Proceedings of the Edinburgh Mathematical Society (2015) **58**, 499–519
DOI:10.1017/S0013091514000212

SOME WEIGHTED GROUP ALGEBRAS ARE OPERATOR ALGEBRAS

HUN HEE LEE[1], EBRAHIM SAMEI[2] AND NICO SPRONK[3]

[1] *Department of Mathematical Sciences, Seoul National University, Gwanak-ro 1 Gwanak-gu, Seoul 151-747, Republic of Korea* (hunheelee@snu.ac.kr)
[2] *Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 5E6, Canada* (samei@math.usask.ca)
[3] *Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada* (nspronk@math.uwaterloo.ca)

(Received 26 June 2012)

Abstract Let G be a finitely generated group with polynomial growth, and let ω be a weight, i.e. a sub-multiplicative function on G with positive values. We study when the weighted group algebra $\ell^1(G,\omega)$ is isomorphic to an operator algebra. We show that $\ell^1(G,\omega)$ is isomorphic to an operator algebra if ω is a polynomial weight with large enough degree or an exponential weight of order $0 < \alpha < 1$. We demonstrate that the order of growth of G plays an important role in this problem. Moreover, the algebraic centre of $\ell^1(G,\omega)$ is isomorphic to a Q-algebra, and hence satisfies a multi-variable von Neumann inequality. We also present a more detailed study of our results when G consists of the d-dimensional integers $\mathbb{Z}^d$ or the three-dimensional discrete Heisenberg group $\mathbb{H}_3(\mathbb{Z})$. The case of the free group with two generators is considered as a counter-example of groups with exponential growth.

Keywords: weighted group algebras; operator algebras; groups with polynomial growth; Littlewood multipliers

2010 *Mathematics subject classification:* Primary 22D15; 47L30
Secondary 47L25

1. Introduction

The motivation for this paper originated from a result of Varopoulos that states that certain weighted group algebras on integers are isomorphic to Q-algebras [**20**]. We recall that a commutative Banach algebra is called a Q-algebra if it is a quotient of a uniform algebra. There are interesting (and non-trivial) classes of Banach algebras that are isomorphic to Q-algebras. For instance, it is shown in [**20**] and [**6**] that the spaces ℓ^p ($1 \leqslant p \leqslant \infty$) with pointwise product are isomorphic to Q-algebras. The case of the Schatten spaces S_p, $1 \leqslant p \leqslant \infty$, endowed with the Schur product, has been considered by many researchers (see [**11**] and [**13**]), and has very recently been covered for full generality (see [**4**]).

Let G be a discrete group, and let $\omega\colon G \to (0,\infty)$ be a weight on G, i.e.

$$\omega(xy) \leqslant \omega(x)\omega(y), \quad x, y \in G.$$

The *weighted group algebra* $\ell^1(G,\omega)$ is the convolution algebra of functions f on G such that $\|f\|_{\ell^1(G,\omega)} = \sum_{x\in G}|f(x)|\omega(x) < \infty$. Varopoulos showed that in the case where $G = \mathbb{Z}$ and $\omega_\alpha(n) = (1+|n|)^\alpha$ ($\alpha \geqslant 0$), $\ell^1(\mathbb{Z},\omega_\alpha)$ is isomorphic to a Q-algebra if and only if $\alpha > 1/2$.

We want to extend Varopoulos's result to other classes of weighted group algebras, possibly on non-abelian groups. However, group algebras are non-commutative in general, so we cannot hope for them to be isomorphic to Q-algebras. Instead, we want to investigate whether a weighted group algebra is isomorphic to an operator algebra. Recall that an *operator algebra* is a closed subalgebra of $B(H)$, the algebra of all bounded operators on a Hilbert space H. Note that any Q-algebra is an operator algebra (see [**6**, Theorem 1.1]). In the proof, Varopoulos actually proved that $\ell^1(\mathbb{Z},\omega_\alpha)$ satisfies one of the sufficient conditions to be isomorphic to a Q-algebra, namely, it is an *injective algebra*. Recall that a Banach algebra $\mathcal{A}$ is called an *injective algebra* if the algebra multiplication map m extends to a bounded map on the injective tensor product:

$$m\colon \mathcal{A} \otimes_\varepsilon \mathcal{A} \to \mathcal{A}.$$

In this paper we also focus on the case where $\ell^1(G,\omega)$ becomes an injective algebra. Using a Littlewood multiplier argument we show that $\ell^1(G,\omega)$ is an injective algebra, and consequently is isomorphic to an operator algebra, if G is a finitely generated group with polynomial growth and ω is a polynomial weight with a large enough degree or a certain exponential weight. Such weights are defined later in this paper.

This paper is organized as follows. In § 2.1 we recall several basic facts about injective algebras and Q-algebras. In § 2.2 we give an equivalence condition for $\ell^1(G,\omega)$ to be isomorphic to an operator algebra. In § 2.3 we recall the definitions of Littlewood multipliers and their consequences. In § 2.4 we give the necessary background on finitely generated groups with polynomial growth and detail how one can use the length function to define various weights such as polynomial and exponential weights on these groups. In §§ 3.1 and 3.2 we show our main results, namely, the case where $\ell^1(G,\omega)$ is isomorphic to an operator algebra. Moreover, we check that the algebraic centre of $\ell^1(G,\omega)$ is a Q-algebra in this case, and, hence, that it satisfies the (δ,L)-multi-variable von Neumann inequality (see § 3.1). We also find estimates for the upper bound of the norm of the multiplication map of the algebra for various weights and use them to determine concrete values of δ and L.

Finally, in § 5 we apply our techniques to study the cases when G consists of the d-dimensional integers $\mathbb{Z}^d$ or the three-dimensional discrete Heisenberg group $\mathbb{H}_3(\mathbb{Z})$. The case of the free group with two generators is examined to give a reasonable explanation of why we focus mainly on groups with polynomial growth.

2. Preliminaries

In this paper, all our groups are discrete.

2.1. p-summing algebras, injective algebras and Q-algebras

We first recall some definitions. Let X and Y be Banach spaces. For $1 \leqslant p < \infty$, a sequence $(x_n)_{n\geqslant 1} \subset X$ is called p-summable (respectively, weakly p-summable) if

$$\|(x_n)\|_p = \left(\sum_{n\geqslant 1} \|x_n\|^p\right)^{1/p} < \infty$$

$$\left(\text{respectively, } \|(x_n)\|_p^w = \sup_{\varphi\in B_{X^*}} \left(\sum_{n\geqslant 1} |\varphi(x_n)|^p\right)^{1/p} < \infty\right).$$

The Chevet–Saphar tensor norms on the algebraic tensor product $X \otimes Y$ are defined by

$$g_p(u) = \inf\left\{\|(x_j)\|_p\|(y_j)\|_{p'}^w \colon u = \sum_{j=1}^n x_j \otimes y_j,\ x_j \in X,\ y_j \in Y\right\},$$

where p' is the conjugate index of p. We denote the completion of $(X\otimes Y, g_p)$ by $X \otimes_{g_p} Y$.

We say that a linear map $T\colon X \to Y$ is p-summing if there exists a constant $C > 0$ such that

$$\|(Tx_n)\|_p \leqslant C\|(x_n)\|_p^w$$

for any sequence $(x_n)_{n\geqslant 1} \subset X$. We denote the infimum of such a C by $\pi_p(T)$, and $\Pi_p(X, Y)$ refers to the Banach space of all p-summing maps with the norm $\pi_p(\cdot)$. It is well known that we have the isometry

$$(X \otimes_{g_p} Y)^* \cong \Pi_{p'}(Y, X^*), \qquad A \otimes B \mapsto T,$$

where $A \in X^*$, $B \in Y^*$ and

$$Ty = \langle y, B\rangle A \quad (x \in X,\ y \in Y).$$

See [**17**, Chapter 6] for the details of p-summing maps and Chevet–Saphar tensor norms.

Standard Banach space theory (see [**17**, Proposition 3.22] and [**19**, Corollary 9.5]) tells us that we have the isometry

$$(\ell^1(G) \otimes_\varepsilon \ell^1(G))^* \cong \Pi_1(\ell^1(G), \ell^\infty(G)), \qquad A \otimes B \mapsto S. \tag{2.1}$$

One more standard fact we use later is that the composition of two 2-summing maps is a 1-summing map (actually, a nuclear map). More precisely, let $T\colon X \to Y$ and $S\colon Y \to Z$ be 2-summing maps between Banach spaces; then $S \circ T$ is 1-summing with

$$\pi_1(S \circ T) \leqslant \pi_2(S)\pi_2(T). \tag{2.2}$$

We say that a Banach algebra $\mathcal{A}$ is a *p-summing algebra* if the algebra multiplication map m extends to a bounded map

$$m\colon \mathcal{A} \otimes_{g_p} \mathcal{A} \to \mathcal{A}.$$

https://doi.org/10.1017/S0013091514000212 Published online by Cambridge University Press

Theorem 2.1 (Tonge [7, Theorem 18.19]). *Every* 2*-summing algebra is isomorphic to an operator algebra.*

Corollary 2.2. *Every injective algebra is isomorphic to an operator algebra.*

Proof. Recall that the injective tensor product is the minimum among Banach space tensor products, so the formal identity $\mathcal{A} \otimes_{g_2} \mathcal{A} \to \mathcal{A} \otimes_{\varepsilon} \mathcal{A}$ is a contraction for a Banach algebra $\mathcal{A}$. Thus, we can conclude that every injective algebra is a 2-summing algebra, which completes the proof. □

Definition 2.3. Let m be the algebra multiplication of a Banach algebra $\mathcal{A}$. In the case that $\mathcal{A}$ is an injective algebra, we define

$$\|m\|_{\varepsilon} := \|m\colon \mathcal{A} \otimes_{\varepsilon} \mathcal{A} \to \mathcal{A}\|.$$

We say that a Banach algebra $\mathcal{A}$ is a *Q-algebra* if it is a quotient of a uniform algebra, which is automatically a commutative algebra. Q-algebras are characterized by a von Neumann-type inequality [**2**, § 5.4.3 (2)].

Theorem 2.4. *Let $\mathcal{A}$ be a commutative Banach algebra. Then, $\mathcal{A}$ is isometrically isomorphic to a Q-algebra if and only if we have*

$$\|p(a_1, \dots, a_n)\| \leqslant \|p\|_{\infty}$$

for any $n \in N$, $\{a_1, \dots, a_n\} \subset A$ with norm less than or equal to 1 *and every polynomial p in n variables without constant terms, where*

$$\|p\|_{\infty} = \sup\{|p(z_1, \dots, z_n)|\, |z_i| \leqslant 1,\ i = 1, \dots, n\}.$$

Motivated by the above, we give the following definition.

Definition 2.5. Let A be a commutative Banach algebra. Then, A is said to satisfy the *multi-variable (δ, L)-von Neumann inequality* provided that, for every $n \in N$, every set of n elements $\{a_1, \dots, a_n\} \subset A$ with $\|a_i\| \leqslant \delta$ $(i = 1, \dots, n)$ and every polynomial p in n variables without constant terms, we have that

$$\|p(a_1, \dots, a_n)\| \leqslant L\|p\|_{\infty}.$$

All commutative injective algebras are Q-algebras (see [**20**]). Actually, a commutative Banach algebra is an injective algebra if and only if it is isomorphic to a quotient of a uniform algebra by a complemented ideal (see [**21**]). A more qualitative result can be found in [**2**] using a modern language of operator spaces.

Theorem 2.6 (Blecher and Le Merdy [2, Theorem 5.4.5, Corollary 5.4.11]). *Let $\mathcal{A}$ be a commutative injective algebra with the multiplication map m. Then, $\mathcal{A}$ satisfies the multi-variable (δ, L)-von Neuman inequality with*

$$\delta = \frac{1}{(1 + \|m\|_{\varepsilon})e} \quad \textit{and} \quad L = 1.$$

2.2. Weighted group algebras

Let G be a group, and let $\omega\colon G \to (0,\infty)$ be a weight on G, i.e.

$$\omega(xy) \leqslant \omega(x)\omega(y) \quad (x, y \in G).$$

The *weighted group algebra* $\ell^1(G,\omega)$ is the convolution algebra of functions f on G such that $\|f\|_{\ell^1(G,\omega)} = \sum_{x\in G} |f(x)|\omega(x) < \infty$. Using the natural duality $\ell^1(G)^* = \ell^\infty(G)$, we can show that $\ell^1(G,\omega)^* = \ell^\infty(G,\omega^{-1})$, where

$$\ell^\infty(G,\omega^{-1}) = \{\varphi \mid \varphi\omega^{-1} \in \ell^\infty(G)\}$$

with

$$\|\varphi\|_{\ell^\infty(G,\omega^{-1})} = \|\varphi\omega^{-1}\|_\infty.$$

In § 2.1 we showed that every injective Banach algebra is isomorphic to an operator algebra. As we show in Theorem 2.8, the converse of the preceding statement is also true in the case of weighted group algebras. However, this requires some operator space knowledge, including the Haagerup tensor product $\otimes_h$ of operator spaces. We refer the reader to [**2**] or [**15**] for details. We first recall the following form of the celebrated Grothendieck theorem.

Theorem 2.7. *If we equip $\ell^1(G)$ with its* max *operator space structure, then the formal identity* $\mathrm{id}\colon \ell^1(G) \otimes_\varepsilon \ell^1(G) \to \ell^1(G) \otimes_h \ell^1(G)$ *has norm less than or equal to K_G, where K_G is the Grothendieck constant.*

Proof. See [**2**, (1.47), (A.7)] and [**16**, (3.11)]. □

Theorem 2.8. *Let G be a group, and let ω be a weight on G. Then, $\ell^1(G,\omega)$ is an injective Banach algebra if and only if it is isomorphic to an operator algebra.*

Proof. The necessary part has been proven in Corollary 2.2. For the sufficient part, suppose that there exist an operator algebra $B \subseteq B(H)$ and a bounded algebra isomorphism $\psi\colon \ell^1(G,\omega) \to B$. This, in particular, implies that $\psi\colon \ell^1(G,\omega) \to B$ is completely bounded when $\ell^1(G,\omega)$ is given its max operator space structure. Now, since from [**2**, Theorem 2.3.2] the multiplication map $m\colon B \otimes_h B \to B$ is completely contractive, we have the bounded map

$$\psi^{-1} \circ m \circ (\psi \otimes \psi)\colon \ell^1(G,\omega)\otimes_h\ell^1(G,\omega) \to \ell^1(G,\omega).$$

But, it is easy to see that $\psi^{-1} \circ m \circ (\psi \otimes \psi)$ is exactly the multiplication map

$$m\colon \ell^1(G,\omega)\otimes_h\ell^1(G,\omega) \to \ell^1(G,\omega).$$

On the other hand, since the mapping

$$\ell^1(G,\omega) \to \ell^1(G), \qquad f \mapsto f\omega$$

is a complete isometric surjection (here, we have again given max operator space structure to both $\ell^1(G,\omega)$ and $\ell^1(G)$), it follows from the Grothendieck theorem (Theorem 2.7) that the formal identity $\mathrm{id}\colon \ell^1(G,\omega)\otimes_\varepsilon\ell^1(G,\omega) \to \ell^1(G,\omega)\otimes_h\ell^1(G,\omega)$ has norm less than or equal to K_G. This implies that the multiplication map $\ell^1(G,\omega)\otimes_\varepsilon\ell^1(G,\omega) \to \ell^1(G,\omega)$ is bounded, and so $\ell^1(G,\omega)$ is injective. □

2.3. Littlewood multiplier

Let G be a (discrete) group. We let the space of *Littlewood multipliers*, denoted by $T^2(G)$, be all the functions $f\colon G\times G\to\mathbb{C}$ for which there exist functions $f_1, f_2\colon G\times G\to\mathbb{C}$ such that

$$f(s,t)=f_1(s,t)+f_2(s,t)\quad (s,t\in G)$$

and

$$\sup_{t\in G}\sum_{s\in G}|f_1(s,t)|^2<\infty,\qquad \sup_{s\in G}\sum_{t\in G}|f_2(s,t)|^2<\infty.$$

We equip this space with the norm

$$\|f\|_{T_2(G)}=\inf\left\{\sup_{t\in G}\left(\sum_{s\in G}|f_1(s,t)|^2\right)^{1/2}+\sup_{s\in G}\left(\sum_{t\in G}|f_2(s,t)|^2\right)^{1/2}\right\},$$

where the infimum is taken over all possible decompositions. Note that the term 'Littlewood functions' has been used for $T^2(G)$ in the literature, but we use the term 'Littlewood multipliers' instead, since it explains the meaning of $T^2(G)$ better.

It easily follows that $T^2(G)$, with the action of pointwise multiplication, is a symmetric Banach $\ell^\infty(G\times G)$-module. Indeed, we have the following contraction:

$$\ell^\infty(G\times G)\otimes_\gamma T^2(G)\to T^2(G),\qquad f\otimes g\mapsto fg, \tag{2.3}$$

where $\otimes_\gamma$ is the projective tensor product of Banach spaces. Moreover, we have the following bounded embedding, which is well known to experts but we have presented its proof for the sake of completeness.

Proposition 2.9. *Let G be a discrete group, and let $I: T_2(G)\to(\ell^1(G)\otimes_\varepsilon\ell^1(G))^*$ be the formal identity. We then have*

$$\|I\|\leqslant K_G.$$

Proof. For simplicity, we write ℓ^1 instead of $\ell^1(G)$, ℓ^2 instead of $\ell^2(G)$, and ℓ^∞ instead of $\ell^\infty(G)$. We first note that, since ℓ^2 is reflexive, we have the isometric isomorphisms

$$B(\ell^1,\ell^2)\cong(\ell^1\otimes^\gamma\ell^2)^*\cong(\ell^1(\ell^2))^*\cong\ell^\infty(\ell^2), \tag{2.4}$$

where $\ell^1(\ell^2)$ and $\ell^\infty(\ell^2)$ are Banach spaces of ℓ^2-valued 1-summable functions and bounded functions, respectively. Now, let $f_1\colon G\times G\to\mathbb{C}$ be a function with

$$\alpha:=\sup_{t\in G}\left(\sum_{s\in G}|f_1(s,t)|^2\right)^{1/2}<\infty.$$

Then, by (2.4), the associated linear map $u\colon\ell^1\to\ell^2,\ g\mapsto u(g)$ given by

$$u(g)(t)=\sum_{s\in G}g(s)f_1(s,t)$$

has the norm $\|u\| = \alpha$, and $I(f_1)$ corresponds to $\mathrm{id}_{2,\infty} \circ u$, where $\mathrm{id}_{2,\infty} \colon \ell^2 \to \ell^\infty$ is the formal identity. We now recall that

$$(\ell^1(G) \otimes_\varepsilon \ell^1(G))^* \cong \Pi_1(\ell^1(G), \ell^\infty(G))$$

and

$$(\ell^1(G) \otimes_h \ell^1(G))^* \cong \Gamma_2(\ell^1(G), \ell^\infty(G)),$$

the space of 2-factorable operators, as Banach spaces (see [**15**, Proposition 5.16], [**7**, Chapter 7]). Then, by Grothendieck's theorem (Theorem 2.7), we have that

$$\|I(f_1)\| = \pi_1(\mathrm{id}_{2,\infty} \circ u) \leqslant K_G \gamma_2(\mathrm{id}_{2,\infty} \circ u) \leqslant K_G \alpha,$$

where $\gamma_2(\cdot)$ is the 2-factorable norm. Similarly, for $f_2 \colon G \times G \to \mathbb{C}$ with

$$\beta := \sup_{s \in G} \left(\sum_{t \in G} |f_1(s,t)|^2 \right)^{1/2} < \infty$$

we get $\|I(f_2)\| \leqslant K_G \cdot \beta$, which gives the desired result. □

2.4. Groups with polynomial growth

Let G be a finitely generated group with a fixed finite symmetric generating set F with the identity of the group G. We say that G has *polynomial growth* if there exists a polynomial f such that

$$|F^n| \leqslant f(n) \quad (n \in \mathbb{N}).$$

Here, $|S|$ is the cardinality of any $S \subseteq G$ and

$$F^n = \{u_1 \cdots u_n \colon u_i \in F,\ i = 1, \dots, n\}.$$

The least degree of any polynomial satisfying the above relation is called *the order of growth* of G and is denoted by $d(G)$. It can be shown that the order of growth of G does not depend on the symmetric generating set F, i.e. it is a universal constant for G.

It is immediate that finite groups are of polynomial growth. More generally, every G with the property that the conjugacy class of every element in G is finite has polynomial growth [**12**, Theorem 12.5.17]. Also, every nilpotent group (hence, abelian group) has polynomial growth [**12**, Theorem 12.5.17]. A deep result of Gromov [**9**] states that every finitely generated group with polynomial growth is virtually nilpotent, i.e. has a nilpotent subgroup of finite index. Moreover, there exist a polynomial f and a constant $0 < \lambda \leqslant 1$ such that

$$\lambda f(n) \leqslant |F^n| \leqslant f(n) \quad \text{for all } n \in \mathbb{N}, \tag{2.5}$$

where $\deg f = d(G)$. If we further assume that G is nilpotent, then, by the *Bass–Guivarch formula* (see [**1**, **10**]), we can actually compute the order of growth of G. More precisely, let G be a finitely generated nilpotent group with lower central series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_m = \{e\}.$$

https://doi.org/10.1017/S0013091514000212 Published online by Cambridge University Press

In particular, the quotient group G_k/G_{k+1} is a finitely generated abelian group. The order of growth of G is then

$$d(G) = \sum_{k=1}^{m-1} k\,\mathrm{rank}(G_k/G_{k+1}), \tag{2.6}$$

where rank denotes the rank of an abelian group, i.e. the largest number of independent and torsion-free elements of the abelian group.

Using the generating set F of G we can define a *length function* $\tau_F\colon G \to [0,\infty)$ by

$$\tau_F(x) = \inf\{n \in \mathbb{N}\colon x \in F^n\} \quad \text{for } x \neq e,\ \tau_F(e) = 0. \tag{2.7}$$

When there is no fear of ambiguity, we write τ instead of τ_F. It is straightforward to verify that τ is a subadditive function on G, i.e.

$$\tau(xy) \leqslant \tau(x) + \tau(y) \quad (x, y \in G). \tag{2.8}$$

Note that, since F is symmetric, for every $x \in G$, $\tau(x) = \tau(x^{-1})$. If we combine this fact with (2.8), then a straightforward calculation shows that

$$|\tau(x) - \tau(y)| \leqslant \tau(xy) \leqslant \tau(x) + \tau(y) \quad (x, y \in G). \tag{2.9}$$

We can use τ to define various weights on G. More precisely, for every $0 \leqslant \alpha \leqslant 1$, $\beta \geqslant 0$ and $C > 0$, we can define the *polynomial weight* ω_β on G of order β by

$$\omega_\beta(x) = (1 + \tau(x))^\beta \quad (x \in G), \tag{2.10}$$

and the *exponential weight* $\sigma_{\alpha,C}$ on G of order (α, C) by

$$\sigma_{\alpha,C}(x) = \mathrm{e}^{C\tau(x)^\alpha} \quad (x \in G). \tag{2.11}$$

3. Weighted group algebras isomorphic to operator algebras

In this section, we use G to denote a finitely generated infinite group with polynomial growth. We denote by F a fixed symmetric generating set of G with the identity, and f and λ refer to the polynomial and the constant satisfying (2.5), respectively.

3.1. The case of polynomial weights

For some weight $\omega\colon G \to (\delta, \infty)$ with $\delta > 0$, we want to check whether $\ell^1(G, \omega)$ is an injective algebra. In order to do so we recall the co-multiplication

$$\begin{aligned} \Gamma\colon \ell^\infty(G) &\to \ell^\infty(G \times G) \\ f &\mapsto \Gamma f, \end{aligned}$$

with $\Gamma f(s,t) = f(st)$, $s, t \in G$. Let $\Gamma_\omega\colon \ell^\infty(G, \omega^{-1}) \to \ell^\infty(G \times G, \omega^{-1} \times \omega^{-1})$ be the extension of Γ to $\ell^\infty(G, \omega^{-1})$. Consider the isometries

$$\begin{aligned} P\colon \ell^\infty(G) &\to \ell^\infty(G, \omega^{-1}) \\ f &\mapsto f\omega \end{aligned}$$

and

$$R\colon \ell^\infty(G\times G,\omega^{-1}\times\omega^{-1})\to \ell^\infty(G\times G)$$
$$F\mapsto F\cdot(\omega^{-1}\times\omega^{-1}).$$

We define the operator $\tilde{\Gamma}\colon \ell^\infty(G)\to\ell^\infty(G\times G)$ so that the following diagram commutes:

$$\begin{array}{ccc} \ell^\infty(G,\omega^{-1}) & \xrightarrow{\Gamma_\omega} & \ell^\infty(G\times G,\omega^{-1}\times\omega^{-1}) \\ {\scriptstyle P}\uparrow & & \downarrow{\scriptstyle R} \\ \ell^\infty(G) & \xrightarrow{\tilde{\Gamma}} & \ell^\infty(G\times G) \end{array}$$

Hence,

$$\tilde{\Gamma}(f)=\Omega\Gamma(f)\quad (f\in\ell^\infty(G)), \tag{3.1}$$

where

$$\Omega:=\frac{\Gamma(\omega)}{\omega\times\omega}. \tag{3.2}$$

Now, $\ell^1(G,\omega)$ is an injective algebra if and only if the multiplication map

$$m\colon \ell^1(G,\omega)\otimes_\varepsilon\ell^1(G,\omega)\to\ell^1(G,\omega)$$

is bounded or, equivalently, $\tilde{\Gamma}$ extends to a bounded map

$$\tilde{\Gamma}\colon \ell^\infty(G)\to(\ell^1(G)\otimes_\varepsilon\ell^1(G))^*.$$

Note that we have

$$\|m\|_\varepsilon=\|\tilde{\Gamma}\|.$$

An application of the Littlewood multiplier argument gives the following positive results on the weighted group algebra $\ell^1(G,\omega_\beta)$, where ω_β is the polynomial weight defined in (2.10).

Theorem 3.1. *$\ell^1(G,\omega_\beta)$ is an injective algebra if one of the following conditions holds.*

(i) $\lambda=1$ *and* $\beta>d(G)/2$.

(ii) $0<\lambda<1$ *and* $\beta>(d(G)+1)/2$.

Moreover, we have that

$$\|m\|_\varepsilon\leqslant K_G\min\{1,2^{\beta-1}\}\left[1+\sum_{n=1}^{\infty}\frac{f(n)-\lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}. \tag{3.3}$$

Proof. Let $\Omega_\beta := \Gamma(\omega_\beta)/(\omega_\beta \times \omega_\beta)$. We first show that $\Omega_\beta \in T^2(G)$. For every $x, y \in G$, we have that

$$\begin{aligned}\Omega_\beta(x,y) &= \frac{\omega_\beta(xy)}{\omega_\beta(x)\omega_\beta(y)} \\ &= \frac{(1+\tau(xy))^\beta}{(1+\tau(x))^\beta(1+\tau(y))^\beta} \\ &\leqslant \frac{(1+\tau(x)+\tau(y))^\beta}{(1+\tau(x))^\beta(1+\tau(y))^\beta} \\ &\leqslant \frac{A_\beta[(1+\tau(x))^\beta+(1+\tau(y))^\beta]}{(1+\tau(x))^\beta(1+\tau(y))^\beta} \qquad (*) \\ &= \frac{A_\beta}{(1+\tau(x))^\beta}+\frac{A_\beta}{(1+\tau(y))^\beta},\end{aligned}$$

where $A_\beta = \min\{1, 2^{\beta-1}\}$ and the inequality $(*)$ follows from the classical inequality

$$(a+b)^\beta \leqslant A_\beta(a^\beta + b^\beta) \quad (a, b \geqslant 0).$$

Hence, there exists the function $u \in \ell^\infty(G \times G)$ with $\|u\|_\infty \leqslant 1$ such that

$$\Omega_\beta(x,y) = u(x,y)\left[\frac{A_\beta}{(1+\tau(x))^\beta}+\frac{A_\beta}{(1+\tau(y))^\beta}\right] \quad (x, y \in G).$$

Thus, by the definition of $T^2(G)$ and (2.3),

$$\|\Omega_\beta\|_{T^2(G)} \leqslant A_\beta\left(\sum_{x\in G}\frac{1}{(1+\tau(x))^{2\beta}}\right)^{1/2}. \tag{3.4}$$

Hence, it suffices to find when $\sum_{x\in G} 1/(1+\tau(x))^{2\beta}$ is finite. To see this, from our hypothesis and (2.5), we have that

$$\begin{aligned}\sum_{x\in G}\frac{1}{(1+\tau(x))^{2\beta}} &= \sum_{n=0}^{\infty}\sum_{\tau(x)=n}\frac{1}{(1+n)^{2\beta}} \\ &= 1+\sum_{n=1}^{\infty}\sum_{x\in F^n\setminus F^{n-1}}\frac{1}{(1+n)^{2\beta}} \\ &\leqslant 1+\sum_{n=1}^{\infty}\frac{f(n)-\lambda f(n-1)}{(1+n)^{2\beta}},\end{aligned}$$

where the series in the last line converges if $\lambda = 1$ and $2\beta > d$ or if $0 < \lambda < 1$ and $2\beta > d+1$. Moreover, in either case, we have that

$$\sum_{x\in G}\frac{1}{(1+\tau(x))^{2\beta}} \leqslant 1+\sum_{n=1}^{\infty}\frac{f(n)-\lambda f(n-1)}{(1+n)^{2\beta}}.$$

Hence, by Proposition 2.9 and (2.3),

$$\begin{aligned}\|\tilde{\Gamma}(f)\|_{(\ell^1(G)\otimes_\varepsilon \ell^1(G))^*} &\leqslant K_G\|\tilde{\Gamma}(f)\|_{T^2(G)}\\ &\leqslant K_G\|\Omega_\beta\|_{T^2(G)}\|\Gamma(f)\|_\infty\\ &\leqslant K_G A_\beta\biggl[1+\sum_{n=1}^{\infty}\frac{f(n)-\lambda f(n-1)}{(1+n)^{2\beta}}\biggr]^{1/2}\|f\|_\infty\end{aligned}$$

for any $f \in \ell^\infty(G)$. □

3.2. The case of exponential weights

In this section we study when the weighted group algebra $\ell^1(G, \sigma_{\alpha,C})$ is an injective algebra, where $\sigma_{\alpha,C}$ is the exponential weight defined in (2.11). If we consider the same additional function $\Omega = \Gamma(\omega)/(\omega \times \omega)$, then it is not clear this time whether we can split the function into two parts with a suitable square summability. However, we can majorize the function with a similar one coming from a polynomial weight. We begin with a technical lemma.

Lemma 3.2. *Let $0 < \alpha < 1$, let $C > 0$ and take $\beta \geqslant \max\{1, 6/C\alpha(1-\alpha)\}$. Define the functions $p\colon [0,\infty) \to \mathbb{R}$ and $q\colon (0,\infty) \to \mathbb{R}$ by*

$$p(x) = Cx^\alpha - \beta\ln(1+x), \qquad q(x) = \frac{p(x)}{x}. \tag{3.5}$$

Then, p is increasing and q is decreasing on $[(\beta^2/C\alpha(1-\alpha))^{1/\alpha}, \infty)$.

Proof. We have

$$p'(x) = C\alpha x^{\alpha-1} - \frac{\beta}{1+x} = \frac{C\alpha x^{\alpha-1} + C\alpha x^\alpha - \beta}{1+x}.$$

Hence, $p'(x) \geqslant 0$ if $C\alpha x^\alpha - \beta \geqslant 0$. This implies that

$$p \text{ is increasing on } \biggl[\biggl(\frac{\beta}{C\alpha}\biggr)^{1/\alpha}, \infty\biggr). \tag{3.6}$$

Now consider $q(x) = Cx^{\alpha-1} - \beta\ln(1+x)/x$. Then,

$$q'(x) = \frac{C(\alpha-1)x^\alpha - \beta x/(1+x) + \beta\ln(1+x)}{x^2} = \frac{h(x) - \beta x/(1+x)}{x^2}, \tag{3.7}$$

where

$$h(x) := C(\alpha-1)x^\alpha + \beta\ln(1+x).$$

Hence, in order to find an interval for which $q'(x) \leqslant 0$, it suffices to find when $h(x) \leqslant 0$. We have

$$h'(x) = \frac{C\alpha(\alpha-1)x^\alpha + C\alpha(\alpha-1)x^{\alpha-1} + \beta}{1+x}.$$

https://doi.org/10.1017/S0013091514000212 Published online by Cambridge University Press

Thus, if we set

$$C_1 = C\alpha(1-\alpha),$$

then $h'(x) \leqslant 0$ whenever $-C_1 x^\alpha + \beta \leqslant 0$, or, equivalently, $x \geqslant (\beta/C_1)^{1/\alpha}$. Hence,

$$h \text{ is decreasing on } \left[\left(\frac{\beta}{C_1}\right)^{1/\alpha}, \infty\right). \tag{3.8}$$

Now, since, by hypothesis, $\beta \geqslant 1$ we have that

$$\left(\frac{\beta^2}{C_1}\right)^{1/\alpha} \geqslant \left(\frac{\beta}{C_1}\right)^{1/\alpha},$$

and so, by (3.8),

$$h(x) \leqslant h\left(\left(\frac{\beta^2}{C_1}\right)^{1/\alpha}\right) \quad \text{whenever } x \geqslant \left(\frac{\beta^2}{C_1}\right)^{1/\alpha}.$$

This implies that if $x \geqslant (\beta^2/C_1)^{1/\alpha}$, then

$$\begin{aligned} h(x) &\leqslant C(\alpha-1)\frac{\beta^2}{C_1} + \beta \ln\left(1 + \left(\frac{\beta^2}{C_1}\right)^{1/\alpha}\right) \\ &= \beta\left[\ln\left(1 + \left(\frac{\beta^2}{C_1}\right)^{1/\alpha}\right) - \frac{\beta}{\alpha}\right]. \end{aligned}$$

On the other hand, since $\beta \geqslant 6/C\alpha(1-\alpha) = 6/C_1$, we have that $\beta^2/C_1 \leqslant \beta^3/6$. Hence, considering the fact that $1/\alpha > 1$,

$$\begin{aligned} 1 + \left(\frac{\beta^2}{C_1}\right)^{1/\alpha} &\leqslant 1 + \left(\frac{\beta^3}{3!}\right)^{1/\alpha} \\ &\leqslant \left(1 + \frac{\beta^3}{3!}\right)^{1/\alpha} \\ &\leqslant \left(\sum_{n=0}^{\infty} \frac{\beta^n}{n!}\right)^{1/\alpha} \\ &= \mathrm{e}^{\beta/\alpha}. \end{aligned}$$

Therefore,

$$\ln\left(1 + \left(\frac{\beta^2}{C_1}\right)^{1/\alpha}\right) - \frac{\beta}{\alpha} \leqslant 0.$$

Hence, $h(x) \leqslant 0$ if $x \geqslant (\beta^2/C_1)^{1/\alpha}$. By (3.7),

$$q(x) \text{ is decreasing on } \left[\left(\frac{\beta^2}{C_1}\right)^{1/\alpha}, \infty\right). \tag{3.9}$$

The final result follows from (3.6) and the fact that $(\beta^2/C_1)^{1/\alpha} \geqslant (\beta/C\alpha)^{1/\alpha}$. □

Theorem 3.3. *Suppose that $0 < \alpha < 1$, $C > 0$ and $\beta \geqslant \max\{1, 6/C\alpha(1-\alpha)\}$. Let p and q be the functions defined in (3.5), and consider the function $\omega\colon G \to (0,\infty)$ defined by*

$$\omega(x) = \mathrm{e}^{p(\tau(x))} = \mathrm{e}^{\tau(x)q(\tau(x))} \quad (x \in G).$$

Then,

$$\omega(xy) \leqslant M\omega(x)\omega(y) \quad (x, y \in G),$$

where

$$M = \max\{\mathrm{e}^{p(t)-p(s)-p(r)}\colon t, s, r \in [0, 4K] \cap \mathbb{Z}\} \tag{3.10}$$

and

$$K = \left(\frac{\beta^2}{C\alpha(1-\alpha)}\right)^{1/\alpha}. \tag{3.11}$$

Proof. By Lemma 3.2, p is increasing and q is decreasing on $[K, \infty)$. We prove the statement of the theorem considering various cases.

Case 1 ($\max\{\tau(x), \tau(y)\} \leqslant 2K$). In this case, $\tau(xy) \leqslant \tau(x) + \tau(y) \leqslant 4K$. Hence,

$$\frac{\omega(xy)}{\omega(x)\omega(y)} = \mathrm{e}^{p(\tau(xy))-p(\tau(x))-p(\tau(y))} \leqslant M.$$

Case 2 ($\max\{\tau(x), \tau(y)\} > 2K$ and $\min\{\tau(x), \tau(y)\} \leqslant K$). Without loss of generality, we can assume that $\tau(x) > 2K$ and $\tau(y) \leqslant K$. Then, by (2.9),

$$\tau(x) + \tau(y) \geqslant \tau(xy) \geqslant \tau(x) - \tau(y) \geqslant 2K - K = K.$$

Thus, by Lemma 3.2,

$$\begin{aligned}
\omega(xy) &= \mathrm{e}^{p(\tau(xy))} \\
&\leqslant \mathrm{e}^{p(\tau(x)+\tau(y))} \\
&= \mathrm{e}^{(\tau(x)+\tau(y))q(\tau(x)+\tau(y))} \\
&= \mathrm{e}^{\tau(x)q(\tau(x)+\tau(y))}\mathrm{e}^{\tau(y)q(\tau(x)+\tau(y))} \\
&\leqslant \mathrm{e}^{\tau(x)q(\tau(x))}\mathrm{e}^{Kq(K)} \\
&= \omega(x)\omega(y)\mathrm{e}^{p(K)-p(\tau(y))} \\
&\leqslant M\omega(x)\omega(y).
\end{aligned}$$

Case 3 ($\min\{\tau(x), \tau(y)\} > K$ and $\tau(xy) \leqslant K$). In this case, we have that

$$\begin{aligned}
\omega(x)\omega(y) &= \mathrm{e}^{p(\tau(x))+p(\tau(y))} \\
&\geqslant \mathrm{e}^{2p(K)} \\
&= \mathrm{e}^{2p(K)-p(\tau(xy))}\omega(xy) \\
&\geqslant \frac{1}{M}\omega(xy).
\end{aligned}$$

Hence,

$$\omega(xy) \leqslant M\omega(x)\omega(y).$$

Case 4 ($\min\{\tau(x), \tau(y), \tau(xy)\} > K$). In this case, by Lemma 3.2, we have that

$$\begin{aligned}\omega(xy) &= \mathrm{e}^{p(\tau(xy))} \\ &\leqslant \mathrm{e}^{p(\tau(x)+\tau(y))} \\ &= \mathrm{e}^{(\tau(x)+\tau(y))q(\tau(x)+\tau(y))} \\ &= \mathrm{e}^{\tau(x)q(\tau(x)+\tau(y))}\mathrm{e}^{\tau(y)q(\tau(x)+\tau(y))} \\ &\leqslant \mathrm{e}^{\tau(x)q(\tau(x))}\mathrm{e}^{\tau(y)q(\tau(y))} \\ &\leqslant \omega(x)\omega(y).\end{aligned}$$

Thus, by comparing the above four cases and considering the fact that $M \geqslant \mathrm{e}^{-p(0)} = 1$, it follows that, for every $x, y \in G$,

$$\omega(xy) \leqslant M\omega(x)\omega(y).$$

□

We are now ready to show when the weighted group algebras of exponential weights are injective algebras.

Theorem 3.4. *Suppose that $0 < \alpha < 1$ and $C > 0$. Then, $\ell^1(G, \sigma_{\alpha,C})$ is a 2-summing algebra. Moreover, we have*

$$\|m\|_\varepsilon \leqslant K_G M 2^{\beta-1}\left[1 + \sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}, \tag{3.12}$$

where

$$\beta = \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d + (1 - \delta_1(\lambda))}{2}\right\}$$

(δ_1 is the Dirac function at 1) and M is the constant (depending on α, β and C) defined in (3.10).

Proof. We define a function $\omega\colon G \to (0, \infty)$ by

$$\omega(x) = \frac{\sigma_{\alpha,C}(x)}{\omega_\beta(x)} = \mathrm{e}^{C\tau(x)^\alpha - \beta\ln(1+\tau(x))} \quad (x \in G),$$

where ω_β is the polynomial weight defined in (2.10). Then, by Theorem 3.3,

$$\omega(xy) \leqslant M\omega(x)\omega(y) \quad (x, y \in G),$$

where M is the constant defined in (3.10). Therefore, if we let

$$\Sigma_{\alpha,C} := \frac{\Gamma(\sigma_{\alpha,C})}{\sigma_{\alpha,C} \times \sigma_{\alpha,C}} \quad \text{and} \quad \Omega_\beta := \frac{\Gamma(\omega_\beta)}{\omega_\beta \times \omega_\beta},$$

then

$$\Sigma_{\alpha,C} \leqslant M\Omega_\beta \leqslant M\left[\frac{2^{\beta-1}}{(1+\tau(x))^\beta} + \frac{2^{\beta-1}}{(1+\tau(y))^\beta}\right].$$

A similar argument to the one presented in the proof of Theorem 3.1 shows that

$$\|m\|_\varepsilon \leqslant K_G \|\Sigma_{\alpha,C}\|_{T^2(G)} \leqslant K_G M 2^{\beta-1}\left[1+\sum_{n=1}^{\infty}\frac{f(n)-\lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}.$$

In particular, $\ell^1(G,\sigma_{\alpha,C})$ is an injective algebra. □

We can actually show exactly when the weighted group algebras of exponential weight are isomorphic to an operator algebra.

Theorem 3.5. *Suppose that $0 \leqslant \alpha \leqslant 1$ and $C > 0$. Then, $\ell^1(G,\sigma_{\alpha,C})$ is isomorphic to an operator algebra if and only if $0 < \alpha < 1$.*

Proof. The case when $0 < \alpha < 1$ has already been proved.

If $\alpha = 0$, then $\ell^1(G,\sigma_{\alpha,C}) \cong \ell^1(G)$, which is known to be non-Arens regular (see [**5**, Theorem 8.11]), so it is not an operator algebra. Now suppose that $\alpha = 1$. For every $m, n \geqslant 2$, take $a_{m,n} \in F^{m+n} \setminus F^{m+n-1}$ (this is possible because G is infinite). Hence, there exist $x_n \in F^n$ and $y_m \in F^m$ such that

$$a_{m,n} = x_n y_m.$$

Moreover, since $a_{m,n} \in F^{m+n} \setminus F^{m+n-1}$, we have

$$x_n \in F^n \setminus F^{n-1} \quad \text{and} \quad y_m \in F^m \setminus F^{m-1}.$$

Therefore,

$$\tau(a_{m,n}) = m+n, \qquad \tau(x_n) = n, \qquad \tau(y_m) = m.$$

Hence,

$$\frac{\sigma_{1,C}(x_n y_m)}{\sigma_{1,C}(x_n)\sigma_{1,C}(y_m)} = \frac{\mathrm{e}^{C\tau(x_n y_m)}}{\mathrm{e}^{C(\tau(x_n)+\tau(y_m))}} = \frac{\mathrm{e}^{Cm+Cn}}{\mathrm{e}^{C(n+m)}} = 1.$$

Thus,

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{\sigma_{1,C}(x_n y_m)}{\sigma_{1,C}(x_n)\sigma_{1,C}(y_m)} = 1,$$

which implies from [**5**, Theorem 8.11] that $\ell^1(G,\sigma_{1,C})$ is not Arens regular, and so is not an operator algebra. □

Remark 3.6. We point out that the upper-bounded estimate obtained in (3.12) goes to ∞ as α approaches either 0 or 1 (this happens because $\beta \to \infty$). This coincides with the result obtained in the statement of Theorem 3.5, since, as $\alpha \to 0$ ($\alpha \to 1$, respectively), the weight $\sigma_{\alpha,C} \to \sigma_{0,C} = \mathrm{e}^C$ ($\sigma_{\alpha,C} \to \sigma_{1,C}$, respectively), and we showed therein that neither $\ell^1(G,\mathrm{e}^C)$ nor $\ell^1(G,\sigma_{1,C})$ is isomorphic to an operator algebra, and so $\|m\|_\varepsilon$ is not bounded.

4. Remarks on Q-algebras and operator space versions

The weighted group algebras in §§ 3.1 and 3.2 are injective algebras, but not isomorphic to Q-algebras, since they are non-commutative in general. However, their algebraic centres are actually isomorphic to Q-algebras. Indeed, the injectivity of the tensor product tells us that the algebraic centre is also an injective algebra with the smaller norm of the multiplication map. Then, the result in [**20**] implies that they are isomorphic to Q-algebras. Moreover, Theorem 2.6 allows us to determine (δ, L) for the corresponding multi-variable von Neumann inequality. Thus, we have the following. We note that, for an algebra A, we denote ZA to be its algebraic centre.

Corollary 4.1. *$Z\ell^1(G, \omega_\beta)$ is isomorphic to a Q-algebra if one of the following conditions holds.*

(i) $\lambda = 1$ *and* $\beta > d(G)/2$.

(ii) $0 < \lambda < 1$ *and* $\beta > (d(G) + 1)/2$.

In this case, $Z\ell^1(G, \omega_\beta)$ satisfies the multi-variable (δ, L)-von Neumann inequality with

$$\delta = \mathrm{e}^{-1}\left(1 + K_G \min\{1, 2^{\beta-1}\}\left[1 + \sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}\right)^{-1} \quad \text{and} \quad L = 1.$$

We have a corresponding result for exponential weights.

Corollary 4.2. *Suppose that $0 < \alpha < 1$ and $C > 0$. Then, $Z\ell^1(G, \sigma_{\alpha,C})$ is isomorphic to a Q-algebra. In this case, $Z\ell^1(G, \sigma_{\alpha,C})$ satisfies the multi-variable (δ, L)-von Neumann inequality with*

$$\delta = \mathrm{e}^{-1}\left(1 + K_G M 2^{\beta-1}\left[1 + \sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}}\right]^{1/2}\right)^{-1} \quad \text{and} \quad L = 1,$$

where

$$\beta = \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d + (1 - \delta_1(\lambda))}{2}\right\}$$

and M is the constant defined in (3.10).

We end this section with a remark on operator space versions. Most of the results in this paper have available operator space versions upon following the approach in [**8**]. For example, the estimates on $\|\Omega_\beta\|_{T^2(G)}$ in Theorem 3.1 tell us that $\ell^1(G, \omega_\beta)$ with the maximal operator space structure is completely isomorphic to an operator algebra. But, in the case of operator spaces we need to show that the algebra multiplication map m extends to a completely bounded map on the Haagerup tensor product, so Littlewood multiplier theory has to be developed up to the level of operator spaces as in [**8**].

5. Examples

5.1. The d-dimensional integers $\mathbb{Z}^d$

A common choice of generating set is

$$F = \{(x_1, \dots, x_d) \mid x_i \in \{-1, 0, 1\}\}.$$

It is straightforward to see that

$$\tau((x_1, \dots, x_d)) = \max\{|x_1|, \dots, |x_d|\}$$

and, for every $n \in \mathbb{N}$,

$$F^n = \{(x_1, \dots, x_d) \mid x_i \in \{-n, \dots, 0, \dots, n\}\}.$$

Thus, we get that

$$|F^n| = (2n+1)^d \quad (n = 0, 1, 2, \dots)$$

and the order of growth of $\mathbb{Z}^d$ is d with $f(n) = (2n+1)^d$ and $\lambda = 1$. It follows from Theorem 3.1 that $\ell^1(\mathbb{Z}^d, \omega_\beta)$ is isomorphic to an operator algebra if $\beta > d/2$. Moreover, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{f(n) - \lambda f(n-1)}{(1+n)^{2\beta}} &= \sum_{n=1}^{\infty} \frac{(2n+1)^d - (2n-1)^d}{(1+n)^{2\beta}} \\ &\leqslant \sum_{n=1}^{\infty} \frac{2d(2n+2)^{d-1}}{(1+n)^{2\beta}} \\ &= d2^d \sum_{n=1}^{\infty} (1+n)^{d-1-2\beta} \\ &\leqslant d2^d \int_1^{\infty} x^{d-1-2\beta} \, \mathrm{d}x \\ &= \frac{d2^d}{2\beta - d}.\end{aligned}$$

Since $\mathbb{Z}^d$ is an abelian group, Theorem 4.1 tells us that $\ell^1(\mathbb{Z}^d, \omega_\beta)$ is actually a Q-algebra and it satisfies the multi-variable von Neumann inequality for $L = 1$ and

$$\delta = \mathrm{e}^{-1} \left\{ 1 + K_G \min\{1, 2^{\beta-1}\} \left[1 + \frac{d2^d}{2\beta - d} \right]^{1/2} \right\}^{-1}.$$

On the other hand, $\ell^1(\mathbb{Z}^d, \omega_\beta)$ fails to be an injective algebra if $\beta \leqslant d/2$ (see [**8**]).

Now, let $\sigma_{\alpha,C}$ be the exponential weight on $\mathbb{Z}^d$ defined in (2.11). Theorem 4.2 tells us that $\ell^1(\mathbb{Z}^d, \sigma_{\alpha,C})$ is a Q-algebra and it satisfies the multi-variable von Neumann inequality for $L = 1$ and

$$\delta = \mathrm{e}^{-1} \left(1 + K_G M 2^{\beta-1} \left[1 + \frac{d2^d}{2\beta - d} \right]^{1/2} \right)^{-1},$$

where

$$\beta = \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d}{2}\right\}$$

and M is the constant defined in (3.10).

A case of particular interest occurs when we let

$$d = 1 \quad \text{and} \quad C = \frac{6}{\alpha(1-\alpha)}.$$

In this case, we can choose $\beta = 1$. Also, if K is the constant defined in (3.11), then it is easy to see that $0 < K < 1/6$. Hence, $M = 1$, and so we get

$$\delta = \frac{1}{\mathrm{e}(1+\sqrt{3}K_G)}.$$

5.2. The three-dimensional discrete Heisenberg group $\mathbb{H}_3(\mathbb{Z})$

We recall that the three-dimensional discrete Heisenberg group $\mathbb{H}_3(\mathbb{Z})$ is a semidirect product of $\mathbb{Z}^2$ with $\mathbb{Z}$, and the product is defined as

$$(a_1, b_1, c_1)\cdot(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, a_1 b_2 + c_1 + c_2), \quad (a_i, b_i, c_i) \in \mathbb{H}_3(\mathbb{Z}).$$

If we identify $\mathbb{Z}$ with the subgroup $\{(0, 0, c) : c \in \mathbb{Z}\}$, then it is easy to see that $\mathbb{H}_3(\mathbb{Z})/\mathbb{Z} \cong \mathbb{Z}^2$. Hence, $\mathbb{H}_3(\mathbb{Z})$ is a 2-step nilpotent group, and by the Bass–Guivarch formula (2.6) we have that

$$d(\mathbb{H}_3(\mathbb{Z})) = 4.$$

Hence, if we let ω_β be the polynomial weight on $\mathbb{H}_3(\mathbb{Z})$, then $\ell^1(\mathbb{H}_3(\mathbb{Z}), \omega_\beta)$ is isomorphic to an operator algebra provided that

$$\beta > \frac{4+1}{2} = \frac{5}{2}.$$

Moreover, $Z\ell^1(\mathbb{H}_3(\mathbb{Z}), \omega_\beta)$ satisfies the multi-variable von Neumann inequality. On the other hand, the restriction of ω_β to $\mathbb{Z}$ is a weight equivalent to the weight $\omega'_\beta(c) = (1 + |c|)^\beta$. Hence, $\ell^1(\mathbb{H}_3(\mathbb{Z}), \omega_\beta)$ has a closed subalgebra that is isomorphic to $\ell^1(\mathbb{Z}, \omega'_\beta)$. Thus, it follows from the result of Varopoulos [**20**] that $\ell^1(\mathbb{H}_3(\mathbb{Z}), \omega_\beta)$ fails to be an injective algebra if $\beta \leqslant 1/2$.

5.3. The free group with two generators $\mathbb{F}_2$

In this subsection we show that $\ell^1(\mathbb{F}_2, \omega_\beta)$ is not an injective algebra for any $\beta > 0$. Since $\mathbb{F}_2$ is one of the typical examples of exponentially growing groups, this gives evidence to suggest that the condition of polynomial growth on the group is necessary for a weighted group to be realizable as an operator algebra.

Recall also the Rudin–Shapiro polynomials defined in the following recursive way (see [**3**, Chapter 4]):

$$P_0(z) := 1, \qquad Q_0(z) := 1$$

and, for $k \geqslant 0$,

$$P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z), \qquad Q_{k+1}(z) := Q_k(z) - z^{2^k} P_k(z).$$

By an induction on k, it is straightforward to check that the coefficients of P_k are ± 1, that $\deg P_k = \deg Q = 2^k - 1$ and

$$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} \quad (z \in \mathbb{T}).$$

Hence,

$$\|P_k\|_{L^\infty(\mathbb{T})} \leqslant \sqrt{2^{k+1}}.$$

Using the contraction (actually, it is a metric surjection due to Nehari's theorem; see, for example, [**14**, § 6])

$$\begin{aligned} Q \colon L^\infty(\mathbb{T}) &\to B(\ell^2) \\ f &\mapsto (\hat{f}(-(i+j)))_{i,j \in \mathbb{Z}}, \end{aligned}$$

we get a sequence of Hankelian matrices

$$A_{2^k} = Q(\bar{P}_k), \quad k \geqslant 0,$$

where A_{2^k} is a $2^k \times 2^k$-matrix with entries ± 1 satisfying

$$\|A_{2^k}\|_{\mathrm{op}} \leqslant \sqrt{2^{k+1}},$$

where $\|\cdot\|_{\mathrm{op}}$ means the operator norm.

Theorem 5.1. *For any $\beta > 0$, $\ell^1(\mathbb{F}_2, \omega_\beta)$ is not an injective algebra.*

Proof. Let g_1 and g_2 be two generators of $\mathbb{F}_2$, and let d be an even positive integer with $d > 2\beta$. Consider the following subsets of $\mathbb{F}_2$:

$$I_n^d = \{g_1^{x_1} g_2^{x_2} g_1^{x_3} \cdots g_2^{x_d} \colon 1 \leqslant x_i \leqslant n \text{ for } i = 1, \dots, d\}.$$

We now recall the function Ω_β defined by

$$\Omega_\beta(g, g') = \frac{\omega_\beta(gg')}{\omega_\beta(g)\omega_\beta(g')} \quad (g, g' \in \mathbb{F}).$$

Let $\Omega_\beta^n = \Omega_\beta 1_{I_n^d \times I_n^d}$. When $g, g' \in I_n^d$ are given by

$$g = g_1^{x_1} g_2^{x_2} g_1^{x_3} \cdots g_2^{x_d} \quad \text{and} \quad g' = g_1^{y_1} g_2^{y_2} g_1^{y_3} \cdots g_2^{y_d}$$

for $x_i, y_j \geqslant 1$, we have

$$\Omega_\beta^n(g, g') = \left(\frac{1 + x_1 + \cdots + x_n + y_1 + \cdots + y_n}{(1 + x_1 + \cdots + x_n)(1 + y_1 + \cdots + y_n)} \right)^\beta.$$

https://doi.org/10.1017/S0013091514000212 Published online by Cambridge University Press

By a similar estimate as that in [**8**, Theorem 6.1] we obtain

$$\|\Omega_\beta^n\|_{\mathrm{op}} \geqslant 2^{-\beta} n^{d/2} \bigg(\sum_{1\leqslant x_1,\dots,x_d\leqslant n} \frac{1}{(1+x_1+\cdots+x_d)^{2\beta}} \bigg)^{1/2}.$$

Now, using the Rudin–Shapiro polynomial, we have a sequence of matrices $A_n \in M_n$, $n = 2^k$ $(k = 1, 2, \dots)$, satisfying the following conditions:

(1) $A_n = (a_{i+j}^n)_{i,j=1}^n$ with $a_i^n \in \{\pm 1\}$,

(2) $\|A_n\|_{\mathrm{op}} \leqslant \sqrt{2n}$.

We consider $b = (b_h)_{h\in\mathbb{F}_2}$ given by

$$\begin{aligned} b_{gg'} &= a_{x_1+y_1}^n \cdots a_{x_d+y_d}^n \quad && \text{for } g = g_1^{x_1} g_2^{x_2} g_1^{x_3} \cdots g_2^{x_d},\ g' = g_1^{y_1} g_2^{y_2} g_1^{y_3} \cdots g_2^{y_d},\ x_i, y_j \geqslant 1, \\ b_h &= 0 && \text{elsewhere.} \end{aligned}$$

In other words, the matrix $[b_{gg'}]_{g,g'\in I_n^d}$ is nothing but the d-tensor power of the matrix $[a_{x+y}^n]_{1\leqslant x,\,y\leqslant n}$. Thus, it follows from [**18**, Theorem 3.1 and Corollary 3.2], (3.1) and $\|b\|_{\mathrm{op}} \leqslant (2n)^{d/2}$ that

$$\begin{aligned} \|\tilde{\Gamma}\| &\geqslant \|\tilde{\Gamma}(b)\|_{(\ell^1(G)\otimes_\varepsilon \ell^1(G))^*} \\ &= \|\Gamma(b)\Omega\|_{(\ell^1(G)\otimes_\varepsilon \ell^1(G))^*} \\ &\geqslant K_G^{-1} \|\Gamma(b)\Omega\|_{(\ell^1(G)\otimes_h \ell^1(G))^*} \\ &= K_G^{-1} \|[b_{gg'}\Omega_\beta^n(g,g')]_{g,g'\in I_n^d}\|_{\mathrm{Schur}} \\ &\geqslant K_G^{-1} \|[b_{gg'}]_{g,g'\in I_n^d}\|_{\mathrm{op}}^{-1} \|\Omega_\beta^n\|_{\mathrm{op}} \qquad (*) \\ &\geqslant K_G^{-1} (2n)^{-d/2} 2^{-\beta} n^{d/2} \bigg(\sum_{1\leqslant x_1,\dots,x_d\leqslant n} \frac{1}{(1+x_1+\cdots+x_d)^{2\beta}} \bigg)^{1/2} \\ &= K_G^{-1} 2^{-d/2} 2^{-\beta} \bigg(\sum_{1\leqslant x_1,\dots,x_d\leqslant n} \frac{1}{(1+x_1+\cdots+x_d)^{2\beta}} \bigg)^{1/2} \\ &\to \infty \quad \text{as } n = 2^k \to \infty \text{ since } 2\beta < d. \end{aligned}$$

Hence, $\ell^1(\mathbb{F}_2, \omega_\beta)$ is not an injective algebra for any $\beta > 0$. Note that in $(*)$ we use the fact that the Schur product of $[b_{gg'}]$ with itself is the matrix with all entries 1, which is the identity in the Schur product. □

Acknowledgements. H.H.L. was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant 2012R1A1A2005963). E.S. was supported by the NSERC (Grant 366066-09). N.S. was supported by the NSERC (Grant 312515-05).

References

1. H. Bass, The degree of polynomial growth of finitely generated nilpotent groups, *Proc. Lond. Math. Soc.* **25**(3) (1972), 603–614.
2. D. Blecher and C. Le Merdy, *Operator algebras and their modules: an operator space approach*, London Mathematical Society Monographs, Volume 30 (Clarendon, Oxford, 2004).
3. P. Borwein, *Computational excursions in analysis and number theory*, CMS Books in Mathematics, Volume 10 (Springer, 2002).
4. J. Briäta, H. Buhrmana, T. Leeb and T. Vidickc, All Schatten spaces endowed with the Schur product are Q-algebras, *J. Funct. Analysis* **262**(1) (2012), 1–9.
5. H. G. Dales and A. T.-M. Lau, *The second duals of Beurling algebras*, Memoirs of the American Mathematical Society, Volume 177 (American Mathematical Society, Providence, RI, 2005).
6. A. M. Davie, Quotient algebras of uniform algebras, *J. Lond. Math. Soc.* **7**(2) (1973), 31–40.
7. J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, Volume 43 (Cambridge University Press, 1995).
8. M. Ghandehari, H. H. Lee, E. Samei and N. Spronk, Some Beurling–Fourier algebras are operator algebras, preprint 2012 (available at http://arxiv.org/abs/1208.4835).
9. M. Gromov, Groups of polynomial growth and expanding maps, *Publ. Math. IHES* **53** (1981), 53–73.
10. Y. Guivarch, Groupes de Lie à croissance polynomiale, *C. R. Acad. Sci. Paris Sér. I* **272** (1971), 1695–1696.
11. C. Le Merdy, The Schatten space S_4 is a Q-algebra, *Proc. Am. Math. Soc.* **126**(3) (1998), 715–719.
12. T. W. Palmer, *Banach algebras and general theorem of *-algebras II* (Cambrige University Press, 2001).
13. D. Pérez-García, The trace class is a Q-algebra, *Annales Acad. Sci. Fenn. Math.* **31**(2) (2006), 287–295.
14. G. Pisier, *Similarity problems and completely bounded maps: includes the solution to 'the Halmos problem'*, Lecture Notes in Mathematics, Volume 1618, 2nd edn (Springer, 2001).
15. G. Pisier, *Introduction to operator space theory* (Cambridge University Press, 2003).
16. G. Pisier, Grothendieck's theorem, past and present, *Bull. Am. Math. Soc.* **49**(2) (2012), 237–323.
17. R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics (Springer, 2002).
18. N. Spronk, Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras, *Proc. Lond. Math. Soc.* **89**(1) (2004), 161–192.
19. N. Tomczak-Jaegermann, *Banach–Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Volume 38 (Longman, New York, 1989).
20. N. T. Varopoulos, Some remarks on Q-algebras, *Annales Inst. Fourier* **22**(4) (1972), 1–11.
21. N. T. Varopoulos, Sur les quotients des algèbres uniformes, *C. R. Acad. Sci. Paris Sér. I* **274** (1972), 1344–1346.