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LINEAR OPERATORS GENERATED BY SONNENSCHEIN MATRICES

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Abstract

An approximation method based on a certain Sonnenschein matrix is studied. Results are obtained for approximation in an interval and in the complex plane. A connection between convergence of the approximation process and regularity of the matrix is also discussed.

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1. Introduction

Let $\phi(z, w)$ be a function which is analytic in w for $|w| < R$ $(R > 1)$ for each fixed z in some set Ω and assume that $\phi(z, 1) = 1$. Sonnenschein (1949), Bajsanski (1956), Clunie and Vermes (1959), Sledd (1962, 1963a, 1963b), Ramanujan (1963) and others have considered sequence to sequence summability methods given by the matrix $A = (a_{nk}(z))$ defined as follows:

$$[\phi(z, w)]^n = \sum_{k=0}^{\infty} a_{nk}(z) w^k, \quad n = 1, 2, \cdots,$$

$$a_{00}(z) = 1, \quad a_{0k}(z) = 0, \quad k = 1, 2, \cdots .$$

The matrix (a_{nk}) is called a *Sonnenschein matrix*. Various important special cases of A are Borel (Cooke (1950)), Euler–Knopp (Agnew (1944), Vermes (1949)), the Hardy–Littlewood–Fekete or circle method, known also as the Taylor method (Hardy (1949)) and Laurent (Vermes (1949)). It has been shown by Vermes (1957) that Sonnenschein matrices are transposes of series to series transformation matrices developed earlier by Perron (1923) and Knopp (1926). Regularity of A has been studied by Bjasanski (1956) and Clunie and Vermes (1959).

It is natural to associate with A linear operators L_n defined by

$$L_n(g, z) = \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right) a_{nk}(z), \quad n = 1, 2, \cdots,$$

for any function g such that the series converges. It is the purpose of this note to investigate the special case when $\phi(z, w) = 1 - z + zf(w)$, where f is a certain analytic function. If $f(w) \equiv w$, A is the Euler–Knopp method and $L_n(g, z)$ is the well known n-th order Bernstein polynomial (Lorentz (1953)). We shall investigate approximation properties of the operators and point out that, when the linear operators satisfy certain convergence requirements, the summability method is regular. See also King (1968) and King and Swetits (1970) for results of this type.

2. Approximation properties

In the sequel let the functions e_k, $k = 0, 1, \cdots$, be defined by $e_k(z) = z^k$. It is easy to see that $L_n(e_k, z)$ is defined for $k = 0, 1, \cdots$ and $n = 1, 2, \cdots$. The following lemma lists some useful properties of the operators. Let $\bar{\Delta}$ denote the closed unit disk.

LEMMA 2.1. *Let f be analytic in $\bar{\Delta}$, $f(1) = f'(1) = 1$ and $f^{(k)}(0) \geqq 0$ for $k = 0, 1, \cdots$. We have the following:*

(i) *If $P(z)$ is a polynomial of degree m $L_n(P, z)$ is a polynomial of degree $\leqq m$ for each n.*

(ii) $\lim_{n\to\infty} L_n(e_k, z) = z^k$, $k = 0, 1, 2, \cdots$, *uniformly in compacta of the finite plane.*

(iii) *If $P(z)$ is a polynomial,* $\lim_{n\to\infty} L_n(P, z) = P(z)$, *uniformly in compacta of the finite plane.*

(iv) *If g is bounded in $[0, \infty)$, $L_n(g, z)$ is entire.*

(v) $L_n^{(\nu)}(e_k, 0) \geqq 0$ *for* $\nu = 0, 1, \cdots$, $k = 0, 1, \cdots$, *and* $n = 1, 2, \cdots$, *and* $L_n^{(\nu)}(e_k, 0) = 0$ *for* $\nu > k$, $k = 0, 1, \cdots$, *and* $n = 1, 2, \cdots$.

(vi) $|L_n(e_k, z)| \leqq L_n(e_k, |z|)$ *for* $k = 0, 1, \cdots$, $n = 1, 2, \cdots$, *and all complex numbers z.*

PROOF. (i) Fix n. We have

$$[1 - z + zf(w)]^n = \sum_{k=0}^{\infty} a_{nk}(z) w^k, \qquad |w| \leqq 1. \tag{2.1}$$

Since L_n is linear, it suffices to prove the result for $e_k(z)$; i.e., it suffices to show that

$$L_n(e_m, z) = \frac{1}{n^m} \sum_{k=0}^{\infty} a_{nk}(z) k^m$$

is a polynomial in z of degree $\leqq m$. Suppose this is true for $e_0, e_1, \cdots, e_{m-1}$. Take the m-th derivative of (2.1) with respect to w and put $w = 1$ in the resulting equation. The left hand side will be a polynomial, $P(z)$, of degree $\leqq m$, while the right hand side will be of the form

$$L_n(e_m, z) + \alpha_1 L_n(e_{m-1}, z) + \cdots + \alpha_{m-1} L_n(e_1, z)$$

where the $\alpha_i(m, n)$ are constants. Transposing yields

$$L_n(e_m, z) = P(z) - \alpha_1 L_n(e_{m-1}, z) - \cdots - \alpha_{m-1} L_n(e_1, z).$$

The proof is completed by observing that $L_n(e_0, z) = 1$ for all z.

(ii) Let m be any positive integer and assume that $\lim_{n\to\infty} L_n(e_k, z) = z^k$ almost uniformly for $k = 0, 1, \cdots, m-1$. As in the proof of (i), take the m-th derivative of (2.1) with respect to w and put $w = 1$ in the resulting equation. After dividing the equation by n^m, transposing and using the facts $f(1) = f'(1) = 1$, it follows that

$$L_n(e_m, z) - z^m = \frac{\alpha_n(m, z)}{n}$$

where, for fixed m, $\{\alpha_n(m, z)\}$ is uniformly bounded in compacta of the plane.

(iii) This follows directly from (ii).

(iv) Let $|g(x)| \leqq M$ for $0 \leqq x < \infty$. Fix n. Then

$$|L_n(g, z)| = \left| \sum_{k=0}^{\infty} a_{nk}(z) g\left(\frac{k}{n}\right) \right| \leqq M \sum_{k=0}^{\infty} |a_{nk}(z)|.$$

Since $a_{nk}(z)$ is a polynomial in z for each k, it suffices to show that $\Sigma_{k=0}^{\infty} |a_{nk}(z)|$ is uniformly convergent in compacta of the plane. Let Ω be a compact subset of the plane. Choose $\alpha > 1$ such that $f(w)$ is analytic for $|w| \leqq \alpha$. Let $M_n = \max\{|1 - z + zf(t)|^n : z \in \Omega \text{ and } |t| = \alpha\}$. It follows that

$$\sum_{k=0}^{\infty} |a_{nk}(z)| \leqq \frac{\alpha M_n}{\alpha - 1}, \qquad z \in \Omega.$$

(v) Fix n and z. Let $f(w) = \Sigma_{k=0}^{\infty} b_k w^k$ for $w \in U$, an open set containing $\bar{\Delta}$. Since f is analytic in U, for $|x|$ sufficiently small it follows that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{x^k}{k!} L_n(e_k, z) &\equiv \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\nu=0}^{\infty} a_{n\nu}(z) \left(\frac{\nu}{n}\right)^k \\
&= \sum_{\nu=0}^{\infty} a_{n\nu}(z) \sum_{k=0}^{\infty} \left(\frac{\nu x}{n}\right)^k \frac{1}{k!} = \sum_{\nu=0}^{\infty} a_{n\nu}(z)(e^{x\nu/n}) \\
&= [1 - z + zf(e^{x/n})]^n = \left[1 - z + z \sum_{k=0}^{\infty} b_k (e^{xk/n})\right]^n \\
&= \left[1 - z + z \sum_{\nu=0}^{\infty} \left(\frac{x}{n}\right)^\nu \frac{1}{\nu!} L_1(e_\nu, 1)\right]^n \\
&= \left[1 + \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu!} \left(\frac{zL_1(e_\nu, 1)}{n^\nu}\right)\right]^n.
\end{aligned}$$

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Therefore, we have

$$(2.2)\qquad \sum_{k=0}^{\infty}\frac{x^k}{k!}L_n(e_k,z)=\left[1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\left(\frac{zL_1(e_k,1)}{n^k}\right)\right]^n.$$

Take the ν-th derivative of (2.2) with respect to z and put $z=0$ in the resulting equation. It follows that

$$(2.3)\qquad \sum_{k=0}^{\infty}x^k\left(\frac{L_n^{(\nu)}(e_k,0)}{k!\,\nu!\binom{n}{\nu}}\right)=\left[\sum_{k=1}^{\infty}x^k\left(\frac{L_1(e_k,1)}{k!\,n^k}\right)\right]^{\nu},\qquad n=1,2,\cdots.$$

When the right hand side of (2.3) is written as a power series in x, all coefficients are non-negative and the leading term is of order x^{ν}. Therefore, result (v) follows.

(vi) This follows directly from results (i) and (v).

THEOREM 2.2. *Let f be as in Lemma 2.1 and g be defined and continuous in $[0,1]$. Define g in $(-\infty,\infty)$ as follows:*

$$g(x)=\begin{cases}g(0), & x\leqq 0\\ g(1), & x\geqq 1.\end{cases}$$

Then $\lim_{n\to\infty}L_n(g,x)=g(x)$ *uniformly in* $[0,1]$.

PROOF. First, $a_{nk}(x)\geqq 0$ for $n=1,2,\cdots$, $k=0,1,\cdots$, and $0\leqq x\leqq 1$ implies $\{L_n\}$ is a sequence of positive linear operators in $[0,1]$. The result now follows from a theorem of Korovkin (1960) and Lemma 2.1(ii).

The proof of the next theorem is similar to that of Theorem 1.6.1 on page 20 of Lorentz (1953) and is omitted.

THEOREM 2.3. *Let f be as in Lemma 2.1 and g be defined and continuous in $[0,1]$. Define $g(x)=g(1)$ for $x\geqq 1$. Let $w(g,\delta)$ denote the modulus of continuity of g; that is,*

$$w(g,\delta)=\max\{|g(x)-g(y)|:0\leqq x,y\leqq 1\quad\textit{and}\quad |x-y|<\delta\}.$$

Then

$$\sup\{|L_n(g,x)-g(x)|:0\leqq x\leqq 1\}\leqq w(g,n^{-1/2})\left[\frac{5}{4}+f^{(2)}(1)\right].$$

THEOREM 2.4. *Let f have the properties described in Lemma 2.1 and $f(z)=\sum_{k=0}^{m}b_kz^k$. Let $g(z)=\sum_{k=0}^{\infty}\alpha_kz^k$ be entire. Then*

$$\lim_{n\to\infty}L_n^{(s)}(g,z)=g^{(s)}(z),\qquad s=0,1,2,\cdots,$$

uniformly in compact subsets of the finite plane.

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PROOF. Write $\Sigma_{\nu=0}^{\infty} b_{n\nu} w^{\nu} = [f(w)]^n$, $n = 1, 2, \cdots$. It follows that

$$a_{nk}(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} (1-z)^{n-\nu} z^{\nu} b_{\nu k}.$$

But $b_k = 0$ for $k > m$ implies $b_{\nu k} = 0$ for $k > \nu m$. Therefore, $a_{nk}(z) = 0$ for $k > nm$. Thus

$$L_n(g, z) = \sum_{k=0}^{nm} a_{nk}(z) g\left(\frac{k}{n}\right)$$

is a polynomial in z for each n. Define the function h as follows:

$$h(x) = \begin{cases} g(-m), & x \leqq -m \\ g(x), & -m \leqq x \leqq m \\ g(m), & x \geqq m. \end{cases}$$

As in the proof of Theorem 2.2, $\lim_{n\to\infty} L_n(h, x) = h(x)$ uniformly on $[0, 1]$. But $L_n(h, x) = L_n(g, x)$ and $h(x) = g(x)$, for $0 \leqq x \leqq 1$. Therefore, $\lim_{n\to\infty} L_n(g, x) = g(x)$ uniformly on $[0, 1]$. Let $|z| \leqq y < \infty$. Using Lemma 2.1(vi) we have

$$\sum_{\nu=0}^{\infty} |\alpha_\nu| \, |L_n(e_\nu, z)| \leqq \sum_{\nu=0}^{\infty} |\alpha_\nu| L_n(e_\nu, y) = \sum_{\nu=0}^{\infty} |\alpha_\nu| \sum_{k=0}^{mn} a_{nk}(y) \left(\frac{k}{n}\right)^{\nu}.$$

Fix n. Just as in the proof of Lemma 2.1(v), we have

$$\sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \sum_{k=0}^{mn} a_{nk}(y) \left(\frac{k}{n}\right)^{\nu} = \left[1 + \sum_{\nu=1}^{\infty} \frac{x^{\nu}}{\nu!} \left(y \sum_{k=0}^{m} b_k \left(\frac{k}{n}\right)^{\nu}\right)\right]^n.$$

Without loss of generality, assume $y \geqq 1$. Since $f(1) = 1$ and $f^{(k)}(0) \geqq 0$, we have

$$0 \leqq y \sum_{k=0}^{m} b_k \left(\frac{k}{n}\right)^{\nu} \leqq \left(\frac{ym}{n}\right)^{\nu}, \qquad \nu = 1, 2, \cdots.$$

Thus the coefficients of x^{ν} in the series

$$1 + \sum_{\nu=1}^{\infty} \frac{x^{\nu}}{\nu!} \left(y \sum_{k=0}^{m} b_k \left(\frac{k}{n}\right)^{\nu}\right)$$

are majorized by those in the series

$$\sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \left(\frac{ym}{n}\right)^{\nu}.$$

But

$$\left[\sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} \left(\frac{ym}{n}\right)^{\nu}\right]^n = \sum_{\nu=0}^{\infty} \frac{x^{\nu}}{\nu!} (ym)^{\nu}.$$

It follows that

$$0 \leqq \sum_{k=0}^{mn} a_{nk}(y)\left(\frac{k}{n}\right)^{\nu} \leqq (ym)^{\nu}.$$

Therefore,

$$\sum_{\nu=0}^{\infty} |\alpha_{\nu}| \sum_{k=0}^{mn} a_{nk}(y)\left(\frac{k}{n}\right)^{\nu} \leqq \sum_{\nu=0}^{\infty} |\alpha_{\nu}|(ym)^{\nu} < \infty$$

for each n. These calculations show that the series

$$\sum_{\nu=0}^{\infty} \alpha_{\nu} \sum_{k=0}^{mn} a_{nk}(z)\left(\frac{k}{n}\right)^{\nu}$$

is uniformly convergent on compacta of the finite plane, for each n, and the sequence

$$F_n(z) \equiv \sum_{\nu=0}^{\infty} \alpha_{\nu} \sum_{k=0}^{mn} a_{nk}(z)\left(\frac{k}{n}\right)^{\nu}$$

is uniformly bounded on compacta of the plane. Since

$$\sum_{k=0}^{mn} a_{nk}(z)\left(\frac{k}{n}\right)^{\nu}$$

is a polynomial in z, $F_n(z)$ is an entire function of z for each n. Also, $L_n(g, z)$ is a polynomial in z and, for $0 \leqq x \leqq 1$, we have $a_{nk}(x) \geqq 0$ and

$$L_n(g, x) = \sum_{k=0}^{mn} a_{nk}(x) \sum_{\nu=0}^{\infty} \alpha_{\nu}\left(\frac{k}{n}\right)^{\nu} = \sum_{\nu=0}^{\infty} \alpha_{\nu} \sum_{k=0}^{mn} a_{nk}(x)\left(\frac{k}{n}\right)^{\nu} = \sum_{\nu=0}^{\infty} \alpha_{\nu} L_n(e_{\nu}, x),$$

since the second double series is absolutely convergent. It now follows that

$$L_n(g, z) = \sum_{\nu=0}^{\infty} \alpha_{\nu} L_n(e_{\nu}, z) = F_n(z)$$

for all complex numbers z. Thus $\{L_n(g, z)\}$ is uniformly bounded on compacta of the finite plane, $L_n(g, z)$ is an entire function of z, and $\lim_{n\to\infty} L_n(g, x) = g(x)$ uniformly on $0 \leqq x \leqq 1$. Theorem 2.4 now follows from the above and Vitali's theorem.

We shall now consider a relationship between convergence of $\{L_n(g, z)\}$ and regularity of $(a_{nk}(z))$. A result of Clunie and Vermes (1959) implies that $A(f, x) = (a_{nk}(x))$ is regular when f satisfies the hypotheses of Lemma 2.1 and $0 < x \leqq 1$. This result depends upon the fact that $a_{nk}(x) \geqq 0$ for all n, k and $x \in [0, 1]$. Theorem 2.6 below gives conditions under which $A(f, x)$ is regular when f is in a more general class of functions. Lemma 2.5 will be needed in the proof of Theorem 2.6.

LEMMA 2.5. *Let f be analytic in $\bar{\Delta}$, $f(1)=1$, f not a constant, and $x>0$. Suppose $\{L_n(g,x)\}$ is almost convergent* [10] *for all $g \in C[0,x]$, where $C[0,x]$ denotes the space of all functions g such that g is continuous in $[0,x]$, defined in $[0,\infty)$ and*

$$\sup_{0 \leqq y < \infty} |g(y)| = \sup_{0 \leqq y \leqq x} |g(y)| \equiv \|g\|.$$

Then

$$\sup\left\{\sum_{k=0}^{\infty} |a_{nk}(x)| : n = 1,2,\cdots\right\} < \infty.$$

PROOF. We have

$$L_n(g,x) = \sum_{k=0}^{\infty} a_{nk}(x) g\left(\frac{k}{n}\right)$$

and

$$[1 - x + xf(w)]^n = \sum_{k=0}^{\infty} a_{nk}(x) w^k, \qquad |w| \leqq 1.$$

Let $\|L_n(\ ,x)\| = \sup_{\|g\| \leqq 1} |L_n(g,x)|$. Then $L_n(\ ,x)$ is a continuous linear operator from $C[0,x]$ to R, where R is the set of real numbers. Our hypothesis implies that $\{L_n(\ ,x)\}$ is pointwise bounded on $C[0,x]$. It follows from the uniform boundedness principle that $\{L_n(\ ,x)\}$ is uniformly bounded. Let n be a fixed positive integer and let m be a positive integer chosen so large that $m/n \geqq x$. Define $h(y)$ as follows:

$$h(y) = \begin{cases} \operatorname{sgn} a_{n0}(x), & y = 0, \\ \dfrac{\operatorname{sgn} a_{n1}(x) - \operatorname{sn} a_{n0}(x)}{n^{-1}}(y-0) + \operatorname{sgn} a_{n0}(x), & 0 < y \leqq \dfrac{1}{n}, \\ \vdots & \\ \dfrac{\operatorname{sgn} a_{n,m-1}(x) - \operatorname{sgn} a_{n,m-2}(x)}{n^{-1}}\left(y - \dfrac{m-2}{n}\right) + \operatorname{sgn} a_{n,m-2}(x), & \dfrac{m-2}{n} < y \leqq \dfrac{m-1}{n}, \\ \dfrac{\operatorname{sgn} a_{nm}(x) - \operatorname{sgn} a_{n,m-1}(x)}{n^{-1}}\left(y - \dfrac{m-1}{n}\right) + \operatorname{sgn} a_{n,m-1}(x), & \dfrac{m-1}{n} < y \leqq \dfrac{m}{n}, \\ 0, & y > \dfrac{m}{n}. \end{cases}$$

Then $h \in C[0, x]$, $\|h\| = 1$ and $L_n(h, x) = \sum_{k=0}^{m} |a_{nk}(x)|$. Therefore, $\sum_{k=0}^{m} |a_{nk}(x)| \leqq \|L_n(\ , x)\|$. It follows that $\sum_{k=0}^{\infty} |a_{nk}(x)| \leqq \|L_n(\ , x)\|$ and

$$\sup\left\{\sum_{k=0}^{\infty} |a_{nk}(x)| : n = 1, 2, \cdots\right\} \leqq \sup\{\|L_n(\ , x)\| : n = 1, 2, \cdots\} < \infty.$$

THEOREM 2.6. *Let f be as in Lemma 2.5. Let $T = \{x : x > 0$ and $f[\bar{\Delta}] \subset \overline{D(x)}$ where $\overline{D(x)}$ denotes the closure of $D(x) \equiv \{w : |1 - x + xw| < 1\}$. Let $\{L_n(g, x)\}$ be almost convergent for all $g \in C[0, x]$. Then $A(f, x) = (a_{nk}(x))$ is regular for $x \in T$.*

PROOF. First $\sum_{k=0}^{\infty} a_{nk}(x) = 1$ for $n = 1, 2, \cdots$. Next, our hypotheses and the Cauchy integral formula show that $\lim_{n\to\infty} a_{nk}(x) = 0$, $k = 0, 1, \cdots$. An application of Lemma 2.5 completes the proof.

THEOREM 2.7. *Let f be as in Lemma 2.5, $f^{(1)}(1) = 1$ and $x_0 > 0$. Let g be continuous in $[0, \infty)$ and $\lim_{x\to\infty} g(x)$ exist. Suppose $\{L_n(h, x_0)\}$ is almost convergent for all $h \in C[0, x_0]$. Then $\lim_{n\to\infty}(g, x_0) = g(x_0)$.*

PROOF. Define the function $\psi(x)$ as follows:

$$\psi(x) = \begin{cases} g\left(\ln \dfrac{1}{x}\right), & 0 < x \leqq 1, \\[2ex] \lim\limits_{t\to\infty} g(t), & x = 0. \end{cases}$$

Then ψ is continuous in $[0, 1]$ and, given $\varepsilon > 0$, there exists a polynomial $p_m(x) = \sum_{i=0}^{m} \alpha_i x^i$ such that $|\psi(x) - p_m(x)| < \varepsilon$ for $0 \leqq x \leqq 1$. Let $R_m(t) = p_m(e^{-t})$ for $0 \leqq t < \infty$. Thus $|g(t) - R_m(t)| < \varepsilon$ for $0 \leqq t < \infty$. We have

$$\begin{aligned} |L_n(g, x_0) - g(x_0)| \leqq |L_n(g - R_m, x_0)| + |L_n(R_m, x_0) - R_m(x_0)| \\ + |R_m(x_0) - g(x_0)|. \end{aligned}$$

It follows from our hypotheses and Lemma 2.5 that

$$\sup\left\{\sum_{k=0}^{\infty} |a_{nk}(x_0)| : n = 1, 2, \cdots\right\} = M < \infty.$$

Thus $|L_n(g - R_m, x_0)| < \varepsilon M$. It only remains to show that $\lim_{n\to\infty} L_n(R_m, x_0) = R_m(x_0)$. Therefore, it suffices to show that $\lim_{n\to\infty} L_n(h_i, x_0) = h_i(x_0)$, where $h_i(t) = e^{-it}$, $i = 0, 1, \cdots, m$. Our hypotheses and the proof of Lemma 2.1 (ii) show that $\lim_{n\to\infty} L_n(e_i, z) = z^i$ uniformly in compacta of the finite plane for $i = 0, 1, 2, \cdots$. Now

$$h_i(t) = e^{-it} = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!}.$$

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Let $\varepsilon > 0$ be given. For fixed $i \geqq 0$, choose $N(i, \varepsilon)$ so large that

$$\left| \sum_{k=n+1}^{\infty} \frac{(-it)^k}{k!} \right| < \varepsilon$$

for $0 \leqq t < \infty$. Write

$$\begin{aligned} L_n(h_i, x_0) &= \sum_{\nu=0}^{\infty} a_{n\nu}(x_0) \left(\sum_{k=0}^{N} + \sum_{k=N+1}^{\infty} \right) \left(\frac{-i\nu}{n} \right)^k \frac{1}{k!} \\ &= \sum_{k=0}^{N} \frac{(-i)^k}{k!} L_n(e_k, x_0) + \sum_{\nu=0}^{\infty} a_{n\nu}(x_0) \sum_{k=N+1}^{\infty} \left(\frac{-i\nu}{n} \right)^k \frac{1}{k!}. \end{aligned}$$

Thus

$$0 \leqq \limsup_{n \to \infty} |L_n(h_i, x_0) - h_i(x_0)| \leqq \left| \sum_{k=0}^{N} \frac{(-ix_0)^k}{k!} - h_i(x_0) \right| + \varepsilon M < \varepsilon(M+1).$$

Therefore, $\lim_{n\to\infty} L_n(h_i, x_0) = h_i(x_0)$ for $i = 0, 1, \cdots, m$ and the proof is complete.

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