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THE PROFINITE COMPLETION OF A PROFINITE PROJECTIVE GROUP

TAMAR BAR-ON*
Department of Mathematics, Bar Ilan University, Ramat Gan 5920002, Israel
e-mail: tamarnachshoni@gmail.com

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Abstract. We prove that the profinite completion of a profinite projective group is projective.

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1. Introduction. Projective groups play a central role in the theory of profinite groups and in Field Arithmetic. We recall that an embedding problem for an abstract group G is a pair

$$(\varphi : G \to A, \ \alpha : B \to A) \tag{1.1}$$

of abstract epimorphisms of abstract groups. Problem (1.1) is finite if B is finite. A weak solution to (1.1) is an abstract homomorphism $\gamma : G \to B$ such that $\alpha \circ \gamma = \varphi$.

Likewise, (1.1) is an embedding problem for profinite groups G, A, B if, in addition, φ and α are continuous. In this case, γ is a weak solution to (1.1) if in addition, γ is continuous.

A profinite group G is profinite projective if every embedding problem of profinite groups for G has a weak solution [**2**, p. 502, Def. 22.3.1].

A basic result of Gruenberg says that G is profinite projective if and only if every finite embedding problem for G is weakly solvable [**2**, p. 503, Lemma 22.3.2]. It follows that G is profinite projective if and only if G is isomorphic to a closed subgroup of a free profinite group [**2**, p. 507, Cor. 22.4.6]. This implies that every closed subgroup of a profinite projective group is also profinite projective.

Most important is the connection of profinite projective groups with Field Arithmetic. By a theorem of Leptin, every profinite group G is isomorphic to the Galois group of some Galois extension of fields. Moreover, G is profinite projective if and only if G is isomorphic to the absolute Galois group of a "PAC field". See [**2**, p. 207, Thm. 11.6.2] for a theorem of Ax with a proof of Haran on the one hand and a theorem of Lubotzky–v.d. Dries [**2**, p. 545, Cor. 23.1.22] on the other hand. Here, a field K is PAC if for every polynomial $f \in K[X, Y]$ which is irreducible over the algebraic closure $\tilde{K}$ of K there exists $(x, y) \in K \times K$ such that $f(x, y) = 0$ [**2**, p. 195, Thm. 11.2.3].

The goal of this note is to add an additional piece of information about profinite projective groups.

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To this end, we recall that the completion of an abstract group G is defined as the inverse limit $\hat{G} = \varprojlim G/N$, where N ranges over all normal subgroups of G of finite index (see [4, p. 42, Example 2.1.6] or [2, p. 341, first paragraph]). The local weight of a profinite G, denoted by $\omega_0(G)$, is the cardinality of the collection of all open subgroups of G. In case G is not finitely generated, it equals to the cardinality of a set of generators of G converging to 1. [4, Section 2.6].

Note that the definition of $\hat{G}$ makes sense also for a profinite group G. As in the abstract case, we allow N to range over all normal subgroups N of finite index of G and not only over open normal subgroups.

A deep result of Nikolov–Segal says that if G is a finitely generated profinite group, then every subgroup of G of finite index is open. Thus, in this case, $\hat{G}$ is isomorphic to G. In particular, if G is projective, so is $\hat{G}$.

The main theorem of this note generalizes the latter result: Let G be a profinite projective group which is not necessarily finitely generated. Then, $\hat{G}$ is also projective.

2. Main result.

We start the proof of the main theorem by an obvious lemma:

LEMMA 1. *Let H be a finitely generated profinite group. Then, every abstract homomorphism of H into a finite group is continuous.*

Proof. Let B be a finite group and let $\psi : H \to B$ be an abstract homomorphism. Then, $\mathrm{Ker}(\psi)$ is a subgroup of H of finite index. Hence, by [3, Thm. 1.1], $\mathrm{Ker}(\psi)$ is an open subgroup of H.

For each $b \in B$ either $\psi^{-1}(b)$ is empty, hence open in H, or there is $h \in H$ with $\psi(h) = b$. In the latter case, $\psi^{-1}(b) = h \cdot \mathrm{Ker}(\psi)$ is also open. It follows that ψ is continuous. □

Denote the set of all finite indexed normal subgroups of G by $\mathcal{N}$. By [2, p. 341, 2nd paragraph], the map $g \mapsto (gN)_{N\in\mathcal{N}}$ is a homomorphism $\iota\colon G \to \hat{G}$ whose kernel is the intersection of all $N \in \mathcal{N}$. In case G is profinite, it is contained in the intersection of all open normal subgroups of G, so by [2, p. 6, Remark 1.2.1(a)], that intersection is the trivial group $\mathbf{1}$ of G. Hence, ι is injective. We call ι the canonical embedding of G into $\hat{G}$. By [2, p. 341, Lemma 17.2.1(a4)], $\iota(G)$ is dense in $\hat{G}$.

Here is a sufficient condition for $\hat{G}$ in terms of G to be a profinite projective group.

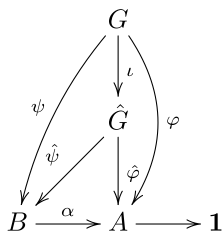
LEMMA 2. *Let G be an abstract group. If every finite abstract embedding problem (1.1) for G is weakly solvable, then $\hat{G}$ is profinite projective.*

Proof. Let

$$(\hat{\varphi}\colon \hat{G} \to A,\ \alpha : B \to A) \tag{2.1}$$

be a finite embedding problem for the profinite group $\hat{G}$. Then, $\varphi := \iota \circ \hat{\varphi}\colon G \to A$ is an abstract homomorphism. By [2, p. 341, Lemma 17.2.1(a4)], $\iota(G)$ is dense in $\hat{G}$. Using that $\hat{\varphi}$ is continuous, we find that $\varphi(G) = \hat{\varphi}(\iota(G))$ is dense in A. Since A is finite, hence discrete, $\varphi(G) = A$. It follows that (1.1) is a finite abstract embedding problem for G.

The assumption of our lemma gives an abstract homomorphism $\psi : G \to B$ such that $\alpha \circ \psi = \varphi$. By [2, p. 343, Lemma 17.2.2], there exists a homomorphism $\hat{\psi} : \hat{G} \to B$ with $\hat{\psi} \circ \iota = \psi$.



For each $g \in G$ we have $\alpha(\hat{\psi}(\iota(g)) = \alpha(\psi(g)) = \varphi(g) = \hat{\varphi}(\iota(g))$. Thus, the continuous homomorphisms $\alpha \circ \hat{\psi}$ and $\hat{\varphi}$ coincide on the dense subset $\iota(G)$ of $\hat{G}$, so $\hat{\varphi} = \alpha \circ \hat{\psi}$. In other words, $\hat{\psi}$ is a solution of embedding problem (2.1), as desired. □

LEMMA 3. *Let G be a profinite projective group. Then, every finite abstract embedding problem (1.1) is weakly solvable.*

Proof. Let $\mathcal{X}$ be the collection of all finite subsets X of G with $\varphi(X) = A$. The rest of the proof breaks up into two parts.

Part A (*Application of the Nikolov–Segal theorem.*) For each $X \in \mathcal{X}$ let H_X be the closed subgroup of G generated by X and let $\varphi_X \colon H_X \to A$ be the restriction of φ to H_X. Then, $\varphi_X(H_X) = A$, so

$$(\varphi_X \colon H_X \to A,\ \alpha \colon B \to A) \tag{2.2}$$

is a finite abstract embedding problem for H_X. Also, $(H_X : \mathrm{Ker}(\varphi_X)) = |A|$ is finite, Hence, by [**3**, Thm. 1.1], $\mathrm{Ker}(\varphi_X)$ is an open subgroup of H_X. It follows from Lemma 1 that φ_X is continuous. Thus, (2.2) is an embedding problem for the profinite group H_X.

By [**2**, p. 507, Prop. 22.4.7], H_X is projective, so there exists an homomorphism $\psi_X \colon H_X \to B$ with $\alpha \circ \psi_X = \varphi_X$. Thus, ψ_X is a solution of embedding problem (2.2). Hence, the set Ψ_X of all solutions of embedding problem (2.2) is non-empty. Since H_X is finitely generated, Ψ_X is finite.

Part B (*Inverse limit of finite sets of homomorphims.*) We order $\mathcal{X}$ by inclusion. If $Y \in \mathcal{X}$ contains X and $\psi_Y \in \Psi_Y$, then $H_X \le H_Y$ and $(\psi_Y)|_X \in \Psi_X$. Thus, restriction to X yields a map $\psi_{Y,X} \colon \Psi_Y \to \Psi_X$. If $Z \in \mathcal{X}$ contains Y, then $\psi_{Y,X} \circ \psi_{Z,Y} = \psi_{Z,X}$. Thus, $(\{\Psi_X \mid X \in \mathcal{X}\}, \psi_{Y,X})_{X \subseteq Y}$ is an inverse system of non-empty finite sets. By [**2**, p. 3, Cor. 1.1.4], the inverse limit of that system is non-empty. Each element ψ in that inverse limit is an abstract homomorphism of G into B. Moreover, for each $g \in G$ we choose a finite subset X of G that contains g with $\varphi(X) = A$. Thus, $X \in \mathcal{X}$ and $\alpha(\psi(g)) = \alpha(\psi|_X(g)) = \varphi_X(g)$, Hence, α is an abstract solution of embedding problem (2.2), as required. □

The combination of Lemmas 2 and 3 yields the main result of our note:

THEOREM 1. *The profinite completion $\hat{G}$ of a profinite projective group is profinite projective.*

Although the converse of Lemma 2 is not needed for the proof of Theorem 1, it is still true.

LEMMA 4. *Let G be an abstract group such that $\hat{G}$ is projective as a profinite group. Then, every finite embedding problem (1.1) for G is weakly solvable.*

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Proof. An application of [2, p. 343, Lemma 17.2.2] to φ (rather than, as in the proof of Lemma 2, to ψ) gives a homomorphism $\hat{\varphi}\colon \hat{G} \to A$ with $\hat{\varphi} \circ \iota = \varphi$. Since φ is surjective, so is $\hat{\varphi}$. The projectivity of $\hat{G}$ yields a homomorphism $\hat{\psi}\colon \hat{G} \to B$ with $\alpha \circ \hat{\psi} = \hat{\varphi}$. Hence, the map $\hat{\psi} \circ \iota\colon G \to B$ satisfies $\alpha \circ \hat{\psi} \circ \iota = \varphi$. Thus, $\hat{\psi} \circ \iota$ is a weak solution of embedding problem (1.1), as desired. □

REMARK 1. (On the local weight of a completion) By [1, Thm. 8] $\omega_0(\hat{G}) = 2^{2^{\omega_0(G)}}$ if $\omega_0(G) > \aleph_0$. In particular, by Theorem 1, if G is projective and $\omega_0(G) > \aleph_0$, then $\hat{G}$ is also projective and $\omega_0(\hat{G}) = 2^{2^{\omega_0(G)}}$.

3. The pro-$\mathcal{C}$ case. We can generalize our result to the category of pro - $\mathcal{C}$ groups, by talking about pro - $\mathcal{C}$ projective groups and the pro - $\mathcal{C}$ completion. Let $\mathcal{C}$ be a *variety* of finite groups, meaning a class of finite groups which is closed under taking quotients, subgroups, and finite direct products. We say that a profinite group is pro - $\mathcal{C}$ if it is isomorphic to an inverse limit of finite groups, all belong to $\mathcal{C}$. The pro - $\mathcal{C}$ completion of an abstract group G is the inverse limit $G_{\hat{\mathcal{C}}} = \varprojlim_{G/N \in \mathcal{C}} G/N$. A pro - $\mathcal{C}$ group is called *pro* - $\mathcal{C}$ projective, if every embedding problem for G, $(\varphi : G \to A,\ \ \alpha : B \to A)$, where A and B are pro -$\mathcal{C}$ are pro - $\mathcal{C}$ groups and all the maps are continuous, is weakly solvable. Similarly to the case of profinite projective groups, pro - $\mathcal{C}$ projective groups are characterized by the ability to weakly solve every *finite* embedding problem. One should be interested for example in pro - p projective groups, as they are precisely the free pro - p groups (see [4, Theorem 7.7.4]) By identical proves to these of Lemmas 2 and 4, we get the following Lemma:

LEMMA 5. *Let G be an abstract group. Then, $G_{\hat{\mathcal{C}}}$ is pro - $\mathcal{C}$ projective iff every abstract embedding problem for G, where $A, B \in \mathcal{C}$ is weakly solvable.*

Since $\mathcal{C}$ is subgroup closed, we get that every closed subgroup H of a pro - $\mathcal{C}$ group G, is a pro - $\mathcal{C}$ group as well, being isomorphic to $\varprojlim_{N \trianglelefteq_o G} H/H \cap N$. Now, assume in addition that $\mathcal{C}$ is extension closed. Then, by [4, Proposition 7.6.3] every closed subgroup of a pro - $\mathcal{C}$ projective group, is pro - $\mathcal{C}$ projective. Thus, by the same proof of Theorem 1, we get the following theorem:

THEOREM 2. *Let $\mathcal{C}$ be an extension closed variety of finite groups and G be pro - $\mathcal{C}$ projective group. Then, $G_{\hat{\mathcal{C}}}$ is pro - $\mathcal{C}$ projective.*

In fact, for every variety $\mathcal{C}$, the pro - $\mathcal{C}$ completion of a pro - $\mathcal{C}$ group is actually equal to its profinite completion. It follows from the following Lemma of Wilson, appeared in his paper [5]:

LEMMA 6. *Let G be a profinite group. Then any abstract finite quotient of G is isomorphic to a quotient of a subgroup of some finite continuous quotient of G.*

It is worth mentioning that actually, we can get Theorem 2 as an immediate consequence of Theorem 1, due to the following proposition:

PROPOSITION 1 [4, Proposition 7.6.7] *Let $\mathcal{C}$ be an extension closed variety. Then the following conditions are equivalent for any pro - $\mathcal{C}$ group G:*

- *G is a profinite projective group.*
- *G is a pro- $\mathcal{C}$ projective group.*

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