (H, G)-bisets, then $t_{U \sqcup U'} = t_U t_{U'}$.*

(iv) *If G, H, and K are finite groups, if U is a finite left inert (H, G)-biset and V is a finite left inert (K, H)-biset, then $t_V \circ t_U = t_{V \times_H U}$.*

(v) *If U is the identity (G, G)-biset, then t_U is the identity map.*

Proof. Since U is left inert, Example A.3 shows that U splits as a disjoint union of (H, G)-bisets,

$$U \cong \bigsqcup_{u \in [U/G]} uG.$$

It follows that the functor $T_U = \mathrm{Hom}_{G\text{-set}}(U, -)$ is isomorphic to the direct product of the functors T_{uG}, for $u \in [U/G]$. Hence, to prove assertion (i), it suffices to consider the case where U is transitive as a right G-set.

In this case, the functor T_U actually induces a *group homomorphism* from $\Xi(G)$ to $\Xi(H)$: to see this, it suffices to check that the defining relations of $\Xi(G)$ are mapped to relations in $\Xi(H)$. Fix $u \in U$, and denote by G_u its stabilizer in G. So $U = uG \cong G/G_u$ as G-set, and for any G-set X, there is a bijection

$$\mathrm{Hom}_{G\text{-set}}(U^{\mathrm{op}}, X) \cong X^{G_u}.$$

This is actually an isomorphism of H-sets if the left action of H on X^{G_u} is defined as follows: for $h \in H$, there is some $g \in G$ such that $hu = ug$, and the action of h on X^{G_u} is defined by $hx = g^{-1}x$, for $x \in X^{G_u}$.

Now let $X \xrightarrow{f} Y$ be a morphism of finite G-sets such that X splits as a disjoint union $X = X_1 \sqcup X_2$ of two G-sets. The image $\pi(X \xrightarrow{f} Y)$ of this morphism in $\Xi(G)$ is equal to the sum of the images of the morphisms

$$X_1 \xrightarrow{f_1} f(X_1) \quad \text{and} \quad X_2 \xrightarrow{f_2} f(X_2),$$

where f_1 and f_2 are the restrictions of f to X_1 and X_2, respectively.

On the other hand, $T_U(X \xrightarrow{f} Y)$ is isomorphic to $X_1^{G_u} \sqcup X_2^{G_u} \xrightarrow{f^{G_u}} Y^{G_u}$ in the category H-Mor, where f^{G_u} is the restriction of f to X^{G_u}. The image $\pi(T_U(X \xrightarrow{f} Y))$ of this morphism in $\Xi(H)$ is equal to the sum of the images of the morphisms

$$X_1^{G_u} \xrightarrow{f_1^{G_u}} f(X_1^{G_u}) \quad \text{and} \quad X_2^{G_u} \xrightarrow{f_2^{G_u}} f(X_2^{G_u}).$$

By Lemma 3.3, since $f(X_1^{G_u}) \subseteq f(X_1)^{G_u}$,

$$\pi(X_1^{G_u} \xrightarrow{f_1^{G_u}} f(X_1^{G_u})) = \pi(X_1^{G_u} \xrightarrow{f_1^{G_u}} f(X_1)^{G_u}),$$

which is equal to $\pi(T_U(X_1 \xrightarrow{f_1} f(X_1)))$. It follows that in $\Xi(H)$,

$$\pi(T_U(X \xrightarrow{f} Y)) = \pi(T_U(X_1 \xrightarrow{f_1} f(X_1))) + \pi(T_U(X_2 \xrightarrow{f_2} f(X_2))),$$

hence T_U induces a group homomorphism $t_U : \Xi(G) \to \Xi(H)$. As the functor T_U maps the direct product of G-sets to the direct product of H-sets, and the trivial G-set to the trivial H-set, the morphism t_U is actually a *unital ring homomorphism* from $\Xi(G)$ to $\Xi(H)$ (recall that U is assumed transitive as a right G-set here).

If U is an arbitrary finite left inert (H, G)-biset, then T_U induces the map

$$t_U = \prod_{u \in [U/G]} t_{uG} : \Xi(G) \to \Xi(H).$$

It follows from Proposition A.5 that $t_U(\Gamma(G)) \subseteq \Gamma(H)$.

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Now assertion (ii) follows, since t_U is equal to a product of unital ring homomorphisms. Assertions (iii), (iv), and (v) are straightforward consequences of the properties of the functor T_U. $\qquad\square$

Remark A.11. It follows that the correspondences $G \mapsto \Xi(G)^\times$ and $G \mapsto \Gamma(G)^\times$ are biset functors for which the biset operations are only defined for *left inert bisets*. Equivalently, these correspondences are *biset functors without induction*: the usual basic operations for biset functors are defined for these correspondences, namely *restriction* to a subgroup, *deflation* from G to a factor group G/N (this is induced by taking *fixed points* by N on G-sets), transport by isomorphism, and inflation from a factor group. But there is no induction from a subgroup.

A.12 The last result of this appendix is the computation of the group $\Xi(G)^\times$. When G is abelian we have the following theorem.

THEOREM A.13. *Let G be a finite abelian group. Then $\Xi(G)^\times (= \Gamma(G)^\times)$ has an $\mathbb{F}_2$-basis consisting of the elements*

$$\begin{bmatrix} -\langle G,G \rangle_G \\ \left. \begin{array}{l} \langle G,G \rangle_G - \langle S,S \rangle_G \\ \langle G,G \rangle_G - \langle G,S \rangle_G \end{array} \right\} \quad \text{for } |G:S| = 2. \end{bmatrix}$$

In particular,

$$\Xi(G)^\times \cong (\mathbb{F}_2)^{2r+1},$$

where r is the number of subgroups of index 2 in G.

Proof. By Theorem 8.4, the group $\Xi(G)^\times$ is isomorphic to the group of sequences $(u_{T,S})_{(T,S)\in\Pi(G)}$ with values in $\{\pm 1\}$, such that, for any $(T,S) \in \Pi(G)$, the map

$$g \in G/S \mapsto u_{\langle gT \rangle, \langle gS \rangle}/u_{T,S}$$

is a group homomorphism. Switching to an additive notation, the group $\Xi(G)^\times$ is isomorphic to the $\mathbb{F}_2$-vector spaces of sequences $(\lambda_{T,S})_{(T,S)\in\Pi(G)}$, with values in $\mathbb{F}_2$, such that, for any $(T,S) \in \Pi(G)$ and any $(g,h) \in G$,

$$\lambda_{\langle ghT \rangle, \langle ghS \rangle} + \lambda_{\langle gT \rangle, \langle gS \rangle} + \lambda_{\langle hT \rangle, \langle hS \rangle} + \lambda_{T,S} = 0.$$

If none of g, h, and gh are in S, this yields an expression

$$\lambda_{T,S} = \lambda_{\langle ghT \rangle, \langle ghS \rangle} + \lambda_{\langle gT \rangle, \langle gS \rangle} + \lambda_{\langle hT \rangle, \langle hS \rangle}$$

of $\lambda_{T,S}$ as a linear combination of $\lambda_{T',S'}$, for $S' > S$. If $|G:S| > 2$, it is always possible to find such elements g and h. It follows that the sequence $(\lambda_{T,S})_{(T,S)\in\Pi(G)}$ is entirely determined by the values $\lambda_{T,S}$, for $|G:S| \leqslant 2$, i.e. the values $\lambda_{G,G}$, $\lambda_{S,S}$ and $\lambda_{G,S}$, where S runs through the set of subgroups of index 2 in G. If there are r such subgroups, this gives $2r+1$ such values, so $\dim_{\mathbb{F}_2} \Xi(G)^\times \leqslant 2r+1$.

To prove the converse, and actually the more precise statement in the theorem, it suffices to show that the elements $-1_{\Xi(G)}$, $u_S^G = \langle G,G \rangle_G - \langle S,S \rangle_G$ and $v_S^G = \langle G,G \rangle_G - \langle G,S \rangle_G$, for $|G:S| = 2$, are linearly independent elements of the $\mathbb{F}_2$-vector space $\Xi(G)^\times$.

First, as $\langle G,G \rangle_G = 1_{\Xi(G)}$, and as $\langle S,S \rangle_G^2 = 2\langle S,S \rangle$ if $|G:S| = 2$, by Proposition 3.8, it follows that $(u_S^G)^2 = (v_S^G)^2 = 1$, hence $\{u_S^G, v_S^G\} \subseteq \Xi(G)^\times$.

Observe now that for $|G : S| = 2$, both elements u_S^G and v_S^G are obtained by inflation from G/S to G of the corresponding elements $u_{\mathbf{1}}^{G/S}$ and $v_{\mathbf{1}}^{G/S}$. The group $C = G/S$ has order two. The additive group $\Xi(C)$ has a $\mathbb{Z}$-basis consisting of $\langle C, C \rangle_C$, $a = \langle C, \mathbf{1} \rangle_C$ and $b = \langle \mathbf{1}, \mathbf{1} \rangle_C$. The element $\langle C, C \rangle_C$ is the identity element $1_{\Xi(C)}$ of $\Xi(C)$, and by Proposition 3.8, the products of the other basis elements are given by $a^2 = 2a$, $b^2 = ab = ba = 2b$. The group of units $\Xi(C)^\times$ has an $\mathbb{F}_2$-basis consisting of $-1_{\Xi(C)}$, $x = 1_{\Xi(C)} - a$ and $y = 1_{\Xi(C)} - b$. Moreover, the set $\{x, y\}$ is an $\mathbb{F}_2$-basis of the kernel $\partial\Xi(G)^\times$ of the deflation map $\mathrm{Def}_{C/C}^C : \Xi(C)^\times \to \Xi(\mathbf{1})^\times = \{\pm 1_{\Xi(\mathbf{1})}\}$ (recall that deflation in this case consists in taking fixed points under C).

Now assume that in $\Xi(G)^\times$, there is a linear relation (with an additive notation) of the form

$$\lambda(-1_{\Xi(G)}) + \sum_{|G:S|=2} (\alpha_S u_S^G + \beta_S v_S^G) = 0,$$

for some coefficients $\lambda, \alpha_S, \beta_S$ in $\mathbb{F}_2$. Fix a subgroup X of index 2 in G, and apply $\mathrm{Def}_{G/X}^G$ to this relation. Observe that

$$\mathrm{Def}_{G/X}^G u_S^G = \mathrm{Def}_{G/X}^G \mathrm{Inf}_{G/S}^G u_{\mathbf{1}}^{G/S} = \mathrm{Inf}_{G/SX}^{G/X} \mathrm{Def}_{G/SX}^{G/S} u_{\mathbf{1}}^{G/S}$$

is equal to 0 if $S \neq X$, and to $u_{\mathbf{1}}^{G/S}$ if $S = X$. Similarly, $\mathrm{Def}_{G/X}^G v_S^G$ is equal to 0 if $S \neq X$, and to $v_{\mathbf{1}}^{G/S}$ if $S = X$. It follows that

$$\lambda(-1_{\Xi(G/X)}) + \alpha_X u_{\mathbf{1}}^{G/X} + \beta_X v_{\mathbf{1}}^{G/X} = 0,$$

hence $\lambda = \alpha_X = \beta_X = 0$. This completes the proof, since this holds for any X of index two in G. □

Remark A.14. Using the inclusion $i_G : B(G) \to \Xi(G)$ of Proposition 3.11, one can identify $B(G)$ with a subring of $\Xi(G)$. Then it is easy to see that $B(G)^\times$ is the subgroup of $\Xi(G)^\times$ generated by the elements $-\langle G, G \rangle_G$ and $\langle G, G \rangle_G - \langle S, S \rangle_G$, for $|G : S| = 2$: this gives another proof of Matsuda's theorem [Mat82], saying that the unit group $B(G)^\times$ of the usual Burnside ring of a finite abelian group G is isomorphic to $(\mathbb{F}_2)^{r+1}$, where r is the number of subgroups of index 2 in G.

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Serge Bouc   serge.bouc@u-picardie.fr

CNRS-LAMFA, Université de Picardie, 33 rue St Leu, 80039, Amiens cedex 1, France