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A sequence algebra associated with distributions

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If $A = (a_{m,n})$ is a regular summability matrix, the sequence $s = \{s_n\}$ is said to be A *uniformly distributed* (see L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, p. 221, John Wiley & Sons, New York, London, Sydney, Toronto, 1974), if

$$\lim_{m\to\infty} \sum a_{m,n} \exp(2\pi i h s_n) = 0$$

$(h = 1, 2, \ldots)$. In this paper we examine sequences belonging to A^*, where $t \in A^*$ if and only if t is bounded and $s + t$ is A uniformly distributed whenever s is A uniformly distributed. By A' are denoted those members t of A^* such that $at \in A^*$ for every real a. The members of A' form a Banach algebra, A^* is not connected under the sup norm, but A' is a component.

1.

In this paper we shall write $e(x)$ for $e^{2\pi i x}$. If $A = (a_{m,n})$ is a regular summability matrix, the sequence $s = \{s_n\}$ is said to be A *uniformly distributed* [1], if[1]

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[1] All summation in this paper is over $n = 1$ to ∞, unless otherwise indicated.

(1) $$\lim_{m\to\infty} \sum a_{m,n} e(hs_n) = 0$$

$(h = 1, 2, \ldots)$. By A_0 we denote the bounded sequences limited to zero by A and write $\xi \in A^0$ if ξ is bounded and $\xi x \in A_0$ for all $x \in A_0$. It is easy to show that A^0 is a Banach algebra; see [3]. In this paper we shall discuss sequences belonging to A^*, where $t \in A^*$ if and only if t is bounded and $s + t$ is A uniformly distributed whenever s is A uniformly distributed. Such sequences are called *admissible* sequences.

It is easy to show [3] that

$$A^* \supset A^0 .$$

Also, if the sequences t^k $(k = 1, 2, \ldots)$, belong to A^* and

$$\lim_{k\to\infty} \|t^k - t\| = 0$$

(where $\|x\| = \sup_n |x_n|$), then

$$\begin{aligned} \left| e\left\{h\left[s_n+t_n^k\right]\right\} - e\left(h\left[s_n+t_n\right]\right) \right| &= |e(hs_n)| \left| e\left(ht_n^k\right) - e(ht_n) \right| \\ &= |e(ht_n)| \left| e\left\{h\left[t_n^k - t_n\right]\right\} - 1 \right| \\ &\le |e(h\varepsilon) - 1| \end{aligned}$$

for a suitable choice of t^k . It is now clear that t is admissible and A^* is closed.

We now prove:

THEOREM 1. *If* $0 \le t_n \le \beta < 1$ *and* $0 \le u_n \le \beta < 1$ $(n = 1, 2, \ldots)$, *and* $t \in A^*$, $u \in A^*$ *then* $ut \in A^*$.

Proof. In the first place, if $t \in A^*$, $2t \in A^*$ and in general $kt \in A^*$ $(k = 1, 2, \ldots)$. Hence

$$\lim_{m\to\infty} \sum a_{mn} e(kt) e(s_n) = 0 ,$$

and the same is true for any trigonometric polynomial, $p_k(t)$. Moreover

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if f is continuous on $(0, \beta)$, f may be approximated uniformly by such a polynomial, so that

$$\left|\sum a_{m,n}|f(t_n)-p_k(t_n)|\right| \leq \varepsilon \sum |a_{m,n}| ,$$

where

$$|f(x)-p_k(x)| < \varepsilon ,$$

$x \in (0, \beta)$.

From this we conclude that

$$\lim_{m\to\infty} \sum a_{m,n} f(t_n) e(s_n) = 0 ,$$

if $t \in A^*$ and f is continuous on $(0, \beta)$. Hence

$$\lim_{m\to\infty} \sum a_{mn} t_n^r e(s_n) = 0$$

$(r = 1, 2, \ldots)$.

If $u \in A^*$, then

$$\lim_{m\to\infty} \sum a_{m,n} t_n^r e(s_n+ku_n) = 0$$

$(r, k = 1, 2, \ldots)$, and so

$$\lim_{m\to\infty} \sum a_{mn} t_n^r p_k(u_n) e(s_n) = 0 .$$

It then follows that

$$\lim_{m\to\infty} \sum a_{mn} t_n^r u_n^r e(s_n) = 0 . \tag{2}$$

If $g(x)$ is a polynomial,

$$\lim_{m\to\infty} \sum a_{m,n} g(t_n u_n) e(s_n) = 0 ,$$

so that, using the Stone-Weierstrass Theorem,

$$\left|\sum a_{m,n}\big(g(t_n u_n)-e(t_n u_n)\big)e(s_n)\right| \leq \varepsilon \sum |a_{m,n}| .$$

From this it follows that

(3) $$\lim_{m\to\infty} \sum a_{m,n} e(s_n+t_n u_n) = 0 .$$

Criterion (1) indicates that if $\{s_n\}$ is A uniformly distributed so are the sequences $\{hs_n\}$ $(h = 1, 2, \ldots)$. Taking this into account and making a slight adjustment to our previous arguments,

$$\lim_{m\to\infty} \sum a_{m,n} h^r t_n^r u_n^r e(hs_n) = 0 ,$$

and so as in (3),

$$\lim_{m\to\infty} \sum a_{m,n} e(hs_n+ht_n u_n) = 0 .$$

This implies that $s + ut$ is A uniformly distributed, $ut \in A^*$.

This proof breaks down for the interval $0 \le x < 1$ or $0 \le x \le 1$.

2.

It turns out there are two types of admissible sequences. If there exists an α , $0 < \alpha \le 1$, such that αt is admissible and $0 < \alpha t_n \le \rho < 1$ $(n = 1, 2, \ldots)$, then t is said to be *non-singular*; if no such α exists then t is said to be *singular*.

THEOREM 2. *If w and t are non-singular admissible sequences, then wt is a non-singular admissible sequence.*

Proof. Since there exists an α , $0 < \alpha \le 1$, such that αt is admissible, and $0 \le \alpha t_n \le \rho < 1$, from Theorem 1 (all constant sequences are admissible), it follows that $\beta\alpha t$ is admissible for any β , $0 \le \beta \le 1$. Hence γt is admissible, $0 \le \gamma \le \alpha$. Moreover, if w and t are non-singular, $\gamma' w$ is admissible, $0 \le \gamma' \le \alpha'$, and $\gamma\gamma' wt$ is admissible $0 \le \gamma\gamma' \le \alpha\alpha'$. Since wt is bounded, there exists an integer k such that $1/k < \alpha\alpha'$, and wt/k is admissible. By adding this k times we have wt is admissible, and of course non-singular.

This proof can also be used to show ηt and ηwt are admissible, $0 \le \eta \le 1$.

We shall write $t \in A'$ if there exists a positive constant δ such that $t + \delta$ is non-singular.

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For any β such that $0 \le \beta \le \alpha$, $0 \le \beta(t_n+\delta) \le \alpha(t_n+\delta) \le \rho < 1$, and if β is chosen so that $0 \le \beta(t_n+\mu) \le \rho$ as well, then $\beta(t+\mu) = \beta(t+\delta) + \beta(\mu-\delta)$ is admissible. This implies that $(t+\mu)$ is non-singular for $\mu \ge \delta$.

THEOREM 3. A' *is a Banach algebra.*

Proof. If $t, u \in A'$, there exist positive constant sequences δ, δ' such that $t + \delta$ and $u + \delta'$ are non-singular. Choose β so that $0 \le \beta t_n+\beta\delta \le \frac{1}{4}$, $0 \le \beta u_n+\beta\delta' \le \frac{1}{4}$ $(n = 1, 2, \ldots)$. Then $0 \le \beta(t_n+u_n+\delta+\delta') \le \frac{1}{2}$ is admissible. This implies that $t + u + \delta + \delta'$ is non-singular and that $t+u \in A'$.

Examination of the real and imaginary parts of (1) shows that if s is A uniformly distributed, $-s$ is A uniformly distributed, and subsequently if $t \in A^*$, then $-t \in A^*$. If $t \in A'$, our remarks at the end of Theorem 2 show $\eta t \in A'$ for $0 \le \eta \le 1$, and hence $\eta t \in A'$ for all positive real η . Choose δ so that $\delta - t$ is a positive sequence and β so that $0 \le \beta$, $0 \le \beta(\delta-t_n) \le \rho < 1$. Then $\beta\delta$ is admissible, $-\beta t$ is admissible, $\beta(\delta-t)$ is admissible and $\delta - t$ is non-singular. It follows that $\eta t \in A'$ for all real η .

If $t, u \in A'$, then if δ, δ' are chosen as before, $(t+\delta)(u+\delta') \in A'$. However $ut = (u+\delta')(t+\delta) - k't - ku - kk'$, and since all four terms are in A' , our linearity condition implies $ut \in A'$.

The unit sequence belongs to A' . We have already seen that A^* is closed. Suppose

$$\lim_{n\to\infty} \|t^n-t\| = 0 ,$$

where $t^n \in A'$; then $t \in A^*$ and is admissible. Also, $\alpha t^n \in A'$ for all real α . Hence

$$\lim_{n\to\infty} \|\alpha t^n-\alpha t\| = 0 ,$$

and $\alpha t \in A^*$ for all real α . A few easy steps now show that $t \in A'$ and A' is a Banach algebra.

We have seen that

$$A^0 \subset A' \subset A^* ,$$

where A^0 and A' are Banach algebras. Of course A^* is not an algebra. In fact, if $t \in A^* \backslash A'$ (we shall continue to call these sequences singular) there are only finitely many α , $0 < \alpha \leq 1$, such that αt is admissible. Otherwise, α_1 and α_2 could be found such that $0 \leq \alpha_1 - \alpha_2 < \varepsilon$ for any $\varepsilon > 0$ and since $(\alpha_1 - \alpha_2)t$ would be admissible, would in fact belong to A' . Also these α must be rational, for $n\alpha - [n\alpha]$ is dense in the unit interval, and if αt is admissible, so is $(n\alpha - [n\alpha])t$. For a finite set of fractions there is always a fraction p/q such that all of the members of the set are integral multiples of p/q . Also p/q is either a member of the set or can be obtained from the set by linear operations. Thus, if t is singular, there exists a t' such that $\alpha t'$ is not admissible, $0 < \alpha < 1$, and nt' $(n = 1, 2, \ldots)$ includes (indeed comprises) all of the admissible multiples of t .

We now see:

THEOREM 4. *If* $B \subset A^*$ *is an algebra that includes the constant sequences,* $B \subset A'$.

Indeed we have just seen that no member of $A^* \backslash A'$ can be part of such an algebra containing all of the constant sequences.

3.

If there are no A uniformly distributed sequences, then A^* has no meaning.

THEOREM 5. *If there is at least one* A *uniformly distributed sequence then* $A^* \backslash A'$ *is non-empty.*

Proof. We can clearly assume that s is A uniformly distributed and bounded. Moreover, all sequences of 1's and 0's belong to A^* . If all of these belong to A' , then all linear combinations or all sequences with finitely many values are in A' (or A^*). Since such sequences are dense in the bounded sequences and A^* is closed then all bounded sequences including $-s$ are in A^* . This is a contradiction and our assertion is proved.

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If $A = (a_{m,n})$ satisfies

$$\lim_{m\to\infty} \sum |a_{m,n}-a_{m,n+1}| = 0 ,$$

for example, all well distributed sequences are A uniformly distributed; see [1].

THEOREM 6. A^* *is non-connected; one of its components is a maximum subalgebra* A' .

Proof. We first show that $A^*\backslash A'$ is a closed set. We already know that if $t^k \in A^*$ and

$$\lim_{k\to\infty} \|t^k-t\| = 0 ,$$

then $t \in A^*$. Suppose

$$\|t^{k_0}-t\| < 1/10 ;$$

then $x \in A'$, where $x = t^{k_0} - t$. If $\alpha t^{k_0} \notin A^*$, $0 \le \alpha \le \beta < 1$, then since $\alpha t^{k_0} = \alpha t + \alpha x$, $\alpha t \notin A^*$ $(\alpha x \in A^*)$. Hence $t \in A^*\backslash A'$. Both $A^*\backslash A'$ and A' are non-empty, A' is closed. This shows that A^* is non-connected.

Since $x \in A'$ implies $\alpha x \in A'$ for all real α , it is easy to show that A' is connected.

4.

Suppose $A = (a_{m,n})$ satisfies (4); then it is said to be strongly regular. A sequence $\{s_n\}$ is said to be *well distributed* if

$$\frac{1}{n+1} \sum_{k=p}^{n+p} e(hs_n) \quad (h = 1, 2, \ldots)$$

has limit zero uniformly in p . The well distributed sequences consist of precisely those which are A uniformly distributed for all strongly regular A ; see [1].

Admissible sequences for well distributed sequences may be defined;

we shall denote these by C^* . In [4], the following theorem is proved:

THEOREM 7. *If* $|t_n - t_{n-1}| \le \frac{1}{2}$ $(n = 1, 2, \ldots)$ *, then* $t \in C^*$ *if and only if*

$$(5) \qquad \frac{1}{n+1} \sum_{k=p}^{n+p} |t_n - t_{n-1}| \to 0$$

uniformly in p *(that is, is almost convergent to zero).*

A sequence $\{s_n\}$ is said to be *thin* with respect to the matrix $A = (a_{m,n})$ if $s_n = 0$, $n \notin E$, where

$$\lim_{m\to\infty} \sum_{n\in E} |a_{m,n}| = 0 .$$

We shall prove:

THEOREM 8. *If* $A = (a_{m,n})$ *is a regular matrix,* $a_{m,n} \ge 0$ $(m, n = 1, 2, \ldots)$ *, which satisfies* (4)*, then if* $|t_n - t_{n-1}| \le \frac{1}{2}$ $(n = 1, 2, \ldots)$ *,* $t \in A^*$ *only if* $t = u + v$ *, where* $u \in C^*$ *and* v *is thin.*

Proof. The matrix $A = (a_{m,n})$ may be adjusted by multiplying the row elements so that

$$\sum a_{m,n} = 1 \quad (m = 1, 2, \ldots) ,$$

without affecting its other properties.

As in [4], we see that if (5) is not satisfied, there is a sequence n_i and a δ such that $t_{n_i} - t_{n_i-1} > \delta$ $(i = 1, 2, \ldots)$. We shall suppose that $\{t_{n_i}\}$ is not thin. Then we choose the well distributed sequences x and y , and construct z as follows:

$$(6) \qquad z_n = \begin{cases} y_j & \text{if } n \in (r_j) , \\ x_i \pmod{\frac{1}{2}} - t_{n_i-1} & \text{if } n \in (n_i) , \\ \frac{1}{2} + x_i \pmod{\frac{1}{2}} - t_{n_i-1} & \text{if } n \in (n_i-1) , \end{cases}$$

where $(r_j) = Z \backslash ((n_i) \cup (n_i - 1))$. The above construction is identical with that in [4], pp. 154, 155, where it is also shown that z is well distributed but $z + t$ is not. This is done by showing that if $I_{(0,\delta)}$ is the characteristic function for $(0, \delta)$, then

$$I_{(0,\delta)}\left(z_{n_i}+t_{n_i}\right) = I_{(0,\delta)}\left(z_{n_i-1}+t_{n_i-1}\right) = 0 \quad (i = 1, 2, \ldots) .$$

It then followed that $z + t$ was not well distributed, and so $t \notin C^*$. We denote $(n_i) \cup (n_i - 1)$ by (g_k). If A satisfies (4), then since z is well distributed it is also A uniformly distributed; see [1]. Let us choose m_ν so that

$$\sum_{k=1}^{\infty} a_{m_\nu, g_k} \geq \varepsilon_0 > 0 \tag{7}$$

$(\nu = 1, 2, \ldots)$; then

$$\sum_{k=1}^{\infty} a_{m_\nu, g_k} I_{(0,\delta)}\left(z_{g_k}+t_{g_k}\right) = 0$$

$(\nu = 1, 2, \ldots)$, and $z + t$ will not be A uniformly distributed, unless

$$\lim_{\nu\to\infty} \sum_{j=1}^{\infty} a_{m_\nu, r_j} I_{(0,\delta)}\left(z_{r_j}+t_{r_j}\right) = \delta . \tag{8}$$

However it is also clear that

$$\limsup_{\nu\to\infty} \sum_{k=1}^{\infty} a_{m_\nu, g_k} I_{(a,b)}\left(z_{g_k}+t_{g_k}\right) \neq 0$$

for all intervals (a, b) , $|b-a| = \delta$, as otherwise a simple addition of finitely many characteristic functions would contradict (7). But then, it is clear from the proof of Theorem 7 in [4] that we can construct z' such that $z'_{r_j} = z_{r_j}$ and $z'_{g_k} = z_{g_k} + \alpha$, where α is some constant; and z' will be well distributed. We can choose this constant α so that

$$\limsup_{\nu\to\infty} \sum_{k=1}^{\infty} a_{m_\nu, g_k} I_{(0,\delta)}\left(z'_{g_k}+t_{g_k}\right) \neq 0 . \tag{9}$$

From (8) and (9), it then follows that

$$\lim_{m\to\infty} \sum a_{m,n} I_{(0,\delta)}(z_n+t_n) \neq \delta ,$$

and t is not admissible.

In [4] it is remarked that if t is admissible, then by translation and addition of integer sequences to t, we obtain t' such that $|t'_n-t'_{n-1}| \le \frac{1}{2}$. The same may be said for members of A^* and we have

THEOREM 9. *If* $A = (a_{mn})$ *is a positive regular matrix satisfying* (4), *then* $t \in A^*$ *only if* $t = u + v$, *where* $u \in C^*$ *and* v *is thin.*

It is also easy to show that if (5) is satisfied then $t = u + v$, where

(10) $$\lim |u_n-u_{n+1}| = 0$$

and v is thin, so that if $t \in A^*$, $|t_n-t_{n-1}| \le \frac{1}{2}$ $(n = 1, 2, \ldots)$, then $t = u + v$, where u satisfies (10) and v is thin.

Of course $A' \subset A^*$, but if $t \in A'$, then there exists a $t' \in A'$, $|t'_n-t'_{n-1}| \le \frac{1}{2}$ obtained from t by *algebraic* operations. From this it follows:

THEOREM 10. *If* $A = (a_{m,n})$ *is positive, regular, and satisfies* (4), *then* $t \in A'$ *only if* $t = u + v$, *where* u *satisfies* (10) *and* v *is thin.*

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