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INVARIANT VALUATIONS ON QUATERNIONIC VECTOR SPACES

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Abstract The spaces of Sp(n)-, Sp(n) · U(1)- and Sp(n) · Sp(1)-invariant, translation-invariant, continuous convex valuations on the quaternionic vector space $\mathbb{H}^n$ are studied. Combinatorial dimension formulae involving Young diagrams and Schur polynomials are proved.

Keywords: valuations; algebraic integral geometry; Hadwiger's theorem; compact symplectic group

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1. Introduction and statement of main results

One of the most important formulae in integral geometry is the principal kinematic formula of Chern–Blaschke–Santaló. Let V be an m-dimensional Euclidean vector space, let $\overline{\mathrm{SO}(V)}$ be the group generated by translations and rotations, endowed with an appropriate Haar measure, and let ω_k be the volume of a k-dimensional unit ball.

Let K, L be compact convex sets. Then the principal kinematic formula reads

$$\int_{\overline{\mathrm{SO}(V)}} \chi(K \cap \bar{g}L)\,\mathrm{d}\bar{g} = \sum_{k=0}^{m} \binom{m}{k}^{-1} \frac{\omega_k \omega_{m-k}}{\omega_m} \mu_k(K)\mu_{m-k}(L), \tag{1.1}$$

where $\mu_0, \ldots, \mu_m$ are the *intrinsic volumes* (see [**22, 25**]) and $\chi = \mu_0$ denotes the *Euler characteristic*, which is defined by $\chi(K) = 1$ for all non-empty compact convex sets K and $\chi(\emptyset) = 0$.

A nice conceptual proof due to Hadwiger goes as follows (see [**22**] for details). First one notes that the intrinsic volumes are *valuations*, i.e. finitely additive maps on the space of compact convex bodies. If $\mathrm{Val}^{\mathrm{SO}(V)}$ denotes the space of continuous, translation and rotation invariant valuations, then the left-hand side of formula (1.1) with K (or L) fixed belongs to $\mathrm{Val}^{\mathrm{SO}(V)}$. Using *Hadwiger's theorem*, which states that $\mu_0, \ldots, \mu_m$ is a basis of $\mathrm{Val}^{\mathrm{SO}(V)}$, one obtains a formula of type (1.1), but with unknown coefficients on the right-hand side. These coefficients may be easily computed by the *template method*, i.e. by plugging in spheres on both sides of the formula and comparing coefficients.

Instead of taking the full rotation group, one may restrict to some subgroup G of $\mathrm{SO}(V)$. Again we let $\bar{G}$ be the group generated by G and translations. Since the main ingredient in the above proof is Hadwiger's theorem, an analogous *G-kinematic formula* exists for all groups G such that the space Val^G of $\bar{G}$-invariant continuous convex valuations is finite dimensional. To write these formulae explicitly may be a challenge, since the template method is in general too weak to determine the constants.

Alesker showed in [**9**] that Val^G is finite dimensional if and only if G acts transitively on the unit sphere. Connected compact groups with this property were classified by Borel and Montgomery–Samelson. There are six infinite series

$$\mathrm{SO}(n), \quad \mathrm{U}(n), \quad \mathrm{SU}(n), \quad \mathrm{Sp}(n), \quad \mathrm{Sp}(n)\cdot \mathrm{U}(1), \quad \mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \tag{1.2}$$

and three exceptional groups

$$\mathrm{G}_2, \quad \mathrm{Spin}(7), \quad \mathrm{Spin}(9). \tag{1.3}$$

A valuation $\mu \in \mathrm{Val}^G$ is called homogeneous of degree k if $\mu(tK) = t^k\mu(K)$ for all compact convex sets and all $t \geqslant 0$. The corresponding subspace is denoted by Val_k^G.

The classical, and most important, case is $G = \mathrm{SO}(n)$. By Hadwiger's theorem, we have $\dim \mathrm{Val}_k^{\mathrm{SO}(n)} = 1$ for $k = 0, 1, \dots, n$.

The case $G = \mathrm{U}(n)$ has been extensively studied in the last few years [**1**–**4**, **16**, **17**, **27**]. In order to compare with our results in the quaternionic cases, we mention just some results.

Alesker showed in [**3**] that

$$\dim \mathrm{Val}_k^{\mathrm{U}(n)} = \min\{\lfloor \tfrac{1}{2}k\rfloor, \lfloor \tfrac{1}{2}(2n-k)\rfloor\} + 1.$$

Let us fix a sequence of holomorphic isometric embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \hookrightarrow \cdots$, they induce a sequence of restriction maps

$$\mathrm{Val}^{\mathrm{U}(1)} \leftarrow \mathrm{Val}^{\mathrm{U}(2)} \leftarrow \mathrm{Val}^{\mathrm{U}(3)} \leftarrow \cdots.$$

Let

$$\mathrm{Val}^{\mathrm{U}(\infty)} := \varprojlim \mathrm{Val}^{\mathrm{U}(n)}$$

be the inverse limit, which is a graded vector space (in fact a graded algebra). Elements of $\mathrm{Val}^{\mathrm{U}(\infty)}$ are called *global valuations*.

Alesker's results imply in particular that the restriction map

$$\mathrm{Val}_k^{\mathrm{U}(\infty)} \to \mathrm{Val}_k^{\mathrm{U}(n)} \tag{1.4}$$

is surjective for all n and k and injective for $n \geqslant k$. Its kernel was described by Fu [**17**].

With respect to the product structure introduced by Alesker [**5**], there is an isomorphism of graded algebras

$$\mathrm{Val}^{\mathrm{U}(\infty)} \cong \mathbb{C}[\![t, s]\!],$$

where t, s are variables of degree 1 and 2 respectively. In particular, the Poincaré series of $\mathrm{Val}^{\mathrm{U}(\infty)}$ is given by

$$\sum_{k=0}^{\infty} \dim \mathrm{Val}_k^{\mathrm{U}(\infty)} x^k = \frac{1}{(1-x)(1-x^2)}. \tag{1.5}$$

Using the $\mathrm{U}(n)$-case, the integral geometry of the groups $\mathrm{SU}(n)$ was studied in [**11**], and two of the three exceptional cases were treated in [**13**].

In the quaternionic cases (i.e. in the cases $G = \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$) relatively little is known. The case $n = 1$ was studied in [**6**] and [**12**]. Since $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, this is also a special case of the general theory for $\mathrm{SU}(n)$, which was developed in [**11**]. Using plurisubharmonic functions in quaternionic variables, Alesker constructed in [**7**] some $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$-invariant valuations on $\mathbb{H}^n$. For instance, the *Alesker–Kazarnovskii-pseudo-volume* (which was called *quaternionic pseudo-volume* in [**7**]) is an element of $\mathrm{Val}_n^{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)}$.

Let us now describe our results in the quaternionic cases. Let $V \cong \mathbb{H}^n$ be an n-dimensional quaternionic vector space. We consider V as a *right* vector space. The *compact symplectic group* $\mathrm{Sp}(n)$ then acts from the left on V by usual matrix multiplication. Moreover, the group $\mathrm{Sp}(1)$ of quaternions of norm 1 and its subgroup of complex numbers of norm 1 act by diagonal multiplication from the right on V. The subgroup of $\mathrm{SO}(4n)$ generated by $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ (respectively $\mathrm{U}(1)$) is denoted by $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ (respectively $\mathrm{Sp}(n) \cdot \mathrm{U}(1)$). More details on quaternionic vector spaces will be given in § 3.

Our first main theorem is similar to (1.4), but the proof will be different. Let $\mathbb{H}^1 \hookrightarrow \mathbb{H}^2 \hookrightarrow \mathbb{H}^3 \hookrightarrow \cdots$ be a sequence of quaternionic isometric embeddings. The corresponding restrictions yield a sequence

$$\mathrm{Val}^{\mathrm{Sp}(1)} \leftarrow \mathrm{Val}^{\mathrm{Sp}(2)} \leftarrow \mathrm{Val}^{\mathrm{Sp}(3)} \leftarrow \cdots$$

whose inverse limit is denoted by

$$\mathrm{Val}^{\mathrm{Sp}(\infty)} := \varprojlim \mathrm{Val}^{\mathrm{Sp}(n)}.$$

The grading on each $\mathrm{Val}^{\mathrm{Sp}(n)}$ induces a grading on $\mathrm{Val}^{\mathrm{Sp}(\infty)}$.

The spaces $\mathrm{Val}^{\mathrm{Sp}(\infty)\cdot\mathrm{U}(1)}$ and $\mathrm{Val}^{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)}$ are defined in an analogous way. Elements in these spaces are called *global valuations*, as opposed to *local valuations* in the spaces $\mathrm{Val}^{\mathrm{Sp}(n)}$, $\mathrm{Val}^{\mathrm{Sp}(n)\cdot\mathrm{U}(1)}$, $\mathrm{Val}^{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)}$.

Our first main theorem is the quaternionic analogue of (1.4).

Theorem 1.1. *The restriction maps*

$$\begin{aligned} \mathrm{Val}_k^{\mathrm{Sp}(\infty)} &\to \mathrm{Val}_k^{\mathrm{Sp}(n)}, \\ \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot\mathrm{U}(1)} &\to \mathrm{Val}_k^{\mathrm{Sp}(n)\cdot\mathrm{U}(1)}, \\ \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)} &\to \mathrm{Val}_k^{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)} \end{aligned}$$

are surjective for all k and n and injective for $n \geqslant k$.

The second main theorem describes the dimensions of the spaces of global valuations. Together with the previous theorem, it also yields the dimensions of the spaces $\mathrm{Val}_k^{\mathrm{Sp}(n)}$, $\mathrm{Val}_k^{\mathrm{Sp}(n)\cdot\mathrm{U}(1)}$, $\mathrm{Val}_k^{\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)}$ when $k \leqslant n$ (and when $k \geqslant 3n$, see Theorem 2.3).

Theorem 1.2. *As formal power series,*

$$\sum_{k=0}^{\infty} \dim \operatorname{Val}_k^{\mathrm{Sp}(\infty)} x^k = \frac{x^4-3x^3+6x^2-3x+1}{(1-x)^7(1+x)^3},$$
$$\sum_{k=0}^{\infty} \dim \operatorname{Val}_k^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)} x^k = \frac{x^6-2x^5+2x^4+2x^2-2x+1}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^6},$$
$$\sum_{k=0}^{\infty} \dim \operatorname{Val}_k^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)} x^k = \frac{x^5+2x^4+x^3+1}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^4}.$$

Explicit formulae and some numerical values for small k can be found at the end of § 5.

In the $\mathrm{U}(n)$-case, (1.4), (1.5) and the Alesker–Fourier transform (see Theorem 2.3) are sufficient to compute the dimension of $\operatorname{Val}_k^{\mathrm{U}(n)}$ for all k. In the $\mathrm{Sp}(n)$-case, however, we only get the dimensions in the ranges $0 \leqslant k \leqslant n$ and $3n \leqslant k \leqslant 4n$. The third main theorem closes this gap, but the resulting formula is of a combinatorial type and far from being closed.

In order to state this theorem, we make the trivial but extremely useful observation that the right action of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ induces the structure of an $\mathrm{SU}(2)$-module on $\operatorname{Val}^{\mathrm{Sp}(n)}$. Since irreducible representations of $\mathrm{SU}(2)$ are indexed by integers, we have a decomposition

$$\operatorname{Val}_k^{\mathrm{Sp}(n)} \cong \bigoplus_{l=0}^{\infty} m_k^l V_l, \tag{1.6}$$

where V_l is the unique irreducible $\mathrm{SU}(2)$-representation of dimension $l+1$ and where the coefficients m_k^l are natural numbers. Knowing these coefficients is equivalent to knowing the character of $\operatorname{Val}_k^{\mathrm{Sp}(n)}$, which is an element in the ring of Laurent polynomials $\mathbb{Z}[t,t^{-1}]$. Also, we can read off the dimensions of the spaces $\operatorname{Val}_k^{\mathrm{Sp}(n)}$, $\operatorname{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{U}(1)}$ and $\operatorname{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)}$ from the decomposition (1.6):

$$\dim \operatorname{Val}_k^{\mathrm{Sp}(n)} = \sum_l (l+1) m_k^l, \tag{1.7}$$
$$\dim \operatorname{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{U}(1)} = \sum_{l \equiv 0(2)} m_k^l, \tag{1.8}$$
$$\dim \operatorname{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)} = m_k^0. \tag{1.9}$$

In order to describe our third main theorem, we recall some terminology and conventions and refer to [**18**] and § 3 for details. A *Young diagram* is an arrangement of a finite number of boxes into rows, with less boxes in lower rows, all aligned to the left. It can be uniquely described by the tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$ where λ_i is the number of boxes in the ith row. Then k is called the *depth* of λ. The *weight* $|\lambda|$ is the number of boxes in λ. We will call a Young diagram *even* if the number of boxes in each row is even.

To each Young diagram λ there is a corresponding *Schur polynomial* s_λ whose definition will be recalled in § 3.

For $n \geqslant 1$ and $m \geqslant 0$, let us define the polynomials (over the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in t)

$$E_n(x) := \sum_\lambda s_\lambda(tx, t^{-1}x), \tag{1.10}$$

$$F_{n,m}(x) := \sum_\lambda s_\lambda(tx, t^{-1}x, t, t^{-1}), \tag{1.11}$$

where the first sum is over all even Young diagrams λ with $\lambda_1 \leqslant 2n$ and the second sum is over all even Young diagrams λ with $\lambda_1 \leqslant 2n$ and $|\lambda| = 2m$.

Theorem 1.3. *The characters of the* $\mathrm{SU}(2)$*-representations* $\mathrm{Val}_k^{\mathrm{Sp}(n)}$ *satisfy*

$$\sum_{k=0}^{4n} \mathrm{char}(\mathrm{Val}_k^{\mathrm{Sp}(n)}) x^k = E_n(x) - F_{n-1,2n}(x) - (1 + x(t^4 + t^{-4}) + x^2) F_{n-1,2n-1}(x) \\ + x(1 + x(t^4 + 1 + t^{-4}) + x^2) F_{n-1,2n-2}(x).$$

From the characters, one can easily obtain the dimensions of the spaces of $\mathrm{Sp}(n)$, $\mathrm{Sp}(n) \cdot \mathrm{U}(1)$, $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$-invariant valuations (see (1.7), (1.8), (1.9)). The values for small dimensions n can be computed with the help of Theorem 1.3 and a computer algebra system, a list is given at the end of § 6.

Plan of the paper

In the next section, we will collect some known facts from the theory of convex valuations. The only new statement is Proposition 2.6, which will be central in the proof of Theorems 1.2 and 1.3.

In § 3, we give some background on quaternionic vector spaces and quaternionic groups and on representation theory for the groups $\mathrm{SU}(2)$ and $\mathrm{GL}(n, \mathbb{C})$.

In § 4 we collect some facts on invariants under the group $\mathrm{Sp}(n)$ which will be used later on.

The proof of Theorem 1.1 is contained in § 5. It uses several tools from the theory of convex valuations: normal cycle, Klain's embedding theorem and the Alesker–Fourier transform. Theorem 1.2 is proved in the same section.

In § 6, we use some computations for $\mathrm{SU}(2)$-representations to prove Theorem 1.3. Finally, in the appendix, we will give the rather technical proof of a lemma which is used in § 5.

2. Background from valuation theory

We refer to [**14**] for a recent survey on the subject of *algebraic integral geometry*, to which the present paper gives a contribution.

In this section, V will be a finite-dimensional oriented Euclidean vector space of dimension m. The set of compact convex subsets in V is denoted by $\mathcal{K}(V)$, it is endowed with the Hausdorff metric.

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Definition 2.1. A (convex) valuation is a map $\mu\colon \mathcal{K}(V) \to \mathbb{C}$ which is finitely additive in the following sense:

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$.

The space of continuous, translation invariant valuations is denoted by Val. With the topology of uniform convergence on compact subsets, it is a Fréchet space, whose dimension is infinite (if $m \geqslant 2$).

The main examples of continuous translation invariant valuations are the volume, the Euler characteristic, the intrinsic volumes and more generally mixed volumes.

The space Val comes with a natural grading found by McMullen [**23**]. An element $\mu \in \mathrm{Val}$ is said to be *homogeneous* of degree k if $\mu(tK) = t^k\mu(K)$ for all $t \geqslant 0$. μ is called *even* if $\mu(-K) = \mu(K)$ and *odd* if $\mu(-K) = -\mu(K)$ for all K. The space of k-homogeneous even/odd valuations is denoted by $\mathrm{Val}_k^{\pm}$.

The McMullen grading is given by

$$\mathrm{Val} = \bigoplus_{\substack{k=0,\dots,m \\ \epsilon=\pm}} \mathrm{Val}_k^{\epsilon}.$$

For a subgroup G of the rotation group $\mathrm{SO}(V)$, Val^G denotes the subspace of Val of G-invariant valuations. If G acts transitively on the unit sphere, then Val^G is finite dimensional [**9**]. The list of connected closed subgroups of $\mathrm{SO}(m)$ with this property was given in the introduction (see (1.2) and (1.3)). If G is such a group, then Val^G consists only of even valuations [**13**] and we may apply *Klain's embedding theorem* [**21**].

More precisely, let $\mu \in \mathrm{Val}_k^+$. For a k-dimensional subspace $E \subset V$, the restriction of μ to $\mathcal{K}(E)$ is a multiple of the k-dimensional volume in E, i.e.

$$\mu(K) = \mathrm{Kl}_\mu(E)\,\mathrm{vol}_k(K), \quad K \in \mathcal{K}(E),$$

for some number $\mathrm{Kl}_\mu(E)$. The function $\mathrm{Kl}_\mu\colon \mathrm{Gr}_k(V) \to \mathbb{C}$ (where Gr_k is the Grassmannian of k-dimensional subspaces in V) is called the *Klain function* of μ.

Theorem 2.2. *The map*

$$\begin{aligned} \mathrm{Kl}\colon \mathrm{Val}_k^+ &\to C(\mathrm{Gr}_k), \\ \mu &\mapsto \mathrm{Kl}_\mu \end{aligned}$$

is injective.

Alesker defined the important dense subspace of smooth translation invariant valuations $\mathrm{Val}^{\mathrm{sm}} \subset \mathrm{Val}$. Its definition will be given below.

Theorem 2.3. *Let $\mu \in \mathrm{Val}_k^{+,\mathrm{sm}}$. Then there exists a unique valuation $\mathbb{F}\mu \in \mathrm{Val}_{m-k}^{+,\mathrm{sm}}$ with*

$$\mathrm{Kl}_{\mathbb{F}\mu}(E) = \mathrm{Kl}_\mu(E^\perp), \quad \forall E \in \mathrm{Gr}_{m-k}.$$

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The map $\mathbb{F}\colon \operatorname{Val}^{+,\mathrm{sm}} \to \operatorname{Val}^{+,\mathrm{sm}}$ is called *Alesker–Fourier transform*. It can be extended to odd valuations [**10**] but we will not need this here. Other notations for the Alesker–Fourier transform of μ are $\mathbb{D}\mu$ and $\hat{\mu}$.

We next describe an important construction of translation invariant valuations, called *normal cycle map* [**28**]. In order to simplify notation, we use the following convention.

All differential forms are assumed to be complex-valued.

By $SV = V \times S(V)$ we denote the unit sphere bundle of V. The product structure on SV induces a bi-grading on the space $\Omega^*(SV)$ of smooth differential forms, and we denote by $\Omega^{k,l}(SV)$ the space of forms of bi-degree (k,l). The subspace of translation invariant forms is denoted by a superscript 'tr', i.e. $\Omega^{k,l}(SV)^{\mathrm{tr}} = \Lambda^k V^* \otimes \Omega^l(S(V))$. If a group G acts on V, then the subspace of $\Omega^{k,l}(SV)$ of translation and G-invariant elements is denoted by a superscript $\bar{G}$.

On SV, there is a canonical 1-form α defined by

$$\alpha|_{(x,v)}(w) = \langle v, d\pi(w)\rangle,$$

where $\pi\colon SV \to V$ is the canonical projection. The kernel of α defines a contact distribution $Q := \ker\alpha$, i.e. SV is a $(2m-1)$-dimensional *contact manifold*.

The *Reeb vector field* T is defined by $T_{(x,v)} = (v,0)$, note that $\alpha(T) = 1$. At each point (x,v), $Q_{(x,v)}$ is the orthogonal sum of two copies of $T_v S(V)$ and we have an orthogonal splitting

$$T_{(x,v)}SV = \mathbb{R}\cdot T_{(x,v)} \oplus T_v S(V) \oplus T_v S(V).$$

To any $K \in \mathcal{K}(V)$, we can associate its *normal cycle* $\operatorname{nc}(K)$. The normal cycle is an $(m-1)$-dimensional Federer–Fleming current in SV. Its support is the set of pairs (x,v), where $x \in \partial K$ and v is an outer normal vector to K at x. The current $\operatorname{nc}(K)$ is a cycle (i.e. $\partial \operatorname{nc}(K) = 0$) and Legendrian (i.e. $\operatorname{nc}(K)\llcorner\alpha = 0$).

Let $0 \leqslant k < m$ and let $\omega \in \Omega^{k,m-1-k}(SV)^{\mathrm{tr}}$. We define a continuous translation invariant valuation of degree k by setting

$$K \mapsto \int_{\operatorname{nc}(K)} \omega.$$

Linear combinations of valuations of this type and of the volume are called *smooth* (see [**8**] for equivalent definitions). The dense subspace of Val of all smooth valuations is denoted by $\operatorname{Val}^{\mathrm{sm}}$.

An important fact (see [**9**]) is that if G is a subgroup of $\mathrm{SO}(V)$ acting transitively on the unit sphere, then

$$\operatorname{Val}^G \subset \operatorname{Val}^{\mathrm{sm}}. \tag{2.1}$$

The normal cycle map may be seen as a surjective map

$$\begin{aligned} \operatorname{nc}\colon \Omega^{k,m-1-k}(SV)^{\mathrm{tr}} &\twoheadrightarrow \operatorname{Val}_k^{\mathrm{sm}}, \\ \omega &\mapsto \left(K \mapsto \int_{\operatorname{nc}(K)} \omega\right). \end{aligned} \tag{2.2}$$

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This map is clearly not injective, since vertical forms (i.e. multiples of the contact form) and exact forms are in the kernel. The kernel was described in [**15**] in terms of the *Rumin operator* D from [**24**]. This second-order differential operator can be defined on any contact manifold, in particular on the sphere bundle SV.

Given $\omega \in \Omega^{k,m-1-k}(SV)^{\mathrm{tr}}$, there exists a unique vertical form ξ such that $\mathrm{d}(\omega+\xi)$ is vertical and $\mathrm{D}\omega := \mathrm{d}(\omega+\xi)$.

It follows immediately that D vanishes on closed forms ω (since $\mathrm{d}\omega = 0$ is vertical), on vertical forms (take $\xi := -\omega$) and on multiples of $\mathrm{d}\alpha$:

$$\mathrm{D}(\mathrm{d}\alpha \wedge \tau) = \mathrm{d}(\mathrm{d}\alpha \wedge \tau + \alpha \wedge \mathrm{d}\tau) = 0.$$

The next theorem is a special case of the main theorem in [**15**].

Theorem 2.4. $\omega \in \ker \mathrm{nc}$ *if and only if* $\mathrm{D}\omega = 0$ *and* $\pi_*\omega = 0$.

Note that the second condition is always satisfied if $k > 0$.

Using arguments as in § 2 in Rumin's paper [**24**], we show that $\mathrm{Val}_k^{\mathrm{sm}}$ fits into some exact sequence.

For this, we define the following spaces:

$$\begin{aligned}
\mathcal{I}^{k,l}(SV)^{\mathrm{tr}} &:= \{\omega \in \Omega^{k,l}(SV)^{\mathrm{tr}} \colon \omega = \alpha \wedge \xi + \mathrm{d}\alpha \wedge \psi, \\
&\qquad\qquad \xi \in \Omega^{k-1,l}(SV)^{\mathrm{tr}},\ \psi \in \Omega^{k-1,l-1}(SV)^{\mathrm{tr}}\}, \\
\Omega_v^{k,l}(SV)^{\mathrm{tr}} &:= \{\omega \in \Omega^{k,l}(SV)^{\mathrm{tr}} \colon \alpha \wedge \omega = 0\}, \\
\Omega_h^{k,l}(SV)^{\mathrm{tr}} &:= \Omega^{k,l}(SV)^{\mathrm{tr}} / \Omega_v^{k,l}(SV)^{\mathrm{tr}}, \\
\Omega_p^{k,l}(SV)^{\mathrm{tr}} &:= \Omega^{k,l}(SV)^{\mathrm{tr}} / \mathcal{I}^{k,l}(SV)^{\mathrm{tr}}.
\end{aligned}$$

Multiplication by the symplectic form $-\mathrm{d}\alpha$ induces an operator

$$L \colon \Omega_h^{k,l}(SV)^{\mathrm{tr}} \to \Omega_h^{k+1,l+1}(SV)^{\mathrm{tr}},$$

which is an injection for $k + l \leqslant m - 2$, moreover

$$\Omega_p^{k,l}(SV)^{\mathrm{tr}} \simeq \Omega_h^{k,l}(SV)^{\mathrm{tr}} / L\Omega_h^{k-1,l-1}(SV)^{\mathrm{tr}}.$$

The exterior derivative induces an operator

$$\mathrm{d}_Q \colon \Omega_p^{k,l}(SV)^{\mathrm{tr}} \to \Omega_p^{k,l+1}(SV)^{\mathrm{tr}}.$$

The Rumin operator vanishes on $\mathcal{I}^{k,m-k-1}(SV)^{\mathrm{tr}}$, hence it induces an operator (which we denote by the same letter) $\mathrm{D} \colon \Omega_p^{k,m-k-1}(SV)^{\mathrm{tr}} \to \Omega_v^{k,m-k}(SV)^{\mathrm{tr}}$.

Lemma 2.5. *Let* $\omega \in \Omega^{k,l}(SV)^{\mathrm{tr}}$ *with* $\mathrm{d}\omega = 0$.

(1) *In the case* $0 < l < m - 1$, *there exists* $\phi \in \Omega^{k,l-1}(SV)^{\mathrm{tr}}$ *with* $\mathrm{d}\phi = \omega$.

(2) *In the case* $l = 0$, $\omega \in \Lambda^k V^* \otimes \mathbb{C}$.

(3) *In the case* $k = 0$, $l = m - 1$, *there exists* $\phi \in \Omega^{0,m-2}(SV)^{\mathrm{tr}}$ *with* $\mathrm{d}\phi = \omega$ *provided that* $\pi_*\omega = 0$.

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Proof. We write

$$\omega = \sum_{i=1}^{q} c_i \kappa_i \wedge \tau_i,$$

where $\kappa_1, \dots, \kappa_q$ is a basis of $\Lambda^k V^*$ and $\tau_1, \dots, \tau_q$ are l-forms on the unit sphere $S(V)$. Then

$$\mathrm{d}\omega = (-1)^k \sum_{i=1}^{q} c_i \kappa_i \wedge \mathrm{d}\tau_i,$$

which shows that all τ_i are closed.

If $0 < l < m-1$, then $H^l_{\mathrm{dR}}(S^{m-1}) = 0$ and we find $\rho_i \in \Omega^{l-1}(S(V))$ with $\mathrm{d}\rho_i = \tau_i$. Then $\phi := (-1)^k \sum_{i=1}^{q} c_i \kappa_i \wedge \rho_i$ satisfies $\mathrm{d}\phi = \omega$. If $l = 0$, then all τ_i are constant and hence $\omega \in \Lambda^k V^* \otimes \mathbb{C}$.

The last statement follows from the fact that $\Omega^{0,m-1}(SV)^{\mathrm{tr}} = \Omega^{m-1}(S(V))$ and that $H^{m-1}_{\mathrm{dR}}(S(V))$ is one dimensional. □

Proposition 2.6. *Let* $0 \leqslant k \leqslant m$. *There is an exact sequence*

$$0 \to \Lambda^k V^* \otimes \mathbb{C} \hookrightarrow \Omega_p^{k,0}(SV)^{\mathrm{tr}} \xrightarrow{\mathrm{d}_Q} \Omega_p^{k,1}(SV)^{\mathrm{tr}} \xrightarrow{\mathrm{d}_Q} \cdots \xrightarrow{\mathrm{d}_Q} \Omega_p^{k,m-k-1}(SV)^{\mathrm{tr}} \xrightarrow{\mathrm{nc}} \mathrm{Val}_k^{\mathrm{sm}} \to 0.$$

The proof follows the arguments in [**24**], with the Poincaré lemma replaced by Lemma 2.5 above.

Proof. Since Val_m is spanned by the Lebesgue measure, there is an exact sequence

$$0 \to \Lambda^m V^* \otimes \mathbb{C} \to \mathrm{Val}_m \to 0,$$

which is the case $k = m$ in the statement.

Let us suppose that $k < m$. It is clear that the sequence is closed. Let us check that it is exact.

- Since $\Omega_p^{k,0}(SV)^{\mathrm{tr}} = \Omega^{k,0}(SV)^{\mathrm{tr}}$, the injectivity on the left-hand side is trivial.
- For $0 \leqslant l < m-k-1$, let $\omega \in \Omega^{k,l}(SV)^{\mathrm{tr}}$ with $\mathrm{d}\omega = \alpha \wedge \xi + \mathrm{d}\alpha \wedge \psi \in \mathcal{I}_{k,l+1}(SV)^{\mathrm{tr}}$. Letting $\omega' := \omega - \alpha \wedge \psi \in \Omega^{k,l}(SV)^{\mathrm{tr}}$ and $\xi' := \xi + \mathrm{d}\psi$, we obtain $\mathrm{d}\omega' = \alpha \wedge \xi'$. Differentiating yields $0 = \mathrm{d}\alpha \wedge \xi' - \alpha \wedge \mathrm{d}\xi'$. In other words, $L(\xi'|_Q) = 0$. By the injectivity of L in degree $k+l < m-1$, we get $\xi'|_Q = 0$ which implies that $\mathrm{d}\omega' = 0$.

  If $l > 0$, Lemma 2.5 implies that there exists $\phi \in \Omega_{k,l-1}(SV)^{\mathrm{tr}}$ with $\mathrm{d}\phi = \omega'$. Hence $[\omega] = [\omega'] = [\mathrm{d}\phi] = \mathrm{d}_Q[\phi]$, which shows that $[\omega]$ is d_Q-exact.
- In the same situation, if $l = 0$, then ω' is a translation invariant k-form on V, hence $[\omega] = [\omega']$ is in the image of $\Lambda^k V^* \otimes \mathbb{C}$.
- The map nc on the right-hand side is surjective by (2.2).

- If $[\omega] \in \Omega_p^{k,m-k-1}(SV)^{\mathrm{tr}}$ lies in the kernel of nc, then $\mathrm{D}[\omega] = \mathrm{d}(\omega + \xi) = 0$ for some form $\xi \in \Omega_v^{k,m-k-1}(SV)^{\mathrm{tr}}$. Then $\omega' := \omega + \xi$ is a closed translation invariant form of bi-degree $(k, m-k-1)$. If $k > 0$, then, by Lemma 2.5, there exists $\phi \in \Omega^{k,m-k-2}(SV)^{\mathrm{tr}}$ with $\mathrm{d}\phi = \omega'$. Hence $[\omega] = [\omega'] = [\mathrm{d}\phi] = \mathrm{d}_Q[\phi]$ is d_Q-exact.
- In the same situation, if $k = 0$, then $\pi_*\omega' = \pi_*\omega = 0$ by Theorem 2.4 (note that $\pi_*\xi = 0$, since ξ is vertical). We may thus apply Lemma 2.5 to find $\phi \in \Omega^{0,m-2}(SV)^{\mathrm{tr}}$ with $\mathrm{d}\phi = \omega'$. Hence $[\omega]$ is d_Q-exact.

□

Corollary 2.7. *Let G be a closed subgroup of $\mathrm{SO}(V)$ acting transitively on the unit sphere of V. Let $0 \leqslant k \leqslant m$. There is an exact sequence*

$$0 \to (\Lambda^k V^* \otimes \mathbb{C})^G \hookrightarrow \Omega_p^{k,0}(SV)^{\bar G} \xrightarrow{\mathrm{d}_Q} \Omega_p^{k,1}(SV)^{\bar G} \xrightarrow{\mathrm{d}_Q} \cdots \xrightarrow{\mathrm{d}_Q} \Omega_p^{k,m-k-1}(SV)^{\bar G} \xrightarrow{\mathrm{nc}} \mathrm{Val}_k^G \to 0.$$

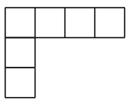
Proof. In Lemma 2.5, if ω is G-invariant, then ϕ may be chosen G-invariant too (just average ϕ with respect to the Haar measure on G). The rest of the proof is analogous. □

3. Background from combinatorics and representation theory

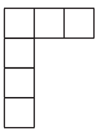
3.1. Young diagrams and Schur functions

Our main reference for Young diagrams is [**18**].

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a given partition, i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$. The *Young diagram* associated to λ has λ_i boxes in the ith row, all aligned to the left. For example, if $\lambda = (4, 1, 1)$, the Young diagram is given by

$\lambda =$ 

The *depth* of a Young diagram is the number of its rows, the *weight* is the number of its boxes. The *conjugate* $\tilde\lambda$ of a Young diagram λ is obtained by reflecting it at the diagonal. In our example

$\tilde\lambda =$ 

The Young diagram λ *dominates* another Young diagram μ if

$$\sum_{i=1}^m \lambda_i \geqslant \sum_{i=1}^m \mu_i$$

for all m. In this case, we write $\lambda \trianglerighteq \mu$.

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A *semi-standard tableau* on λ is given by putting one of the numbers $1, 2, \dots, m$ (where m may be different from the weight of λ) into each box of λ in such a way that the numbers weakly increase in each row from left to right and strictly increase in each column from top to bottom.

For instance the semi-standard tableau on

$$\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

with $m = 3$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Let λ be a Young diagram and let $\mu = (\mu_1, \dots, \mu_m)$ be a multi-index. The number of semi-standard tableaux on λ with exactly μ_1 1s, μ_2 2s and so on is denoted by $K_{\lambda\mu}$ and is called the *Kostka number*. A non-trivial fact is that $K_{\lambda\mu}$ does not depend on the order of μ, hence we may assume that μ is itself a Young diagram. Since in any semi-standard tableau, the numbers in the boxes of the ith row are at least i, we have $K_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$.

For a given Young diagram λ and a number m, the polynomial

$$s_\lambda(x_1, x_2, \dots, x_m) := \sum_\mu K_{\lambda\mu} x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m},$$

where μ ranges over all multi-indices, is called the *Schur polynomial* of λ. By convention, if $\lambda = (0, \dots, 0)$ is the empty Young diagram, then $s_\lambda(x_1, \dots, x_m) = 1$.

In our example, $\lambda = (2, 1)$ and $m = 3$, and we get

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

In general, $s_\lambda(x_1, \dots, x_m)$ is a symmetric polynomial. Moreover, the set of all Schur polynomials $s_\lambda(x_1, \dots, x_m)$, as λ ranges over all Young diagrams of depth not greater than m, is a basis of the vector space of all symmetric polynomials in $x_1, \dots, x_m$.

3.2. Representations of SU(2)

Recall that a maximal torus in SU(2) is $\mathrm{U}(1) = S^1$, which we embed by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

(see also (3.5)). If (V, ϕ) is a finite-dimensional representation of SU(2), then

$$V = \bigoplus_\alpha V_\alpha,$$

where α ranges over all weights of V (which are integers) and

$$V_\alpha = \{v \in V \mid \phi(z)(v) = z^\alpha v\}$$

is the corresponding *weight space*. A non-zero vector $v \in V_\alpha$ is called *vector of weight* α.

The *character* of (V,ϕ) is defined by

$$\operatorname{char}(V)=\sum_{\alpha=-\infty}^{\infty}\dim V_\alpha t^\alpha\in\mathbb{Z}[t,t^{-1}].$$

Let $V\cong\mathbb{C}^2$ be the standard representation of SU(2) and set $V_k:=\operatorname{Sym}^k V$. Then V_k is an irreducible SU(2)-representation of dimension $k+1$. Any finite-dimensional irreducible representation of SU(2) is isomorphic to V_k for some k. We have

$$\operatorname{char}(V_k)=\sum_{j=0}^{k}t^{2j-k}\in\mathbb{Z}[t,t^{-1}].$$

We will frequently use the following rules, whose proofs can be found in [**19**]:

$$V_k\otimes V_l\simeq V_{k+l}\oplus V_{k+l-2}\oplus\cdots\oplus V_{|k-l|}\quad\text{(Clebsch–Gordan rule)},\tag{3.1}$$

$$\operatorname{Sym}^k V_2\simeq\bigoplus_{l=0}^{\lfloor k/2\rfloor}V_{2k-4l}.\tag{3.2}$$

It will be convenient to state many formulae in the representation ring $R\,\mathrm{SU}(2)$, which is the free $\mathbb{Z}$-module generated by variables $V_0,V_1,V_2,\ldots,$ with multiplication given by the Clebsch–Gordan rule.

The spaces of differential forms which are considered in this paper are *graded commutative* SU(2)-*algebras*, i.e. graded SU(2)-modules W endowed with a graded-commutative product map $W\otimes W\to W$ which is an SU(2)-morphism and which is compatible with the grading.

3.3. Representations of $\mathbf{GL}(n,\mathbb{C})$

Irreducible complex representations of $\mathrm{GL}(n,\mathbb{C})$ are classified by sequences $\lambda=(\lambda_1,\ldots,\lambda_n)$ of integers with $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$. The standard ($n$-dimensional) representation corresponds to $\lambda=(1)$, while the one-dimensional determinantal representation corresponds to $\lambda=(1,1,\ldots,1)$.

An explicit description of the representation Γ_λ corresponding to an arbitrary sequence λ can be found in any standard text on representation theory, e.g. [**19**] or [**18**]. If $\lambda_n\geqslant 0$, then the character of Γ_λ equals the Schur function $s_\lambda(x_1,\ldots,x_n)$.

Let V be the standard representation of $\mathrm{GL}(n,\mathbb{C})$. A standard fact from representation theory of the general linear group [**18**, § 8.3] tells us that

$$\Lambda^{k_1}V\otimes\cdots\otimes\Lambda^{k_s}V\simeq\bigoplus_{\nu}K_{\tilde\nu\mu}\Gamma_\nu,\tag{3.3}$$

where $\mu=(k_1,\ldots,k_s)$ and ν ranges over all Young diagrams of depth less than or equal to n, $\tilde\nu$ is the conjugate of ν and $K_{\tilde\nu\mu}$ is the Kostka number from § 3.1.

3.4. Symplectic groups

Let $\mathbb{H}$ be the four-dimensional real vector space of quaternions. This space has an $\mathbb{R}$-basis given by 1, i, j, k. The algebra structure of $\mathbb{H}$ is defined by $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$, $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$. The conjugate of a quaternion $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ is the quaternion $\bar{q} := a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$. The norm of q is the real number $q\bar{q}$.

We will consider $\mathbb{H}$ as a complex vector space, i.e. we write

$$\mathbb{H} = \mathbb{C} \oplus \mathrm{j}\mathbb{C}. \tag{3.4}$$

Multiplication from the left on $\mathbb{H} = \mathbb{C} \oplus \mathrm{j}\mathbb{C} = \mathbb{C}^2$ defines an embedding of $\mathbb{H}$ into $M_2\mathbb{C}$. Explicitly,

$$\begin{aligned} \mathbb{H} &\hookrightarrow M_2\mathbb{C}, \\ z_1 + \mathrm{j}z_2 &\mapsto \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \end{aligned} \tag{3.5}$$

The quaternions of norm 1 form a subgroup $\mathrm{Sp}(1)$ of $\mathbb{H}^*$. It contains the subgroup $\mathrm{U}(1)$ of all complex numbers of norm 1 (i.e. quaternions of the form $a + \mathrm{i}b$ with $a^2 + b^2 = 1$). The embedding (3.5) identifies $\mathrm{Sp}(1)$ with $\mathrm{SU}(2)$, which is homeomorphic to a three-dimensional sphere.

Let V be a quaternionic (right) vector space of dimension n. We endow V with a quaternionic Hermitian form K, i.e. an $\mathbb{R}$-bilinear form

$$K \colon V \times V \to \mathbb{H}$$

such that

(1) K is conjugate $\mathbb{H}$-linear in the first and $\mathbb{H}$-linear in the second factor, i.e.

$$K(vq, wr) = \bar{q}K(v,w)r, \quad q, r \in \mathbb{H},$$

(2) K is Hermitian in the sense that

$$K(w,v) = \overline{K(v,w)},$$

(3) K is positive definite, i.e.

$$K(v,v) > 0, \quad \forall v \neq 0.$$

The standard example of such a form is given by

$$V = \mathbb{H}^n, \qquad K(v,w) = \sum_{i=1}^n \bar{v}_i w_i, \quad v = (v_1, \dots, v_n),\ w = (w_1, \dots, w_n) \in \mathbb{H}^n.$$

Recall that $\mathrm{GL}(V, \mathbb{H}) = \mathrm{GL}(n, \mathbb{H})$ is the group of $\mathbb{H}$-linear automorphisms of V. The subgroup of $\mathrm{GL}(V, \mathbb{H})$ of all elements preserving K is called *compact symplectic group* and denoted by $\mathrm{Sp}(V, K)$ or $\mathrm{Sp}(n)$. It acts from the left on V.

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With respect to the decomposition (3.4), we can decompose

$$K(v,w) = H(v,w) + \mathrm{j}Q(v,w).$$

Then H is a complex Hermitian form and Q a skew-symmetric complex linear form. Moreover, $Q(v,w) = H(v\mathrm{j},w)$ and $H(v\mathrm{j},w\mathrm{j}) = \overline{H(v,w)}$. If $\mathrm{U}(2n)$ denotes the unitary group with respect to H, and $\mathrm{Sp}_{2n}\mathbb{C}$ the symplectic group with respect to Q (i.e. the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ consisting of all elements preserving Q) then

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}_{2n}\mathbb{C}.$$

Hence $\mathrm{Sp}(n)$ is a compact form of $\mathrm{Sp}_{2n}\mathbb{C}$, a fact which will be used in § 4.

We let $\mathrm{Sp}(1)$ and its subgroup $\mathrm{U}(1)$ act on V from the right. The actions of $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ and $\mathrm{Sp}(n)\times\mathrm{U}(1)$ on V are not faithful: in both cases the kernel consists of the two elements $(\mathrm{Id},1)$ and $(-\mathrm{Id},-1)$ and is isomorphic to $\mathbb{Z}_2$. Hence the factor groups

$$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) := \mathrm{Sp}(n)\times\mathrm{Sp}(1)/\mathbb{Z}_2,$$
$$\mathrm{Sp}(n)\cdot\mathrm{U}(1) := \mathrm{Sp}(n)\times\mathrm{U}(1)/\mathbb{Z}_2$$

act faithfully on V. Since $\mathrm{Sp}(n)$ acts transitively on the unit sphere, the same holds true for $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ and $\mathrm{Sp}(n)\cdot\mathrm{U}(1)$.

Let $v_0 \in V$ be a unit vector. Let us assume that v_0 is the first standard coordinate vector. Then a pair $(A,z)\in\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ stabilizes v_0 if and only if A is of the form

$$A = \begin{pmatrix} z^{-1} & 0 \\ 0 & A' \end{pmatrix}, \quad A' \in \mathrm{Sp}(n-1).$$

It follows that the stabilizer of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ at v_0 is isomorphic to $\mathrm{Sp}(n-1)\cdot\mathrm{Sp}(1)$.

The tangent space $T_{v_0}S^{4n-1}$ splits as $U\oplus\tilde{V}$, where $\tilde{V}$ is the quaternionic orthogonal complement of v_0 and U is of dimension 3. Then the action of $\mathrm{Sp}(n-1)\cdot\mathrm{Sp}(1)$ on $\tilde{V}\cong\mathbb{H}^{n-1}$ is the usual one (i.e. $\mathrm{Sp}(n-1)$ acts from the left, $\mathrm{Sp}(1)$ from the right), while the action on U is the adjoint action of $\mathrm{Sp}(1)$ (i.e. (A',z) acts by multiplication by z^{-1} from the left followed by right multiplication by z).

3.5. Spherical representations of $\mathbf{SL}(2n,\mathbb{C})$

We will need the following proposition which is a consequence of known facts from representation theory. We refer to [**20**, § 12] for the terminology in the proof.

Proposition 3.1. *Let λ be a Young diagram and Γ_λ the corresponding irreducible representation of $\mathrm{GL}(2n,\mathbb{C})$. Let $\Gamma_\lambda^{\mathrm{Sp}_{2n}\mathbb{C}}$ be the subspace of $\mathrm{Sp}_{2n}\mathbb{C}$-fixed elements. Then*

$$\dim \Gamma_\lambda^{\mathrm{Sp}_{2n}\mathbb{C}} = \begin{cases} 1 & \text{if } \tilde{\lambda} \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

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Proof. Since $\mathrm{Sp}_{2n}\mathbb{C} \subset \mathrm{SL}(2n,\mathbb{C})$, the determinantal representation of $\mathrm{GL}(2n,\mathbb{C})$ restricts to the trivial representation of $\mathrm{Sp}_{2n}\mathbb{C}$. We may thus suppose that $\lambda_{2n} = 0$ and consider Γ_λ as a representation of $\mathrm{SL}(2n,\mathbb{C})$.

Since the pair $G = \mathrm{SL}(2n,\mathbb{C}), K = \mathrm{Sp}_{2n}\mathbb{C}$ is a *spherical pair*, the dimension of Γ_λ^K is at most 1. If Γ_λ contains a non-zero K-fixed vector, Γ_λ is called *spherical*.

By Theorem 12.3.13 of [**20**] and the explicit computation in [**20**, § 12.3.2, Type AII], we may conclude that Γ_λ is spherical if and only if $\tilde{\lambda}$ is even. □

4. Invariant theory of Sp(n)

Let us fix a sequence of quaternionic isometric embeddings

$$\mathbb{H}^1 \overset{\iota_{12}}{\hookrightarrow} \mathbb{H}^2 \overset{\iota_{23}}{\hookrightarrow} \mathbb{H}^3 \overset{\iota_{34}}{\hookrightarrow} \cdots .$$

Then we get a sequence of restriction maps

$$(\Lambda^*\mathbb{H}^* \otimes \mathbb{C})^{\mathrm{Sp}(1)} \overset{\iota_{12}^*}{\longleftarrow} (\Lambda^*(\mathbb{H}^2)^* \otimes \mathbb{C})^{\mathrm{Sp}(2)} \overset{\iota_{23}^*}{\longleftarrow} (\Lambda^*(\mathbb{H}^3)^* \otimes \mathbb{C})^{\mathrm{Sp}(3)} \overset{\iota_{34}^*}{\longleftarrow} \cdots$$

and define

$$(\Lambda^*(\mathbb{H}^\infty)^* \otimes \mathbb{C})^{\mathrm{Sp}(\infty)} := \varprojlim (\Lambda^*(\mathbb{H}^n)^* \otimes \mathbb{C})^{\mathrm{Sp}(n)}.$$

The grading, algebra structure and SU(2)-action on each of the spaces $(\Lambda^*(\mathbb{H}^n)^* \otimes \mathbb{C})^{\mathrm{Sp}(n)}$ induce the structure of a graded SU(2)-algebra on $(\Lambda^*(\mathbb{H}^\infty)^* \otimes \mathbb{C})^{\mathrm{Sp}(\infty)}$.

Proposition 4.1. *The graded* SU(2)*-algebra* $(\Lambda^*(\mathbb{H}^\infty)^* \otimes \mathbb{C})^{\mathrm{Sp}(\infty)}$ *is freely generated by one copy of* V_2 *in degree* 2*. The restriction map*

$$u_n\colon (\Lambda^k(\mathbb{H}^\infty)^* \otimes \mathbb{C})^{\mathrm{Sp}(\infty)} \to (\Lambda^k(\mathbb{H}^n)^* \otimes \mathbb{C})^{\mathrm{Sp}(n)}$$

commutes with the SU(2)*-action, is surjective for all* n *and injective for* $n \geqslant k$*.*

Proof. Let V be an n-dimensional quaternionic vector space with a quaternionic Hermitian form K as in the previous subsection. We have an isomorphism between $\mathrm{Sp}_{2n}\mathbb{C}$-modules

$$\begin{aligned} \phi\colon V &\cong V^*, \\ v &\mapsto [w \mapsto Q(v,w)]. \end{aligned} \tag{4.1}$$

In the following, we consider V as a complex vector space. To each (complex) polynomial

$$f\colon \underbrace{V \times \cdots \times V}_{m} \to \mathbb{C}$$

in the real and imaginary coordinates, we associate a polynomial

$$\tilde{f}\colon \underbrace{V \times \cdots \times V}_{m} \times \underbrace{V^* \times \cdots \times V^*}_{m} \to \mathbb{C}$$

in the following way.

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(1) If $f(v_1,\dots,v_m)=v_{ij}$, then $\tilde f(v_1,\dots,v_m,\xi_1,\dots,\xi_m):=v_{ij}$.

(2) If $f(v_1,\dots,v_m)=\bar v_{ij}$, then $\tilde f(v_1,\dots,v_m,\xi_1,\dots,\xi_m):=\xi_{ij}$.

(3) $f\mapsto \tilde f$ is an algebra homomorphism.

We get another polynomial

$$\hat f\colon \underbrace{V\times\cdots\times V}_{2m}\to\mathbb{C}$$

by composing with the isomorphism (4.1), i.e.

$$\hat f(v_1,\dots,v_m,w_1,\dots,w_m):=\tilde f(v_1,\dots,v_m,\phi(w_1),\dots,\phi(w_m)).$$

It follows as in [**26**, Chapter 13, Addendum 1] that if f is invariant under $\mathrm{Sp}(n)$, then $\tilde f$ and $\hat f$ are invariant under $\mathrm{Sp}_{2n}\mathbb{C}$.

Now the first fundamental theorem (FFT) for $\mathrm{Sp}_{2n}\mathbb{C}$ [**19**, Proposition F.13] tells us that $\hat f$ is a polynomial in the *basic invariants*

$$(v_1,\dots,v_{2m})\mapsto Q(v_i,v_j),\quad 1\leqslant i<j\leqslant 2m.$$

Unravelling the identifications, we get that f is a polynomial in the three basic invariants

$$(v_1,\dots,v_m)\mapsto\begin{cases}Q(v_i,v_j) & 1\leqslant i<j\leqslant m,\\ \overline{Q(v_i,v_j)} & 1\leqslant i<j\leqslant m,\\ H(v_i,v_j) & 1\leqslant i,j\leqslant m.\end{cases}$$

Since Q and $\bar Q$ are antisymmetric and since $H(v,w)-H(w,v)=2\mathrm{i}\,\mathrm{Im}\,H(v,w)$, we deduce that the algebra $(\Lambda^*V^*\otimes\mathbb{C})^{\mathrm{Sp}(n)}$ is generated by the three basic 2-forms Q, $\bar Q$ and $\mathrm{Im}\,H$. They span an irreducible three-dimensional $\mathrm{SU}(2)$-representation.

The second fundamental theorem (SFT) for $\mathrm{Sp}_{2n}\mathbb{C}$ [**20**, Theorem 12.2.15] yields that there are no relations between the basic invariants in degree less than or equal to n, hence the above restriction map is injective if $k\leqslant n$. ☐

Let

$$S\mathbb{H}^1\overset{\tilde\iota_{12}}{\hookrightarrow}S\mathbb{H}^2\overset{\tilde\iota_{23}}{\hookrightarrow}S\mathbb{H}^3\overset{\tilde\iota_{34}}{\hookrightarrow}\cdots$$

be the induced sequence of embeddings. We get a sequence of restriction maps

$$(\Omega_h^{*,*}(S\mathbb{H}^1))^{\mathrm{Sp}(1)}\overset{\tilde\iota_{12}^*}{\longleftarrow}(\Omega_h^{*,*}(S\mathbb{H}^2))^{\mathrm{Sp}(2)}\overset{\tilde\iota_{23}^*}{\longleftarrow}(\Omega_h^{*,*}(S\mathbb{H}^3))^{\mathrm{Sp}(3)}\overset{\tilde\iota_{34}^*}{\longleftarrow}\cdots$$

and define

$$(\Omega_h^{*,*}(S\mathbb{H}^\infty))^{\mathrm{Sp}(\infty)}:=\varprojlim(\Omega_h^{*,*}(S\mathbb{H}^n))^{\mathrm{Sp}(n)},$$

which is a bi-graded $\mathrm{SU}(2)$-algebra.

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Proposition 4.2. *The bi-graded* SU(2)*-algebra* $(\Omega_h^{*,*}(S\mathbb{H}^\infty))^{\mathrm{Sp}(\infty)}$ *is generated by five copies of* V_2 *in bi-degrees* $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$ *and one copy of* V_0 *in bi-degree* $(1,1)$. *The restriction maps*

$$\tilde{u}_n\colon (\Omega_h^{k,l}(S\mathbb{H}^\infty))^{\mathrm{Sp}(\infty)} \to (\Omega_h^{k,l}(S\mathbb{H}^n))^{\mathrm{Sp}(n)}$$

are surjective for all k, l, n *and injective for* $k+l \leqslant n-1$.

Proof. The proof is based on the same argument as the proof of Proposition 4.1.

Fix a unit vector $v_0 \in V$. The orthogonal complement $v_0^\perp \subset V$ splits as

$$v_0^\perp = U \oplus \tilde{V},$$

where $\tilde{V}$ is the quaternionic orthogonal complement of v_0 and $U := v_0 \cdot \mathbb{H} \cap v_0^\perp$ is three dimensional. The stabilizer of $\mathrm{Sp}(n) \cong \mathrm{Sp}(V,K)$ at v_0 is $\mathrm{Sp}(n-1) \cong \mathrm{Sp}(\tilde{V}, K|_{\tilde{V}})$, it acts trivially on U. As SU(2)-representation $U \simeq V_2$, the adjoint representation (compare the discussion in § 3.4).

The horizontal hyperplane at the point $(0, v_0) \in SV$ is then given by

$$Q_{(0,v_0)} = U \oplus \tilde{V} \oplus U \oplus \tilde{V},$$

and $\mathrm{Sp}(n-1)$ acts trivially on the U-factors and diagonally on the second and fourth factors.

As SU(2)-representations we thus have the following isomorphisms:

$$\begin{aligned}
\Omega_h^{*,*}(SV)^{\mathrm{Sp}(n)} &\simeq \Lambda^{*,*} Q^*_{(0,v_0)} \otimes \mathbb{C} \\
&\simeq \Lambda^{*,*}(U^* \oplus \tilde{V}^* \oplus U^* \oplus \tilde{V}^*)^{\mathrm{Sp}(n-1)} \otimes \mathbb{C} \\
&\simeq (\Lambda^*(U^* \oplus \tilde{V}^*) \otimes \Lambda^*(U^* \oplus \tilde{V}^*))^{\mathrm{Sp}(n-1)} \otimes \mathbb{C} \\
&\simeq \sum_{k,l=0}^{3} \Lambda^k U^* \otimes \Lambda^l U^* \otimes (\Lambda^{*-k}\tilde{V}^* \otimes \Lambda^{*-l}\tilde{V}^*)^{\mathrm{Sp}(n-1)} \otimes \mathbb{C}.
\end{aligned}$$

Now the algebra $\Lambda^* U^* \otimes \mathbb{C}$ is generated by $U^* \simeq V_2$. Using arguments as in the previous proof, we get that $(\Lambda^*\tilde{V}^* \otimes \Lambda^*\tilde{V}^* \otimes \mathbb{C})^{\mathrm{Sp}(n-1)}$ is generated by three copies of V_2 in degrees $(2,0)$, $(1,1)$ and $(0,2)$ and one copy of V_0 corresponding to the symplectic form on SV.

These invariant forms are defined on $S\mathbb{H}^n$ for each n, behave well under the restriction maps and can therefore be considered as elements of $\Omega_h^{*,*}(S\mathbb{H}^\infty)^{\mathrm{Sp}(\infty)}$. The surjectivity of $\tilde{u}_n$ follows.

Since by the SFT for $\mathrm{Sp}_{2n-2}\,\mathbb{C}$ there are no relations of degree less than or equal to $n-1$ between the basic $\mathrm{Sp}(n-1)$-invariant forms, the injectivity of $\tilde{u}_n$ in the case $k+l \leqslant n-1$ follows. □

5. Local and global valuations

Let us fix again a sequence of quaternionic embeddings

$$\mathbb{H}^1 \overset{\iota_{12}}{\hookrightarrow} \mathbb{H}^2 \overset{\iota_{23}}{\hookrightarrow} \mathbb{H}^3 \overset{\iota_{34}}{\hookrightarrow} \cdots .$$

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Correspondingly, we have embeddings of the sphere bundles

$$S\mathbb{H}^1 \overset{\tilde\iota_{12}}{\hookrightarrow} S\mathbb{H}^2 \overset{\tilde\iota_{23}}{\hookrightarrow} S\mathbb{H}^3 \overset{\tilde\iota_{34}}{\hookrightarrow} \cdots$$

and restrictions

$$\mathrm{Val}^{\mathrm{Sp}(1)} \xleftarrow{r_{21}} \mathrm{Val}^{\mathrm{Sp}(2)} \xleftarrow{r_{32}} \mathrm{Val}^{\mathrm{Sp}(3)} \xleftarrow{r_{43}} \cdots.$$

Let

$$\mathrm{Val}^{\mathrm{Sp}(\infty)} := \varprojlim \mathrm{Val}^{\mathrm{Sp}(n)}$$

be the inverse limit. An element of $\mathrm{Val}^{\mathrm{Sp}(\infty)}$ is thus a sequence $(\mu_1, \mu_2, \dots)$ with $\mu_n \in \mathrm{Val}^{\mathrm{Sp}(n)}$ and $r_{n+1,n}(\mu_{n+1}) = \mu_n$ for $n = 1, 2, \dots$. The inverse limit comes with restriction maps

$$r_n \colon \mathrm{Val}^{\mathrm{Sp}(\infty)} \to \mathrm{Val}^{\mathrm{Sp}(n)},$$
$$(\mu_1, \mu_2, \dots) \mapsto \mu_n,$$

which satisfy $r_n = r_{n+1,n} \circ r_{n+1}$ for all n.

Similarly, we denote by

$$\mathrm{Val}^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)} := \varprojlim \mathrm{Val}^{\mathrm{Sp}(n)\,\mathrm{U}(1)},$$
$$\mathrm{Val}^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)} := \varprojlim \mathrm{Val}^{\mathrm{Sp}(n)\,\mathrm{Sp}(1)}$$

the corresponding inverse limits.

Proof of Theorem 1.1. We will prove the statement only in the case $G = \mathrm{Sp}(n)$, the two other cases are similar.

We have to show that the restriction map

$$r_n \colon \mathrm{Val}_k^{\mathrm{Sp}(\infty)} \to \mathrm{Val}_k^{\mathrm{Sp}(n)}$$

is surjective for all k and n and injective for $n \geqslant k$. The case $k = 0$ is trivial (each of the spaces $\mathrm{Val}_0^{\mathrm{Sp}(n)}$ and $\mathrm{Val}_0^{\mathrm{Sp}(\infty)}$ is spanned by the Euler characteristic χ). We thus assume that $k > 0$.

Injectivity. Let us first show that r_n is injective for $n \geqslant k$. For this, it is enough to show that

$$r_{n+1,n} \colon \mathrm{Val}_k^{\mathrm{Sp}(n+1)} \to \mathrm{Val}_k^{\mathrm{Sp}(n)}$$

is injective for all $n \geqslant k$.

Let $\mu \in \mathrm{Val}_k^{\mathrm{Sp}(n+1)}$ be a valuation whose restriction to $\mathbb{H}^n$ vanishes. Take a space $E \in \mathrm{Gr}_k(\mathbb{H}^{n+1})$. Let $E^{\mathbb{H}}$ be the quaternionic vector space generated by E. Note that $\dim_{\mathbb{H}} E^{\mathbb{H}} \leqslant k \leqslant n$.

We may find an element $g \in \mathrm{Sp}(n+1)$ with $g(E^{\mathbb{H}}) \subset \mathbb{H}^n$. Then $F := g(E) \in \mathrm{Gr}_k(\mathbb{H}^n)$ and hence $\mu|_F = 0$. Since E and F are in the same $\mathrm{Sp}(n+1)$-orbit and μ is $\mathrm{Sp}(n+1)$invariant, it follows that $\mu|_E = 0$. Since E was arbitrary, the Klain function of μ vanishes. By Theorem 2.2, μ vanishes. This proves injectivity of $r_{n+1,n}$ and hence injectivity of r_n for $n \geqslant k$.

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Surjectivity. Let n be arbitrary. We have the following maps

$$\Omega^{k,k-1}(S\mathbb{H}^n)^{\mathrm{Sp}(n)} \xrightarrow[*_1]{\cong} \Omega^{4n-k,k-1}(S\mathbb{H}^n)^{\mathrm{Sp}(n)} \xrightarrow{\mathrm{nc}} \mathrm{Val}_{4n-k}^{\mathrm{Sp}(n)} \xrightarrow[\mathbb{F}]{\cong} \mathrm{Val}_k^{\mathrm{Sp}(n)}.$$

The first map is $*_1$, the Hodge star operator acting on the first component of

$$\Omega^{*,*}(SV)^{\mathrm{Sp}(n)} = (\Lambda^* V \otimes \Omega^*(S^{4n-1}))^{\mathrm{Sp}(n)}.$$

The second map is integration over the normal cycle from (2.2). Since $k > 0$, this map is surjective by (2.1) and (2.2). The third map is the Alesker–Fourier transform $\mathbb{F}$ (see Theorem 2.3), which is an isomorphism. The composition of these maps will be denoted by

$$q_n \colon \Omega^{k,k-1}(S\mathbb{H}^n)^{\mathrm{Sp}(n)} \twoheadrightarrow \mathrm{Val}_k^{\mathrm{Sp}(n)},$$

which is a surjective map.

Recall the definition of the surjective map $\tilde{u}_n$ from Proposition 4.2. We claim that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Omega^{k,k-1}(S\mathbb{H}^{n+1})^{\mathrm{Sp}(n+1)} & \xrightarrow{q_{n+1}} & \mathrm{Val}_k^{\mathrm{Sp}(n+1)} \\
 & \overset{\tilde{u}_{n+1}}{\nearrow} & \Big\downarrow \tilde{\iota}^*_{n,n+1} & & \Big\downarrow r_{n+1,n} \\
\Omega^{k,k-1}(S\mathbb{H}^\infty)^{\mathrm{Sp}(\infty)} & & & & \\
 & \underset{\tilde{u}_n}{\searrow} & & & \\
 & & \Omega^{k,k-1}(S\mathbb{H}^n)^{\mathrm{Sp}(n)} & \xrightarrow{q_n} & \mathrm{Val}_k^{\mathrm{Sp}(n)}
\end{array}$$

The commutativity of the left-hand triangle follows from the definitions. To prove the commutativity of the right-hand square, let $E \in \mathrm{Gr}_k(\mathbb{H}^n)$, $\omega \in \Omega^{k,k-1}(S\mathbb{H}^{n+1})^{\mathrm{Sp}(n+1)}$ and let B_E (respectively $B_{E^\perp}$, S_E) be the unit ball in E (respectively in $E^\perp$, respectively the unit sphere). Let us abbreviate $\iota := \iota_{n,n+1}$, $\tilde{\iota} := \tilde{\iota}_{n,n+1}$, $r := r_{n+1,n}$. Then

$$\begin{aligned}
\mathrm{Kl}_{r \circ q_{n+1}\omega}(E) &= \mathrm{Kl}_{q_{n+1}\omega}(\iota E) \\
&= \mathrm{Kl}_{\mathrm{nc}(*_1\omega)}((\iota E)^\perp) \\
&= \frac{1}{\omega_{4n+4-k}} \int_{\mathrm{nc}(B_{(\iota E)^\perp})} *_1\omega \\
&= \frac{1}{\omega_{4n+4-k}} \int_{B_{(\iota E)^\perp} \times S_{(\iota E)}} *_1\omega \quad \text{since } \omega \text{ is of bi-degree } (k, k-1) \\
&= \frac{1}{\omega_k} \int_{B_{(\iota E)} \times S_{(\iota E)}} \omega.
\end{aligned}$$

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On the other hand,

$$\begin{aligned}\mathrm{Kl}_{q_n\circ\tilde\iota^*\omega}(E) &= \mathrm{Kl}_{\mathrm{nc}(*_1\tilde\iota^*\omega)}(E^\perp)\\ &= \frac{1}{\omega_{4n-k}}\int_{\mathrm{nc}(B_E^\perp)} *_1\tilde\iota^*\omega\\ &= \frac{1}{\omega_{4n-k}}\int_{B_E^\perp\times S_E} *_1\tilde\iota^*\omega \quad \text{since } \tilde\iota^*\omega \text{ is of bi-degree } (k,k-1)\\ &= \frac{1}{\omega_k}\int_{B_E\times S_E}\tilde\iota^*\omega\\ &= \frac{1}{\omega_k}\int_{B_{(\iota E)}\times S_{(\iota E)}}\omega.\end{aligned}$$

The claim now follows from Klain's embedding theorem 2.2.

The claim implies that for each $\omega \in \Omega^{k,k-1}(S\mathbb{H}^\infty)^{\mathrm{Sp}(\infty)}$, the element

$$\mu_\omega = (\mu_1,\mu_2,\dots)\in \mathrm{Val}_k^{\mathrm{Sp}(\infty)}, \qquad \mu_n := q_n\circ\tilde u_n(\omega)$$

is well-defined.

Now we get surjectivity of r_n: let $\tau\in\mathrm{Val}_k^{\mathrm{Sp}(n)}$. Since $\tilde u_n$ and q_n are surjective (see Proposition 4.2), there exists $\omega\in\Omega^{k,k-1}(S\mathbb{H}^\infty)^{\mathrm{Sp}(\infty)}$ with $\tau = q_n\circ\tilde u_n(\omega) = r_n(\mu_\omega)$. □

Proof of Theorem 1.2. The proof in the three cases $G=\mathrm{Sp}(\infty),\mathrm{Sp}(\infty)\cdot\mathrm{U}(1),\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)$ is similar and we first give an outline. We let $G(n)$ be the corresponding group (i.e. if $G=\mathrm{Sp}(\infty)$ then $G(n)=\mathrm{Sp}(n)$, etc.). Set

$$\begin{aligned} b_k &:= \dim(\Lambda^k\mathbb{H}^\infty\otimes\mathbb{C})^G,\\ b_{k,l} &:= \dim\Omega_h^{k,l}(S\mathbb{H}^\infty)^{\bar G}\end{aligned}$$

and

$$\begin{aligned} b_k^n &:= \dim(\Lambda^k\mathbb{H}^n\otimes\mathbb{C})^{G(n)},\\ b_{k,l}^n &:= \dim\Omega_h^{k,l}(S\mathbb{H}^n)^{\bar G(n)}.\end{aligned}$$

We have the following symmetries (see also Lemma 6.3)

$$b_k^n = b_{4n-k}^n, \qquad b_{k,l}^n = b_{l,k}^n = b_{4n-1-k,l}^n = b_{k,4n-1-l}^n.$$

Moreover,

$$b_{k,l} = \lim_{n\to\infty} b_{k,l}^n, \qquad b_k = \lim_{n\to\infty} b_k^n.$$

In fact, the first sequence stabilizes at $n=k+l+1$, the second at $n=k$ (see Propositions 4.2 and 4.1). Let us introduce the formal power series

$$\begin{aligned} f^G(x) &= \sum_{k=0}^\infty b_k x^k,\\ h^G(x,y) &= \sum_{k,l=0}^\infty b_{k,l}x^k y^l.\end{aligned}$$

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It turns out that in all three cases there is a rational function $\tilde{h}^G$ such that

$$(x-y)h^G(x,y) = \tilde{h}^G(x,xy) - \tilde{h}^G(y,xy). \tag{5.1}$$

Let us fix some $n > k+l$. By Theorems 1.1, 2.3 and Corollary 2.7, we have

$$\begin{aligned}
\dim \mathrm{Val}_k^G &= \dim \mathrm{Val}_k^{G(n)} \\
&= \dim \mathrm{Val}_{4n-k}^{G(n)} \\
&= \sum_{l=0}^{k-1} (-1)^{k+l+1} \dim \Omega_p^{4n-k,l}(S\mathbb{H}^n)^{\bar{G}(n)} + (-1)^k \dim(\Lambda^{4n-k}\mathbb{H}^n \otimes \mathbb{C})^{G(n)} \\
&= \sum_{l=0}^{k-1} (-1)^{k+l+1} (b^n_{4n-k,l} - b^n_{4n-k-1,l-1}) + (-1)^k b^n_{4n-k} \\
&= \sum_{l=0}^{k-1} (-1)^{k+l+1} (b^n_{k-1,l} - b^n_{k,l-1}) + (-1)^k b^n_k \\
&= \sum_{l=0}^{k-1} (-1)^{k+l+1} (b_{k-1,l} - b_{k,l-1}) + (-1)^k b_k.
\end{aligned}$$

We multiply by x^k and sum over k to obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \dim \mathrm{Val}_k^G x^k &= -\sum_{k=0}^{\infty} \left(\sum_{l=0}^{k-1} (-1)^l (b_{k-1,l} - b_{k,l-1}) \right) (-x)^k + \sum_{k=0}^{\infty} (-1)^k b_k x^k \\
&= -\tilde{h}^G(-x,x) + f^G(-x).
\end{aligned}$$

Let us compute the function f^G. By Proposition 4.1 and (3.2), $(\Lambda^k\mathbb{H}^\infty \otimes \mathbb{C})^{\mathrm{Sp}(\infty)} = 0$ if k is odd and

$$(\Lambda^k\mathbb{H}^\infty \otimes \mathbb{C})^{\mathrm{Sp}(\infty)} \simeq \mathrm{Sym}^{k/2} V_2 \simeq \bigoplus_{l=0}^{\lfloor k/4 \rfloor} V_{k-4l}$$

if k is even. Therefore,

$$\begin{aligned}
\dim(\Lambda^k\mathbb{H}^\infty \otimes \mathbb{C})^{\mathrm{Sp}(\infty)} &= \sum_{l=0}^{\lfloor k/4 \rfloor} (k-4l+1) = \binom{\frac{1}{2}k+2}{2}, \\
\dim(\Lambda^k\mathbb{H}^\infty \otimes \mathbb{C})^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)} &= \lfloor \tfrac{1}{4}k \rfloor + 1, \\
\dim(\Lambda^k\mathbb{H}^\infty \otimes \mathbb{C})^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)} &= \begin{cases} 1 & k \equiv 0 \mod 4, \\ 0 & k \not\equiv 0 \mod 4, \end{cases}
\end{aligned}$$

from which we deduce that

$$f^{\mathrm{Sp}(\infty)}(x) = \frac{1}{(1-x^2)^3},$$
$$f^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}(x) = \frac{1}{(1-x^2)(1-x^4)},$$
$$f^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)}(x) = \frac{1}{1-x^4}.$$

Next, we compute the function h^G. By Proposition 4.2, there is an isomorphism of SU(2)-representation

$$\Omega_h^{k,l}(S\mathbb{H}^\infty)^{\overline{\mathrm{Sp}(\infty)}} = \bigoplus \mathrm{Sym}^{k_1} V_2 \otimes \mathrm{Sym}^{k_2} V_2 \otimes \mathrm{Sym}^{k_3} V_0 \otimes \mathrm{Sym}^{k_4} V_2 \otimes \mathrm{Sym}^{k_5} V_2 \otimes \mathrm{Sym}^{k_6} V_2, \quad (5.2)$$

where the sum is over all tuples $(k_1, \dots, k_6)$ with $k_1 + k_3 + k_5 + 2k_6 = k$ and $k_2 + k_3 + 2k_4 + k_5 = l$.

Claim. The SU(2)-representation $\Omega_h^{k,l}(S\mathbb{H}^\infty)^{\overline{\mathrm{Sp}(\infty)}}$ does not contain V_{2s+1} for any s and contains V_{2s} with a multiplicity which is given by the coefficient of $x^k y^l$ in the power series

$$\frac{1}{(x-y)^2(x+y)(1-x^4)(1-y^4)(1-x^2y^2)(1-xy)} \times \left(\frac{(1+x)^2(1+y)(1+x^3)(1+y^2)(x^2+y)(x^2y+1)}{(1-xy)(1-x^3y)} x^{2s} - \frac{(1+x)^2(1+y)^2(1+y^2)(1+x^2)(x^2y+1)(y^2x+1)(x+y)}{(-1+xy^3)(-1+x^3y)} x^s y^s + \frac{(1+y)^2(1+y^3)(1+x^2)(y^2+x)(y^2x+1)(1+x)}{(1-xy)(1-xy^3)} y^{2s} \right). \quad (5.3)$$

We defer the technical proof of the claim, which only uses the Clebsch–Gordan rule, to the appendix. Multiplying (5.3) by $2s+1$ and summing over all s we obtain

$$h^{\mathrm{Sp}(\infty)}(x,y) = \frac{1}{(1-x)^3(1-xy)^4(1-y)^3}.$$

Setting

$$\tilde{h}^{\mathrm{Sp}(\infty)}(x,y) := \frac{x(1-xy)}{(1-x)^3(1-y)^7},$$

(5.1) is satisfied. It follows that

$$\sum_{k=0}^\infty \dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)} x^k = f^{\mathrm{Sp}(\infty)}(-x) - \tilde{h}^{\mathrm{Sp}(\infty)}(-x,x) = \frac{x^4 - 3x^3 + 6x^2 - 3x + 1}{(1-x)^7(1+x)^3}.$$

Similarly, summing (5.3) over all s, we obtain

$$h^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}(x,y)=\frac{A}{B},$$

where

$$\begin{aligned} A &:= -2y^2+y-1-x(3y-1)(y+1)(y^2-y+1)+x^2(y^4-5y^3-2)\\ &\quad -x^3y(2y^4+5y-1)+x^4y(y-3)(1+y)(y^2-y+1)-x^5y^3(y^2-y+2),\\ B &:= (-1+y)^2(-1+xy^3)(-1+x)^2(-1+xy)^3(xy+1)(1+y^2)(1+x^2)(-1+x^3y). \end{aligned}$$

Now one can check that

$$\tilde{h}^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}(x,y):=\frac{\tilde{A}(x,y)}{\tilde{B}(x,y)}$$

with

$$\begin{aligned} \tilde{A}(x,y) &:= -x(1+3y^2+3y^3+y^5)+x^2(3y^6-3y+y^2-y^4-y^5+1)\\ &\quad +x^3(-4y^2+2y-2y^6+2y^7-2+4y^5)\\ &\quad +x^4(-y^5+y^2-3y+3y^6+y^3-y^7)+x^5(y^7+y^2+3y^5+3y^4),\\ \tilde{B}(x,y) &:= (-1+y)^6(-1+x^2y)(-1+x)^2(1+x^2)(1+y)^2(1+y^2)(y^2+y+1) \end{aligned}$$

satisfies (5.1).

It follows that

$$\begin{aligned} \sum_{k=0}^{\infty}\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}x^k &= f^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}(-x)-\tilde{h}^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}(-x,x)\\ &= \frac{x^6-2x^5+2x^4+2x^2-2x+1}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^6}. \end{aligned}$$

Finally, setting $s=0$ in (5.3) we obtain

$$h^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)}(x,y)=\frac{A}{B},$$

where

$$\begin{aligned} A &:= (-1+x^2y^2-3x^3y^3-x^4y^4+x^6y^6-x-x^2-xy-y-y^2+x^6y^4\\ &\quad +x^3+y^3+2x^3y-x^3y^5+2xy^3-xy^5-2x^2y^4-x^5y^3+x^4y^6-y^4x^3\\ &\quad -x^4y^3-2x^4y^2-x^5y+2x^3y^2+2x^2y+2y^2x+x^4y+2x^2y^3+y^4x)(x-y),\\ B &:= (-1+xy^3)(-1+x^3y)(-1+x^2y^2)(y^3-y^2+y-1)(x^3-x^2+x-1)(-1+xy)^2. \end{aligned}$$

Now one can check that

$$\tilde{h}^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)}(x,y):=\frac{\tilde{A}(x,y)}{\tilde{B}(x,y)}$$

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with

$$\begin{aligned}\tilde{A}(x,y) := {} & x(1+y)(y^5+y^4+2y^3+2y^2+1)+x^2(2y^5+y^3-1)\\ & +x^3(-y^7-y^5+y^4+y^3+y^2-1)+x^4(y+1)(2y^4+3y^2+y+1)\\ & -x^5y(-1+y)(1+y)(y^2+1)(y^2+y+1),\end{aligned}$$

$$\tilde{B}(x,y) := (-x^3+x^2-x+1)(-1+y^2)(-1+y)^2(-1+x^2y)(y^6+y^5+y^4-y^2-y-1)$$

satisfies (5.1).

It follows that

$$\begin{aligned}\sum_{k=0}^\infty \dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)} x^k &= f^{\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)}(-x) - \tilde{h}^{\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)}(-x,x)\\ &= \frac{1+x^3+2x^4+x^5}{(x^2+1)(x^2+x+1)(1+x)^2(1-x)^4}.\end{aligned}$$

□

Corollary 5.1. *The dimension of the space of k-homogeneous translation G-invariant continuous valuations is given by*

$$\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)} = \frac{(k+4)(k+2)(2k^4+24k^3+100k^2+168k+405+315(-1)^k)}{5760}, \tag{5.4}$$

$$\begin{aligned}\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot\mathrm{U}(1)} = {} & \frac{1}{8}(\mathrm{i}^k+(-\mathrm{i})^k)+\frac{4\sqrt{3}(-1)^{k+1}}{81}\sin\left(\frac{\pi k}{3}\right)\\ & +\frac{(15+5k)(-1)^k}{64}+\frac{33}{64}+\frac{421}{960}k+\frac{5}{24}k^2\\ & +\frac{7}{108}k^3+\frac{1}{96}k^4+\frac{1}{1440}k^5,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot\mathrm{Sp}(1)} = {} & \frac{3}{16}(\mathrm{i}^k+(-\mathrm{i})^k)+\frac{2\sqrt{3}(-1)^{k+1}}{27}\sin\left(\frac{\pi k}{3}\right)\\ & +\frac{(5+k)(-1)^k}{32}+\frac{15}{32}+\frac{37}{96}k+\frac{3}{16}k^2+\frac{5}{144}k^3.\end{aligned} \tag{5.6}$$

Proof. This follows from Theorem 1.2 by a standard technique. □

We list the first few values of these dimensions in Table 1.

6. A character formula for $\mathrm{Val}^{\mathrm{Sp}(n)}$

The aim of this section is to prove Theorem 1.3. Let us fix some notation first. Let V be a quaternionic vector space of dimension n, endowed with a quaternionic Hermitian form K. The group $\mathrm{Sp}(V,K)$ is then isomorphic to $\mathrm{Sp}(n)$. Next, we fix some unit vector $v_0 \in V$ and write

$$V = \mathbb{R}\cdot v \oplus U \oplus \tilde{V},$$

where $\tilde{V}$ is the quaternionic complement of $v_0 \cdot \mathbb{H}$ and U is three dimensional.

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Table 1. *Dimensions of spaces of invariant global valuations.*

k	$\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)}$	$\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot \mathrm{U}(1)}$	$\dim \mathrm{Val}_k^{\mathrm{Sp}(\infty)\cdot \mathrm{Sp}(1)}$
0	1	1	1
1	1	1	1
2	7	3	2
3	14	6	4
4	42	14	8
5	84	24	11
6	182	44	17
7	330	72	24
8	603	117	34
9	1001	177	44
10	1645	265	58

The stabilizer of $\mathrm{Sp}(n) \cong \mathrm{Sp}(V, K)$ at v_0 is $\mathrm{Sp}(n-1) \cong \mathrm{Sp}(\tilde{V}, K|_{\tilde{V}})$, it acts trivially on U. As $\mathrm{SU}(2)$-representation $U \simeq V_2$, the adjoint representation.

In the following, we will consider real and complex vector spaces. In order to distinguish between both cases, we will write $\Lambda_{\mathbb{R}}$ and $\Lambda_{\mathbb{C}}$ referring to the base field $\mathbb{R}$ and $\mathbb{C}$ respectively.

Lemma 6.1.

$$\dim(\Lambda_{\mathbb{C}}^{k_1} V \otimes \cdots \otimes \Lambda_{\mathbb{C}}^{k_s} V)^{\mathrm{Sp}_{2n}\mathbb{C}} = \sum_{\lambda} K_{\lambda\mu},$$

where $\mu = (k_1, \ldots, k_s)$ and the sum is over all even Young diagrams λ with $\lambda_1 \leqslant 2n$.

Proof. We can consider V as a representation of the larger group $\mathrm{GL}(2n, \mathbb{C})$. By (3.3), we have

$$\Lambda_{\mathbb{C}}^{k_1} V \otimes \cdots \otimes \Lambda_{\mathbb{C}}^{k_s} V = \bigoplus_{\tilde{\nu}} K_{\tilde{\nu}\mu} \Gamma_{\nu},$$

where ν ranges over all partitions of depth less than or equal to $2n$, $\tilde{\nu}$ is the conjugate partition of ν and $K_{\tilde{\nu}\mu}$ is the Kostka number.

Proposition 3.1 and the fact that $K_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$ thus imply that

$$\dim(\Lambda_{\mathbb{C}}^{k_1} V \otimes \cdots \otimes \Lambda_{\mathbb{C}}^{k_s} V)^{\mathrm{Sp}_{2n}\mathbb{C}} = \sum_{\substack{\tilde{\nu} \text{ even} \\ \text{depth } \nu \leqslant 2n}} K_{\tilde{\nu}\mu} = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2n}} K_{\lambda\mu}.$$

□

Lemma 6.2. *Let*

$$S_k := (\Lambda_{\mathbb{R}}^k V)^{\mathrm{Sp}(n)} \otimes \mathbb{C},$$

which is an $\mathrm{SU}(2)$*-representation. Then*

$$\sum_{k=0}^{4n} \mathrm{char}(S_k) x^k = E_n(x),$$

where $E_n(x)$ is the function defined in (1.10).

Proof. We have seen in § 5 that

$$(\Lambda^k_{\mathbb{R}} V)^{\mathrm{Sp}(n)} \otimes \mathbb{C} = \bigoplus_{k_1+k_2=k} (\Lambda^{k_1}_{\mathbb{C}} V \otimes \Lambda^{k_2}_{\mathbb{C}} V^*)^{\mathrm{Sp}_{2n}\mathbb{C}}.$$

Elements of $(\Lambda^{k_1}_{\mathbb{C}} V \otimes \Lambda^{k_2}_{\mathbb{C}} V^*)^{\mathrm{Sp}_{2n}\mathbb{C}}$ are of weight $k_1 - k_2$, hence

$$\begin{aligned}
\sum_{k=0}^{4n} \operatorname{char}(S_k) x^k &= \sum_{k=0}^{4n} \operatorname{char}((\Lambda^k_{\mathbb{R}} V)^{\mathrm{Sp}(n)} \otimes \mathbb{C}) x^k \\
&= \sum_{k=0}^{4n} \sum_{k_1+k_2=k} \dim(\Lambda^{k_1}_{\mathbb{C}} V \otimes \Lambda^{k_2}_{\mathbb{C}} V^*)^{\mathrm{Sp}_{2n}\mathbb{C}} t^{k_1-k_2} x^k \\
&= \sum_{\lambda} \sum_{k_1,k_2} K_{\lambda,(k_1,k_2)} (xt)^{k_1} (xt^{-1})^{k_2} \\
&= \sum_{\lambda} s_\lambda(xt, xt^{-1}) \\
&= E_n(x),
\end{aligned}$$

where λ in the above sums ranges over all even Young diagrams of depth less than or equal to 2 with $\lambda_1 \leqslant 2n$. □

Note that $S_k = 0$ if k is odd and that $S_k = S_{4n-k}$.

Lemma 6.3. *Set*

$$R_{k,l} := (\Lambda^{k,l}_{\mathbb{R}} (\tilde{V} \oplus \tilde{V}))^{\mathrm{Sp}(n-1)} \otimes \mathbb{C},$$

which is an SU(2)*-representation. Then for each* m

$$\sum_{k=0}^{2m} \operatorname{char} R_{k,2m-k} x^k = F_{n-1,m}(x).$$

Proof.

$$\begin{aligned}
&\sum_{k=0}^{2m} \operatorname{char}(R_{k,2m-k}) x^k \\
&\quad = \sum_{k=0}^{2m} \sum_{\substack{k_1+k_2=k \\ l_1+l_2=2m-k}} \dim(\Lambda^{k_1}_{\mathbb{C}} \tilde{V} \otimes \Lambda^{k_2}_{\mathbb{C}} \tilde{V}^* \otimes \Lambda^{l_1}_{\mathbb{C}} \tilde{V} \otimes \Lambda^{l_2}_{\mathbb{C}} \tilde{V}^*)^{\mathrm{Sp}_{2n-2}\mathbb{C}} x^k t^{k_1-k_2+l_1-l_2} \\
&\quad = \sum_{\lambda} \sum_{k} \sum_{\substack{k_1+k_2=k \\ l_1+l_2=2m-k}} K_{\lambda,(k_1,k_2,l_1,l_2)} x^k t^{k_1-k_2+l_1-l_2} \\
&\quad = \sum_{\lambda} \sum_{\substack{k_1,k_2,l_1,l_2 \\ k_1+k_2+l_1+l_2=2m}} K_{\lambda,(k_1,k_2,l_1,l_2)} (xt)^{k_1} (xt^{-1})^{k_2} t^{l_1-l_2} \\
&\quad = \sum_{\lambda} s_\lambda(tx, t^{-1}x, t, t^{-1}) \\
&\quad = F_{n-1,m}(x).
\end{aligned}$$

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Here λ ranges over all even Young diagrams of depth less than or equal to 4, weight $2m$ with $\lambda_1 \leqslant 2n-2$. $\square$

Note that we have isomorphisms of SU(2)-representations $R_{k,l} = R_{l,k} = R_{k,4n-4-l}$ and that $R_{k,l} = 0$ if $k+l$ is odd.

Proposition 6.4. *In $R\,\mathrm{SU}(2)$, we have the following equation*

$$\begin{aligned}\mathrm{Val}_k^G = S_k - R_{k,k-2} - R_{k,k-4} + R_{k-1,k-1} + (-V_4+V_2)R_{k-1,k-3} \\ + (V_4 - V_2 + V_0)R_{k-2,k-2} - R_{k-2,k-4} + R_{k-3,k-3}.\end{aligned} \tag{6.1}$$

Proof. By Proposition 2.6,

$$\mathrm{Val}_k^{\mathrm{Sp}(n)} = (-1)^k(\Lambda_{\mathbb{R}}^k V)^{\mathrm{Sp}(n)} \otimes \mathbb{C} + \sum_{l=0}^{4n-k-1} (-1)^{k+l+1} \Omega_p^{k,l}(SV)^{\mathrm{Sp}(n)}. \tag{6.2}$$

By definition,

$$\begin{aligned}\Omega_p^{k,l}(SV)^{\mathrm{Sp}(n)} &= \Omega_h^{k,l}(SV)^{\mathrm{Sp}(n)} - \Omega_h^{k-1,l-1}(SV)^{\mathrm{Sp}(n)} \\ &= [\Lambda_{\mathbb{R}}^k(U \oplus \tilde{V})^* \otimes \Lambda_{\mathbb{R}}^l(U \oplus \tilde{V})^* \otimes \mathbb{C}]^{\mathrm{Sp}(n-1)} \\ &\quad - [\Lambda_{\mathbb{R}}^{k-1}(U \oplus \tilde{V})^* \otimes \Lambda_{\mathbb{R}}^{l-1}(U \oplus \tilde{V})^* \otimes \mathbb{C}]^{\mathrm{Sp}(n-1)}.\end{aligned}$$

Note that for all j

$$\Lambda_{\mathbb{R}}^j(U \oplus \tilde{V}) = \Lambda_{\mathbb{R}}^j \tilde{V} \oplus V_2 \otimes \Lambda_{\mathbb{R}}^{j-1}\tilde{V} \oplus V_2 \otimes \Lambda_{\mathbb{R}}^{j-2}\tilde{V} \oplus \Lambda_{\mathbb{R}}^{j-3}\tilde{V}.$$

In the alternating sum (6.2), most of the terms cancel out and we get, for any given k, Equation (6.1). $\square$

Proof of Theorem 1.3. Using Proposition 6.4 and Lemmas 6.2 and 6.3 we obtain

$$\begin{aligned}&\sum_{k=0}^{4n} \mathrm{char}(\mathrm{Val}_k^G)x^k \\ &\quad= \sum_{k=0}^{4n} \mathrm{char}(S_k)x^k - \sum_{k=0}^{4n} \mathrm{char}(R_{k,4n-k-2})x^k - \sum_{k=0}^{4n} \mathrm{char}(R_{k,4n-k})x^k \\ &\qquad + \sum_{k=0}^{4n} \mathrm{char}(R_{k-1,4n-k-3})x^k + \sum_{k=0}^{4n} \mathrm{char}(-V_4+V_2)\,\mathrm{char}(R_{k-1,4n-k-1})x^k \\ &\qquad + \sum_{k=0}^{4n} \mathrm{char}(V_4 - V_2 + V_0)\,\mathrm{char}(R_{k-2,4n-k-2})x^k - \sum_{k=0}^{4n} \mathrm{char}(R_{k-2,4n-k}) \\ &\qquad + \sum_{k=0}^{4n} \mathrm{char}(R_{k-3,4n-k-1})x^k \\ &\quad= E_n(x) - F_{n-1,2n}(x) - (1 + x(t^4 + t^{-4}) + x^2)F_{n-1,2n-1}(x) \\ &\qquad + x(1 + x(t^4 + 1 + t^{-4}) + x^2)F_{n-1,2n-2}(x).\end{aligned}$$

$\square$

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As an example, we work out the cases $n = 1$ and $n = 2$.

Case $n = 1$

The only Young diagram with $\lambda_1 \leqslant 2n - 2 = 0$ is $\lambda = (0, \dots, 0)$ and it has weight 0. Hence $F_{0,m} = 0$ for $m \neq 0$ and $F_{0,0} = 1$. The polynomial E_1 equals

$$E_1(x) = x^4 + x^2(t^2 + 1 + t^{-2}) + 1.$$

From our formula it follows that

$$\begin{aligned}\sum_{k=0}^{4} \operatorname{char}(\mathrm{Val}_k^{\mathrm{Sp}(1)}) x^k &= x^4 + x^2(t^2 + 1 + t^{-2}) + 1 + x(1 + x(t^4 + 1 + t^{-4}) + x^2) \\ &= x^4 + x^3 + x^2(t^4 + t^2 + 2 + t^{-2} + t^{-4}) + x + 1,\end{aligned}$$

hence

$$\begin{aligned}\mathrm{Val}_k^{\mathrm{Sp}(1)} &= V_0 \quad \text{for } k = 0, 1, 3, 4, \\ \mathrm{Val}_2^{\mathrm{Sp}(1)} &= V_4 + V_0.\end{aligned}$$

These decompositions have been obtained in [**11**] (note that $\mathrm{Sp}(1) = \mathrm{SU}(2)$).

Case $n = 2$

The polynomials E_2 and $F_{1,m}, m = 2, 3, 4$ are given as follows:

$$\begin{aligned}E_2 &= x^8 + x^6(t^2 + 1 + t^{-2}) + x^4(t^4 + t^2 + 1 + t^{-2} + t^4) + x^2(t^2 + 1 + t^{-2}) + 1, \\ F_{1,2} &= x^4 + x^3(t^2 + 2 + t^{-2}) + x^2(t^4 + 2t^2 + 4 + 2t^{-2} + t^{-4}) + x(t^2 + 2 + t^{-2}) + 1, \\ F_{1,3} &= x^4(t^{-2} + 1 + t^2) + x^3(t^{-2} + 2 + t^2) + x^2(t^{-2} + 1 + t^2), \\ F_{1,4} &= x^4.\end{aligned}$$

Putting these values in the formula from Theorem 1.3, we obtain

$$\begin{aligned}\sum_{k=0}^{8} \operatorname{char}(\mathrm{Val}_k^{\mathrm{Sp}(2)}) x^k = x^8 + x^7 &+ x^6(t^4 + t^2 + 3 + t^{-2} + t^{-4}) \\ &+ x^5(2t^4 + 2t^2 + 5 + 2t^{-2} + 2t^{-4}) \\ &+ x^4(t^8 + t^6 + 4t^4 + 4t^2 + 9 + 4t^{-2} + 4t^{-4} + t^{-6} + t^{-8}) \\ &+ x^3(2t^4 + 2t^2 + 5 + 2t^{-2} + 2t^{-4}) \\ &+ x^2(t^4 + t^2 + 3 + t^{-2} + t^{-4}) + x + 1,\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{Val}_0^{\mathrm{Sp}(2)} &= \mathrm{Val}_1^{\mathrm{Sp}(2)} = \mathrm{Val}_7^{\mathrm{Sp}(2)} = \mathrm{Val}_8^{\mathrm{Sp}(2)} = V_0, \\ \mathrm{Val}_2^{\mathrm{Sp}(2)} &= \mathrm{Val}_6^{\mathrm{Sp}(2)} = V_4 + 2V_0, \\ \mathrm{Val}_3^{\mathrm{Sp}(2)} &= \mathrm{Val}_5^{\mathrm{Sp}(2)} = 2V_4 + 3V_0, \\ \mathrm{Val}_4^{\mathrm{Sp}(2)} &= V_8 + 3V_4 + 5V_0.\end{aligned}$$

Table 2. *Dimensions of spaces of invariant valuations.*

n	$\dim \mathrm{Val}_k^{\mathrm{Sp}(n)}$, $k = 0, \ldots, 4n$
1	1,1,6,1,1
2	1,1,7,13,29,13,7,1,1
3	1,1,7,14,41,71,111,71,41,14,7,1,1
4	1,1,7,14,42,83,169,259,344,259,169,83,42,14,7,1,1
5	1,1,7,14,42,84,181,317,532,742,903,742,532,317,181,84,42,14,7,1,1

n	$\dim \mathrm{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{U}(1)}$, $k = 0, \ldots, 4n$
1	1,1,2,1,1
2	1,1,3,5,9,5,3,1,1
3	1,1,3,6,13,19,25,19,13,6,3,1,1
4	1,1,3,6,14,23,39,53,64,53,39,23,14,6,3,1,1
5	1,1,3,6,14,24,43,67,98,124,141,124,98,67,43,24,14,6,3,1,1

n	$\dim \mathrm{Val}_k^{\mathrm{Sp}(n)\cdot \mathrm{Sp}(1)}$, $k = 0, \ldots, 4n$
1	1,1,1,1,1
2	1,1,2,3,5,3,2,1,1
3	1,1,2,4,7,8,9,8,7,4,2,1,1
4	1,1,2,4,8,10,14,16,18,16,14,10,8,4,2,1,1
5	1,1,2,4,8,11,16,21,26,28,30,28,26,21,16,11,8,4,2,1,1

Similar computations yield the decomposition of each $\mathrm{Val}_k^{\mathrm{Sp}(n)}$, from which we may compute the dimensions in Table 2.

Note that, for $k \leqslant n$, these numbers are consistent with those in Table 1 and Theorem 1.1.

Appendix A. Proof of (5.3)

Let $R := R\,\mathrm{SU}(2)$ and define

$$T_s := \sum_{k=0}^{\infty} V_0 (xy)^k \in R[\![x, y]\!],$$

$$T_0 := \sum_{k=0}^{\infty} \mathrm{Sym}^k V_2 y^{2k} \in R[\![y]\!],$$

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$$T_1 := \sum_{k=0}^{\infty} \operatorname{Sym}^k V_2 (xy)^k \in R[\![x,y]\!],$$

$$T_2 := \sum_{k=0}^{\infty} \operatorname{Sym}^k V_2 x^{2k} \in R[\![x]\!],$$

$$B := V_0 + V_2 x + V_2 x^2 + V_0 x^3 \in R[\![x]\!],$$

$$G := V_0 + V_2 y + V_2 y^2 + V_0 y^3 \in R[\![y]\!].$$

Then (5.2) can be rewritten as

$$\sum_{k,l} \Omega_h^{k,l}(S\mathbb{H}^\infty)^{\overline{\mathrm{Sp}(\infty)}} x^k y^l \cong B \cdot G \cdot T_s \cdot T_0 \cdot T_1 \cdot T_2. \tag{A.1}$$

Set

$$p := \sum_{k=0}^{\infty} V_{2k} x^{2k} \in R[\![x]\!],$$

$$q := \sum_{k=0}^{\infty} V_{2k} y^{2k} \in R[\![y]\!],$$

$$r := \sum_{k=0}^{\infty} V_{2k} (xy)^k \in R[\![x,y]\!].$$

Lemma A.1.

$$T_0 = \frac{q}{1-y^4}, \tag{A.2}$$

$$T_1 = \frac{r}{1-x^2y^2}, \tag{A.3}$$

$$T_2 = \frac{p}{1-x^4}, \tag{A.4}$$

$$T_s = \frac{V_0}{1-xy}, \tag{A.5}$$

$$(x^2-y^2) p \otimes q = \frac{x^2(1+y^2)p - y^2(1+x^2)q}{1-x^2y^2}, \tag{A.6}$$

$$(x-y) p \otimes r = \frac{x(1+xy)p - y(1+x^2)r}{1-x^3y}, \tag{A.7}$$

$$(x-y) r \otimes q = \frac{x(1+y^2)r - y(1+xy)q}{1-xy^3}, \tag{A.8}$$

$$xB \otimes p = (1+x)^2(1+x^3)p - (1+x)(1+x^2)V_0, \tag{A.9}$$

$$yG \otimes q = (1+y)^2(1+y^3)q - (1+y)(1+y^2)V_0. \tag{A.10}$$

Proof. We have

$$\operatorname{Sym}^k V_2 = \sum_{s=0}^{\lfloor k/2 \rfloor} V_{2k-4s}$$

and hence

$$\begin{aligned}
T_0 &= \sum_{k=0}^{\infty} \operatorname{Sym}^k V_2 y^{2k} \\
&= \sum_{k=0}^{\infty} \sum_{s=0}^{\lfloor k/2 \rfloor} V_{2k-4s} y^{2k} \\
&= \sum_{s=0}^{\infty} y^{4s} \sum_{k=2s}^{\infty} V_{2k-4s} y^{2k-4s} \\
&= \frac{1}{1-y^4} \sum_{k=0}^{\infty} V_{2k} y^{2k} \\
&= \frac{q}{1-y^4}.
\end{aligned}$$

This shows (A.2); the proofs of (A.3) and (A.4) are similar. Equation (A.5) is trivial. Next, for (A.6) we compute

$$\begin{aligned}
(x^2-y^2) p \otimes q &= (x^2-y^2) \sum_{k,l=0}^{\infty} V_{2k} \otimes V_{2l} x^{2k} y^{2l} \\
&= (x^2-y^2) \sum_{k,l=0}^{\infty} \sum_{s=|k-l|}^{k+l} V_{2s} x^{2k} y^{2l} \\
&= (x^2-y^2) \sum_{s=0}^{\infty} V_{2s} \sum_{k,l:\ |k-l| \leqslant s \leqslant k+l} x^{2k} y^{2l} \\
&= (x^2-y^2) \sum_{s=0}^{\infty} V_{2s} \sum_{k=0}^{\infty} x^{2k} \sum_{l=|s-k|}^{k+s} y^{2l} \\
&= (x^2-y^2) \sum_{s=0}^{\infty} V_{2s} \sum_{k=0}^{\infty} x^{2k} \frac{y^{2|k-s|} - y^{2(k+s+1)}}{1-y^2} \\
&= \frac{x^2-y^2}{1-y^2} \sum_{s=0}^{\infty} V_{2s} \left(\frac{y^{2s+2} - x^{2s+2}}{y^2 - x^2} + \frac{x^{2s+2} y^2}{1 - x^2 y^2} - \frac{y^{2s+2}}{1 - x^2 y^2} \right) \\
&= \frac{x^2(y^2+1)}{(1-x^2y^2)} p - \frac{y^2(1+x^2)}{(1-x^2y^2)} q.
\end{aligned}$$

Equations (A.7) and (A.8) are proved in a similar way.

Finally, Equations (A.9) and (A.10) follow easily from the Clebsch–Gordan rule. □

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From the lemma, we deduce that

$$\begin{aligned} B\cdot G\cdot T_s\cdot T_0\cdot T_1\cdot T_2 &= c\frac{(1+x)(1+x^3)(1+y^2)(x^2+y)(x^2y+1)}{(1-xy)(x^2-y^2)(1-x^3y)(x-y)}p \\ &\quad - c\frac{(1+y^2)(1+x^2)(1+x)(1+y)(x^2y+1)(y^2x+1)}{(-1+xy^3)(y-x)^2(-1+x^3y)}r \\ &\quad + c\frac{(1+y)(1+y^3)(1+x^2)(y^2+x)(y^2x+1)}{(1-xy)(x^2-y^2)(1-xy^3)(x-y)}q, \end{aligned}$$

where

$$c := \frac{(1+x)(1+y)}{(1-xy)(1-x^4)(1-y^4)(1-x^2y^2)}.$$

Comparing the coefficient of V_{2s} on both sides, taking into account (A.1), yields (5.3).

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