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ANALYTIC METHODS FOR SELECT SETS

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We analyze the asymptotic number of items chosen in a selection procedure. The procedure selects items whose rank among all previous applicants is within the best $100p$ percent of the number of previously selected items. We use analytic methods to obtain a succinct formula for the first-order asymptotic growth of the expected number of items chosen by the procedure.

1. INTRODUCTION

This study responds to Krieger, Pollak, and Samuel-Cahn [2], which analyzes a selection rule in which a number of items are sequentially observed. Some of the items are retained; the others are permanently discarded. None are revisited. The values of the first n items are random variables $X_1, X_2, \ldots, X_n$, such that the $n!$ orderings are equally likely (no ties allowed). The selection procedure only utilizes the relative rank of the random variables. The random variable of interest is L_n, the number of the first n items that are retained.

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As in [2], "'better' is equivalent to 'smaller'". Inheriting their notation, we let R_i^n be the rank of the ith item among the first n items, that is,

$$R_i^n := \sum_{j=1}^{n} I\{X_j \le X_i\} = \#\{j \mid X_j \le X_i, \text{with } 1 \le j \le n\},$$

where $I\{A\}$ is an indicator for event A. The first item is always retained, so $L_1 = 1$. For $n \ge 2$, the nth item is retained if its rank among the first n applicants is within the best $100p$ percent of L_{n-1}, that is, if $R_n^n \le \lceil pL_{n-1} \rceil$. (The value $0 < p \le 1$ is fixed.) We illustrate the first few cases

1. Since $L_1 = 1$, and $\lceil pL_1 \rceil = 1$, item 2 is retained iff $R_2^2 = 1$, that is, when $X_2 < X_1$. So $P(L_2 = 2) = P(L_2 = 1) = 1/2$.

2a. If $L_2 = 1$, we have $\lceil pL_2 \rceil = 1$, so item 3 is retained iff $R_3^3 = 1$, that is, if $X_3 < \min\{X_1, X_2\}$. So $P(L_3 = 2 \mid L_2 = 1) = 1/3$, and $P(L_3 = 1 \mid L_2 = 1) = 2/3$.

2b. If $L_2 = 2$:
   (a) For $0 < p \le 1/2$, we have $\lceil pL_3 \rceil = 1$, so item 3 is retained iff $R_3^3 = 1$, that is, if $X_3 < \min\{X_1, X_2\}$. So $P(L_3 = 3 \mid L_2 = 2) = 1/3$ and $P(L_3 = 2 \mid L_2 = 2) = 2/3$.
   (b) For $1/2 < p \le 1$, we have $\lceil pL_3 \rceil = 2$, so item 3 is retained iff R_3^3 is 1 or 2, that is, if $X_3 \not> \max\{X_1, X_2\}$. So $P(L_3 = 3 \mid L_2 = 2) = 2/3$ and $P(L_3 = 2 \mid L_2 = 2) = 1/3$.

Another way to view a recursive definition of the L_n's is given in (2) of Section 4.

2. MOTIVATION

The first main result proved by Krieger et al. [2] is that, for $0 < p \le 1$, there exists a constant $c_p > 0$ such that $E(L_n)/n^p \to c_p$ as $n \to \infty$ (Theorem 4.1 of [2]). Krieger et al. only state $c_1 = 1/2$; they do not give any other values of c_p. Furthermore, they state, on page 366 of [2], that "It seems impossible to determine c_p analytically, except for $p = 1$." In this study, however, we accomplish this task: We use analytic methods to reveal the values c_p for all p.

Krieger et al. also used the simulation to estimate the values of c_p, but several of these estimations were inaccurate; we give precise values for all c_p in this report. When p is rational, we can use symbolic algebra to evaluate the c_p.

3. MAIN RESULTS

THEOREM 1: *As $n \to \infty$, we have $E(L_n)/n^p \to c_p$, where*

$$c_p = \frac{1 + \sum_{k \ge 1} \frac{\lceil pk \rceil - pk}{\lceil pk \rceil} \prod_{j=1}^{k} \frac{1}{1 + \frac{p}{\lceil pj \rceil}}}{(p+1)\Gamma(p+1)}. \tag{1}$$

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TABLE 1. Some representative values of c_p

p	c_p
1	$\frac{1}{2}$
1/2	$\frac{2\sqrt{\pi}}{3}$
1/3	$\frac{\pi^2}{3(\Gamma(2/3))^2}$
2/3	$\frac{2^{1/3}\pi\sqrt{3}}{5\Gamma(2/3)}$
1/4	$\frac{\sqrt{2}\pi^3}{10(\Gamma(3/4))^3}$
3/4	$\frac{4\pi 3^{1/4}\sqrt{2}}{21\Gamma(3/4)}$
1/5	$\frac{16\pi^4}{375(\Gamma(4/5))^4(3-\sqrt{5})}$
2/5	$\frac{4\pi^{3/2}(\sqrt{5}+1)2^{3/5}\Gamma(7/10)}{7(\sqrt{5}-1)(5+\sqrt{5})(\Gamma(4/5))^2}$
3/5	$\frac{5\pi 3^{3/10}\Gamma(3/5)}{12\Gamma(8/15)\Gamma(2/3)}$
4/5	$\frac{5\Gamma(1/5)2^{2/5}}{36}$
1/6	$\frac{4\pi^5}{189(\Gamma(5/6))^5}$
5/6	$\frac{12\pi 5^{1/6}}{55\Gamma(5/6)}$

When $p \in \mathbb{Q}$, for example, $p = r/s$, then c_p has a form we can symbolically evaluate:

$$c_p = \frac{1 + \sum_{\ell \geq 0} \left(\prod_{\sigma=1}^{\ell} \mu_{r,s}(\sigma) \right) \left(\sum_{b=1}^{s-1} \nu_{r,s}(\ell, b) \right)}{(p+1)\Gamma(p+1)},$$

where $\mu_{r,s}(\sigma) = \prod_{j=1}^{s} \frac{1}{1+\frac{p}{(\sigma-1)r+\lceil pj \rceil}}$ *and* $\nu_{r,s}(\ell, b) = \frac{\lceil pb \rceil - pb}{r\ell + \lceil pb \rceil} \prod_{i=1}^{b} \frac{1}{1+\frac{p}{r\ell+\lceil pi \rceil}}$.

This theorem yields succinct values of c_p. To demonstrate the intimate relation to the Gamma function, we list several c_p's in Table 1.

In Table 2, we improve upon the values from Table 1 of Krieger et al. [2]. (Their values cover the case $n = 10{,}000$, and our values correspond to the asymptotic case,

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Table 2. Values of c_p (compare with Table 1 of [2], which lists values for $n = 10{,}000$)

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
c_p	5.803	2.961	2.193	1.671	1.182	1.202	1.048	0.841	0.693	0.500

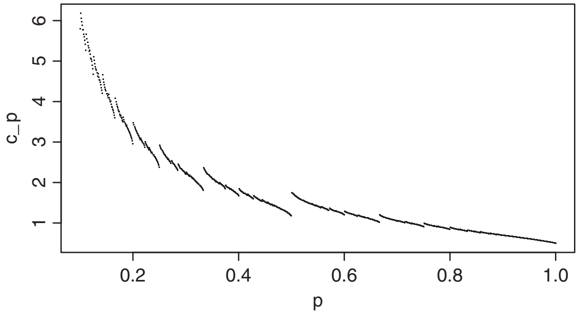

Figure 1. Values of c_p for $p = j/1,000$, where $100 \le j \le 1,000$.

that is, as $n \to \infty$.) In Figure 1 we graph c_p. [We conjecture c_p is continuous at each irrational p but only left-continuous (not right-continuous) at each rational p.]

4. LEMMAS AND PROOFS

The L_n's are defined recursively, as in Lemma 2.1(i) of [2]:

$$L_1 = 1 \quad \text{and} \quad L_{n+1} = \begin{cases} L_n + 1 & \text{with probability } \lceil pL_n \rceil/(n+1), \\ L_n & \text{otherwise.} \end{cases} \tag{2}$$

In particular, L_n is an integer-valued random variable with mass on $[1, n]$.

For succinctness, we use the notation

$$P_{n,k} := P(L_n = k).$$

We use generating functions as a key tool in the proofs. Thus, we define

$$g(z) = \sum_{n\ge1} E(L_n) z^n \quad \text{and} \quad f(z) = \sum_{n\ge1} E(\lceil pL_n \rceil - pL_n) z^n.$$

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The fundamental recurrence is that $L_1 = 1$ and, for $n > 1$,

$$P_{n+1,k+1} = \frac{\lceil pk \rceil}{n+1} P_{n,k} + \left(1 - \frac{\lceil p(k+1) \rceil}{n+1}\right) P_{n,k+1}. \quad \textbf{(3)}$$

LEMMA 2: *For each $n \geq 1$,*

$$E(L_{n+1}) - E(L_n) = \frac{pE(L_n) + E(\lceil pL_n \rceil - pL_n)}{n+1}.$$

PROOF OF LEMMA 2: The lemma basically follows from the fundamental recurrence given in (3). The recurrence gives

$$E(L_{n+1}) - E(L_n) = \sum_k (kP_{n+1,k} - kP_{n,k}) = \frac{\sum k \left(\lceil p(k-1) \rceil P_{n,k-1} - \lceil pk \rceil P_{n,k}\right)}{n+1}.$$

We can shift the values of k by 1 in the first part, to obtain

$$E(L_{n+1}) - E(L_n) = \frac{\sum_k \left((k+1)\lceil pk \rceil P_{n,k} - k\lceil pk \rceil P_{n,k}\right)}{n+1} = \frac{\sum_k \lceil pk \rceil P_{n,k}}{n+1}.$$

The numerator is $E(\lceil pL_n \rceil)$, so the lemma follows. ■

We turn Lemma 2 into a differential equation, using generating functions. Multiplying by z^{n+1}, summing over $n \geq 1$, and differentiating yields

$$(1-z)g'(z) - 1 = (p+1)g(z) + f(z).$$

Noting that $g(0) = 0$, this differential equation has solution

$$g(z) = \frac{\int_0^z (1+f(t))(1-t)^p\, dt}{(1-z)^{p+1}}.$$

We handle $g(z)$ with analytic methods, that is, with $z \in \mathbb{C}$, as espoused in [1,3]. Since $f(t)$ has real-valued coefficients between 0 and 1, then $\int_0^z (1+f(t))(1-t)^p dt$ does not have singularities that are strictly inside the unit circle in $\mathbb{C}$. Also, $\int_0^1 (1+f(t))(1-t)^p\, dt$ is a constant (to be determined below). Thus, the singularity of $g(z)$ at $z = 1$ is a pole of order $p+1$; any other singularity located directly on the boundary of the unit circle could only be a pole of order 1 or less. Thus, the singularity at $z = 1$ completely determines the first-order asymptotic growth of the coefficients in the Maclaurin representation of $g(z)$. This is the result of Krieger et al., namely

$$E(L_n)/n^p \sim c_p,$$

but we have the additional fact that

$$c_p = \frac{\int_0^1 (1 + f(t))(1-t)^p \, dt}{\Gamma(p+1)}.$$

Of course $\int_0^1 (1-t)^p \, dt = \frac{1}{p+1}$, so $c_p = \frac{1}{(p+1)\Gamma(p+1)} + \frac{\int_0^1 f(t)\,(1-t)^p \, dt}{\Gamma(p+1)}$. The Maclaurin series of $(1-t)^p$ is $(1-t)^p = \sum_{n\geq 0} \frac{\Gamma(n-p)}{\Gamma(-p)\Gamma(n+1)} t^n$, and thus

$$f(t)(1-t)^p = \sum_{k\geq 1} (\lceil pk \rceil - pk) \sum_{m\geq 1} P_{m,k} \sum_{n\geq 0} \frac{\Gamma(n-p)}{\Gamma(-p)\Gamma(n+1)} t^{n+m}.$$

Next we evaluate the corresponding definite integral

$$\int_0^1 f(t)(1-t)^p \, dt = \sum_{k\geq 1} (\lceil pk \rceil - pk) \sum_{m\geq 1} P_{m,k} \sum_{n\geq 0} \frac{\Gamma(n-p)}{\Gamma(-p)\Gamma(n+1)} \frac{1}{n+m+1}.$$

To simplify, we note

$$\sum_{n\geq 0} \frac{\Gamma(n-p)}{\Gamma(-p)\Gamma(n+1)} \frac{1}{n+m+1} = \frac{m!\Gamma(p+1)}{\Gamma(m+p+2)},$$

and thus

$$c_p = \frac{1}{(p+1)\Gamma(p+1)} + \sum_{k\geq 1} (\lceil pk \rceil - pk) \sum_{m\geq 1} \frac{m! P_{m,k}}{\Gamma(m+p+2)}. \tag{4}$$

LEMMA 3: *For* $k > 1$,

$$P_{m,k} = \lceil p(k-1) \rceil \sum_{n<m} \frac{P_{n,k-1}}{n+1} \prod_{\ell=n+2}^{m} \left(1 - \frac{\lceil pk \rceil}{\ell}\right).$$

PROOF OF LEMMA 3: If $L_m = k$, there must be a *largest* value $n < m$ such that $L_n = k-1$. Since n is the largest such value, $L_\ell = k$ for $n < \ell \leq m$. Thus

$$\begin{aligned} P_{m,k} &= P(L_m = k) \\ &= \sum_{n<m} P(L_n = k-1) P(L_{n+1} = L_{n+2} = \cdots = L_m = k \mid L_n = k-1) \\ &= \sum_{n<m} P_{n,k-1} \frac{\lceil p(k-1) \rceil}{n+1} \prod_{\ell=n+2}^{m} \left(1 - \frac{\lceil pk \rceil}{\ell}\right). \end{aligned}$$

Factoring out $\lceil p(k-1) \rceil$ completes the proof of the lemma. ∎

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COROLLARY 4: *For $k > 1$, we have*

$$\sum_{m\geq 1} \frac{m!P_{m,k}}{\Gamma(m+p+2)} = \frac{\lceil p(k-1)\rceil}{\lceil pk\rceil + p} \sum_{n\geq 1} \frac{n!P_{n,k-1}}{\Gamma(n+p+2)}.$$

PROOF OF COROLLARY 4: By Lemma 3,

$$\begin{aligned}
\sum_{m\geq 1} \frac{m!P_{m,k}}{\Gamma(m+p+2)} &= \sum_{m\geq 1} \frac{m!\lceil p(k-1)\rceil \sum_{n<m} \frac{P_{n,k-1}}{n+1} \prod_{\ell=n+2}^{m} \left(1 - \frac{\lceil pk\rceil}{\ell}\right)}{\Gamma(m+p+2)} \\
&= \lceil p(k-1)\rceil \sum_{n\geq 1} \frac{P_{n,k-1}}{n+1} \sum_{m>n} \frac{m!\prod_{\ell=n+2}^{m} \left(1 - \frac{\lceil pk\rceil}{\ell}\right)}{\Gamma(m+p+2)} \\
&= \lceil p(k-1)\rceil \sum_{n\geq 1} \frac{P_{n,k-1}}{n+1} \frac{(n+1)!}{(\lceil pk\rceil + p)\Gamma(n+p+2)}.
\end{aligned}$$

This completes the proof of the corollary. ■

Applying Corollary 4, a total of $k-1$ times to (4) yields

$$c_p = \frac{1}{(p+1)\Gamma(p+1)} + \sum_{k\geq 1} (\lceil pk\rceil - pk) \left(\prod_{j=2}^{k} \frac{\lceil p(j-1)\rceil}{\lceil pj\rceil + p} \right) \sum_{m\geq 1} \frac{m!P_{m,1}}{\Gamma(m+p+2)}. \quad \textbf{(5)}$$

Simplifying, we have

$$\prod_{j=2}^{k} \frac{\lceil p(j-1)\rceil}{\lceil pj\rceil + p} = \frac{1+p}{\lceil pk\rceil} \prod_{j=1}^{k} \frac{\lceil pj\rceil}{\lceil pj\rceil + p} = \frac{1+p}{\lceil pk\rceil} \prod_{j=1}^{k} \frac{1}{1 + \frac{p}{\lceil pj\rceil}}. \quad \textbf{(6)}$$

Also $L_m = 1$ iff the 2nd, 3rd, … , mth items are not retained, so

$$P_{m,1} = \prod_{j=2}^{m} \left(1 - \frac{\lceil p\rceil}{j}\right) = \prod_{j=2}^{m} \left(1 - \frac{1}{j}\right) = 1/m.$$

So

$$\sum_{m\geq 1} \frac{m!P_{m,1}}{\Gamma(m+p+2)} = \sum_{m\geq 1} \frac{(m-1)!}{\Gamma(m+p+2)} = \frac{1}{(p+1)^2\Gamma(p+1)}. \quad \textbf{(7)}$$

Substituting (6) and (7) into (5) gives (1), the main equation of the theorem. Finally, in the rational case, $p = r/s$, so we simplify (1) by grouping the numerator's terms

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according to the value of k mod s. Writing $k = \ell s + b$ yields

$$\lceil pk \rceil - pk = \lceil p(\ell s + b) \rceil - p(\ell s + b) = r\ell + \lceil pb \rceil - r\ell - pb = \lceil pb \rceil - pb.$$

So

$$\sum_{k \geq 1} \frac{\lceil pk \rceil - pk}{\lceil pk \rceil} \prod_{j=1}^{k} \frac{1}{1 + \frac{p}{\lceil pj \rceil}} = \sum_{b=1}^{s-1} \sum_{\ell \geq 0} \frac{\lceil pb \rceil - pb}{r\ell + \lceil pb \rceil} \prod_{j=1}^{\ell s + b} \frac{1}{1 + \frac{p}{\lceil pj \rceil}}$$

and

$$\prod_{j=1}^{\ell s + b} \frac{1}{1 + \frac{p}{\lceil pj \rceil}} = \left(\prod_{j=1}^{\ell s} \frac{1}{1 + \frac{p}{\lceil pj \rceil}} \right) \left(\prod_{i=\ell s+1}^{\ell s + b} \frac{1}{1 + \frac{p}{\lceil pi \rceil}} \right)$$

$$= \left(\prod_{\sigma=1}^{\ell} \prod_{j=1}^{s} \frac{1}{1 + \frac{p}{(\sigma - 1)r + \lceil pj \rceil}} \right) \left(\prod_{i=1}^{b} \frac{1}{1 + \frac{p}{r\ell + \lceil pi \rceil}} \right).$$

Defining $\mu_{r,s}(\sigma)$ and $\nu_{r,s}(\ell, b)$ as in the theorem statement, and substituting, yields Theorem 1.

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