Can. J. Math., Vol. XXV, No. 4, 1973, pp. 840–841

METRIZABILITY OF M-SPACES

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An unsolved problem in metrization theory is whether every Hausdorff M-space with a G_δ-diagonal is metrizable. There are several recent results which have a bearing upon this question. In [**9**], P. Zenor showed that an M-space is metrizable if and only if it has a regular G_δ-diagonal; in [**1**], Borges showed that a regular meta-Lindelöf M-space is metrizable if and only if it has a $\bar{G}_\delta$-diagonal. In [**5**], T. Ishii introduced the notion of a wM-space, which generalizes M-spaces, and in [**6**] showed that a wM-space is metrizable if and only if it has a $\bar{G}_\delta(2)$-diagonal and that a normal wM-space is metrizable if and only if it has a $\bar{G}_\delta(1)$-diagonal. In [**8**], T. Shiraki showed that a Hausdorff wM-space is metrizable if and only if it is a $\sigma^\#$-space. In view of Theorems 1 and 2 and the implications $\bar{G}_\delta(2)$-diagonal $\Rightarrow$ regular G_δ-diagonal $\Rightarrow$ $\bar{G}_\delta$-diagonal $\Rightarrow$ $\bar{G}_\delta(1)$-diagonal, for regular spaces all five of the foregoing metrization theorems are special cases of Corollary 5 below.

A *c-semi-stratification* for a topological space X is a system $\{g_n(x) : x \in X; n = 1, 2, \ldots\}$ of open subsets of X which satisfies the following conditions:

(1) $x \in g_n(x)$;

(2) $g_{n+1}(x) \subset g_n(x)$;

(3) if A is a closed compact subset of X and $x \in X - A$, then there exists n such that $x \notin g_n(a)$ for every $a \in A$.

A space X is said to be *c-semi-stratifiable* if X has a c-semi-stratification. If $\{g_n(x)\}$ is a c-semi-stratification for X and $S \subset X$, then we let

$$g_n(S) = \bigcup \{g_n(x) : x \in S\}.$$

A c-semi-stratification $\{g_n(x)\}$ is, of course, a semi-stratification if $F = \bigcap \{g_n(F) : n = 1, 2, \ldots\}$ for every closed subset F of X [**3**]. A study of c-semi-stratifiable spaces and, in particular, proofs of Theorems 1 and 2, may be found in [**7**].

THEOREM 1. *Any space with a $\bar{G}_\delta(1)$-diagonal is c-semi-stratifiable.*

THEOREM 2. *Every $\sigma^\#$-space is c-semi-stratifiable.*

A topological space X is a *β-space* provided that X has a system $\{g_n(x) : x \in X; n = 1, 2, \ldots\}$ of open subsets such that $x \in g_n(x)$ for all $x \in X$ and all n and if $x \in g_n(x_n)$ for some $x \in X$ and a sequence $\{x_n\}$ in X, then $\{x_n\}$ has a cluster point. The notion of a β-space is due to R. Hodel [**4**], who proved that a space X is semi-stratifiable if and only if it is both a β-space

Received April 10, 1972 and in revised forms, July 24, 1972 and April 2, 1973.

https://doi.org/10.4153/CJM-1973-086-0 Published online by Cambridge University Press

and a $\sigma^{\#}$-space. In the presence of regularity, Hodel's factorization theorem remains true when $\sigma^{\#}$-spaces are replaced by c-semi-stratifiable spaces.

THEOREM 3. *A regular space is semi-stratifiable if and only if it is a c-semi-stratifiable β-space.*

Proof. Let X be a regular c-semi-stratifiable β-space. X has a c-semi-stratification $\{g_n(x):x \in X;\ n = 1, 2, \dots\}$ such that $\mathrm{cl}(g_{n+1}(x)) \subset g_n(x)$ for all $x \in X$ and all n and such that if $a \in g_n(b_n)$ for $n = 1, 2, \dots$, then the sequence $\{b_n\}$ has a cluster point. Let $x, x_n \in X$ such that $x \in g_n(x_n)$ for $n = 1, 2, \dots$; we shall show that $x_n \to x$, thereby showing that $\{g_n(x)\}$ is a semi-stratification and completing the proof.

The sequence $\{x_n\}$ has at least one cluster point; moreover, it is easy to show that every subsequence of $\{x_n\}$ also has at least one cluster point. Suppose y is a cluster point of $\{x_n\}$ and that $y \neq x$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \in g_i(y)$ for $i = 1, 2, \dots$ and $x \neq x_{n_i}$ for all i. Since $\mathrm{cl}(g_{i+1}(y)) \subset g_i(y)$, y is the only possible cluster point of $\{x_{n_i}\}$. Since every subsequence of $\{x_{n_i}\}$ has a cluster point, it follows that $x_{n_i} \to y$ so that the set $S = \{y\} \cup \{x_{n_i}:i = 1, 2, \dots\}$ is compact. There exists m such that $x \notin g_m(S)$; choose $k > m$ such that $x_k \in S$; then $x \notin g_m(x_k) \supset g_k(x_k)$, which is a contradiction. It follows that x is the only cluster point of $\{x_n\}$; since every subsequence of $\{x_n\}$ has a cluster point, necessarily $x_n \to x$, completing the proof.

COROLLARY 4. *A regular space is developable if and only if it is a c-semi-stratifiable $w\Delta$-space.*

Proof. Let X be a regular, c-semi-stratifiable $w\Delta$-space. Any $w\Delta$-space is a β-space so that X is semi-stratifiable, hence a $\sigma^{\#}$-space. Burke has shown that any regular, $\sigma^{\#}$, $w\Delta$-space is developable [**2**].

COROLLARY 5. *A regular space is metrizable if and only if it is a c-semi-stratifiable wM-space.*

Proof. Let X be a regular c-semi-stratifiable wM-space. Every wM-space is a β-space so that X is semi-stratifiable, hence a $\sigma^{\#}$-space. Shiraki has shown that every regular, $\sigma^{\#}$, wM-space is metrizable [**8**], completing the proof.

REFERENCES

1. C. Borges, *On metrizability of topological spaces*, Can. J. Math. *20* (1968), 795–804.
2. D. Burke, *On p-spaces and wΔ-spaces*, Pacific J. Math. *35* (1970), 285–296.
3. G. Creede, *Concerning semi-stratifiable spaces*, Pacific J. Math. *32* (1970), 47–54.
4. R. E. Hodel, *Moore spaces and wΔ-spaces*, Pacific J. Math. *38* (1971) 641–652.
5. T. Ishii, *On wM-spaces. I*, Proc. Japan Acad. *46* (1970), 6–10.
6. ——— *On wM-spaces. II*, Proc. Japan Acad. *46* (1970), 11–15.
7. H. Martin, *Topological spaces in which compacta are uniformly G_δ* (to appear).
8. T. Shiraki, *On some metrization theorems* (to appear).
9. P. Zenor, *On spaces with regular G_δ-diagonals*, Pacific J. Math. *40* (1972), 759–763.

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https://doi.org/10.4153/CJM-1973-086-0 Published online by Cambridge University Press