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Conversion of magnetic energy and the width of current sheets

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Abstract. Magnetic reconnection is one of the most efficient ways of transforming magnetic into kinetic and thermal energies. We prove a general identity relating the energy transfer in a neighborhood of a current sheet, where reconnection is assumed to occur. With some reasonable hypotheses regarding the geometry of stream and field lines, we prove that for a constant rate of transformation of magnetic energy, the width of the current sheet must grow with the plasma conductivity. Hence an enhanced diffusivity seems necessary for certain classical models of fast reconnection to work.

1. Introduction

Some spectacular astrophysical phenomena, such as solar flares and geomagnetic substorms, occur because of the sudden conversion of magnetic energy due to reconnection of magnetic field lines. While this is a practically undisputed fact, the details of how reconnection takes place are an active area of research. One of the main problems is to identify configurations that permit the fast rate of conversion of magnetic in kinetic and thermal energy observed in real phenomena. The most obvious way to release magnetic tension is to connect field lines going in opposite senses, which naturally leads to a large gradient in the tangential component of the field and therefore to a current sheet. The presence of a positive resistivity precludes a true discontinuity of the field (and makes reconnection possible), but nevertheless the gradient of the field is sharp enough to call the set with vanishing field a current sheet. This configuration was the first to be studied and still is the usual object of modelling. The Sweet–Parker model [1–3] of two masses of plasma colliding head-on, while possessing a slow reconnection rate, did present some of the main features of later studies: plasma flow approaching a sheet and escaping laterally, with a magnetic field tangent to the sheet and of opposite senses at both sides of it. Petschek [4] presented a variant where the reconnection region is narrow and slow magnetosonic shocks occur, obtaining a faster reconnection rate. This model was later extended and made rigorous [5–9]. However, some authors [10–12] argued that Petschek-like geometries cannot hold in the highly relevant limit of small resistivity, since the current sheet will rapidly grow in size; others disagree [13], but most numerical models seem to present this feature. It is not clear, however, how dependent these results are on the choice of boundary conditions. It is also likely that several physical processes may enhance the diffusivity near the sheet: in this case, the presence of fast reconnection is very convincing both in modelling and observation [14]. Still, the problem of finding a geometry where the size of the current sheet does not increase with (uniform) conductivity is still an open one. This is not the same as saying that the sheet width must be constant in time for a certain resistivity, which is not likely to happen [14]. Also, the models described before are naturally as simplified as possible: they are mostly two-dimensional and quasistatic. We will instead study a general configuration, obtaining rigorously an integral identity relating the main parameters of the induction equation. The study of its terms will tell us whose characteristics govern the sheet size. Boundary conditions at the domain boundary will not play any role, since we will localize the main identity into a neighbourhood of the current sheet.

2. General estimates

Assume that the magnetic field $\mathbf{B}$ in a plasma of velocity $\mathbf{u}$ and uniform resistivity η in an N-dimensional domain $\boldsymbol{\Omega}$ satisfies the MHD induction equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} - \eta\nabla^2\right)\mathbf{B} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{B}\boldsymbol{\nabla}\cdot\mathbf{u}. \tag{1}$$

Let Φ be any real-valued smooth function of N variables, and f a real smooth function defined in $\boldsymbol{\Omega}$. Define $F = f(\Phi\circ\mathbf{B})$. After some tedious but essentially straightforward algebra [15], it is found that F satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} - \eta\nabla^2\right)F \\ = f(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot(\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u}) - f(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{u}) + (\Phi\circ\mathbf{B})(\mathbf{u}\cdot\boldsymbol{\nabla}f) \\ -\eta(\Phi\circ\mathbf{B})\nabla^2 f - 2\eta\boldsymbol{\nabla}f\cdot[(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot\boldsymbol{\nabla}\mathbf{B}] - \eta f\boldsymbol{\nabla}\mathbf{B}\cdot(\Phi''\circ\mathbf{B})\cdot\boldsymbol{\nabla}\mathbf{B},\end{aligned} \tag{2}$$

where the last term means $-\eta f\sum_{i,j,k}(\partial^2_{j,k}\Phi\circ\mathbf{B})\partial_i B_j\partial_i B_k$. Since f and F have compact support contained in $\boldsymbol{\Omega}$, the integral of the left-hand term is

$$\frac{\partial}{\partial t}\int_\Omega F\,dV,$$

whereas

$$\int_\Omega(\Phi\circ\mathbf{B})\nabla^2 f\,dV = -\int_\Omega\boldsymbol{\nabla}f\cdot[(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot\boldsymbol{\nabla}\mathbf{B}]\,dV.$$

Also,

$$\begin{aligned}\int_\Omega f(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot(\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u})\,dV \\ = -\int_\Omega(\mathbf{B}\cdot\boldsymbol{\nabla}f)\cdot[(\boldsymbol{\nabla}\Phi\circ\mathbf{B})\cdot\mathbf{u}] + f(\mathbf{B}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\Phi\circ\mathbf{B}))\cdot\mathbf{u}\,dV.\end{aligned}$$

Let us choose as Φ a smooth function g of the field magnitude B. Notice that $\mathbf{B}\to B$ is not differentiable at 0, but, by taking g with $g(0) = g'(0) = g''(0) = 0$, Φ is of class $\mathscr{C}^2$, as needed. After some algebra, one gets

$$\boldsymbol{\nabla}\Phi\circ\mathbf{B} = g'(B)\mathbf{b}, \tag{3a}$$

$$\boldsymbol{\nabla}\mathbf{B}\cdot(\Phi''\circ\mathbf{B})\cdot\boldsymbol{\nabla}\mathbf{B} = g''(B)|\boldsymbol{\nabla}B|^2 + g'(B)B|\boldsymbol{\nabla}\mathbf{b}|^2, \tag{3b}$$

where $\mathbf{b}$ is the unit magnetic field vector $\mathbf{B}/B$; notice that wherever $B = 0$ and $\mathbf{b}$ is not defined, it is multiplied by $g'(0) = 0$, and although $\boldsymbol{\nabla}B$ does not exist there either, it is multiplied by $g''(0) = 0$, so that the expressions in (3) are admissible everywhere.

A well-known theorem [16] asserts that the level surfaces (or curves, if $N = 2$) $S_r : B = r$ are smooth for almost every r and

$$\int_\Omega f g''(B)|\nabla B|^2\, dV = \int_0^\infty g''(r)\, dr \int_{S_r} f|\nabla B|\, d\sigma, \tag{4}$$

where $d\sigma$ stands for the area measure for $N = 3$ and the arclength for $N = 2$. Assume that the set S_0 is a true surface ($N = 3$) or curve ($N = 2$), so that its N-dimensional measure is 0, and that the function

$$r \to \int_{S_r} f|\nabla B|\, d\sigma$$

tends to

$$2\int_{S_0} f|\nabla B|\, d\sigma.$$

This means that since S_r enfolds the surface (curve) S_0, as it tends to S_0 the integral should tend to twice the integral at S_0 (one for every side of it).

Let g now approach the function $g(r) = 0$ for $r \leqslant 0$, $g(r) = r$ for $r > 0$. Then g' tends in the sense of distributions towards the Heaviside function, whereas g'' tends to the Dirac measure δ_0. Therefore $\nabla\Phi \circ \mathbf{B}$ tends weakly to $\mathbf{b}$ outside the measure-zero set S_0. In particular, the term $f(\mathbf{B} \cdot \nabla(\nabla\Phi \circ \mathbf{B})) \cdot \mathbf{u}$ tends to $f(\mathbf{B} \cdot \nabla\mathbf{b}) \cdot \mathbf{u} = fB\mathbf{k} \cdot \mathbf{u}$, where $\mathbf{k}$ is the normal vector to the magnetic field lines; $|\mathbf{k}| = \kappa$ is their curvature. Finally, let f approach the characteristic function of a neighbourhood W of the current sheet S_0. Then the terms where ∇f occurs tend to become boundary integrals at the (smooth) boundary ∂W of the neighbourhood, and ∇f tends to the inner normal $\mathbf{n}_i$. In the limit, therefore, integrating the identity (2) in the whole domain, we get

$$\begin{aligned} &2\eta\int_{S_0} |\nabla B|\, d\sigma + \eta\int_W B|\nabla\mathbf{b}|^2\, dV \\ &\quad = -\frac{\partial}{\partial t}\int_W B\, dV + \int_{\partial W}\left(\frac{1}{B}\mathbf{S} - \eta\nabla B\right)\cdot \mathbf{n}_i\, d\sigma - \int_W B(\mathbf{k}\cdot\mathbf{u} + \nabla\cdot\mathbf{u})\, dV, \end{aligned} \tag{5}$$

where $\mathbf{S} = B^2\mathbf{u} - (\mathbf{B}\cdot\mathbf{u})\mathbf{B} = -(\mathbf{u}\times\mathbf{B})\times\mathbf{B}$ denotes the (ideal) Poynting vector, i.e. the flux of electromagnetic energy due to the plasma motion.

3. Analysis of the main equation (5)

Now take W as a neighbourhood of the current sheet where transformation of electromagnetic into kinetic and thermal energy occurs. Although B does not represent exactly the density of magnetic energy, its integral is a certain measure of it, so if magnetic energy decreases through reconnection we may expect $(\partial/\partial t)\int_W B\, dV \leqslant 0$; of course in static configurations, the term vanishes. The second, and most important, term on the right-hand side of (5) represents the flux of magnetic energy from the surrounding plasma into the vicinity of the current sheet due to the inductive effect of the plasma velocity. There is another component of the full Poynting vector due to diffusion, but it is multiplied by the small constant η and therefore is not significant. Since W is near S_0, where B vanishes, the size of B inside W should

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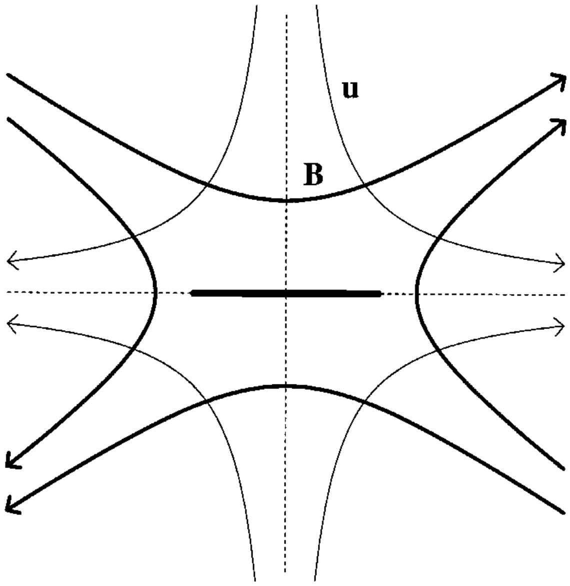

Figure 1. Classical configuration of magnetic field and velocity near a current sheet.

be smaller than outside, so that ∇B should point outwards at ∂W and $\nabla B \cdot \mathbf{n}_i \leqslant 0$. This term, however, is also multiplied by η, and again its value is not likely to be relevant.

As for the remaining term on the right-hand side of (5), let us accept the view that plasma flows towards both sides of S_0 and escapes through the sides. This configuration should, if anything, compress the plasma as it approaches S_0 (i.e. in most of the volume), so that $\nabla \cdot \mathbf{u} \leqslant 0$ there. Hence the integral of $B\nabla \cdot \mathbf{u}$ should be negative. Of course, if the plasma is incompressible, it escapes more rapidly through the sides and the term vanishes. The term $B\mathbf{k}\cdot\mathbf{u}$ is negative if the streamlines cut the field lines towards the concave side. Since field lines are transported by the plasma (and experience diffusion), these lines tend to be in the front sets of the plasma jet approaching the current sheet, which is concave towards it. This only fails for field lines ejected from the sheet, which form a smaller portion of the volume. This certainly happens in all classical plane configurations [11, 14] (Fig. 1).

We may therefore conclude that the right-hand side is the flux of magnetic energy due to convection plus a positive term. The value of the geometric third summand is probably not too high – at least for incompressible plasmas, since most models predict field lines not highly curved. Apparently its most significant term is the input of electromagnetic energy to be converted near the current sheet.

Let us now consider the left-hand side of (5). The size of $B|\nabla \mathbf{b}|^2$ is larger if field lines are more highly curved: the factor in B cannot add much if we assume that $\int_W B\,dV$ does not increase. Certainly this is a possibility when the resistivity decreases (E. R. Priest, personal communication, 2001). However, it is unlikely that this effect may account for the whole factor of order $1/\eta$ needed to balance the right-hand term when η decreases. Such high curvature does not seem to occur in simulations, except in small regions of W. We are led to conclude that $\int_{S_0} |\nabla B|\,dV$ is mostly responsible for the growth of order $1/\eta$. This could happen because S_0 keeps its size while $|\nabla B|$ grows in this order. Even without assuming a growth of B (which we have rejected), this could happen in principle because smaller diffusivity allows for larger gradients of the plasma field: essentially this depends on the thickness of the diffusion region around S_0. However, one could expect a decrease of this thickness of order at most $1/\sqrt{\eta}$; even this moderate growth does not seem to occur, according to simulations [11]. The most likely factor to account for the increase is the measure of S_0, which, provided that the rate of energy input within W does not decrease, should be at least of order $1/\eta$.

4. Conclusions

An integral identity relating the main magnetohydrodynamic parameters in the vicinity of a current sheet has been obtained. Essentially it balances the measure of the sheet plus the mean square curvature of field lines with a term involving the rate of transformation of magnetic energy, the compression of plasma near the sheet, and the geometric shape of the incoming plasma flow. Provided that this term does not decrease with a larger plasma conductivity, either field lines become highly curved or more likely the current sheet increases its size at least with the order of the conductivity. This coincides with the result of most simulations performed in simplified cases, and suggests that an enhanced diffusivity near the sheet is necessary for an effective reconnection to occur in near-ideal plasmas.

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