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EXISTENCE OF K-MULTIMAGIC SQUARES AND MAGIC SQUARES OF kth POWERS WITH DISTINCT ENTRIES

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Abstract

We demonstrate the existence of K-multimagic squares of order N consisting of N^2 distinct integers whenever $N > 2K(K+1)$. This improves our earlier result [D. Flores, 'A circle method approach to K-multimagic squares', preprint (2024), arXiv:2406.08161] in which we only required $N+1$ distinct integers. Additionally, we present a direct method by which our analysis of the magic square system may be used to show the existence of $N \times N$ magic squares consisting of distinct kth powers when

$$N > \begin{cases} 2^{k+1} & \text{if } 2 \leqslant k \leqslant 4, \\ 2\lceil k(\log k + 4.20032)\rceil & \text{if } k \geqslant 5, \end{cases}$$

improving on a recent result by Rome and Yamagishi ['On the existence of magic squares of powers', preprint (2024), arxiv:2406.09364].

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1. Introduction

An $N \times N$ matrix $\mathbf{Z} = (z_{i,j})_{1 \leqslant i,j \leqslant N}$ is a *magic square* of order N if the sums of the entries in each of its rows, columns and two main diagonals are equal. Given $K \geqslant 2$, we say a matrix $\mathbf{Z} \in \mathbb{Z}^{N \times N}$ is a K-multimagic square of order N, or an $\mathbf{MMS}(K, N)$ for short, if the matrices

$$\mathbf{Z}^{\circ k} := (z_{i,j}^k)_{1 \leqslant i,j \leqslant N},$$

remain magic squares for $1 \leqslant k \leqslant K$. Here, we expand on our previous investigation [5], where we saw that given any $K \geqslant 2$ and $N \in \mathbb{N}$, there exists many trivial examples of $\mathbf{MMS}(K, N)$ using at most N distinct integers. Given K, we previously focused on the problem of finding a lower bound for N such that there exists an $\mathbf{MMS}(K, N)$ using

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TABLE 1. Best known results for K-multimagic squares.

K	Smallest N for which an $\mathbf{MMS}(K,N)$ with distinct entries is known to exist	Attributed to
2	6	J. Wroblewski [1]
3	12	W. Trump [9]
4	243	P. Fengchu [1]
5	729	L. Wen [1]
6	4096	P. Fengchu [1]
$K \geqslant 2$	$(4K-2)^K$	Zhang *et al.* [11]

$N+1$ or more digits. Via the circle method, we proved in [5] that $N > 2K(K+1)$ is a suitable lower bound on N for this problem.

However, this may not be satisfactory for those familiar with magic squares as the typical parlance usually refers to magic squares with any repeated entries as *trivial*. A more satisfactory result would be to determine a lower bound on N in terms of K for which there exists an $\mathbf{MMS}(K,N)$ with distinct entries.

This question has been considered in the past by several authors (see [1, 4, 9–11]) via constructive methods. We give a brief overview of the best known results in Table 1. The curious reader is encouraged to read the introductions of both [5, 7] for more information on the history of magic squares.

Although the circle method tells us there exists an $\mathbf{MMS}(K,N)$ with at least N distinct entries when $N > 2K(K+1)$, it could be the case that to establish the existence of an $\mathbf{MMS}(K,N)$ with all distinct entries, we may require N to be even larger relative to $2K(K+1)$. This difficulty may be seen in the recent work by Rome and Yamagishi [7], where they tackle the simpler problem of showing the existence of magic squares of distinct kth powers. Via the techniques in [7], one may obtain an asymptotic formula for the cardinality of the subset of $N \times N$ magic squares of kth powers (with potential repeats) as soon as

$$N \geqslant \begin{cases} 2^{k+2} + \Delta & \text{if } 2 \leqslant k \leqslant 4, \\ 4\lceil k(\log k + 4.20032)\rceil + \Delta & \text{if } 5 \leqslant k, \end{cases}$$

with $\Delta = 12$. However, Rome and Yamagishi end up requiring $\Delta = 20$ to ensure that all entries of these magic squares are *distinct* kth powers. To understand why this increase in N is required in [7], we first need to establish the notion of a *partitionable matrix*.

DEFINITION 1.1. We say a matrix $C = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{rn}]$ of dimensions $r \times rn$ is partitionable if there exist disjoint sets $J_l \subset \{1, 2, \ldots, rn\}$ of size r for $1 \leqslant j \leqslant n$ satisfying

$$\operatorname{rank}(C_{J_l}) = r \quad \text{for } 1 \leqslant l \leqslant n,$$

where C_J denotes the submatrix of C consisting of columns indexed by J.

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Upon examination of the methods used in [7], one sees that N is required to be slightly larger because of the difficulty of finding a large enough partitionable submatrix for the family of coefficient matrices associated with magic squares with particular repeated entries.

In [5], we define the notion of a matrix *dominating* a function as follows.

DEFINITION 1.2. We say that a matrix $C \in \mathbb{C}^{r\times s}$ dominates a function $f : \mathbb{N} \to \mathbb{R}^+$ if for all $J \subset \{1, \ldots, s\}$,

$$\operatorname{rank}(C_J) \geqslant \min\{f(|J|), r\},$$

where $C_J = [\mathbf{c}_j]_{j\in J}$.

Then, by [6, Lemma 1], if a matrix C dominates a certain function, we obtain information regarding its partitionable submatrices. This allows us to circumvent several difficulties encountered in [7] and prove the following result.

THEOREM 1.3. *Given $K \geqslant 2$, there exist infinitely many* **MMS**(K, N) *consisting of N^2 distinct integers as soon as $N > 2K(K + 1)$.*

It is important to note here that our lower bound on N remains unchanged. Additionally, just as in [5], one may easily obtain the following result via the Green–Tao theorem.

COROLLARY 1.4. *Given $K \geqslant 2$, there exist infinitely many* **MMS**(K, N) *consisting of N^2 distinct prime numbers as soon as $N > 2K(K + 1)$.*

Finally, we present an analogous argument for finding magic squares of distinct kth powers. This approach uses the notion of our matrix of coefficients dominating a particular function, allowing us to establish the following result.

THEOREM 1.5. *Given $k \geqslant 2$, there exist infinitely many $N \times N$ magic squares of distinct kth powers as soon as*

$$N > \begin{cases} 2^{k+1} & \text{if } 2 \leqslant k \leqslant 4, \\ 2\lceil k(\log k + 4.20032)\rceil & \text{if } k \geqslant 5. \end{cases}$$

This improves the recent result of Rome and Yamagishi [7]. It is worth noting, just as Rome and Yamagishi did in [8], that Theorem 1.5 is not entirely optimal for $4 \leqslant k \leqslant 20$, we simply choose this representation of our theorem for convenience (see the introduction of [7] for more detail).

REMARK 1.6. The application of the circle method to this problem has been part of the mathematical folklore for at least 30 years, with discussions dating back to the early 90s in talks by Bremner (see [2, 3]). The recent breakthrough in this area lies in achieving a refined understanding of the coefficient matrix associated with the magic square system. Viewing a matrix as dominating a function appears to be the appropriate perspective, as it provides insight into the partitionability of submatrices.

2. Finding solutions of additive systems with distinct entries

Our basic parameter, P, is always assumed to be a large positive integer. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for every $\varepsilon > 0$. Implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε, r, s, k, K and the elements of the matrix C.

Let $C = [\mathbf{c}_1, \dots, \mathbf{c}_s] = (c_{i,j})_{\substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant s}} \in \mathbb{Z}^{r\times s}$ be given, and consider the diagonal system

$$\sum_{1\leqslant j\leqslant s} c_{i,j}x_j^k = 0, \quad 1 \leqslant i \leqslant r. \tag{2.1}$$

We define $S_k(P; C)$ to be the set of solutions $\mathbf{x} \in \mathbb{Z}^s$ to (2.1), where $\max_j |x_j| \leqslant P$. Given i, j with $1 \leqslant i < j \leqslant s$, it is not difficult to see that

$$\#\{\mathbf{x} \in S_k(P; C) : x_i = x_j\} = \#S_k(P; C^{(i,j)}),$$

where $C^{(i,j)}$ is the matrix obtained by substituting the ith column of C with $\mathbf{c}_i + \mathbf{c}_j$ and deleting the jth column of C.

Thus, if $S_k^*(P; C)$ denotes the subset of $S_k(P; C)$ with distinct entries,

$$\#\bigcap_{1\leqslant k\leqslant K} S_k^*(P; C) = \#\bigcap_{1\leqslant k\leqslant K} S_k(P; C) + O\Big(\sum_{1\leqslant i<j\leqslant s} \#\bigcap_{1\leqslant k\leqslant K} S_k(P; C^{(i,j)})\Big). \tag{2.2}$$

Separately, one may deduce from [5, Lemma 3.4] and [6, Lemma 1] the following result.

LEMMA 2.1. *Let $K \geqslant 2$ and $C \in \mathbb{Z}^{r\times s}$ with $s \geqslant rK(K+1)$. If C contains a partitionable submatrix of size $r \times rK(K+1)$, then one has the bound*

$$\#\bigcap_{1\leqslant k\leqslant K} S_k(P; C) \ll P^{s-rK(K+1)/2+\varepsilon}$$

for any $\varepsilon > 0$.

If C dominates the function

$$\frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/2\rfloor},$$

then, for $1 \leqslant i < j \leqslant s$, the matrix $C^{(i,j)}$ contains a submatrix of C of size $r \times (s-2)$. By [6, Lemma 1], this matrix contains a partitionable submatrix of size $r \times r\lfloor (s-2)/r\rfloor$. Combining this with (2.2), Lemma 2.1 and [5, Theorem 2.2], we deduce the following general result.

LEMMA 2.2. *Let $K \geqslant 2$ and suppose that $C \in \mathbb{Z}^{r\times s}$ satisfies $s \geqslant rK(K+1) + 2$. If C dominates the function*

$$F(x) = \max\left\{\frac{x - r\{s/r\}}{\lfloor s/r\rfloor}, \frac{x - r\{(s-1)/r\}}{\lfloor (s-1)/r\rfloor}, \frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/r\rfloor}\right\},$$

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then

$$\# \bigcap_{1\leqslant k\leqslant K} S_k^*(P;C) = P^{s-rK(K+1)/2}(\sigma_K(C)+o(1)),$$

where $\sigma_K(C)\geqslant 0$ *is a real number depending only on* K *and* C*. Additionally,* $\sigma_K(C)>0$ *if there exist nonsingular real and p-adic simultaneous solutions to the system (2.1) for* $1\leqslant k\leqslant K$*.*

3. K-multimagic squares with distinct entries

A matrix $\mathbf{Z}=(z_{i,j})_{1\leqslant i,j\leqslant N}$ is an $\mathbf{MMS}(K,N)$ if and only if, for all k with $1\leqslant k\leqslant K$, it satisfies the simultaneous conditions

$$\sum_{1\leqslant i\leqslant N} z_{i,j}^k = \sum_{1\leqslant i\leqslant N} z_{i,i}^k \quad \text{for } 1\leqslant j\leqslant N, \tag{3.1}$$

$$\sum_{1\leqslant j\leqslant N} z_{i,j}^k = \sum_{1\leqslant j\leqslant N} z_{j,N-j+1}^k \quad \text{for } 1\leqslant i\leqslant N. \tag{3.2}$$

One may wonder if these equations are equivalent to those of an $\mathbf{MMS}(K,N)$. Indeed, at a first glance, it does not seem clear that the main diagonal and anti-diagonal are equal. One can show that this is implied by simply summing over all j in (3.1) and noting that this is equal to summing over all i in (3.2). Upon dividing out a factor of N, one deduces that (3.1) and (3.2) imply

$$\sum_{1\leqslant i\leqslant N} z_{i,i}^k = \sum_{1\leqslant j\leqslant N} z_{j,N-j+1}^k.$$

Before we construct a matrix corresponding to this system, we must first establish some notational shorthand. Let $\mathbf{1}_n$ denote an n-dimensional vector of all ones and $\mathbf{0}_n$ an n-dimensional vector of all zeros. Let $\mathbf{e}_n(m)$ denote the mth standard basis vector of dimension n. For a fixed N, we define

$$D_1(N)=\{(i,j)\in([1,N]\cap\mathbb{Z})^2 : i=j\}$$

and

$$D_2(N)=\{(i,j)\in([1,N]\cap\mathbb{Z})^2 : i+j=N+1\}.$$

For each $(i,j)\in([1,N]\cap\mathbb{Z})^2$, we define the $2N$-dimensional vectors

$$\mathbf{d}_{i,j}=\begin{cases}(\mathbf{e}_N(i)-\mathbf{1_N},\mathbf{e}_N(j)-\mathbf{1_N}) & \text{for } (i,j)\in D_1(N)\cap D_2(N),\\ (\mathbf{e}_N(i)-\mathbf{1_N},\mathbf{e}_N(j)) & \text{for } (i,j)\in D_1(N)\backslash D_2(N),\\ (\mathbf{e}_N(i),\mathbf{e}_N(j)-\mathbf{1_N}) & \text{for } (i,j)\in D_2(N)\backslash D_1(N),\\ (\mathbf{e}_N(i),\mathbf{e}_N(j)) & \text{otherwise.}\end{cases}$$

Let $\phi:[1,N^2]\cap\mathbb{Z}\to([1,N]\cap\mathbb{Z})^2$ be any fixed bijection. Then, the $2N\times N^2$ matrix,

$$C_N^{\text{magic}}=C_N^{\text{magic}}(\phi)=[\mathbf{d}_{\phi(1)},\ldots,\mathbf{d}_{\phi(N^2)}],$$

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corresponds to the system defined by (3.1) and (3.2) up to some arbitrary relabelling of variables defined by the bijection ϕ. We now establish that one may apply Lemma 2.2 with $C = C_N^{\text{magic}}$.

LEMMA 3.1. *For $N \geqslant 4$, the matrix C_N^{magic} dominates the function*

$$F(x) = \max\left\{\frac{x - r\{s/r\}}{\lfloor s/r\rfloor}, \frac{x - r\{(s-1)/r\}}{\lfloor (s-1)/r\rfloor}, \frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/r\rfloor}\right\},$$

with $s = N^2$ and $r = 2N$.

PROOF. We show in [5, Section 4] that C_N^{magic} satisfies the rank condition

$$\operatorname{rank}((C_N^{\text{magic}})_J) \geqslant \begin{cases} \lceil 2\sqrt{|J|}\rceil - 1 & \text{if } 1 \leqslant |J| \leqslant N(N-1)-1, \\ |J| - N^2 + 3N - 1 & \text{if } N(N-1)-1 \leqslant |J| \leqslant N(N-1)+1, \\ 2N & \text{if } N(N-1)+1 \leqslant |J| \leqslant N^2, \end{cases}$$

when $|J| \neq (N-1)^2 + 1$ and

$$\operatorname{rank}((C_N^{\text{magic}})_J) \geqslant 2N - 3,$$

when $|J| = (N-1)^2 + 1$. Given this, if $|J| > N(N-1)$, trivially,

$$\min\{F(|J|), 2N\} = 2N \leqslant \operatorname{rank}((C_N^{\text{magic}})_J).$$

Let us now focus on the case in which N is even and $|J| \leqslant N(N-1)$. When N is even, one may show that

$$F(x) = \begin{cases} \dfrac{2x}{N} & \text{if } 0 \leqslant x \leqslant N(N-1), \\ \dfrac{2x-4}{N-2} - 4 & \text{if } N(N-1) \leqslant x. \end{cases}$$

Thus, if $N(N-1) - 1 \leqslant |J| \leqslant N(N-1)$,

$$\min\{F(|J|), 2N\} = \frac{2|J|}{N} \leqslant |J| - N^2 + 3N - 1 \leqslant \operatorname{rank}((C_N^{\text{magic}})_J).$$

Let us now suppose that $1 \leqslant |J| < N(N-1) - 1$ and $|J| \neq (N-1)^2 + 1$. Similarly, in this range,

$$\min\{F(|J|), 2N\} = \frac{2|J|}{N} \leqslant \lceil 2\sqrt{|J|}\,\rceil - 1 \leqslant \operatorname{rank}((C_N^{\text{magic}})_J).$$

Let us now consider the case $|J| = (N-1)^2 + 1$. Since $N \geqslant 4$,

$$\min\{F(|J|), 2N\} = 2N - 4 + 4/N \leqslant 2N - 3 \leqslant \operatorname{rank}((C_N^{\text{magic}})_J).$$

The case in which N is odd may be established in a similar fashion. □

By Lemma 2.2 and using the existence of nonsingular solutions to the magic square system proved in [5], we deduce the following asymptotic formula.

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THEOREM 3.2. *For $K \geqslant 2$ and $N > 2K(K+1)$, let $M^*_{K,N}(P)$ denote the number of* $\boldsymbol{MMS}(K,N)$ *consisting of N^2 distinct integers bounded in absolute value by P. Then, there exists a constant $c > 0$ for which one has the asymptotic formula*

$$M^*_{K,N}(P) \sim cP^{N(N-K(K+1))}.$$

This immediately implies Theorem 1.3.

4. Magic squares of distinct kth powers

Let us now consider the problem of showing the existence of magic squares of distinct kth powers. One may repeat the arguments of Rome and Yamagishi [7], where instead of requiring a lower bound on the size of the largest partitionable submatrix, we assume that the matrix of coefficients dominates the function

$$F(x) = \max\left\{\frac{x - r\{s/r\}}{\lfloor s/r\rfloor}, \frac{x - r\{(s-1)/r\}}{\lfloor (s-1)/r\rfloor}, \frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/r\rfloor}\right\}.$$

It is clear that all arguments of [7] follow from this assumption by [6, Lemma 1], assuming $\lfloor (s-2)/r\rfloor$ is large enough in terms of k. How large this term needs to be, as seen in Lemma 2.1, is directly determined by when one can establish a mean value estimate of the type

$$\int_0^1 \left|\sum_{x\in A} e(\alpha x^k)\right|^s d\alpha \ll (\#A)^{s-k+\varepsilon},$$

where A is the set from which the solutions may come. Then, by a direct analogue of the arguments in Section 2, we obtain a version of [7, Theorem 1.4] which provides an asymptotic for the number of solutions with distinct entries.

LEMMA 4.1. *Let $k \geqslant 2$ and $C \in \mathbb{Z}^{r\times s}$, where $s \geqslant r\min\{2^k, k(k+1)\} + 2$. If C dominates the function*

$$F(x) = \max\left\{\frac{x - r\{s/r\}}{\lfloor s/r\rfloor}, \frac{x - r\{(s-1)/r\}}{\lfloor (s-1)/r\rfloor}, \frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/r\rfloor}\right\},$$

then

$$\#S^*_k(P;C) = P^{s-rk}(\sigma_k(C) + o(1)),$$

where $\sigma_k(C) \geqslant 0$ is a real number depending only on k and C. Additionally, $\sigma_k(C) > 0$ if there exist nonsingular real and p-adic solutions to the system (2.1).

Let

$$\mathscr{A}(Q) = \{\mathbf{x} \in \mathbb{Z}^s : \text{prime } p \mid x_i \text{ for any } i \text{ with } 1 \leqslant i \leqslant s, \text{implies } p \leqslant Q\}.$$

Then, the same may be done to obtain an analogue of [7, Theorem 1.5], which provides an asymptotic for the number of smooth solutions with distinct entries.

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LEMMA 4.2. *Let $k \geqslant 2$ and $C \in \mathbb{Z}^{r\times s}$, where $s \geqslant r\lceil k(\log k + 4.20032)\rceil + 2$. If C dominates the function*

$$F(x) = \max\left\{\frac{x - r\{s/r\}}{\lfloor s/r\rfloor}, \frac{x - r\{(s-1)/r\}}{\lfloor (s-1)/r\rfloor}, \frac{x - r\{(s-2)/r\}}{\lfloor (s-2)/r\rfloor}\right\},$$

then, providing $\eta > 0$ is sufficiently small in terms of s, r, k and C,

$$\#(S_k^*(P; C) \cap \mathscr{A}(P^\eta)) = c(\eta)P^{s-rk}(\sigma_k(C) + o(1)),$$

where $\sigma_k(C)$ is the same quantity as in Lemma 4.1 and $c(\eta) > 0$ depends only on η.

Applying Lemmas 4.1 and 4.2 with $C = C_N^{\text{magic}}$ and noting that the analysis in [5, Section 5] implies $\sigma_k(C_N^{\text{magic}})$ is positive, we deduce Theorem 1.5.

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