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Naturally bounded plumes

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This paper investigates theoretically the vertical evolution of a turbulent plume into a linearly stratified ambient fluid, by regarding it as composed of two distinct regions. In the first region, called the positive buoyant region, the plume buoyancy and the plume momentum act in the same upward direction, whereas in the second region, called the negative buoyant region, they act in opposite directions. In a first step, analytical expressions for the plume variables at the transition height (i.e. between the two regions) are obtained from one-dimensional conservation equations, using the plume entrainment model and under the Boussinesq approximation. In a second step, these variables are used in order to determine analytically the buoyancy and volume fluxes as well as the density deficit of the plume at its top. In this investigation, the transition height (denoted z_t) and the total plume height (denoted z_p) are obtained in the form of two integrals. These integrals are evaluated asymptotically in three different cases associated with particular flow regimes. Finally, the limit of the Boussinesq assumption for such flows is discussed.

Key words: convection, geophysical and geological flows, plumes/thermals

1. Introduction

A turbulent plume that develops in a stable density-stratified environment has a natural bounded limit. Indeed, owing to the decrease of the ambient density with respect to the vertical coordinate z, the plume *de facto* turns into a rising fountain (i.e. a negatively buoyant plume) at a certain height. As an example, figure 1, based on qualitative air–helium experiments, illustrates this phenomenon, which can be described as follows.

(*a*) At the initial development of the plume (figure 1*a*), owing to the mixing and/or entrainment process, the density difference between the plume and the ambient fluid decreases with z until it becomes null. Hereafter, this particular vertical location will be called the transition height and will be denoted z_t. Note that, in the literature, this height is sometimes called the zero (or neutral) buoyancy height.

(*b*) Above this height, the released fluid continues to rise as a result of its momentum. However, the buoyancy now acts in the opposite direction, so that the plume behaves like a rising fountain and naturally reaches a maximal height, here denoted z_p, as soon as its momentum becomes null (figure 1*b*).

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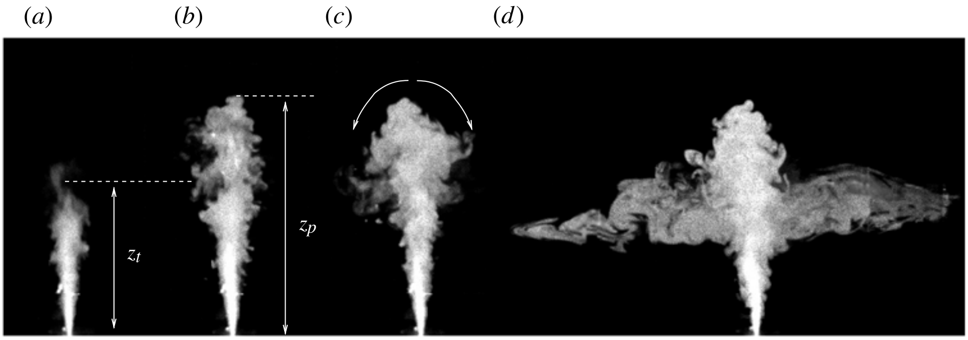

FIGURE 1. Typical evolution of a turbulent plume in a stratified environment (photos by the authors taken from qualitative air–helium experiments).

(*c*) Once this maximal height is reached, the released fluid falls down around the central up-flow (figure 1*c*).

(*d*) This down-flow then spreads horizontally as a radial gravity current at a vertical location usually called the spreading height (figure 1*d*).

These naturally bounded plumes are widely encountered in environmental and geophysical flows, as, for example, volcanic eruptions (Woods 1995; Suzuki *et al.* 2005; Kaminski *et al.* 2011), submarine pyroclastic eruptions (Head & Wilson 2003) and hydrothermal plumes (Speer & Rona 1989), because the oceans and the atmosphere are generally density-stratified.

During the past fifty years, many studies focusing on this kind of flow have been carried out. The first experimental studies measured the height z_p, since this value was the most accessible one (see e.g. Briggs 1969). Also, theoretical studies have been performed, allowing us to gain insight into the physics of the phenomenon. For example, Woods (1997) investigated the non-Boussinesq effects on plumes in a stratified ambient and showed that they are confined to a region close to the source. Caulfield & Woods (1998) studied Boussinesq plumes in nonlinear stratified surroundings in order to determine whether a given stratification (not necessarily stable) is responsible for the occurrence of a natural bounded limit or not. Scase, Caulfield & Dalziel (2006), using a point source formalism, gave the evolution of the plume variables as a function of the vertical coordinate for a linearly stratified ambient. More recently, Kaye & Scase (2011) studied the stratification conditions under which the plume is straight-sided. The interested reader can also refer to the reviews by Turner (1986), Kaye (2008) and Woods (2010).

Most of the theoretical studies are based on the formalism due to Morton, Taylor & Turner (1956). In their article, these authors proposed a one-dimensional model in which the velocity of the ambient fluid entrained at the edge of the plume is basically assumed to be proportional to the mean centreline velocity of the plume. The constant of proportionality is generally denoted α and called the entrainment coefficient. This approach leads us to a set of coupled differential equations for buoyancy, momentum and volume fluxes, which are given, under the Boussinesq approximation and considering top-hat profiles for the velocity and density, as follows:

$$\frac{\mathrm{d}(ub^2)}{\mathrm{d}z} = 2\alpha bu, \quad \frac{\mathrm{d}(u^2b^2)}{\mathrm{d}z} = g\eta b^2, \quad \frac{\mathrm{d}(\eta ub^2)}{\mathrm{d}z} = -b^2uN^2, \tag{1.1}$$

where g is the gravitational acceleration and $b(z)$, $u(z)$ and $\eta(z)$ are, respectively, the radius, the mean vertical velocity and the mean density deficit of the plume at the vertical location z. The density deficit reads as $\eta(z) = (\rho_0(z) - \rho(z))/\rho_{ref}$, where $\rho(z)$ and $\rho_0(z)$ are, respectively, the plume density and the ambient density and ρ_{ref} is a reference density, which is typically taken, under the Boussinesq approximation, as the ambient density at the source level ($z = 0$). The parameter N that characterizes the stratification of the ambient fluid is defined as follows:

$$N^2 = -\frac{1}{\rho_{ref}} \frac{\mathrm{d}\rho_0}{\mathrm{d}z}. \tag{1.2}$$

Note that $\sqrt{gN^2}$ corresponds to the well-known Brunt–Väisälä frequency.

The aim of this paper is to investigate theoretically turbulent plumes in a linearly stratified ambient ($\mathrm{d}\rho_0/\mathrm{d}z = \text{const.} < 0$). To do so, we shall benefit from recent advances in plume theoretical modelling, where two dimensionless functions $\Gamma(z)$ and $\sigma(z)$ have been introduced:

$$\Gamma(z) = \frac{5g\eta(z)b(z)}{8\alpha u(z)^2} \quad \text{and} \quad \sigma(z) = \frac{N^2 u(z)^2}{g\eta(z)^2}. \tag{1.3}$$

The function $\Gamma(z)$, also called the plume function, corresponds to a normalized Richardson number. It was used by Hunt & Kaye (2005) and Michaux & Vauquelin (2008) in order to rewrite the conservation equations of Morton *et al.* (1956) in the case $N = 0$ (i.e. homogeneous ambient). Remarkably, the plume function enables the inflow regimes to be classified as forced when the plume is initially momentum-dominated ($\Gamma_i < 1$) or as lazy when the plume is initially buoyancy-dominated ($\Gamma_i > 1$). Note that the subscript i stands for *initial* and denotes quantities at the source level ($z = 0$).

The function $\sigma(z)$ (sometimes called the buoyancy frequency parameter) has been introduced in order to account for the stratification of the ambient. According to Bloomfield & Kerr (1998), the function $\sigma(z)$ quantifies the relative magnitude between the two buoyancy fluxes that control the inflow behaviour.

By using these two functions, Mehaddi, Vauquelin & Candelier (2012) have investigated theoretically negatively buoyant releases (rising fountains) in linearly stratified ambients. Herein, we propose to generalize their approach by dealing with an initially positive buoyant release (plumes) in the case of a linear stable stratification ($N = \text{const.}$). More precisely, the two regions depicted in figure 1(*b*), which correspond to a positive buoyant region ($0 < z < z_t$) and a negative buoyant one ($z_t < z < z_p$), will be investigated successively in the next two sections.

2. Positive buoyant region

The purpose of this section is to provide us with the value of the transition height z_t, as well as analytical expressions for the plume variables (b_t, u_t and η_t) at this location, as functions of the initial conditions. By using the plume function $\Gamma(z)$ and the function $\sigma(z)$ given by (1.3), the conservation equations (1.1) can be combined in order to get the derivative of the primary variables as follows:

$$\frac{\mathrm{d}b}{\mathrm{d}z} = \frac{4\alpha}{5}\left(\frac{5}{2} - \Gamma\right), \tag{2.1}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{8}{5}\frac{\alpha u}{b}\left(\Gamma - \frac{5}{4}\right), \tag{2.2}$$

$$\frac{d\eta}{dz} = -\frac{2\alpha\eta}{b}\left(\frac{4}{5}\Gamma\sigma + 1\right). \tag{2.3}$$

Note that (2.3) indicates that the derivative of η is strictly negative so that the density deficit decreases (from a finite value at the source to zero at the transition height). We can also note from (2.1) and (2.2) the existence of a neck (minimal radius for $\Gamma = 5/2$) as well as a maximum of velocity (for $\Gamma = 5/4$) so that we recover results that are well known for a plume in a homogeneous environment (Fannelop & Webber 2003; Michaux & Vauquelin 2008).

By using (2.1)–(2.3), it may be shown that $\Gamma(z)$ and $\sigma(z)$ are solutions of a set of coupled differential equations:

$$\frac{d\Gamma}{dz} = \frac{4\alpha\Gamma}{b}\left[1 - \left(1 + \frac{2}{5}\sigma\right)\Gamma\right], \quad \Gamma(0) = \Gamma_i, \tag{2.4}$$

$$\frac{d\sigma}{dz} = \frac{16}{5}\frac{\alpha\sigma\Gamma}{b}(\sigma + 1), \quad \sigma(0) = \sigma_i. \tag{2.5}$$

Further, by combining the five equations (2.1)–(2.5) and after some algebra, the plume variables (divided by their values at the source level) can be expressed as

$$\frac{b}{b_i} = \left(\frac{\Gamma}{\Gamma_i}\right)^{1/2}\left(\frac{\sigma}{\sigma_i}\right)^{3/8}\left(\frac{\sigma_i + 1}{\sigma + 1}\right)^{1/8}, \tag{2.6}$$

$$\frac{u}{u_i} = \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{\sigma_i}{\sigma}\right)^{1/8}\left(\frac{\sigma_i + 1}{\sigma + 1}\right)^{1/8}, \tag{2.7}$$

$$\frac{\eta}{\eta_i} = \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{\sigma_i}{\sigma}\right)^{5/8}\left(\frac{\sigma_i + 1}{\sigma + 1}\right)^{1/8}. \tag{2.8}$$

By combining (2.4) and (2.5) we also obtain

$$\Gamma = \left[I(\sigma) - I(\sigma_i) + \frac{\Gamma_i\sigma_i^{5/4}}{(\sigma_i + 1)^{3/4}}\right]\frac{(\sigma + 1)^{3/4}}{\sigma^{5/4}}, \tag{2.9}$$

where $I(\sigma) = (5/4)\int_0^\sigma [t^{1/4}/(t+1)^{7/4}]\,dt$ is an incomplete beta function.

Thus, by substituting b given by (2.6) and Γ given by (2.9) into (2.5), the differential equation for $\sigma(z)$ reads as

$$\frac{d\sigma}{dz} = \frac{\Lambda_i}{b_i}\left(I(\sigma) - I(\sigma_i) + \frac{\Gamma_i\sigma_i^{5/4}}{(\sigma_i + 1)^{3/4}}\right)^{1/2}(\sigma + 1)^{3/2}, \tag{2.10}$$

with

$$\Lambda_i = \frac{16\alpha}{5}\frac{\Gamma_i^{1/2}\sigma_i^{3/8}}{(\sigma_i + 1)^{1/8}}. \tag{2.11}$$

Note that, formally, the (numerical) integration of this differential equation would allow the vertical evolution of the plume variables to be derived.

In the present case, however, we are only interested in obtaining the plume variables at the transition height z_t. To do so, we first note that, at this height, the density deficit is equal to zero, so that the functions Γ and σ tend, respectively, to zero and to infinity. To avoid this problem, it is convenient to introduce another function $\Delta = \Gamma^2\sigma$

that is independent of η and has a finite value at the transition height. This finite value can be obtained by multiplying (2.9) by σ^2 and subsequently by using the fact that $\sigma \to \infty$ when $z \to z_t$. This yields

$$\Delta_t = \left[\frac{5}{4}\beta\left[\frac{1}{2},\frac{5}{4}\right] - I(\sigma_i) + \frac{\Gamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} \right]^2, \tag{2.12}$$

where the beta function $\beta[1/2, 5/4] = 1.748$.

Using this value in (2.6)–(2.8) yields

$$\frac{b_t}{b_i} = \frac{(\sigma_i+1)^{1/8}}{\Gamma_i^{1/2}\sigma_i^{3/8}}\Delta_t^{1/4}, \quad \frac{u_t}{u_i} = \frac{\Gamma_i^{1/2}\sigma_i^{1/8}(\sigma_i+1)^{1/8}}{\Delta_t^{1/4}}, \quad \eta_t = 0. \tag{2.13}$$

At this stage, the expressions for the plume variables at the transition height have been drawn but the value of z_t remains to be determined. Equation (2.10) is then integrated between $z = 0$ and $z = z_t$ to obtain

$$\frac{z_t}{b_i} = \frac{1}{\Lambda_i}\int_{\sigma_i}^{\infty} \left(I(\sigma) - I(\sigma_i) + \frac{\Gamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} \right)^{-1/2} (\sigma+1)^{-3/2}\,\mathrm{d}\sigma. \tag{2.14}$$

The complexity of this equation does not allow analytical solutions to be derived. Nevertheless, to go further, an asymptotic analysis can be performed in order to obtain simple relations for the transition height z_t.

In the case $\sigma_i \gg 1$, it may be shown that

$$I(\sigma) - I(\sigma_i) = \tfrac{5}{2}(\sigma_i^{-1/2} - \sigma^{-1/2}) + O(\sigma_i^{-3/4}) \tag{2.15}$$

and

$$\frac{\Gamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} = \Gamma_i \sigma_i^{1/2} + O(\Gamma_i \sigma_i^{1/4}). \tag{2.16}$$

The term in (2.15) can be neglected in comparison with that in (2.16) provided the condition $\Gamma_i\sigma_i \gg 1$ is satisfied. In this case, (2.14) turns to be

$$\frac{z_t}{b_i} = \frac{5}{8\alpha\Gamma_i\sigma_i} + O(\Gamma_i^{-1}\sigma_i^{-5/4}). \tag{2.17}$$

For the other limiting case $\sigma_i \ll 1$, similar developments yield

$$I(\sigma_i) = \sigma_i^{5/4} + O(\sigma_i^{9/4}) \tag{2.18}$$

and

$$\frac{\Gamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} = \Gamma_i \sigma_i^{5/4} + O(\Gamma_i \sigma_i^{9/4}). \tag{2.19}$$

These two terms can be neglected in comparison with $I(\sigma)$, provided that $\Gamma_i\sigma_i^{5/4} \ll 1$, and we obtain

$$\frac{z_t}{b_i} = \frac{5A}{16\alpha\Gamma_i^{1/2}\sigma_i^{3/8}} + O(\Gamma_i^{-1/2}\sigma_i^{5/8}), \tag{2.20}$$

where A is a constant given by

$$A = \lim_{\sigma_i \to 0} \int_{\sigma_i}^{\infty} (I(\sigma))^{-1/2} (\sigma + 1)^{-3/2} \, \mathrm{d}\sigma \simeq 3.3. \tag{2.21}$$

Note that the relation (2.20), which is valid for small values of σ_i and when $\Gamma_i \sigma_i^{5/4} \ll 1$, has already been obtained by several authors (see e.g. Turner 1986; Malin 1989) with a slightly different formalism. However, the relation (2.17), which is valid for large values of σ_i and when $\Gamma_i \sigma_i \gg 1$, constitutes, to our knowledge, a new result.

Having obtained the plume variables at the transition height given by (2.13), the problem corresponding to the negative buoyant region ($z_t < z < z_p$) can now be tackled.

3. Negative buoyant region

Beyond the transition height, the problem corresponds to that of an initially non-buoyant release (since $\eta_t = 0$) in a linearly stratified environment. Such a problem is actually a particular case of the recent theoretical investigation by Mehaddi *et al.* (2012), which deals with rising fountains in a stratified ambient. For this reason, the details of the calculations will not be given here. However, for more clarity, we briefly recall the basis of their analysis. Starting from (1.1), the negative buoyant region of the inflow can be modelled after taking the following points into account:

(i) The buoyancy acts in the opposite direction since the density of the plume is greater than that of the ambient. So, for mathematical convenience, the variable η is redefined as $\eta = (\rho - \rho_0)/\rho_{ref}$ (to remain positive) and now corresponds to a density excess.

(ii) The initial conditions (found in the previous section) are b_t, u_t and η_t (=0).

(iii) At the top of the plume (i.e. $z = z_p$), $\Gamma(z)$ and $\sigma(z)$ tend, respectively, towards infinity and zero.

When the release reaches its maximal height z_p, the buoyancy flux $B = ub^2\eta$ and the volume flux $Q = ub^2$ have finite values even if the velocity is equal to zero and the radius is infinite at this location. From the conservation equations, it can actually be shown that

$$M^2 + \frac{g}{N^2} B^2 = \text{const.} = M_i^2 + \frac{g}{N^2} B_i^2, \tag{3.1}$$

where $M = u^2b^2$ is the momentum flux. According to the fact that M tends to zero at the top of the plume, (3.1) allows the buoyancy flux at $z = z_p$ to be obtained as

$$B_p = \left(\frac{N^2}{g} M_i^2 + B_i^2 \right)^{1/2} = B_i \left(1 + \sigma_i\right)^{1/2}, \tag{3.2}$$

where the subscript p refers to quantities at $z = z_p$.

In all cases, the volume flux Q reaches its maximal value at the top of the plume. This value can be first expressed as a function of u_t, b_t and Δ_t, by considering the problem of an initially non-buoyant release (Mehaddi *et al.* 2012). This maximal volume flux reads as

$$\frac{Q_p}{u_t b_t^2} = \frac{\left(\Delta_t^{1/2} + \frac{5}{4} \beta \left[\frac{1}{2}, \frac{5}{4} \right] \right)^{1/2}}{\Delta_t^{1/4}}, \tag{3.3}$$

where $\varDelta_t$, u_t and b_t have to be replaced by their expressions (2.12) and (2.13). We are led to

$$\frac{Q_p}{Q_i} = \frac{(\sigma_i+1)^{3/8}}{\varGamma_i^{1/2}\sigma_i^{5/8}} \left[\frac{5}{2}\beta\left[\frac{1}{2},\frac{5}{4}\right] - I(\sigma_i) + \frac{\varGamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} \right]^{1/2}. \tag{3.4}$$

The density excess at the top of the rising fountain can also be readily obtained since $\eta_p = B_p/Q_p$ and its value reads as

$$\frac{\eta_p}{\eta_i} = \frac{\varGamma_i^{1/2}\sigma_i^{5/8}(\sigma_i+1)^{1/8}}{\left[\frac{5}{2}\beta\left[\frac{1}{2},\frac{5}{4}\right] - I(\sigma_i) + \frac{\varGamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} \right]^{1/2}}. \tag{3.5}$$

This last quantity is of practical interest since it allows us to obtain an estimate of the dilution rate in the upper part of the plume and, subsequently, to obtain an order of magnitude of the dilution rate in the layer that spreads horizontally.

The rising fountain height z_f $(= z_p - z_t)$ can also be expressed as a function of $\varDelta_t$ and b_t (Mehaddi *et al.* 2012):

$$\frac{z_f}{b_t} = \frac{5}{16\alpha}\varDelta_t^{-1/4} \int_0^\infty \left[\varDelta_t^{1/2} + \frac{5}{4}\beta\left[\frac{1}{2},\frac{5}{4}\right] - I(\sigma) \right]^{-1/2} (\sigma+1)^{-3/2}\,\mathrm{d}\sigma. \tag{3.6}$$

Bearing in mind that $\varDelta_t$ and b_t are functions of the initial conditions σ_i and $\varGamma_i$, this yields

$$\frac{z_f}{b_i} = \frac{1}{\varLambda_i}\int_0^\infty \left[\frac{5}{2}\beta\left[\frac{1}{2},\frac{5}{4}\right] + \frac{\varGamma_i \sigma_i^{5/4}}{(\sigma_i+1)^{3/4}} - I(\sigma) - I(\sigma_i) \right]^{-1/2} (\sigma+1)^{-3/2}\ \mathrm{d}\sigma. \tag{3.7}$$

This equation cannot be solved analytically in the general case, but by using an asymptotic analysis, three relations can be derived:

$$\frac{z_f}{b_i} = \frac{5}{8\alpha\varGamma_i\sigma_i^{1/2}} + O(\varGamma_i^{-1}\sigma_i^{-3/4}), \quad \text{for} \quad \sigma_i \gg 1 \quad \text{and} \quad \varGamma_i\sigma_i^{1/2} \gg 1, \tag{3.8}$$

$$\frac{z_f}{b_i} = \frac{5D}{16\alpha\varGamma_i^{1/2}\sigma_i^{1/4}} + O(\varGamma_i^{-1/2}\sigma_i^{-5/4}), \quad \text{for} \quad \sigma_i \gg 1 \quad \text{and} \quad \varGamma_i\sigma_i^{1/2} \ll 1, \tag{3.9}$$

$$\frac{z_f}{b_i} = \frac{5C}{16\alpha\varGamma_i^{1/2}\sigma_i^{3/8}} + O(\varGamma_i^{-1/2}\sigma_i^{5/8}), \quad \text{for} \quad \sigma_i \ll 1 \quad \text{and} \quad \varGamma_i\sigma_i^{5/4} \ll 1, \tag{3.10}$$

where $C \approx 1.1388$ and $D \approx 2.5563$.

The total height of the plume z_p is given by the sum of z_t and z_f. Expressions for this height are given and discussed in the following section.

4. Plume total height

By using (2.14) and (3.7), the plume total height z_p is given in the form of two integrals, which have to be integrated numerically in the general case. In the present section, we focus on the asymptotic cases for which analytical expressions have previously been carried out. Three cases, in particular, can be considered that correspond to different flow behaviours.

4.1. *Plume-like behaviour*

Let us consider the case $\sigma_i \ll 1$ and $\Gamma_i \sigma_i^{5/4} \ll 1$, which typically corresponds to a plume (forced or moderately lazy) emitted into a weakly stratified ambient. From (2.20) and (3.10) the ratio between the transition height and the rising fountain height can be straightforwardly obtained as

$$\frac{z_t}{z_f} = \frac{\alpha_n}{\alpha_p} \frac{A}{C}, \tag{4.1}$$

where α_n and α_p are, respectively, the entrainment coefficients of the negative buoyant region and of the positive buoyant region. As a first approximation, if we consider that $\alpha_n \simeq \alpha_p$, then the ratio between the transition height and the plume height (z_t/z_p) is constant and remains around 0.75. In other words, the positive buoyant region represents 75% of the total plume height, and this is the reason why the inflow may be considered as plume-like.

The total plume height can also be expressed as

$$\frac{z_p}{b_i} = \frac{5}{16} \left(\frac{C}{\alpha_n} + \frac{A}{\alpha_p} \right) \frac{1}{\Gamma_i^{1/2} \sigma_i^{3/8}} \tag{4.2}$$

or, alternatively, in terms of the source buoyancy flux as

$$z_p = \frac{1}{4} \left(\frac{5}{2} \right)^{1/2} \left(\frac{A}{\alpha_p^{1/2}} + \frac{C \alpha_p^{1/2}}{\alpha_n} \right) B_i^{1/4} \left(g^{1/3} N^2 \right)^{-3/8}. \tag{4.3}$$

We therefore recover a relation that has already been proposed and used by several authors (Turner 1986; Malin 1989; Woods 2010; Kaminski *et al.* 2011). An interesting feature of this relation is that it allows the source buoyancy flux B_i to be calculated as soon as the plume height has been estimated.

To compare (4.2) with experimental data, we use the laboratory experimental results of Fan (1967), Abraham & Eysink (1969) and Sneck & Brown (1974) reported by Carazzo, Kaminski & Tait (2008). The entrainment coefficients α_n and α_p have been set, respectively, to 0.07 and 0.12, as proposed by Carazzo *et al.* (2008). Such values are consistent with the fact that a negatively buoyant release entrains less than a positively buoyant one (Kaminski, Tait & Carazzo 2005; Kaye 2008). Results are plotted in figure 2. Although in this approach the entrainment process in the vicinity of the transition height may not be well represented (the entrainment coefficient is sharply modified between the two regions), a good agreement is observed between the theory and experiments.

Additional comparisons with numerical results can be made for the particular case of a pure plume ($M_i = 0$), which has recently been studied by Devenish, Rooney & Thomson (2010) using large-eddy simulations. In spite of the fact that $\Gamma_i \to \infty$ and $\sigma_i \to 0$, the theory can still be applied in this case since the terms $\Gamma_i \sigma_i^{5/4}$ and b_i/Λ_i remain finite:

$$\Gamma_i \sigma_i^{5/4} = \frac{5}{8\alpha_p g^{1/4}} \frac{Q_i^2 N^{5/2}}{B_i^{3/2}} \quad \text{and} \quad \frac{b_i}{\Lambda_i} = \frac{1}{4} \left(\frac{5}{2\alpha_p} \right)^{1/2} \frac{B_i^{1/4}}{g^{1/8} N^{3/4}}. \tag{4.4}$$

The plume height and also the transition height are plotted in figure 3. A good agreement is again observed, except for the numerical simulation corresponding to $\Gamma_i \sigma_i^{5/4} = 0.006$, for which a discrepancy is noticeable. Anyway, the values assigned to

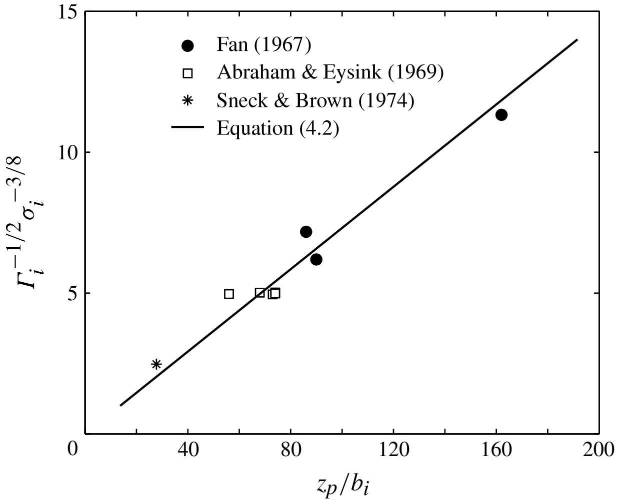

FIGURE 2. Comparisons between experimental data of Fan (1967), Abraham & Eysink (1969) and Sneck & Brown (1974) and the relation (4.2).

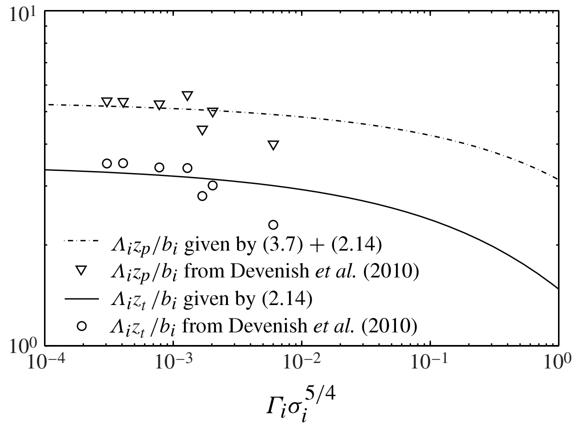

FIGURE 3. Comparisons between the theory ($\alpha_n = 0.07$ and $\alpha_p = 0.12$) and the numerical results of Devenish *et al.* (2010) for the transition height and the plume height.

the entrainment coefficients in the model provide us with fairly good results for this asymptotic regime.

4.2. *Non-buoyant forced (rising) fountain behaviour*

The second asymptotic case considered corresponds to $\sigma_i \gg 1$, $\Gamma_i\sigma_i \gg 1$ and $\Gamma_i\sigma_i^{1/2} \ll 1$. A typical case that illustrates this regime is a highly forced plume in a strongly stratified ambient. According to these assumptions, (2.17) and (3.8) show that $z_t \ll z_f$ so that $z_p \simeq z_f$. In terms of the initial source momentum flux $M_i = u_i^2 b_i^2$, the total plume height can be rewritten as follows:

$$z_p = \frac{D}{4}\left(\frac{5}{2\alpha}\right)^{1/2}\left(\frac{M_i}{gN^2}\right)^{1/4}. \tag{4.5}$$

This equation is similar to the relation given by Mehaddi *et al.* (2012) and by Scase *et al.* (2006) for a non-buoyant forced (rising) fountain ($\eta_i = 0$) in a linearly stratified ambient.

Furthermore, by comparing (4.5) with the correlation given by Malin (1989), the entrainment coefficient α should be around 0.07, which is close to that of a forced jet (Turner 1986).

4.3. *Non-buoyant weak (rising) fountain behaviour*

In the case $\sigma_i \gg 1$ and $\sigma_i^{1/2}\Gamma_i \gg 1$, corresponding to a lazy plume released in a strongly stratified ambient, the ratio between the rising fountain height and the plume height only depends on the parameter σ_i. By using (2.17) and (3.8), this ratio reads as

$$\frac{z_t}{z_f} = \frac{1}{\sigma_i^{1/2}}. \tag{4.6}$$

For this regime, and similarly as in the previous case, the rising fountain height is much greater than the transition height (i.e. $z_f \gg z_t$) and, remarkably, these two heights are independent of the entrainment coefficient. Indeed, in this case, the entrainment of the ambient fluid is weak and the evolution of the density difference is primarily driven by the stratification of the ambient.

Alternatively, as the problem is defined by two constants related to the ambient conditions (g and N^2), it is possible to build intrinsic length and velocity scales, respectively, $1/N^2$ and $\sqrt{g/N^2}$. Then, (2.17) and (3.8) can be rewritten as

$$N^2 z_t = \eta_i(1 + O(\sigma_i^{-1/4})), \tag{4.7}$$

$$N^2 z_f = \frac{u_i}{\sqrt{g/N^2}}(1 + O(\sigma_i^{-1/4})). \tag{4.8}$$

With this new representation, (4.7) shows that the transition height z_t depends in a very simple way on the source density and on the strength of stratification, independently of the other parameters. It can also be remarked from (4.8) that the dimensionless rising fountain height z_f is the ratio between the initial velocity of the release and the intrinsic velocity defined by the ambient conditions.

5. Discussion and conclusion

This paper has considered the vertical evolution of a Boussinesq plume in a linearly stratified ambient. Particular attention has been paid to the total plume height z_p, which has been estimated as a function of the initial release conditions and of the ambient stratification. Furthermore, the mathematical approach used in this study has allowed the plume buoyancy B_p and volume Q_p fluxes at the total plume height location to be obtained analytically.

In contrast with the vertical evolution of the density difference of a plume developing in a homogeneous ambient, which is continuously decreasing, this evolution is systematically non-monotonic in a stratified ambient. Indeed, in the positive buoyant region, the density deficit $\eta(z)$ decreases with respect to the vertical

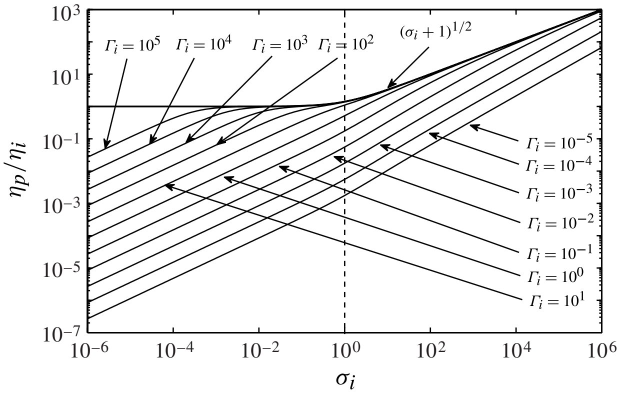

FIGURE 4. The dimensionless dilution rate η_p/η_i as a function of σ_i for given values of Γ_i.

coordinate from η_i at the source to zero at the transition height z_t. Above this location (i.e. in the negative buoyant region), the opposite trend is observed, since the density excess increases from zero to a finite value η_p at the top of the plume z_p.

Under certain conditions, this finite value η_p may be greater than the initial density deficit η_i, and thus the problem might fall outside the Boussinesq assumption.

In order to investigate this point, the ratio η_p/η_i has been plotted in figure 4 for several values of Γ_i and σ_i. Furthermore, from (3.5), it can be readily shown that

$$\lim_{\Gamma_i \to \infty} \frac{\eta_p}{\eta_i} = (\sigma_i + 1)^{1/2}, \tag{5.1}$$

which corresponds to the limiting (bold) curve plotted in figure 4. It can be seen that, for $\sigma_i \ll 1$, the order of magnitude of η_p will never exceed that of η_i, so the Boussinesq approximation will remain valid throughout the plume. In the case $\sigma_i \gg 1$ and $\Gamma_i \ll 1$, an asymptotic form of (3.5) reads as

$$\frac{\eta_p}{\eta_i} \simeq 0.68\Gamma_i^{1/2}\sigma_i^{3/4}. \tag{5.2}$$

Then, returning to the three regimes identified in § 4, we observe that, in the first one (the plume-like regime), the plume will always remain within the Boussinesq approximation. In the second one (non-buoyant forced rising fountain-like regime) and in the third one (non-buoyant weak rising fountain-like regime), it is advised to verify, by using the relation (5.2), whether the density excess at the top of the plume η_p remains compatible with the Boussinesq assumption.

REFERENCES

ABRAHAM, G. & EYSINK, W. D. 1969 Jets issuing into fluid with a density gradient. *Delft Hydraul. Lab. Publ.* **66**, 145–175.

BLOOMFIELD, L. J. & KERR, R. C. 1998 Turbulent fountains in a stratified fluid. *J. Fluid Mech.* **358**, 335–356.

BRIGGS, G. A. 1969. *Plume Rise.* US Atomic Energy Commission Critical Review Series.

CAULFIELD, C. P. & WOODS, A. W. 1998 Turbulent gravitational convection from a point source in a non-uniformly stratified environment. *J. Fluid Mech.* **360**, 229–248.

CARAZZO, G., KAMINSKI, E. & TAIT, S. 2008 On the rise of turbulent plumes: quantitative effects of variable entrainment for submarine hydrothermal vents, terrestrial and extra terrestrial explosive volcanism. *J. Geophys. Res.* **113**, B09201.

DEVENISH, B. J., ROONEY, G. G. & THOMSON, D. J. 2010 Large-eddy simulation of a buoyant plume in uniform and stably stratified environments. *J. Fluid Mech.* **652**, 75–103.

FAN, L. 1967. Turbulent buoyant jets into stratified or flowing ambient fluids. *Report* KH-R-15, California Inst. of Technology, Pasadena, California, USA.

FANNELOP, T. K. & WEBBER, D. M. 2003 On buoyant plumes from area sources in a calm environment. *J. Fluid Mech.* **497**, 319–334.

HEAD, J. & WILSON, L. 2003 Deep submarine pyroclastic eruptions: theory and predicted landforms and deposits. *J. Volcanol. Geotherm. Res.* **121**, 155–193.

HUNT, G. R. & KAYE, N. B. 2005 Lazy plumes. *J. Fluid Mech.* **533**, 329–338.

KAMINSKI, E., CHENET, A. L., JAUPART, C. & COURTILLOT, V. 2011 Rise of volcanic plumes to the stratosphere aided by penetrative convection above large lava flows. *Earth Planet. Sci. Lett.* **301**, 171–178.

KAMINSKI, E., TAIT, S. & CARAZZO, G. 2005 Turbulent entrainment in jets with arbitrary buoyancy. *J. Fluid Mech.* **526**, 361–376.

KAYE, N. B. 2008 Turbulent plumes in stratified environments: a review of recent work. *Atmos. Ocean* **46** (4), 433–441.

KAYE, N. B. & SCASE, M. M. 2011 Straight-sided solutions to classical and modified plume flux equations. *J. Fluid Mech.* **680**, 564–573.

MALIN, M. R. 1989 Analysis of turbulent forced plumes into a stable environment. *Appl. Math. Model.* **13**, 122–126.

MEHADDI, R., VAUQUELIN, O. & CANDELIER, F. 2012 Analytical solutions for Boussinesq fountains in a linearly stratified environment. *J. Fluid Mech.* **691**, 487–497.

MICHAUX, G. & VAUQUELIN, O. 2008 Solutions for turbulent buoyant plumes rising from circular sources. *Phys. Fluids* **20**, 066601.

MORTON, B. R., TAYLOR, G. I. & TURNER, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proc. R. Soc. Lond.* A **234**, 1–23.

SCASE, M. M., CAULFIELD, C. P. & DALZIEL, S. B. 2006 Boussinesq plumes and jets with decreasing source strengths in stratified environments. *J. Fluid Mech.* **563**, 463–472.

SNECK, H. J. & BROWN, D. H. 1974 Plume rise from large thermal sources such as dry cooling towers. *Tans. ASME: J. Heat Transfer* **96**, 232–238.

SPEER, K. G. & RONA, P. A. 1989 A model of an Atlantic and Pacific hydrothermal plume. *J. Geophys. Res.* **94**, 6213–6220.

SUZUKI, Y. J., KOYAGUCHI, T., OGAWA, M. & HACHISU, I. 2005 A numerical study of turbulent behaviour in eruption clouds using a 3-D fluid-dynamics model. *J. Geophys. Res.* **110**, B08201.

TURNER, J. S. 1986 Turbulent entrainment: the development of the entrainment assumption, and its application to geophysical flows. *J. Fluid Mech.* **173**, 431–471.

WOODS, A. W. 1995 The dynamics of explosive volcanic eruptions. *Rev. Geophys.* **33** (4), 495–530.

WOODS, A. W. 1997 A note on non-Boussinesq plumes in an incompressible stratified environment. *J. Fluid Mech.* **345**, 347–356.

WOODS, A. W. 2010 Turbulent plumes in nature. *Annu. Rev. Fluid Mech.* **42**, 391–412.