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INTRODUCTION

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It is a great pleasure for me to introduce this special issue in honor of Clifford Wymer. This special issue (which has been jointly edited by Giovanni Di Bartolomeo, Daniela Federici, and Enrico Saltari) is a fitting tribute to a great scholar. We all know him well, so there is no need for me to eulogize him.

I will only note that he has published surprisingly few papers, most of which have been extorted from him by his friends. Is this excessive modesty or simple laziness? I do not know: posterity will judge. In any case, all are seminal papers that have deeply influenced the field after circulating for many years as mimeos.

I first met Clifford when he was giving a series of lectures at Confindustria's research center on the econometric estimation of continuous time dynamic models. This was exactly what I needed.

In fact, following Richard Goodwin, I was firmly convinced that dynamic economic models should use differential equations, hence continuous time. I was also convinced that in the standard way of discretizing a continuous model for estimation purposes, there was something wrong (see Section 1 below). Clifford's method provided the correct answer, and since then I became an adept of the sect of "continuists."

The use of Clifford's software requires a deep knowledge of the theory behind it (see Section 2 below), and this in turn requires time and effort. Maybe our combined wishes will succeed in extorting from Clifford the creation of such a user-friendly software. Although not all the papers contained in this special issue use Clifford's software, it may be useful for the reader to have a brief summary of the procedure to be followed in estimating continuous time models according to such a software. A selected list of references is presented at the end [Bergstrom (1976, 1984), Gandolfo (1992, 1993), Wymer (1972, 1987, 1993, 1995, 1997, 2012)].

1. DISCRETIZATION OF CONTINUOUS TIME MODELS

Continuous time models explicitly take into account the fact that a flow variable cannot be measured instantaneously, so that what we actually observe is the integral of such a variable over the observation period, while stocks are measured

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and observed at a point in time (this allows us a correct treatment of stock-flow models).

To show the importance of the stock-flow distinction, let us consider the following example, where consumption is assumed to depend on income and wealth:

$$C(t) = b_0 + b_1 Y(t) + b_2 W(t) + u(t), \tag{1}$$

where $C(t)$, $Y(t)$ are flows, whereas $W(t)$ is a stock variable, and $u(t)$ is a disturbance term. As stated above, $C(t)$, $Y(t)$ cannot be measured instantaneously, whereas $W(t)$ can. Hence, using the superscript ^{o} to denote observed variables, we have

$$C^o(t) = \frac{1}{\delta}\int_0^\delta C(t)\mathrm{d}t,\ Y^o(t) = \frac{1}{\delta}\int_0^\delta Y(t)\mathrm{d}t,\ W^o(t) = W(t). \tag{2}$$

The standard way of proceeding to estimate (1) is to use

$$C^o(t) = b_0 + b_1 Y^o(t) + b_2 W^o(t) + u(t). \tag{3}$$

This would however be incorrect, since to obtain an observationally equivalent to (1) we must integrate through, thus obtaining (the time interval δ can be assumed equal to unity without loss of generality)

$$\int_0^1 C(t)\mathrm{d}t = \int_0^1 b_0\mathrm{d}t + b_1\int_0^1 Y(t)\mathrm{d}t + b_2\int_0^1 W(t)\mathrm{d}t + \int_0^1 u(t)\mathrm{d}t, \tag{4}$$

i.e.,

$$C^o(t) = b_0 + b_1 Y^o(t) + b_2\int_0^1 W^o(t)\mathrm{d}t + \int_0^1 u(t)dt, \tag{5}$$

which is quite different from (3), due to the presence of the integral of the stock variable and of the disturbance term (which introduces serial correlation). Continuous time econometrics offers the proper way of dealing with these problems.

2. THE ESTIMATION METHOD

Let us start from a generic continuous time first-order[1] nonlinear dynamic model

$$\mathrm{D}\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{z}, \Theta) + \mathbf{u}(t), \tag{6}$$

where D denotes the differential operator d/dt, $\mathbf{x}$ is a vector of endogenous variables, $\mathbf{z}$ is a vector of exogenous variables, Θ is a vector of parameters, and $\mathbf{u}(t)$ is a vector of disturbances of the white noise type, conventionally assumed to satisfy the equation:

$$\mathbf{u}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\zeta(t), \tag{7}$$

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where $\zeta(t)$ is generated by a stochastic process with mean zero, with uncorrelated increments, and homogeneous, that is,

$$\begin{aligned} E[\zeta(t)] &= 0, \qquad \forall t, \\ E\{[\zeta(t_1)-\zeta(t_2)][\zeta(t_3)-\zeta(t_4)]^T\} &= 0, \ \forall t_1 > t_2 \geq t_3 > t_4, \\ E\{[\zeta(t+h)-\zeta(t)][\zeta(t+h)-\zeta(t)]^T\} &= \Omega\,|h|, \end{aligned} \tag{8}$$

where Ω is a matrix of constants.

2.1. Linear Estimation

The linear estimation method consists of the following steps:

(L1) Linearize the model analytically (expansion in Taylor's series, etc.) to obtain

$$\mathrm{D}\mathbf{x}(t) = \mathbf{Ax}(t)+\mathbf{Bz}(t)+\mathbf{u}(t), \tag{9}$$

where $\mathbf{A}, \mathbf{B}$ are constant matrices whose elements are functions of the parameters. If the model is linear, then we directly start from (9). Note that when $\zeta(t)$ is nonmean square differentiable, so that the process $\mathbf{u}(t)$ cannot be rigorously defined, it is possible to consider the system:

$$\mathrm{d}\mathbf{x}(t)= \mathbf{Ax}(t)\mathrm{d}t+\mathbf{Bz}(t)\mathrm{d}t + \mathrm{d}\zeta(t), \tag{10}$$

where $\mathbf{u}(t)$ is replaced by the mean square of the process $\zeta(t)$. System (10) has the same properties as system (9).

(L2) Determine the stochastically equivalent discrete analogue to (9). For this purpose, we proceed as follows. First, integrating system (9), we obtain

$$\mathbf{x}(t) = \mathbf{x}(0)\mathrm{e}^{\mathbf{A}t} + \int_0^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathbf{Bz}(\omega)\mathrm{d}\omega + \int_0^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathbf{u}(\omega)\mathrm{d}\omega, \tag{11}$$

or, given (7) and (10),

$$\mathbf{x}(t) = \mathbf{x}(0)\mathrm{e}^{\mathbf{A}t} + \int_0^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathbf{Bz}(\omega)\mathrm{d}\omega + \int_0^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathrm{d}\zeta(\omega). \tag{12}$$

Suppose now that the continuous variables are observed at equispaced intervals, say every δ units of time, where δ is the length of the observation interval in terms of the basic time unit of the system. Evaluating (12) at time $t-\delta$, we have

$$\mathbf{x}(t-\delta) = \mathbf{x}(0)\mathrm{e}^{\mathbf{A}(t-\delta)} + \int_0^{t-\delta} \mathrm{e}^{\mathbf{A}(t-\delta-\omega)}\mathbf{Bz}(\omega)\mathrm{d}\omega + \int_0^{t-\delta} \mathrm{e}^{\mathbf{A}(t-\delta-\omega)}\mathrm{d}\zeta(\omega). \tag{13}$$

By simple manipulations, we obtain

$$\mathbf{x}(t) - \mathrm{e}^{\delta\mathbf{A}}\mathbf{x}(t-\delta) = \int_{t-\delta}^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathbf{Bz}(\omega)\mathrm{d}\omega + \int_{t-\delta}^t \mathrm{e}^{\mathbf{A}(t-\omega)}\mathrm{d}\zeta(\omega), \tag{14}$$

from which, by a suitable change of variable,

$$\mathbf{x}(t) = \mathrm{e}^{\delta \mathbf{A}}\mathbf{x}(t-\delta) + \int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathbf{B}\mathbf{z}(t-\omega)\mathrm{d}\omega + \int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathrm{d}\zeta(t-\omega). \qquad \textbf{(15)}$$

Let now $\mathbf{y}_\tau$ be the discrete observation of a continuous variable $\mathbf{y}(t)$ at time $\tau\delta$, τ being an integer, that is, $\mathbf{y}_\tau = \mathbf{y}(\tau\delta)$. Setting $t = \tau\delta$ in (15) and letting $\delta = 1$, we have

$$\mathbf{x}_\tau = \mathrm{e}^{\delta \mathbf{A}}\mathbf{x}_{\tau-\delta} + \int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathbf{B}\mathbf{z}(\tau-\omega)\mathrm{d}\omega + \int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathrm{d}\zeta(\tau-\omega). \qquad \textbf{(16)}$$

System (16) can be considered as a set of stochastic difference equations generating the discrete-time data $\mathbf{x}_\tau$ over some sample period $\tau = 1, 2, \ldots, T$. Since the observations generated by the differential equation system (9) will satisfy the exact discrete model (16) irrespective of the length of the observation interval, the sampling properties of the differential system can be studied by considering the sampling properties of the exact discrete model.

(L3) The treatment so far assumes that all the variables can be measured at an instant in time, that is to say, that they are *point* (or *instantaneous*) variables. This is true of certain variables like stocks, interest rates, etc., but if the model contains flow variables, we must remember that they are measured as an integral over the observation interval. Therefore, a model containing flow variables needs to be integrated over the observation interval to produce a model defined in terms of variables that are measurable. If we integrate equation (15), we obtain

$$\begin{aligned}
\frac{1}{\delta}\int_{t-\delta}^{t} \mathbf{x}(\omega)\mathrm{d}\omega &= \mathrm{e}^{\delta \mathbf{A}}\frac{1}{\delta}\int_{t-\delta}^{t} \mathbf{x}(\omega-\delta)\mathrm{d}\omega + \frac{1}{\delta}\int_{t-\delta}^{t}\int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathbf{B}\mathbf{z}(s-\omega)\mathrm{d}\omega\mathrm{d}s \\
&\quad + \frac{1}{\delta}\int_{t-\delta}^{t}\int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathrm{d}\zeta(s-\omega)\mathrm{d}s \\
&= \mathrm{e}^{\delta \mathbf{A}}\frac{1}{\delta}\int_{t-\delta}^{t} \mathbf{x}(\omega-\delta)\mathrm{d}\omega + \int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\frac{1}{\delta}\int_{t-\delta}^{t}\mathbf{B}\mathbf{z}(s-\omega)\mathrm{d}s\mathrm{d}\omega \\
&\quad + \frac{1}{\delta}\int_{t-\delta}^{t}\int_0^\delta \mathrm{e}^{\mathbf{A}\omega}\mathrm{d}\zeta(s-\omega)\mathrm{d}s. \qquad \textbf{(17)}
\end{aligned}$$

Now, apart from a model only containing flow variables, the integration carried out has also transformed the instantaneous variables into integrals and these we must deal with. In fact, if we now let $y(t)$ denote a generic instantaneous variable, the integral

$$\frac{1}{\delta}\int_{t-\delta}^{t} y(\omega)\mathrm{d}\omega \qquad \textbf{(18)}$$

is not observable and has to be evaluated (using, for example, the trapezoidal rule).

But there is a more serious problem: the integration that we have carried out in (17) to obtain measurable variables implies that the disturbances in (17) are no longer serially uncorrelated. It is however possible to prove [Gandolfo (1981, Appendix II to Chapter 3)] that—denoting by $\zeta(t)$ the disturbance vector in (17)—this can be approximated by the stochastic process:

$$\zeta(t) \simeq (1 + 0.268\mathrm{L})\epsilon(t), \tag{19}$$

where L is the lag operator and $\epsilon(t)$ a serially uncorrelated random disturbance vector. Since the moving average process (19) is independent of the parameters of the model, the model can be transformed by using the inverse of this process to obtain a model with serially uncorrelated disturbances.

(L4) In practice, the apparently forbidding procedure described in (L3) amounts (when we have a mixed stock-flow model) to replacing instantaneous variables with their integral, and to prewhiten all the data using the process $(1 + 0.268\mathrm{L})^{-1}$. If we denote by an asterisk the variables so transformed, and let $\delta = 1$, we have the final expression:

$$\mathbf{x}_\tau^* = \mathrm{e}^{\mathbf{A}}\mathbf{x}_{\tau-1}^* + \int_0^1 \mathrm{e}^{\mathbf{A}\omega}\mathbf{B}\mathbf{z}^*(\tau - \omega)\mathrm{d}\omega + \epsilon(t). \tag{20}$$

(L5) The final step is to estimate the parameters of model (20) by a full information maximum-likelihood (FIML) method.

2.2. Nonlinear Estimation

The nonlinear estimation method of a nonlinear model such as (6) consists of the following steps:

(NL1) Given the sample of discrete observations, say $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$, assume given values of the parameters (say Θ_0), and solve the nonlinear differential equation (d.e.) system (6) using $\mathbf{x}_0, \mathbf{z}_0$ as initial point (the solution is calculated by a numerical variable-order, variable-step Adams method).

(NL2) Using this solution compute $\hat{\mathbf{x}}_1$ and hence compute the residuals between the calculated and actual values at time 1, $\mathbf{u}_1 = \mathbf{x}_1 - \hat{\mathbf{x}}_1$.

(NL3) Repeat the process for time 2 reinitializing the d.e. system, namely using the same set of parameters but the actual $\mathbf{x}_1$ as the new starting point. Hence, compute the vector of residuals $\mathbf{u}_2$, and so on (Figure 1 presents a simplified representation of this process).

(NL4) In this way we obtain a set of residuals $\mathbf{u}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t;\Theta)$, $t = 1, 2, \ldots, n$. We can then determine the residual covariance matrix and form the logarithm of the determinant of this matrix:

$$L(\Theta) = \ln\det\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\mathbf{x}(t) - \hat{\mathbf{x}}(t;\Theta)\right]\left[\mathbf{x}(t) - \hat{\mathbf{x}}(t;\Theta)\right]^T\right\}.$$

(NL5) $L(\Theta)$ is then minimized by an iterative procedure, which consists of the following steps:

(NL5a) change the values of the parameters as suggested by the optimization algorithm (a quasi-Newton method), say into Θ_1;

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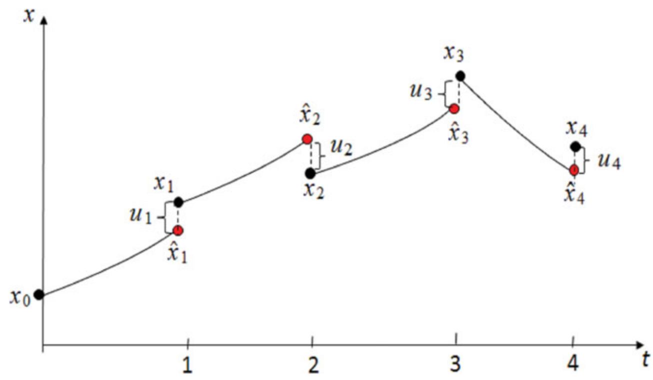

FIGURE 1. A simplified representation of the nonlinear estimation method.

(NL5b) repeat the procedure NL1–NL3 using the new set of parameters Θ_1;
(NL5c) from Θ_1 go to a new set Θ_2 and so on, until the procedure converges to a minimum.

It is clear that the nonlinear procedure is much more intensive in computer time than the linear procedure. The latter will in any case be helpful to determine the initial parameter vector Θ_0. In other words, if one does not have a good guess for Θ_0, the best way is to set it equal to the parameter vector estimated with the linear method.

NOTE

1. Higher order models can always be reduced to first-order models by means of appropriate substitutions.

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3. THE SPECIAL ISSUE

This section was written by Giovanni Di Bartolomeo, Daniela Federici, and Enrico Saltari, who are the editors of the special issue.

The contributions collected in this special issue in honor of Clifford R. Wymer are related to his work from several perspectives. Some of them focus on continuous time economics and econometrics from both a theoretical and an empirical point of view. In particular, one work presents an application of WYSEA, the program developed by Clifford to estimate continuous time models. All the authors provide researches in the heterogeneous fields of interest of Clifford: productivity dynamics, financial markets, money demand, exchange rate dynamics, and temporary deviations from the rational expectation paradigm.

In the first contribution, William Barnett and Liting Su explore another topic dear to Clifford: financial market. They focus on the aggregation and index number theory needed to measure the joint services of credit cards and money. The authors generalize the model to permit risk aversion in the decision of the representative consumer. By their approach, credit cards can be included in measures of the money supply without the limiting need of assuming risk neutrality.

Clifford Wymer, Daniela Federici, and Enrico Saltari then present and estimate a nonlinear general disequilibrium model for the Italian economy for the sample period 1980–2010. They explicitly consider the key role of the information and communication technologies (ICT) in improving the Italian productivity performance. They address this issue in a multisector approach: two intermediate goods, a traditional and an ICT intermediate good, are combined to produce a final good. The traditional good is produced by capital and labor, whereas the ICT good is produced by innovative capital and skilled labor. Their results bring forth some light concerning the Italian productivity paradox, i.e., an increasing weight of the traditional sector and the difficulties encountered by the Italian economy in exiting from its actual worst recession since the 1930s. They found that in both the traditional and innovative sectors elasticity of substitution is well below one, indicating input complementarity. Moreover, the input complementarity is tighter in the ICT than in the traditional sector. The input complementarity in the traditional sector, which is predominant in the production of the final good, helps explain most of the labor share decline in the Italian economy because of the slowdown in the growth of capital intensity. Technological progress in the ICT sector showed a negative evolution over the sample period with an obvious adverse impact on the Italian total factor productivity.

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In the third contribution, Elton Beqiraj, Giovanni Di Bartolomeo, and Carolina Serpieri explore the consequences for stabilization policies of parsimonious deviations from the rational expectation paradigm. Clifford in several places highlighted that long-run properties of the economies are consistent with rational expectations, but such assumptions may be far too restrictive to provide a satisfactory representation of their short-run dynamics. The authors thus build a stylized simple sticky-price New Keynesian model where agents' beliefs are not homogeneous. Agents choose optimal plans forecasting macroeconomic variables over an infinite horizon, but not all are "rational," some of them use (biased) heuristics to forecast the future. In such a framework, they study optimal policies consistent with a second-order approximation of the policy objective from the consumers' utility function. They find that bounded rationality significantly raises the welfare cost of economic fluctuations. The presence of heterogeneity in fact introduces a new dimension in the policy maker's policy trade-off; the central banker should now account for the cross-sectional variability of consumption (i.e., inequality). They consider different policy regimes, for realistic calibrations; they show that the costs of ignoring inequality are relatively high. The cost of ignoring inequality is lower when the central bank targets inflation instead of price dispersion.

In the last contribution, Marco Grossy and Willi Semmler include in a continuous time inflation-targeting model a financial stabilization goal. The model includes regime-switching features and its state equations comprise a Phillips curve, the dynamics of the output gap, and (possibly destabilizing) loan aggregates. In their setup, the authors explore the stabilizing or destabilizing effects of price and nonprice (credit volume) drivers of the output gap, inflation, and credit flows. The model is solved by using a new global solution algorithm (NMPC); in contrast, the authors substantiate the theoretical part of the paper by approaching the subject empirically with local methods, relying to that end on a regime-switching structural VAR for the euro area. The empirical results are used as guidance for the calibration of the theoretical model variants that explore conventional monetary policy shocks and unconventional ones, loan supply, and loan demand shocks under different regime assumptions to reveal the state-dependent effects of both interest rate and volume-based policies.