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LOCAL CONNECTEDNESS OF THE STONE-ČECH COMPACTIFICATION

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ABSTRACT. A uniform space X is said to be uniformly locally connected if given any entourage U there exists an entourage $V \subset U$ such that $V[x]$ is connected for each $x \in X$. It is said to have property S if given any entourage U, X can be written as a finite union of connected sets each of which is U-small.

Based on these two uniform connection properties, another proof is given of the following well known result in the theory of locally connected spaces: The Stone-Čech compactification βX is locally connected if and only if X is locally connected and pseudocompact.

1. **Introduction.** It is the purpose of this paper to give yet another proof of the following result: The Stone-Čech compactification βX of a Tychonoff space X is locally connected if and only if X is locally connected and pseudocompact. The necessity part of the above result was first shown by Banaschewski [2] in his theory of local connectedness of extension spaces, where the concepts of 'trace' filters and 'connected' filters play an important role. Later, Henriksen and Isbell [7] showed that Banaschewski's necessary conditions were also sufficient, using techniques from the theory of uniform spaces. Then Wulbert [9] presented a normed linear lattice characterization of $C^*(X)$ for spaces X having locally connected Stone-Čech compactifications, from which the above result was deduced.

We present yet another proof based on the concepts of uniform local connectedness and property S, two uniform connection properties introduced into the theory of uniform spaces by Gleason [6] and Collins [3] respectively, and which are well known in the setting of metric spaces (see e.g. [10]). The methods developed here also have application to the study of perfect locally connected compactifications in general as we shall show in a later paper.

All topological spaces considered are completely regular Hausdorff and all uniform spaces are Hausdorff. $(X, \mathscr{U})$ shall denote a uniform space with $\mathscr{U}$ the family of entourages of X.

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2. **Preliminaries.** Here we present the main definitions, list known results and obtain some new ones which we shall need in Section 3.

DEFINITION 2.1. (a) ([3]) *A uniform space* $(X, \mathscr{U})$ *is said to have property* S *if given* $U \in \mathscr{U}$, $\exists$ *a finite collection of connected* U*-small subsets of* X *covering* X

(b) ([6]) *A uniform space* $(X, \mathscr{U})$ *is said to be uniformly locally connected if given* $\mathscr{U} \in \mathscr{U}$, $\exists V \in \mathscr{U}$, $V \subset U$ *such that* $V[x]$ *is connected for each* $x \in X$.

THEOREM 2.2. ([3]) (a) *Every totally bounded uniformly locally connected space* $(X, \mathscr{U})$ *has property* S

(b) *Each of the above uniform connection properties imply local connectedness of the underlying space.*

(c) *For a compact Hausdorff space* X, *the following are equivalent*

(i) X *is uniformly locally connected*

(ii) X *has property* S

(iii) X *is locally connected.*

In [6] A. M. Gleason showed that the category of uniformly locally connected spaces and uniformly continuous maps is coreflective in the category of uniform spaces and uniformly continuous maps. The construction of this coreflection is described in the following theorem.

THEOREM 2.3. ([6]) *Let* $(X, \mathscr{U})$ *be a uniform space and let* $\mathscr{T}$ *be the topology of* X. *Let* $(X, \mathscr{T}^*)$ *be the locally connected coreflection of* $(X, \mathscr{T})$. *For each* $U \in \mathscr{U}$, *let*

$V_U = \{ (x, y) \in X \times X |$ *there exists a* $\mathscr{T}^*$*-connected subset* K *of* X *containing both* x *and* y *such that* $K \times K \subset U\}$.

Then $\{V_U | U \in \mathscr{U}\}$ *is a basis for a uniformity* $\mathscr{V}$ *on* X, *and* $(X, \mathscr{V})$ *is the uniformly locally connected coreflection of* $(X, \mathscr{U})$ *with the associated topology* $\mathscr{T}^*$.

REMARK 2.4. (a) Note that $\mathscr{V} \supset \mathscr{U}$.

(b) Although not stated by Gleason it is easy to prove that $(X, \mathscr{U})$ and $(X, \mathscr{V})$ have the same topology if and only if $(X, \mathscr{U})$ is locally connected.

(c) If $(X, \mathscr{U})$ is locally connected, then $(X, \mathscr{V})$ has property S if and only if $(X, \mathscr{U})$ has property S. This follows easily from the definitions, or else see [1] Theorem 2.1.

We shall need the following lemma due to Henriksen and Isbell [7] in Section 3.

LEMMA 2.5. *An open subset* U *of* βX *is connected if and only if* $U \cap X$ *is connected.*

We also need the following results on property S.

PROPOSITION 2.6. (a) *A uniform space* $(X, \mathscr{U})$ *has property* S *if and only if for* *each $U \in \mathscr{U}$, X can be written as a finite union of open connected sets each of which is U-small.*

(b) *Any uniformly continuous image of a space having property S also has property S*

(c) *If A is a dense subspace of a uniform space $(X, \mathscr{U})$ and A has property S (in the relative uniformity) then $(X, \mathscr{U})$ has property S.*

PROOF. (b) and (c) follow easily from the definitions and we omit the proof. To prove (a) let $U \in \mathscr{U}$ and find open symmetric $V \in \mathscr{U}$ such that $V \circ V \subset U$. Then by property S, find connected A_i, $i = 1, 2, \ldots, n$ such that $X = \cup_{i=1}^{n} A_i$ and $A_i \times A_i \subset V$ for each i. For each i, choose $x_i \in A_i$ and let C_i be the component of x_i in $V[x_i]$. Since X is locally connected (Theorem 2.2 (b)) and $V[x_i]$ is open, C_i must be open. Also, $A_i \subset C_i$ for each i which implies $X = \cup_{i=1}^{n} C_i$. Furthermore

$$C_i \times C_i \subset V[x_i] \times V[x_i] \subset V \circ V \subset U \text{ completing the proof.}$$

Finally we state the following result of Doss [4] which we require in Section 3.

THEOREM 2.7. *X is totally bounded with respect to each of its uniformities compatible with its topology if and only if X is pseudocompact.*

3. **The main results.** If $\mathscr{U}_F$ is the fine uniformity on a locally connected space X, then by Theorem 2.3 and Remark 2.4 (a) and (b), its coreflection $(X, \mathscr{V})$ must be $(X, \mathscr{U}_F)$. Thus we have

THEOREM 3.1. *The following conditions are equivalent for a space X.*

(a) *X is locally connected*

(b) *X is uniformly locally connected with respect to its fine uniformity.*

For spaces which are locally connected and pseudocompact we have the following criterion

THEOREM 3.2. *The following conditions are equivalent for a space X.*

(a) *X has property S with respect to each uniformity compatible with its topology*

(b) *X has property S with respect to its fine uniformity*

(c) *X is locally connected and pseudocompact.*

PROOF. The equivalence of (a) and (b) is immediate. (a) $\Rightarrow$ (c) follows from Theorem 2.2 (b) and Theorem 2.7. Finally to show (c) $\Rightarrow$ (a) note that by Theorem 3.1, $(X, \mathscr{U}_F)$ is uniformly locally connected. Since by Theorem 2.7, $(X, \mathscr{U}_F)$ is totally bounded as well, we have by Theorem 2.2 (a) that $(X, \mathscr{U}_F)$ has property S.

The following result which follows immediately from Theorem 2.7, Theorem

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3.1, and Theorem 3.2 strengthens Theorem 2.2 (c) since, of course, every compact Hausdorff space admits a unique compatible uniformity.

THEOREM 3.3. *If X is a space having a unique compatible uniformity then the following are equivalent.*

(a) *X is locally connected*

(b) *X has property S*

(c) *X is uniformly locally connected.*

Our methods give an easy deduction of

THEOREM 3.4 [7]. *Let X be locally connected and pseudocompact. Then any space Y containing X as a dense subspace is locally connected. In particular βX is locally connected.*

PROOF. Let $\mathscr{U}$ be any compatible uniformity on Y, and $\mathscr{U}_X$ the relative uniformity on X. By Theorem 3.2, $(X, \mathscr{U}_X)$ has property S. By Proposition 2.6 (c), $(Y, \mathscr{U})$ has property S, and thus by Theorem 2.2 (b) Y must be locally connected.

We now prove that if βX is locally connected, then X is locally connected and pseudocompact. To this end recall that βX may be described as the completion of X with respect to the uniformity on X having as a subbase the collection

$$\mathscr{S} = \{U_{f,\epsilon} | f \in C^*(X), \epsilon > 0\}$$

where $U_{f,\epsilon} = \{(x, y) \in X \times X \mid |f(x) - f(y)| < \epsilon\}$ and $C^*(X)$ is the set of bounded real-valued continuous functions on X, (see [5], pp. 225-226). As βX has a unique compatible uniform structure each element of $\mathscr{S}$ must belong to the relative uniformity on X induced by the unique uniformity on βX.

We then have

LEMMA 3.5. *βR cannot have property S.*

PROOF. For otherwise using Lemma 2.5 and Proposition 2.6 (a) R would have property S with respect to the relative uniformity on R induced by βR. Now consider any $f \in C^*(R)$ which takes the value 1 at all the odd integers and the value 0 at all the even integers. Then $U_{f,1/2}$ belongs to the relative uniformity on R, and as R has property S, R can be written as a finite union of connected sets A_i, $i = 1, \ldots, n$, such that $A_i \times A_i \subset U_{f,1/2}$ for each i. Now at least one A_i, say A_1, is an unbounded interval. Thus there exists an odd integer s and an even integer t in A_1. But then $|f(s) - f(t)| = 1 > 1/2$, contradicting $A_i \times A_i \subset U_{f,1/2}$. Thus βR cannot have property S.

THEOREM 3.6 [2]. *If βX is locally connected, then X is locally connected and pseudocompact.*

PROOF. That X is locally connected is an immediate consequence of Lemma

2.5. Suppose now that X is not pseudocompact. Then X contains a C-embedded copy of N, $S = \{x_n\}_{n=1}^{\infty}$, say, ([5], p. 20). Let $Q = \{q_n\}_{n=1}^{\infty}$ be the rationals, and let $f:S \to Q$ be defined by $f(x_n) = q_n$. Then f is continuous as S is discrete. Since S is C-embedded in X, f has a continuous extension $f':X \to R$. Let $\bar{f}':\beta X \to \beta R$ be the Stone extension of f'. Now $\bar{f}'$ is clearly uniformly continuous. Since βX is locally connected, βX must have property S by Theorem 2.2 (c). By Proposition 2.6 (b), $\bar{f}'(\beta X)$ must have property S. Now $Q \subset \bar{f}'(\beta X) \subset \beta R$, and as Q is dense in βR and $\bar{f}'(\beta X)$ is compact we must have $\bar{f}'(\beta X) = \beta R$. This shows βR has property S contradicting the above lemma. Hence X is pseudocompact.

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